Review of Fourier transforms

Note - a lot of what we discuss here applies to Laplace transforms as well by a 90 degree rotation in the complex plane. I will focus on Fourier here as it this directly relevant to the Heston model to be considered later. But watch out for issues to do with deciding which half-plane the inversion contour needs to be placed in (left or right for Laplace, upper or lower for Fourier), as this is problematic in financial applications. I will come back to that if I get time.

Definition of the Fourier transform

Their are many different definitions of the Fourier transform in the literature, all of which are related by numerical normalization factors involving 2π and -1. Try not to let this worry you - you just need to be careful about compiling information from different sources. Given a function $f(x)$ defined for $-\infty < x < \infty$, we set

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx \tag{17.1}$$

One of the main things we want to think about is trying to work this out for different classes of function f. Option-pricing requires us to think outside the usual "box" of functions for which the transform exists (e.g. square-integrable is sometimes stated). We shall need to get a proper grip on this before talking about Heston's model, and in particular along the way we shall confront the initially surprising fact that vanilla call and put payoffs have functionally identical Fourier transforms!

An informal look at the delta-function

Those of you with a background in vector spaces may be aware of the concept of a "dual space". This is the set of linear mappings from a vector space to the underlying field, e.g. the set of real numbers. When dealing with finite-dimensional vector spaces the dual space has the same dimension as the original space, and if we have some notion of distance we can make an isomorphism between a space and its dual (think about ordinary dot products in three real dimensions).

When dealing with infinite-dimensional function spaces, it all gets more complicated. The dual space is much bigger than the original function space (which is already infinite in dimension), and it is called the set of 'distributions'. There are things in the dual space that are not like ordinary functions. This takes us straight to the notion of a 'delta-function', which is not really a function at all! Pick a point a in the interval on which the functions are defined, and set

$$\Delta_a[f] = f(a) \tag{17.2}$$

This is a perfectly valid linear mapping that does not arise through the integration of f against any other continuous function (the dot product for function space). What we do now is to invent a representation of such distributions that *looks like* an ordinary function, and call it

$$\delta(x-a) \tag{17.3}$$

the *delta-function, or δ-function, with support at a*. It has the following properties:

$$\delta(x-a) = 0 \quad \text{if } x \neq a \tag{17.4}$$

$$\int_{\alpha}^{\beta} f(x)\delta(x-a)\, dx = f(a) \tag{17.5}$$

provided $\alpha < a < \beta$. You can think of $\delta(x-a)$ as being an 'infinite spike' located at a that integrates to unity.

■ The δ-function as the limit of a sequence of functions

Although this object is not really a function at all, it is very convenient to think of it in terms of the limit of a sequence of functions. This 'limit' is not the limit in any of the senses of classical analysis - it is only to be applied under integration. It does not actually matter what particular sequence of functions is employed, provided that they are continuous and integrate to unity over the entire real line. Let's introduce two sequences that do the job. First, let's define Δ_1 as follows.

```
Δ1[ϵ_, x_, a_] := 1 / (ϵ Sqrt[2 Pi]) Exp[-(x - a)^2 / (2 ϵ^2)]
```

In standard mathematical notation (we have also converted the output) it is

```
TraditionalForm[Δ1[ϵ, x, a]]
```

$$\frac{e^{-\frac{(x-a)^2}{2\epsilon^2}}}{\sqrt{2\pi}\,\epsilon} \tag{17.6}$$

So this is just a Gaussian function. It integrates to unity provided ϵ is real and positive:

```
Integrate[Δ1[ϵ, x, a], {x, -Infinity, Infinity}, Assumptions → ϵ > 0]
```

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Students of probability or statistics will recognize this as the density function associated with a normal distribution with mean a and standard deviation ϵ. The second function we shall introduce is

```
Δ2[ϵ_, x_, a_] := ϵ / Pi / ((x - a)^2 + ϵ^2)
```

In mathematical notation (the output is converted here also) this is

$$\frac{\epsilon}{\pi\left((x-a)^2+\epsilon^2\right)} \tag{17.7}$$

```
Integrate[Δ2[ϵ, x, a], {x, -Infinity, Infinity},
 Assumptions → {Im[a] == 0, ϵ > 0}]
```

