Exercises on characteristic classes

1. Let C be a smooth complex curve of genus $g \geq 2$ and let T_C be its (holomorphic) tangent bundle. Compute $\dim_{\mathbb{C}} H^1(C, T_C) - \dim_{\mathbb{C}} H^0(C, T_C)$. (This is equal to the dimension $\dim_{\mathbb{C}} H^1(C, T_C)$ of the moduli space $\mathcal{M}_g$ of curves of genus $g \geq 2$, since $H^0(C, T_C) = 0$ if $g \geq 2$.)

2. Recall that the cohomology ring of $\mathbb{P}^n = \mathbb{CP}^n$ is given by $H^*(\mathbb{P}^n; \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^{n+1})$, with $\int_{\mathbb{P}^n} \alpha^n = 1$ (or equivalently $\alpha = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$, which is also equal to the Poincaré dual of the hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$). Prove that the total Chern class $c(T_{\mathbb{P}^n})$ of the tangent bundle $T_{\mathbb{P}^n}$ is given by $c(T_{\mathbb{P}^n}) = (1 + \alpha)^{n+1}$; this formula was used in Agustin's example last time. (Hint: use the Euler exact sequence $0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n} \to 0$.)

3. Let $X_5 \subset \mathbb{P}^4$ be a quintic hypersurface (e.g. $X_5 = \{\sum_{i=0}^4 Z_i^5 = 0\}$ where $[Z_0 : \cdots : Z_4]$ are the homogeneous coordinates on $\mathbb{P}^4$). Prove that the topological Euler characteristic (i.e. alternating sum of Betti numbers) of X_5 is -200. Compute all the Hodge numbers of X_5. (Hint: use $c(T_{\mathbb{P}^n}) = (1+\alpha)^{n+1}$ above and proceed as Agustin did last time, using the Lefschetz hyperplane theorem; observe also that X_5 is Calabi-Yau and use Serre duality.)

4. Let E be a rank r complex vector bundle over a smooth manifold X. Prove $c_1(E) = c_1(\bigwedge^r E)$.

5. Let E be a rank 3 complex vector bundle over a smooth manifold X. Compute $c(\bigwedge^2 E)$ and $c(\mathrm{Sym}^5 E)$ in terms of $c_i(E)$'s (at least in terms of Chern roots).

6. Let L be an ample line bundle over a smooth projective variety X of complex dimension n. Prove

$$\lim_{k \to \infty} \frac{\dim_{\mathbb{C}} H^0(X, L^{\otimes k})}{k^n} = \frac{1}{n!} \int_X c_1(L)^n.$$

   This quantity is called the *volume*[1] of L. (Hint: use Kodaira-Serre vanishing.)

7. (OPTIONAL, requires Chern-Weil theory) We have the exponential exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 1$ of sheaves over a smooth projective variety X; the first map is given by the inclusion of the constant sheaf $\mathbb{Z}$ consisting of integer-valued constant functions to the sheaf $\mathcal{O}_X$ of holomorphic functions, and the second by the exponential map $\exp(2\pi\sqrt{-1}\cdot)$ to the sheaf $\mathcal{O}_X^*$ of non-zero holomorphic functions. Consider the associated exact sequence in cohomology groups $0 \to H^0(X, \mathbb{Z}) \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X^*) \to H^1(X, \mathbb{Z}) \to \cdots$. Show that the connecting homomorphism (given by the snake lemma) $\delta : H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$ is given by -1 times the first Chern class. (To make sense of this, observe first $H^1(X, \mathcal{O}_X^*) \cong \mathrm{Pic}(X)$, where $\mathrm{Pic}(X)$ is the set[2] of isomorphism classes of holomorphic line bundles on X. The statement to be proved is $\delta : \mathrm{Pic}(X) \ni L \mapsto -c_1(L) \in H^2(X, \mathbb{Z})$.)

[1] The above limit exists if L is ample, but in general we have to replace $\lim_{k\to\infty}$ by $\limsup_{k\to\infty}$ to define the volume.
[2] $\mathrm{Pic}(X)$ has the structure of abelian group given by the tensor product.

Since $\ker(\delta)$ is isomorphic to $H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$, the torus $\mathrm{Pic}^0(X) := H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$, called the *Picard variety* of X, classifies the isomorphism classes of holomorphic line bundles with the first Chern class 0.

8. (OPTIONAL, requires Chern-Weil theory) The above argument applies word by word to the exponential exact sequence $0 \to \mathbb{Z} \to \mathcal{A}_X \to \mathcal{A}_X^* \to 1$ of sheaves over a smooth manifold X, where $\mathcal{A}_X$ is now the sheaf of C^∞ complex functions and $\mathcal{A}_X^*$ is the sheaf of C^∞ non-zero complex functions. The difference is that now the sheaf $\mathcal{A}_X$ is *fine*, i.e. the partition of unity is possible, and hence $H^i(X, \mathcal{A}_X) = 0$ for all $i > 0$ by the general theory of sheaf cohomology. In particular, we have $H^1(X, \mathcal{A}_X) = H^2(X, \mathcal{A}_X) = 0$, and hence the connecting homomorphism $\delta = -c_1(\cdot)$ is an isomorphism $\delta : H^1(X, \mathcal{A}_X^*) \xrightarrow{\sim} H^2(X, \mathbb{Z})$, where $H^1(X, \mathcal{A}_X^*)$ is isomorphic to the abelian group $\mathrm{Pic}^\infty(X)$ of C^∞-isomorphism classes of complex line bundles on X. In other words, *C^∞-isomorphism classes of complex line bundles are classified by the first Chern class.*

The reference for the optional problems[3] is Proposition 4.4.12 in the textbook *Complex Geometry* by Huybrechts. Any questions or comments to be sent to Yoshi Hashimoto at `yoshinori.hashimoto.12@ucl.ac.uk`

[3] Those who are annoyed by the negative sign should also see the discussion in pp200-201.