Characteristic classes of vector bundles*

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1 Introduction

Let X be a smooth, connected, compact manifold of dimension n without boundary, and $p : E \to X$ be a (real or complex) vector bundle of rank k over x. *Characteristic classes* associate to each vector bundle E over X a cohomology class $c(E) \in H^k(X; R)$ (R is an "appropriate" coefficient ring as we will see later) such that the association is *natural*, i.e. if we consider a map $f : X \to Y$ between smooth manifolds and a pull-back bundle f^*E over Y, the characteristic class of the pull-back bundle $c(f^*E) \in H^k(Y; R)$ is equal to the pull-back of the cohomology class $f^*c(E) \in H^k(Y; R)$.

The power of characteristic classes is that it can define an *invariant* of an isomorphism class of vector bundles over X. Suppose $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ are two isomorphic vector bundles over X, with the bundle isomorphism given by $f : E_1 \to E_2$, so that $E_1 = f^*E_2$. Suppose also (for the sake of simplicity) that $c(\cdot)$ is in the top degree with $\mathbb{Z}$ as the coefficient ring, i.e. $c(E_1),\ c(E_2) \in H^n(X; \mathbb{Z})$. Then, by coupling with the fundamental class of X, we get a number $\int_X c(E_1)$ and $\int_X c(E_2)$ corresponding to each vector bundle E_1 and E_2. Since we assumed $E_1 = f^*E_2$, we see that (noting that f acts as an identity on the base X)

$$\int_X c(E_1) = \int_X c(f^*E_2) = \int_X f^*c(E_2) = \int_X c(E_2)$$

by virtue of naturality. This means that, if there are two vector bundles $p : E \to X$ and $p' : E' \to X$ which have different numbers $\int_X c(E)$ and $\int_X c(E')$ they cannot possibly be isomorphic. Thus, a *number* $\int_X c(E)$ can be powerful enough to tell an isomorphism class of a vector bundle. Often, a number obtained by integrating a characteristic class is called a *characteristic number*. Later, we will see that the tautological bundle over $\mathbb{CP}^1$ cannot be isomorphic to the hyperplane bundle over $\mathbb{CP}^1$ by using this principle.

Of course, we have not shown if such characteristic classes exist at all, let alone explaining how to construct such an association $E \mapsto c(E) \in H^k(X; R)$. But an example of characteristic classes was already known in 1935, called the *Stiefel-Whitney classes* $w_i(E) \in H^i(X; \mathbb{Z}/2\mathbb{Z})$. In 1940's, *Pontrjagin classes* $p_i(E) \in H^{4i}(X; \mathbb{Z})$ were discovered for a real vector bundle E, and *Chern classes* $c_i(E) \in H^{2i}(X; \mathbb{Z})$ were discovered for a complex vector bundle E. Another famous characteristic class is the *Euler class* $e(E) \in H^k(X; \mathbb{Z})$, integration of which is equal to the Euler characteristic $\chi(X)$ if $E = TX$.

All of these characteristic classes are not independent of each other, i.e. there are certain relationships between various characteristic classes, and the way of defining these characteristic classes is not unique. The approach we take in this section is differential-geometric, which is called *Chern-Weil theory* in a wider context. Mainly because of the limitation on length, we will only focus on the Chern classes mentioned above. One reason for doing so is that Chern classes are indispensable in complex differential and algebraic

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geometry, whereas other characteristic classes have more to do with much more subtle topological structure on a vector bundle, e.g. existence of spin structure, signature of the intersection form, to mention just a few. Another reason is that once we define the Chern classes by means of the Chern-Weil theory, the definition of the Pontrjagin classes goes parallel with only minor modification. The definition of Euler class also follows a similar strategy. We also have to remark that the Chern-Weil theory cannot be used to define the Stiefel-Whitney classes, since the Chern-Weil theory goes through de Rham theory and the Stiefel-Whitney classes are defined over $\mathbb{Z}/2\mathbb{Z}$.

