Introduction to Noncommutative geometry

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Abstract

The objective of non-commutative geometry(NCG) is to find the link between the spectrum of operator algebra and the geometrical space. The motivation of the study came from Gelfand-Naimark Theorem which will be the first topic of this talk. Then, I will give the definition of Spectral Triple and I will demonstrate(for commutative case) how this triple characterised the geometry. After that, I will give the example of non-commutative geometry and then say a few words about the (Fredholm) index of this spectral triple.

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1 Motivation

The motivation of non commutative geometry (NCG) came from the study of spectral theory, in particular, the spectrum arises from C*-algebra.

1.1 Gelfand-Naimark Theorem

Definition 1.1.1. *Let $\mathcal{A}$ be an associative Banach algebra over, $\mathbb{C}$(an algebra equips with norm and complete with respect to such norm). $\mathcal{A}$ is called C*-algebra if*

(i). there exists the involution map, $*: \mathcal{A} \to \mathcal{A}$ such that for any $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$

- $(a^*)^* = a$
- $(a.b)^* = b^*.a^*$
- $(\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*$

(ii). for any $a \in \mathcal{A}, ||a.a^*|| = ||a||^2$.

If a C*-algebra is nonunital, one can add a unit element by considering $\mathcal{A}^+ := \mathcal{A} \times \mathbb{C}$ with the multiplication rule $(a, \lambda)(b, \mu) := (ab + \lambda b + \mu a, \lambda\mu)$. This abstractly define algebra can be identified with a concrete operator algebra $B(\mathcal{H})$ through GNS construction.

Theorem 1.1.2. *(Gelfand-Naimark-Segal Construction) Any C*-algebra is isometrically isomorphic to a norm closed, adjoint closed subalgebra of bounded linear operator on Hilbert space, $B(\mathcal{H})$*

The aim of this section is to provide the proof of the following theorem.

Theorem 1.1.3. *(Gelfand-Naimark)*
Let $\mathcal{A}$ be the commutative C-algebra, then there exists a locally compact Hausdorff space X such that $\mathcal{A}$ is isometrically $*$-isomorphic to the algebra of continuous function on X denoted by, C(X).*

We will go through the construction of this topological space X and prove the Gelfand-Naimark theorem. Roughly speaking, it is construct from the spectrum of C*-algebra.

Definition 1.1.4. *For an element of C*-algebra $a \in \mathcal{A}$ (C*-algebra), the set $Sp(a) = \{\lambda \in \mathbb{C} | a - \lambda 1$ is not invertible $\}$ is the spectrum of a. The collection of these sets denoted by $Sp(\mathcal{A})$.*

Definition 1.1.5. *Let $\mathcal{A}$ be a Banach algebra (C*-algebra). A **character** of $a \in \mathcal{A}$ is a non-zero homomorphism $\mu : \mathcal{A} \to \mathbb{C}$, which is surjective. The set of all characters denoted by $M(\mathcal{A})$.*

For an example, let Y be a locally compact space and $\mathcal{A} = C_0(Y)$ the algebra of continuous functions vanishing at infinity. The evaluation map $\epsilon_y : f \mapsto f(y)$ at $y \in Y$ defines a character.

One can see that $\mu(a) \in Sp(a),\ \forall a \in \mathcal{A}$. Since $\mu(a - \mu(a)1) = 0 \Rightarrow a - \mu(a)1$ is not invertible. From spectral theory, all $\lambda \leq ||a||$ (the equality holds for self-adjoint elements) and therefore $|\mu(a)| \leq ||a||$. Moreover, $||\mu|| = 1$ because $\mu(a) = \mu(1 \cdot a) = \mu(1)\mu(a)$.

Consider the weak* topology on $\mathcal{A}^*$, the Banach space of bounded linear functional. The Banach-Alaoglu theorem says that the unit ball A_1^* in $\mathcal{A}^*$ is compact in weak* topology. Therefore, we can define the **Gelfand topology** on $M(\mathcal{A})$ by the inclusion $M(\mathcal{A}) \hookrightarrow A_1^*$.