1

Students of probability or statistics will recognize this as the density function associated with a Cauchy distribution centred on a, parametrized by ϵ. Both of these functions have the property that they integrate to unity, and are peaked at a. (You might like to plot these functions using *Mathematica*.) As the parameter ϵ tends to zero, these functions become more strongly peaked at a. The idea is that their limiting form is precisely that of a δ-function. You can get a better grip on this by considering integrating either function against a 'test-function' f. We can write (from now on the integration range is fixed as the entire real line):

$$\begin{aligned}
&\int_{-\infty}^{\infty} f(x)\Delta_i(\epsilon, x, a)\,dx \\
&= \int_{-\infty}^{\infty} (f(x)-f(a))\Delta_i(\epsilon, x, a)\,dx + \int_{-\infty}^{\infty} f(a)\,\Delta_i(\epsilon, x, a)\,dx \\
&= f(a)\int_{-\infty}^{\infty} \Delta_i(\epsilon, x, a)\,dx + \int_{-\infty}^{\infty} (f(x)-f(a))\Delta_i(\epsilon, x, a)\,dx \\
&= f(a) + \int_{-\infty}^{\infty} (f(x)-f(a))\Delta_i[\epsilon, x, a]\,dx
\end{aligned} \tag{17.8}$$

Now consider what happens as $\epsilon \to 0$. The last integral has an integrand that is zero at $x = a$, because of the factor $f(x) - f(a)$, but the Δ-function concentrates itself at this point, becoming zero elsewhere. Some careful analysis shows that this latter term tends to zero, leaving us with just $f(a)$ when $\epsilon = 0$. So these Δ-functions do have a limit that is a δ-function, when all 'limits' are taken assuming an integration is being carried out.

Having got a grip on the δ-function, we need to say what it has to do with Fourier transforms. The concept we are after is to define the Fourier transform of unity, i.e,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 1\, e^{i\omega x}\, dx \tag{17.9}$$

This does not exist at all in the usual sense, but it has a very simple interpretation once distributions are introduced. The way to get at this is very simple. We replace '1' in the integral by a function whose limit is unity. There are several choices, but the one that is most convenient is to consider, for $\epsilon > 0$,

$$\frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} e^{-\epsilon |x|}\, e^{i\omega x}\, dx \tag{17.10}$$

This can be done by pen-and-paper in two pieces, or we can get *Mathematica* to sort it out. Note that we have said that the imaginary part of ω mustg be less than ϵ in magnitude. Think about why this must be true for the integral in Eq. (17.15) to converge.

```
1/Sqrt[2 Pi]Integrate[Exp[-ϵ*Abs[x]]*Exp[I*ω*x],
  {x, -Infinity, Infinity}, Assumptions → {ϵ>0, -ϵ < Im[ω] < ϵ}]
```

$$\frac{\sqrt{\frac{2}{\pi}}\ \epsilon}{\epsilon^2 + \omega^2}$$

which is precisely

$$\sqrt{2\pi}\ \Delta_2(\epsilon, \omega, 0) \tag{17.11}$$

That is, we have established that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\epsilon |x|} e^{i\omega x}\, dx = \sqrt{2\pi}\ \Delta_2(\epsilon, \omega, 0) \tag{17.12}$$

Under any subsquent integration over ω, we can let $\epsilon \to 0$, and hence assert that, *as a distribution*,

$$\int_{-\infty}^{\infty} e^{i\omega x} dx = 2\pi\delta(\omega) \tag{17.13}$$

Eq. (17.13) gives the fundamental link between the Fourier transform and the δ-function.

Inversion, convolution, shifting and differentiation

We are now in a position to give informal distributional proofs of the key results - the inversion theorem and the convolution theorem. These are stated with our definitional convention given in Eq. (17.1).