2 Chern classes

Let $p : E \to X$ be a complex vector bundle of rank k (i.e. each fibre is a $\mathbb{C}$-vector space with dimension k over $\mathbb{C}$) defined on the base manifold X. We define a connection ∇ on E, which exists by the Exercise 3.B.2. Let $R^\nabla \in \Gamma(\mathrm{End}_{\mathbb{C}}(E) \otimes \wedge^2 T^*X)$ be the curvature, i.e. $R^\nabla(V,W) = \nabla_V \nabla_W - \nabla_W \nabla_V - \nabla_{[V,W]}$. Since R^∇ is a 2-form on X with coefficients in $\mathrm{End}_{\mathbb{C}}(E)$, we see that the following

$$\det\left(\lambda I - \frac{R^\nabla}{2\pi\sqrt{-1}}\right) \tag{1}$$

makes sense, where I is the identity matrix in the endomorphism ring $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^k)$, by noting that det is the usual determinant taken over the $\mathrm{End}_{\mathbb{C}}(E)$ sector of R^∇, and we may treat (1) as a matrix polynomial since R^∇ is a 2-form and hence commutes with itself. The presence of the numerical constant $1/2\pi\sqrt{-1}$ will be explained later.

We give a provisional definition of the Chern classes as follows.

Definition 2.1. *We define differential $2i$-forms $c_i(E,\nabla) \in \Gamma(\wedge^{2i}T^*X)$ as*

$$\det\left(\lambda I - \frac{R^\nabla}{2\pi\sqrt{-1}}\right) = \sum_{i=0}^{k} c_i(E,\nabla)\lambda^{k-i},$$

which later will turn out to be what is known as Chern classes.

Examples

$$\begin{aligned} c_0(E,\nabla) &= 1 \in \mathbb{R}. \\ c_1(E,\nabla) &= -\mathrm{tr}\left(\frac{R^\nabla}{2\pi\sqrt{-1}}\right) \in \Gamma(\wedge^2 T^*X). \end{aligned} \tag{2}$$

The main theorem that we wish to prove is the following.

Theorem 2.2. *$c_i(E,\nabla)$, for $0 \le i \le k$, is a closed form, and the cohomology class $[c_i(E,\nabla)] \in H^{2i}(X;\mathbb{C})$ is independent of the connection ∇ initially chosen to define $c_i(E,\nabla)$. Moreover, $[c_i(E,\nabla)]$ is in fact a real class, i.e. $[c_i(E,\nabla)] \in H^{2i}(X;\mathbb{R}) \subseteq H^{2i}(X;\mathbb{C})$*

Remark This theorem has a far-reaching generalisation, which is called the *Chern-Weil theory.* The key feature is that the determinant is adjoint-action invariant (adjoint action in the sense of Lie group theory). However, it is not possible to discuss this in its full details here, due to the limitation on length.

In fact, there are 4 axioms which uniquely characterise Chern classes as defined in algebraic topology [H], [K]. In particular, these axioms characterise Chern classes as an integral class. We can show that $[c_i(E,\nabla)]$ defined above satisfy all these axioms, although we do not provide details in this section. We refer to [K] for details. An important consequence of this is the following theorem.

Theorem 2.3. $[c_i(E,\nabla)]$ *is in fact an integral class, i.e.* $[c_i(E,\nabla)] \in H^{2i}(X;\mathbb{Z}) \subseteq H^{2i}(X;\mathbb{R})$

Remark This is the reason for the numerical constant $1/2\pi\sqrt{-1}$ appearing in the definition of the Chern classes - it ensures that $[c_i(E,\nabla)]$ takes value in the integral class.

These theorems finally enable us to define the Chern classes.

Definition 2.4. *For a complex vector bundle of rank k over X, i-th* Chern class *of E, written $c_i(E)$, is the integral cohomology class defined by $[c_i(E,\nabla)] \in H^{2i}(X;\mathbb{Z})$. This is well-defined because $[c_i(E,\nabla)] = [c_i(E,\nabla')]$ for any two connections ∇ and ∇' on E, by Theorem 2.2.*

Examples Let $X = \mathbb{CP}^1$ and E be the tautological line bundle L. Then, we can show $\int_{\mathbb{CP}^1} c_1(L) = -1$. This is in fact one of the 4 axioms in algebraic topology which characterise Chern classes. Let H be the hyperplane bundle over $\mathbb{CP}^1$, i.e. the dual of L. By noting that the curvature of the dual bundle is given by $-(R^\nabla)^t$ and recalling (2), we see that $\int_{\mathbb{CP}^1} c_1(H) = -\int_{\mathbb{CP}^1} c_1(L) = +1$. This shows that H is not isomorphic to L, as mentioned in §1.