Lemma 1.1.6. *For a commutative Banach algebra $\mathcal{A}$(can be non-unital), $M(\mathcal{A})$ endowed with the Gelfand topology is a locally compact space.*

As I mentioned that if the C*-algebra is non-unital one can consider $\mathcal{A}^+$. Then the character $M(\mathcal{A}^+) = M((\mathcal{A}) \cup \{0\}$ is compact in weak* topology. Therefore the set of character of non-unital C*-algebra is locally compact. Indeed, the locally compact Hausdorff space $M(\mathcal{A})$ will become the space X in GN theorem. But what about the isomorphism?

Definition 1.1.7. *Let $\mathcal{A}$ be a commutative Banach algebra. The **Gelfand transform** of $a \in \mathcal{A}$ is the function $\hat{a} : M(\mathcal{A}) \to \mathbb{C}$ given by the evaluation at a*

$$\hat{a}(\mu) := \mu(a). \tag{1}$$

*The **Gelfand Transformation** is a map $\mathcal{G} : a \mapsto \hat{a}$ from $\mathcal{A}$ to $C_0(M(\mathcal{A}))$.*

In general Banach algebra, the Gelfand transformation is not just a homomorphism but the situation improves greatly when considering C*-algebra. If we can show that $\mathcal{G}$ is isometric $*-$isomorphism then the proof of GN theorem is complete and we shall need two more lemmas and the Stone-Weierstrass theorem to show this.

Lemma 1.1.8. *Let $a \in \mathcal{A}$ be a self-adjoint element in C*-algebra. Then $\mu(a) \in \mathbb{R}$ for all $\mu \in M(\mathcal{A})$.*

I will skip the proof of this lemma. Note that any element of C*-algebra can be written as the combination of self-adjoint elements $a = a_1 + ia_2$ with $a_1 := \frac{1}{2}(a^* + a)+,\ a_2 := \frac{i}{2}(a^* - a)$. As a consequence of above theorem

$$\mu(a^*) = \mu(a_1 - ia_2) = \mu(a) - i\mu(a) = \overline{\mu(a)}, \tag{2}$$

or equivalently, $\widehat{a^*}(\mu) = \overline{\hat{a}(\mu)}$. One concludes that the Gelfand transformation is a $*-$homomorphism.

Lemma 1.1.9. *Let $\mathcal{A}$ be a commutative C*-algebra, and let $\lambda \in Sp(a),\ a \in \mathcal{A}$. If $\lambda \neq 0$, there is a character μ such that $\mu(a) = \lambda$.*

The last ingredient is the Stone-Weierstrass theorem which states that, if X is a locally compact space, and if B is a closed subalgebra of $C_0(X)$ such that

(i) for $p \neq q \in X$, there is some $y \in B$ with $y(p) \neq y(q)$

(ii) $y \in B$ vanishes identically at no point of X

(iii) B is closed under complex conjugation

Then $B = C_0(X)$

The proof of Gelfand-Naimark theorem

The relation (2) shows that $\mathcal{G} : \mathcal{A} \to C_0(M(\mathcal{A}))$ is a $*-$homomorphism. It is isometric because for any $a \in \mathcal{A}$

$$\begin{aligned} ||\hat{a}||^2 &= ||\hat{a}^* a|| = ||\widehat{a^* a}|| = \sup \frac{|\mu(a^* a)|}{||\mu||} \\ &= \sup\{\lambda \in Sp(a^* a)\} = r(a^* a) = ||a^* a|| = ||a||^2. \end{aligned} \tag{3}$$

The fourth equality is given by Lemma 1.1.9. The isometric property requires that $\mathrm{Ker}(\mathcal{G}) = 0$, therefore, it is injective. Now $\mathcal{G}(\mathcal{A})$ is a subalgebra of $C_0(M(\mathcal{A}))$ that is complete since $\mathcal{A}$ is complete and $\mathcal{G}$ is isometric, and therefore is closed (complete metric space). The evaluation maps in $\mathcal{G}(\mathcal{A})$ separate the characters, do not all vanish at any point and (2) shows that $\mathcal{G}(\mathcal{A})$ is closed under complex conjugation; $\mathcal{G}$ is surjective.