■ The inversion theorem

Suppose that Eq. (17.1) *holds, i.e.*

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx$$

Then the inversion theorem states that

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega x}\, d\omega \tag{17.14}$$

Assuming the distributional result from Eq. (17.13) and that some reordering of integrals is possible, we can give a simple proof. We have:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{-i\omega x}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(y)e^{i\omega y}\,dy\right)e^{-i\omega x}\,d\omega$$
$$= \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega(y-x)}\,d\omega\right)f(y)\,dy = \int_{-\infty}^{\infty}\delta(y-x)\,\phi(y)\,dy = f(x) \tag{17.15}$$

■ The convolution theorem

Suppose we have two transforms:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{i\omega x}\,dx;\ \hat{g}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(x)e^{i\omega x}\,dx \tag{17.16}$$

The convolution of f with g, $[f*g](x)$, is defined by,

$$[f*g](x) = \int_{-\infty}^{\infty}f(y)\,g(x-y)\,dy \tag{17.17}$$

If $h(x) = [f*g](x)$, then the Fourier transform of the convolution is

$$\hat{h}(\omega) = \sqrt{2\pi}\ \hat{f}(\omega)\,\hat{g}(\omega) \tag{17.18}$$

Here is the proof.

$$\sqrt{2\pi}\ \hat{h}(\omega) = \int_{-\infty}^{\infty}e^{i\omega x}h(x)\,dx$$
$$= \int_{-\infty}^{\infty}e^{i\omega x}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(p)e^{-ipy}dp\right)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(q)e^{-iq(x-y)}dq\right)dy\,dx \tag{17.19}$$
$$= \frac{1}{(2\pi)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dp\,dq\,dy\,dx\,e^{i\omega x}e^{-ipy}\,e^{-iq(x-y)}\hat{f}(p)\hat{g}(q)$$

Doing the y-integration reduces this to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dp\,dq\,d\mathrm{x}e^{i\omega x}\,e^{-iqx}\,\hat{f}(p)\hat{g}(q)\delta(q-p) \tag{17.20}$$

We now use the δ-function to do the q integration, leaving us with

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dp\,d\mathrm{x}e^{i(\omega-p)x}\,\hat{f}(p)\hat{g}(p) \tag{17.21}$$

Now we integrate by x, to obtain

$$2\pi\int_{-\infty}^{\infty}dp\delta(\omega-p)\,\hat{f}(p)\hat{g}(p)\ = 2\pi\,\hat{f}(\omega)\hat{g}(\omega) \tag{17.22}$$

So we have established that

$$\hat{h}(\omega) = \sqrt{2\pi}\ \hat{f}(\omega)\,\hat{g}(\omega) \tag{17.23}$$

That is, the Fourier transform of the convolution is essentially (up to a normalization) the product of the transforms. Similarly, if we have the product of two functions in x terms, the Fourier transform of such a product can be written as the convolution of the transforms.

■ The shift and scaling theorems

The shift theorem. This is simply the observation that *if*

$$h(x) = e^{ixa}\,f(x) \tag{17.24}$$

then

$$\hat{h}(\omega) = \hat{f}(a+\omega) \tag{17.25}$$

The scaling theorem. This is the result that *if*

$$h(x) = f(x/a) \tag{17.26}$$

then

$$\hat{h}(\omega) = a\hat{f}(\omega a) \tag{17.27}$$

Note also that the Fourier transform is linear. If

$$h(x) = \alpha\, f(x) + \beta\, g(x) \tag{17.28}$$

then the transform of h satisfies

$$\hat{h}(\omega) = \alpha\,\hat{f}(\omega) + \beta\,\hat{g}(\omega) \tag{17.29}$$

■ The differentiation theorem

For applications in mathematical physics and finance the critial observation is the manner in which differentiation with respect to x becomes simple multiplication by $(-i\omega)$ on the transform. *Let*

$$h(x) = f'(x) \tag{17.30}$$

then

$$\hat{h}(\omega) = -i\omega\hat{f}(\omega) \tag{17.31}$$

To prove this, we note that this is a point at which we must be a little less cavalier about the class of functions we are dealing with. From the definition

$$\hat{h}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)\, e^{i\omega x}\, dx \tag{17.32}$$

and we need to be able to write:

$$\hat{h}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{\partial\left(f(x)\, e^{i\omega x}\right)}{\partial x} - i\omega f(x)\, e^{i\omega x}\right) dx \tag{17.33}$$

and use integration to kill the first term - this requires of course that $f \to 0$ as $x \to \pm\infty$. Then we obtain

$$\hat{h}(\omega) = -i\,\omega\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx = -i\omega\hat{f}(\omega) \tag{17.34}$$