We finally comment on how this definition of the Chern classes satisfy the naturality[1] mentioned in §1. Let $f : X \to Y$ be a map between smooth manifolds and consider the pull-back bundle f^*E over Y. We define an *induced connection* $f^*\nabla$ on f^*E as follows. For any local section s of E, we have a local section f^*s of f^*E and any section of f^*E arises this way, by the definition of the pull-back bundle. Then, for any vector field v on Y, we define $f^*\nabla_v(f^*s) := f^*(\nabla_{f_*v}s)$. We can prove that the curvature of $f^*\nabla$ is given by f^*R^∇ (pull-back of a differential form), by considering a *connection 1-form* of ∇ (connection 1-form is defined, for example, in [K]). Thus

$$\sum_{i=0}^{k} c_i(f^*E, f^*\nabla)\lambda^{k-i} = \det\left(\lambda I - \frac{R^{f^*\nabla}}{2\pi\sqrt{-1}}\right) = \det\left(\lambda I - \frac{f^*R^\nabla}{2\pi\sqrt{-1}}\right) = \sum_{i=0}^{k} f^*c_i(E,\nabla)\lambda^{k-i}$$

and we see that the naturality of $c_i(E,\nabla)$ is satisfied, even at the level of differential forms (i.e. even before taking the cohomology class).

3 Proof of Theorem 2.2.

We follow the exposition in [K].

1. $c_i(E,\nabla)$ is closed.

   Proof. We remark that $\mathrm{End}_{\mathbb{C}}(\wedge^k E)$ is a trivial line bundle, since an endomorphism bundle of any line bundle is trivial. Thus, the covariant exterior derivative d^∇ on $\mathrm{End}_{\mathbb{C}}(\wedge^k E)$ agrees with the usual

[1] In fact, the naturality is one of the 4 axioms in algebraic topology which define the Chern classes.

exterior derivative d. Thus, by noting that $\det\left(\lambda I - R^\nabla/2\pi\sqrt{-1}\right)$ is an $\mathrm{End}_\mathbb{C}(\wedge^k E)$-valued differential form on X, we see that

$$d\left(\det\left(\lambda I - \frac{R^\nabla}{2\pi\sqrt{-1}}\right)\right) = d^\nabla\left(\det\left(\lambda I - \frac{R^\nabla}{2\pi\sqrt{-1}}\right)\right) = 0 \tag{3}$$

where we used the Bianchi identity $d^\nabla R^\nabla = 0$ and the Leibniz rule of d^∇ to derive the second equality. This shows $dc_i(E,\nabla) = 0$ for $\forall i$. $\square$

2. $[c_i(E,\nabla)] \in H^{2i}(X;\mathbb{C})$ is independent of the connection ∇ chosen.

Proof. Let ∇^0 and ∇^1 be two connections defined on E. Then, the difference $\nabla^1 - \nabla^0$ is an $\mathrm{End}_\mathbb{C}(E)$-valued 1-form which we call α. Let $\nabla^t := (1-t)\nabla^0 + t\nabla^1 = \nabla^0 + t\alpha$ be a family of connections connecting ∇^0 and ∇^1. Let d^t be the covariant exterior derivative defined by ∇^t. Then, the curvature $R^t := R^{\nabla^t}$ is given by $R^t = d^t \circ d^t = (d^{\nabla^0} + t\alpha) \circ (d^{\nabla^0} + t\alpha)$. Then,

$$\frac{dR^t}{dt} = \frac{d(d^t)}{dt} \circ d^t + d^t \circ \frac{d(d^t)}{dt} = \alpha \circ d^t + d^t \circ \alpha = d^t\alpha \tag{4}$$

where the last equality is the one as an element of $\Gamma(\mathrm{End}_\mathbb{C}(E) \otimes \wedge^2 T^*X)$, i.e. $(\alpha \circ d^t)\xi + (d^t \circ \alpha)\xi = (d^t\alpha)\xi$ for $\xi \in \Gamma(E)$.

Note that $\det\left(\lambda I - R^\nabla/2\pi\sqrt{-1}\right)$ acts on a section $\xi_1 \wedge \cdots \wedge \xi_k$ of $\wedge^k E$ by

$$\det\left(\lambda I - \frac{R^\nabla}{2\pi\sqrt{-1}}\right)\xi_1 \wedge \cdots \wedge \xi_k = \left(\lambda I - \frac{R^\nabla}{2\pi\sqrt{-1}}\right)\xi_1 \wedge \cdots \wedge \left(\lambda I - \frac{R^\nabla}{2\pi\sqrt{-1}}\right)\xi_k.$$