2 Spectral Triple

Definition 2.0.10. *Let π be a representation of C^*-algebra $\mathcal{A}$ on $\mathcal{H}$ as a left A-module. A self-adjoint operator (not necessary bounded), $\mathcal{D}$ which is densely define on $\mathcal{H}$ is called Dirac operator if,*

(i). for all $a \in \mathcal{A}, [\mathcal{D}, \pi(a)]$ is a bounded operator on Domain of $\mathcal{D}$(thus extends to bounded operator on $\mathcal{H}$).

(ii). for all $a \in \mathcal{A}, \pi(a)(1 + \mathcal{D}^2)^{-1/2}$ is a compact operator.

The algebra, the Hilbert space and the Dirac operator are three important ingredients that define a spectral geometry, together $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is called a *spectral triple.* In particular, we are interested in a **real even spectral triple**

Definition 2.0.11. *Let $(\mathcal{A}, \mathcal{H}.\mathcal{D})$ be a spectral triple*

i). $(\mathcal{A}, \mathcal{H}.\mathcal{D})$ is 'even' if there exists a self-adjoint operator $\gamma \in B(\mathcal{H})$ such that $\gamma^2 = 1,\ \mathcal{D}\gamma + \gamma\mathcal{D} = 0$ and for all $a \in \mathcal{A}, \gamma\pi(a) = \pi(a)\gamma$.

ii). $(\mathcal{A}, \mathcal{H}.\mathcal{D})$ is 'real' if there exists an operator J on $\mathcal{H}$ such that

$$J^2 = \epsilon, \quad J\mathcal{D} = \epsilon' \mathcal{D} J. \tag{4}$$

If the spectral triple is even, then we also need, $J\gamma = \epsilon''\gamma J$. The constant $\epsilon, \epsilon', \epsilon''$ can be either $+1$ or -1.

Since $\epsilon, \epsilon', \epsilon''$ can only be $\{\pm 1\}$, there are only 8 possible choices of real structure one can impose on a spectral triple (bear in mind that ϵ''only define for even spectral triple).

	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1	-	-1	-	1	-	-1	-

The numbers on the top row denote KO dimension which, for the canonical triple, is the dimension of the manifold modulo 8. In commutative spectral triple, the number 8 relates to the 8 different algebras B such that $M_k(B) = \mathrm{Cl}_n$, a Clifford algebra on a manifold M^n. The numbers 0-7 can be paired up with the general real spectral triple as well, but to do that one needs to understand the correspondent between K-theory and K-homology (KK-theory). The existence of real structure allows us to define $b^0 := Jb^*J^*, b \in \mathcal{A}$ the *right action* on Hilbert space. The right action needs to satisfy the following conditions

$$[a, b^0] = 0, \tag{5}$$

$$\big[[\mathcal{D}, a], b^0\big] = 0. \tag{6}$$

2.1 The canonical triple

Let (M, g) be a compact oriented Riemannian spin manifold and $(C^\infty(M), L^2(M, S), \nabla\!\!\!/)$ be a canonical triple. This triple consisting of

- Algebra of smooth complex value function $C^\infty(M)$,
- Hilbert space of squared integrable spinor $L^2(M, S)$,
- A Dirac operator of spin^C-connection.

In addition, since M is a spin^C manifold there is a natural grading operator, $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ and the Charge conjugation operator, J_M of spinor (These two operator will be important later on). Therefore the canonical triple is a real even spectral triple. One can define a function

$$d_{D\!\!\!/}(x, y) = \sup_f \{|\epsilon_x(f) - \epsilon_y(f)|; f \in C^\infty(M), ||[\nabla\!\!\!/, f]|| \leq 1\}, \tag{7}$$

is equal to the geodesic from x to y. Note that $||[\not\nabla, f]||$ is the supremum norm.
To show that the function define above is indeed a geodesic, let $z, y \in M$ and consider

$$\begin{aligned} 1 \geq ||[\not\nabla, f]||_{\sup} &= \sup_{\psi \neq 0} \frac{\langle [\not\nabla, f]\psi, [\not\nabla, f]\psi \rangle^{1/2}}{\langle \psi, \psi \rangle^{1/2}} \\ &= \sup_{\psi \neq 0} \frac{\langle \psi, \gamma^\nu \gamma^\mu \nabla_\nu \bar{f} \nabla_\mu f \psi \rangle^{1/2}}{\langle \psi, \psi \rangle^{1/2}} \\ &= \sup_{x \in M} ||\nabla f||. \end{aligned} \tag{8}$$