Provided higher derivatives tend to zero at $\pm\infty$ and the derivatives of the function remain integrable (continuous will do), repeated application of this result can be used to show that the Fourier transform of the nth derivative is given by

$$\widehat{\left(f^{(n)}\right)}[\omega] = (-i\omega)^n\,\hat{f}(\omega) \tag{17.35}$$

Note that if the opposite sign convention is employed for the exponent, the right side of this becomes $(i\omega)^n\,\hat{f}(\omega)$. There is a corresponding inverse result that multiplication by x corresponds to differentiation with respect to ω - see Exercise 17.3 for the details.

Jordan's lemma: semicircle theorem II

It is evident that we need general methods for computing integrals of the form

$$\int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx$$

A key result for the evaluation of such integrals is Jordan's lemma, which gives very useful conditions under which the integration region may be completed by a large semicircle in the upper (or lower) half-plane (UHP or LHP), and hence evaluated by the calculus of residues. Let's work with the upper half-plane version, where we assume that $\omega > 0$.

Theorem 17.6: Jordan's lemma. *Consider the semicircular path*

$$\Phi_R[t] = R e^{it} \qquad 0 \le t \le \pi \tag{17.36}$$

and let M_R be the maximum (formally, the supremum) of $|f(z)|$ on the image of Φ_R in $\mathbb{C}$. Suppose that $\omega > 0$ and that as $R \to \infty$,

$$M_R \to 0 \tag{17.37}$$

Then Jordan's lemma states that

$$\int_{\Phi_R} f(z) \, e^{i\omega z} \, dz \to 0 \tag{17.38}$$

as $R \to \infty$.

■ Comments

(1) The condition on f is very weak - we just need that the function tends to zero.

(2) If $\omega > 0$ then as the imaginary part of z becomes large the integrand is exponentially damped - this is why we need to associate positive ω with the *upper* half-plane. If $\omega < 0$ there is an obvious corresponding result for the *lower* half-plane.

(3) Once we have this result the answer for the integral for $\omega > 0$ is just

$$2\pi i \sum_{\text{UHP}} \text{Res}\left[f(z) e^{i\omega z}\right] \tag{17.39}$$

and for $\omega < 0$ it is

$$-2\pi i \sum_{\text{LHP}} \text{Res}\left[f(z) e^{i\omega z}\right] \tag{17.40}$$

Note the additional minus sign in Eq. (17.44) - we traverse the LHP contour clockwise. These can give quite different functional forms for the answer, and this is important.

■ Proof of Jordan's lemma

Let's first write the integration in terms of an integral over the path parameter t. We apply the integration inequality from Section 11.4:

$$\left| \int_{\Phi_R} F(z) e^{i\omega z} \, dz \right| \leqslant M_R \int_0^{\pi} \left| e^{i\omega R e^{it}} i R e^{it} \right| dt \tag{17.41}$$

But

$$\left| e^{i\omega R e^{it}} i R e^{it} \right| = R \left| e^{i\omega R(\cos(t) + i\sin(t))} \right| = R e^{-\omega R \sin(t)} \tag{17.42}$$

so that

$$\left| \int_{\Phi_R} F(z) e^{i\omega z} \, dz \right| \leqslant R M_R \int_0^{\pi} e^{-\omega R \sin(t)} \, dt = 2 R M_R \int_0^{\pi/2} e^{-\omega R \sin(t)} \, dt \tag{17.43}$$