Similarly, we define an $\mathrm{End}_\mathbb{C}(\wedge^k E)$-valued differential form φ by

$$\varphi := -\frac{k}{2\pi\sqrt{-1}}\int_0^1 \alpha \wedge \left(\lambda I - \frac{R^t}{2\pi\sqrt{-1}}\right) \wedge \cdots \wedge \left(\lambda I - \frac{R^t}{2\pi\sqrt{-1}}\right) dt.$$

Therefore, by recalling (3), the Bianchi identity, Leibniz rule, (4), and the fundamental theorem in calculus, we compute

$$\begin{aligned}
d\varphi &= -\frac{k}{2\pi\sqrt{-1}}\int_0^1 d\left(\alpha \wedge \left(\lambda I - \frac{R^t}{2\pi\sqrt{-1}}\right) \wedge \cdots \wedge \left(\lambda I - \frac{R^t}{2\pi\sqrt{-1}}\right)\right) dt \\
&= -\frac{k}{2\pi\sqrt{-1}}\int_0^1 d^t\left(\alpha \wedge \left(\lambda I - \frac{R^t}{2\pi\sqrt{-1}}\right) \wedge \cdots \wedge \left(\lambda I - \frac{R^t}{2\pi\sqrt{-1}}\right)\right) dt \\
&= -\frac{k}{2\pi\sqrt{-1}}\int_0^1 (d^t\alpha) \wedge \left(\lambda I - \frac{R^t}{2\pi\sqrt{-1}}\right) \wedge \cdots \wedge \left(\lambda I - \frac{R^t}{2\pi\sqrt{-1}}\right) dt \\
&= -\frac{k}{2\pi\sqrt{-1}}\int_0^1 \left(\frac{dR^t}{dt}\right) \wedge \left(\lambda I - \frac{R^t}{2\pi\sqrt{-1}}\right) \wedge \cdots \wedge \left(\lambda I - \frac{R^t}{2\pi\sqrt{-1}}\right) dt \\
&= k\int_0^1 \left(\frac{d}{dt}\left(\lambda I - \frac{R^t}{2\pi\sqrt{-1}}\right)\right) \wedge \left(\lambda I - \frac{R^t}{2\pi\sqrt{-1}}\right) \wedge \cdots \wedge \left(\lambda I - \frac{R^t}{2\pi\sqrt{-1}}\right) dt \\
&= \int_0^1 \frac{d}{dt}\left(\left(\lambda I - \frac{R^t}{2\pi\sqrt{-1}}\right) \wedge \left(\lambda I - \frac{R^t}{2\pi\sqrt{-1}}\right) \wedge \cdots \wedge \left(\lambda I - \frac{R^t}{2\pi\sqrt{-1}}\right)\right) dt \\
&= \det\left(\lambda I - \frac{R^{\nabla^1}}{2\pi\sqrt{-1}}\right) - \det\left(\lambda I - \frac{R^{\nabla^0}}{2\pi\sqrt{-1}}\right)
\end{aligned}$$

which proves the claim. $\square$

3. $c_i(E, \nabla)$ is a real form.

Proof. Define a hermitian metric on E and take a connection ∇ which is compatible with the metric. They do exist on any complex vector bundle, as explained in Chapter 5 of [M]. Then, by Gram-Schmidt process, we can reduce the bundle structure group from $GL_k(\mathbb{C})$ to $U(k)$, and the curvature R^∇ takes value in the Lie subalgebra $\mathfrak{u}(k) = \mathrm{Lie}(U(k))$ of the Lie algebra $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^k) = \mathfrak{gl}_k(\mathbb{C})$. Thus, R^∇ is a differential 2-form with values in skew-hermitian matrices.

Thus, assuming $\lambda \in \mathbb{R}$ without loss of generality,

$$\begin{aligned}\det\left(\lambda I - \frac{R^\nabla}{2\pi\sqrt{-1}}\right) &= \det\left(\lambda I - \frac{R^\nabla}{2\pi\sqrt{-1}}\right)^t = \det\left(\lambda I - \frac{(R^\nabla)^t}{2\pi\sqrt{-1}}\right) \\ &= \det\left(\lambda I + \frac{\overline{R^\nabla}}{2\pi\sqrt{-1}}\right) = \det\overline{\left(\lambda I - \frac{R^\nabla}{2\pi\sqrt{-1}}\right)} = \overline{\det\left(\lambda I - \frac{R^\nabla}{2\pi\sqrt{-1}}\right)}\end{aligned}$$

which proves the claim, and hence completes the proof of Theorem 2.2. □

4 References

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