To obtain the final equality, one need to use the fact that the operator norm of multiplicative operator $T_f \psi = f\psi$ and the supremum norm are the same i.e. $||T_f||_{op} = ||f||_{L^\infty}$. Then from Cauchy-Schwarz inequality, we have

$$\begin{aligned} |df| \leq ||\nabla f|| \cdot ||dx|| \quad &\Rightarrow \quad |\int df| \leq \int |df| \leq \int_y^x ||dx|| \\ &\Rightarrow \quad \sup_f |f(x) - f(y)| \leq \inf_\gamma \int_\gamma dt \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}. \end{aligned} \tag{9}$$

The equality holds when chosen $f = d(x, .) \Rightarrow |d(x, x) - d(x, y)| = d(x, y)$.

Connes also showed that for any commutative spectral triple (with some additional conditions) there exists a smooth compact oriented spin^c manifold X, such that $\mathcal{A} \cong C^\infty(X)$ [1]. It is important to note that, Connes' distance formula is not limited on commutative algebra but makes sense on any spectral triple. In non-commutative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ the notion of points is replaced by the pure states and the distance between them is given by

$$d_{\mathcal{D}}(\phi, \psi) = \sup\{|\phi(a) - \psi(a)|; a \in \mathcal{A}, ||[D, a]|| < 1\}, \tag{10}$$

where ϕ, ψ characters (or pure states) of $\mathcal{A}$. It should be clear now that the term NCG is defined by the non-commutative spectral triple.

2.2 Finite spectral triple

Now we will choose $\mathcal{A}$ to be non-commutative algebra. Consider the matrix algebra, which is a simple extension of commutative algebra to non-commutative

$$F := (\mathcal{A}_F = M_n(\mathbb{C}), \mathcal{H}_F, \mathcal{D}_F) \tag{11}$$

This is called Finite spectral triple. It contains the commutative case where the algebra given by $\mathbb{C}$ as the centre of $M_n(\mathbb{C})$. Although the finite spectral triple seems to be a very trivial choice of NCG, the its product with the canonical has many interesting properties:

$$(C^\infty(M, \mathcal{A}_F), L^2(M, S) \otimes \mathcal{H}_F, \not\nabla \otimes \mathbb{1} + \gamma_5 \otimes \mathcal{D}_F). \tag{12}$$

This triple is called the Almost commutative spectral triple. As one can show that $\mathrm{Diff}(M) \cong \mathrm{Aut}(C^\infty(M))$; let $\phi \in \mathrm{Diff}(M)$, then by setting $\alpha_\phi : f \mapsto f \circ \phi^{-1}$. Analogously one may define $\mathrm{Diff}(M \times F) := \mathrm{Aut}(C^\infty(M, \mathcal{A}_F))$ and $\alpha_\phi : a \mapsto a \circ \phi^{-1}$. By doing this we enlarge the symmetry of manifold M i.e. includes the symmetry of the algebra $\mathcal{A}_F$ such as the unitary group $\Rightarrow$ gauge symmetry.

3 Index of Dirac operator

An invariant one can obtain from spectral triple is the index of Dirac operator

3.1 Fredholm index of Dirac operator

The first problem one encounters in trying to compute the index of $\mathcal{D}$ is that it is not a bounded operator, therefore, the usual definition of Fredholm operator is not applicable to $\mathcal{D}$!