Now we have the inequality, valid for $0 \leqslant t \leqslant \pi/2$,

$$\sin(t) \geq \frac{2\,t}{\pi} \tag{17.44}$$

(some plots with *Mathematica*, and the concave nature of the sine function in this range of t, should quickly convince you of this) and so we can state that

$$\left|\int_{\Phi_R} F(z) e^{i\omega z}\, dz\right| \leq 2RM_R \int_0^{\pi/2} e^{-2\omega R t/\pi}\, dt \;=\; 2RM_R\left(\frac{\pi\left(1-e^{-R\omega}\right)}{2R\omega}\right) \tag{17.45}$$

$$\left|\int_{\Phi_R} F(z) e^{i\omega z}\, dz\right| \leq \frac{\pi M_R\left(1-e^{-R\omega}\right)}{\omega} \leq \frac{\pi M_R}{\omega} \tag{17.46}$$

which tends to zero as required.

Examples of transforms

A large number of Fourier transforms can now be generated by combining together:

(1) the basic distributional results;
(2) the shift, scaling and differentiation theorems;
(3) applications of Jordan's lemma and the calculus of residues;
(4) other contours invented for various special cases;
(5) *Mathematica* calculations.

It must be appreciated that one or a combination of these methods must be employed.

■ An example using the basic distributional properties

What is the Fourier transform of $f(x) = \sin(a\,x)$? This, by definition, is

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(ax) e^{i\omega x}\, dx \tag{17.47}$$

This is best approached by writing the integrand in terms of pure exponential functions:

$$\begin{aligned}
\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\left(\frac{e^{iax}-e^{-iax}}{2i}\right) e^{i\omega x}\, dx \\
&= \frac{1}{2i}\,\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\left(e^{i(\omega+a)x} - e^{i(\omega-a)x}\right) dx \\
&= -i\,\pi\,\frac{1}{\sqrt{2\pi}}\left(\delta(a+\omega)-\delta(\omega-a)\right) = i\sqrt{\frac{\pi}{2}}\left(\delta(\omega-a)-\delta(a+\omega)\right)
\end{aligned} \tag{17.48}$$

■ Example using Jordan's lemma

Let's consider the function we have used before in defining δ-functions:

$$\frac{\epsilon}{\pi\left(x^2+\epsilon^2\right)} \tag{17.49}$$

We can now evaluate its transform very quickly, for this function tends to zero for large z, and hence we can apply Jordan's lemma immediately. For $\omega > 0$ the transform is given by

$$\frac{1}{\sqrt{2\pi}}\, 2\pi i \sum_{\text{UHP}} \text{Res}\Big[\frac{\epsilon}{\pi\left(z^2+\epsilon^2\right)}\, e^{i\omega z}\Big] = \frac{1}{\sqrt{2\pi}} \sum_{\text{UHP}} \text{Res}\Big[\frac{2i\epsilon\, e^{i\omega z}}{(z-i\,\epsilon)\,(z+i\,\epsilon)}\Big] \tag{17.50}$$

In the UHP there is only one pole, at $z = i\epsilon$, and it is simple. So we cover up the one singular factor and evaluate the remaining expression to obtain

$$\frac{1}{\sqrt{2\pi}}\, e^{-\epsilon\omega} \tag{17.51}$$

For $\omega < 0$ a similar calculation can be done in the LHP to obtain $e^{\epsilon\,\omega}$. Thus the answer for all real ω (it integrates to $1\big/\sqrt{2\pi}$ if ω=0) is just

$$\frac{1}{\sqrt{2\pi}}\, e^{-\epsilon|\omega|} \tag{17.52}$$

■ An inverse transform using Jordan's lemma

Consider the transform function

$$\hat{f}(\omega) = \frac{i}{\omega + i a} \tag{17.53}$$

What is the associated $f(x)$? It is given by the inversion formula

$$f(x) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\omega + i a}\, e^{-i\omega x}\, d\omega \tag{17.54}$$

Note that now we are integrating over ω and the parameter in Jordan's Lemma is $-x$. The non-exponential part of the integral tends to zero at infinity so we can go ahead and apply the lemma. For $x > 0$ we must complete in the *lower* half-plane. There is one simple pole and we get the answer

$$f(x) = \frac{i}{\sqrt{2\pi}}\, (-2\pi i)\, e^{-i\,(-i a) x} \;=\; \sqrt{2\pi}\;\, e^{-a x} \tag{17.55}$$

But for $x < 0$ we must complete in the upper half-plane, where there are no poles at all! Hence the answer is then zero.