Consider a subspace $\mathcal{H}_1 = \{\xi \in \mathcal{H} | \mathcal{D}\xi \in \mathcal{H}\}$ with inner product

$$\langle \xi, \eta \rangle_1 = \langle \xi, \eta \rangle + \langle \mathcal{D}\xi, \mathcal{D}\eta \rangle . \tag{13}$$

Observes that for $\xi \in \mathcal{H}_1$

$$||\mathcal{D}\xi||^2 \leq ||\xi||^2 + ||\mathcal{D}\xi||^2 = ||\xi||_1^2. \tag{14}$$

Therefore it is the bounded from $\mathcal{H}_1 \to \mathcal{H}$. The subspace $\mathcal{H}_1$ is a Hilbert space. Suppose $\{\xi_n\} \in \mathcal{H}_1$ is a Cauchy sequence, thus $\xi_n \to \xi \in \mathcal{H}$. Since $\mathcal{D}$ is bounded on $\mathcal{H}_1 \Rightarrow$ continuous i.e.

$$\lim_{n\to\infty} \mathcal{D}\xi_n = \mathcal{D}\xi \in \mathcal{H}. \tag{15}$$

Hence $\xi \in \mathcal{H}_1$. We can think of $\mathcal{D}$ as a bounded operator from $\mathcal{H}_1 \to \mathcal{H}$. Any unbounded operator satisfies such property is called an unbounded Fredholm operator
Next we need to show that this bounded operator is really Fredholm operator. One may use the following theorem as the alternative definition of Fredholm operators

Theorem 3.1.1. *(Atkinson's theorem)*
$T \in B(\mathcal{H}_1, \mathcal{H}_2)$ is a Fredholm operator iff there exists $S_1, S_2 \in B(\mathcal{H}_2, \mathcal{H}_1)$ and compact operators $K_i \in B(\mathcal{H}_i)$ such that

$$S_1 T = 1_{\mathcal{H}_1} + K_1, \quad T S_2 = 1_{\mathcal{H}_2} + K_2. \tag{16}$$

Lemma 3.1.2. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple. Then $\mathcal{D}$ is unbounded Fredholm operator.*
Proof. *Defines an operator*

$$\mathcal{D}(1 + \mathcal{D}^2)^{-1} : \mathcal{H} \to \mathcal{H}_1. \tag{17}$$

It follows that

$$\mathcal{D} \cdot \mathcal{D}(1 + \mathcal{D}^2)^{-1} = (-1 + 1 + \mathcal{D}^2)(1 + \mathcal{D}^2)^{-1} = 1 - (1 + \mathcal{D}^2)^{-1} \tag{18}$$

For the right approximate inverse, one chooses $(1 + \mathcal{D}^2)^{-1}\mathcal{D}$ then follows the similar calculation.

Although we manage to get a well defined index from $\mathcal{D}$, it is not very useful because $\mathcal{D}$ is self-adjoint $\to$ index$(\mathcal{D}) = 0$. However, when $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is even, we define

$$\mathcal{D}^+ = \frac{1-\gamma}{2}\mathcal{D}\frac{1+\gamma}{2}, \quad \mathcal{D}^+ : \mathcal{H}_1^+ \to \mathcal{H}^- \tag{19}$$

Theorem 3.1.3. *(McKean-Singer Formula)*
Let $\mathcal{D}$ be an unbounded self-adjoint operator with compact resolvent. Let γ be a grading operator which anti-commutes with $\mathcal{D}$. Finally, let $f(s)$, $s \in \mathbb{R}$ be a continuous even function with $f(0) \neq 0$ and f($\mathcal{D}$) trace-class. Then $\mathcal{D}^+$ is Fredholm and

$$\mathrm{Index}(D^+) = \frac{1}{f(0)} \mathrm{Tr}\left(\gamma f\left(\mathcal{D}\right)\right). \tag{20}$$

The traditional choice is $f(s) = e^{-ts^2}$, $t > 0$, so we have the operator $e^{-t\mathcal{D}^2}$ which is called the **heat operator**.

It requires the relation between K-theory and K-homology to prove that the index formula also compatible with non-commutative geometry. Suppose this is true, we may observe that the Laplacian operator of almost commutative spectral triple can written as $\mathcal{D}^2 = \Delta^E + F$. For simplicity, Let M^m be a manifold without boundary, the asymptotic expansion of heat operator is given by [6]

$$\mathrm{Tr} e^{-t\mathcal{D}^2} \sim \sum_{n=0}^{\infty} t^{\frac{n-m}{2}} \int_M a_n(x, \mathcal{D}^2) \mathrm{d}V, \tag{21}$$

where the first three non-vanishing coefficients are(all odd coefficients vanish)

$$\begin{aligned}
a_0(x, \mathcal{D}^2) &= (4\pi)^{m/2} \mathrm{Tr}\left(\mathrm{Id}\right), \\
a_0(x, \mathcal{D}^2) &= (4\pi)^{m/2} \mathrm{Tr}\left(F - \frac{R}{6}\right), \\
a_4(x, \mathcal{D}^2) &= \frac{(4\pi)^{m/2}}{360} \mathrm{Tr}\left(12\Delta R + 5R^2 - 2(R_{\mu\nu})^2 + 2(R_{\mu\nu\rho\sigma})^2 - 60RF + 180F^2 - 60\Delta F + 30(\Omega^E_{\mu\nu})^2\right)
\end{aligned} \tag{22}$$

3.2 The Standard Model of particle physics

The Bosonic action is defined by $S_b := f(0)\mathrm{Index}(D_A^+)$. If we consider asymptotic expansion of this invariant [6, 2], the spectral action becomes

$$\begin{aligned}
S_b := \mathrm{Tr}_{L^2} f\left(\frac{D_A^2}{\Lambda^2}\right) \;\sim\; & \frac{24}{\pi^2} f_4 \Lambda^4 \int d^4x\sqrt{g} \\
& - \frac{2}{\pi^2} f_2 \Lambda^2 \int d^4x \sqrt{g}\left(R + \frac{1}{2}a\bar{H}H + \frac{1}{4}c\sigma^2\right) \\
& + \frac{1}{2\pi^2} f(0) \int dx^4 \sqrt{g}\left(\frac{11}{30}R^*R^* - \frac{3}{5}C^2_{\mu\nu\rho\sigma} + \frac{5}{3}g_1^2 B^2_{\mu\nu} + g_2^2 (W^i_{\mu\nu})^2 + g_3^2(V^a_{\mu\nu})^2\right. \\
& \left. + \frac{1}{6}aR\bar{H}H + b(\bar{H}H)^2 + a|\nabla_\mu H|^2 + 2e\bar{H}H\sigma^2 + \frac{1}{2}d\sigma^4 + \frac{1}{12}cR\sigma^2 + \frac{1}{2}c(\partial_\mu\sigma)^2\right) \\
& + O(\Lambda^{-2}),
\end{aligned} \tag{23}$$

where coefficients f_k defined by $f_{4-k} := \int_0^\infty t^{4-k-1}f(t)dt$, for $0 \leq k < 4$. The quantity $R^*R^* := C^2_{\mu\nu\rho\sigma} - 2(R^2_{\mu\nu} - \frac{1}{3}R^2)$ is called a Gauss-Bonnet term. The integration of this term is the Euler characteristic which is a constant if the manifold does not change its topology. The tensor $C_{\mu\nu\rho\sigma}$ is Weyl tensor, which will be our main interest in the following sections, and $B_{\mu\nu}, W^i_{\mu\nu}, V^a_{\mu\nu}$ are field strength tensor of $U_Y(1), SU(2)$ and $SU(3)$ gauge fields respectively. The field H is the $SU(2)$ doublet or the Higgs-Englert field and σ is the singlet scalar field. Finally, the constants

$$\begin{aligned}
b &= \mathrm{tr}\left((k^{*\nu}k^\nu)^2 + (k^{*e}k^e)^2 + 3\left((k^{*u}k^u)^2 + (k^{*d}k^d)^2\right)\right), \\
d &= \mathrm{tr}\left((k^{*\nu R}k^{\nu R})^2\right), \\
e &= \mathrm{tr}\left(k^{*\nu}k^\nu k^{*\nu R}k^{\nu R}\right),
\end{aligned} \tag{24}$$

are computed from Yukawa matrices. This asymptotic expansion only valid for compact boundary less manifold. For the case of compact manifold with boundary the action are more complicated [3]. Note that, this action is asymptotically expanded in the Euclidean signature manifold, one needs to perform Wick rotation to get the action in Minkowski signature manifold (space-time). This transition between Euclidean spectral triple and Lorentzian spectral triple via Wick rotation was discussed in [5].

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