■ A special contour for Gaussian functions

The transforms of some functions require special treatment. Consider the Gaussian function

$$\frac{e^{-(x-\mu)^2/\left(2\sigma^2\right)}}{\sigma\sqrt{2\pi}} \tag{17.56}$$

We have written it this way in order to make the link with the characteristic function for the normal distribution with mean μ and standard deviation σ. The Fourier transform of this is

$$\hat{f}(\omega) = \frac{1}{2\,\pi\,\sigma} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/\left(2\sigma^2\right)}\, e^{i\,\omega\,x}\, dx \tag{17.57}$$

The term in the exponential is

$$i\omega x - \frac{(x-\mu)^2}{2\sigma^2} \;= -\frac{x^2}{2\sigma^2} + \left(\frac{\mu}{\sigma^2} + i\omega\right)x - \frac{\mu^2}{2\sigma^2}$$
$$= \frac{-x^2 - \mu^2 + 2x\big(\mu + i\sigma^2\omega\big)}{2\sigma^2}$$

$$= \frac{-\left(x-\left(\mu+i\sigma^2\omega\right)\right)^2-\mu^2+\left(\mu+i\sigma^2\omega\right)^2}{2\sigma^2}$$

$$= \frac{-\left(x-\left(\mu+i\sigma^2\omega\right)\right)^2}{2\sigma^2}+i\mu\omega-\frac{\sigma^2\omega^2}{2}$$

$$= \frac{-p^2}{2\sigma^2}+i\mu\omega-\frac{\sigma^2\omega^2}{2}$$

where p is the complex shifted variable

$$p = x-\left(i\omega\sigma^2+\mu\right) \tag{17.59}$$

We can make a real change of variables to eliminate μ, but what do we do about the imaginary shift? We can write the result so far as

$$e^{i\omega\mu-\sigma^2\omega^2/2}\,J \tag{17.60}$$

where the quantity J is

$$J = \frac{1}{2\pi\sigma}\int_{-\infty-i\omega\sigma^2}^{\infty-i\omega\sigma^2} e^{-q^2/(2\sigma^2)}\,dx \tag{17.61}$$

Now consider a rectangular contour obtained by taking the contour in the definition of J, and adding a piece coming backwards along the real axis, joined at both ends, to form a rectangle. We observe:

(1) there is no contribution from the vertical contours, as the integrand tends to zero;
(2) there are no poles inside the rectangle.

By Cauchy's theorem, the total integral must be zero. Hence we note that

$$J = \frac{1}{2\pi\sigma}\int_{-\infty}^{\infty} e^{-q^2/(2\sigma^2)}\,dx = \frac{1}{\sqrt{2\pi}}\,\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-q^2/(2\sigma^2)}\,dx = \frac{1}{\sqrt{2\pi}} \tag{17.62}$$

and hence that

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\,e^{i\omega\mu-\sigma^2\omega^2/2} \tag{17.63}$$

This is almost the 'characteristic function' for the normal distribution, which omits the factor $1\big/\sqrt{2\pi}$.

Expanding the setting to a fully complex picture

Complex numbers actually play other roles in the management of Fourier transforms. It is not just a matter of finding the values of integrals using Jordan's lemma and applying the calculus of residues. In fact, we really need to see transforms as being defined for complex values of ω. Why should we bother with this? In fact it allows us to cope with transforms of a rather larger class of functions, and furthermore to regard Laplace transforms and Fourier transforms as being related in a rather trivial fashion, by a 90-degree rotation in the complex plane. Furthermore, the application of Jordan's lemma requires that the function being integrated is actually holomorphic - when is this so?

In this subsection the independent variable will be taken to be t rather than x. This is partly to make the link with Laplace transforms easier, as we shall be concerned here mainly with Fourier transforms of functions that are identically zero for $t < 0$.

As a motivating example, consider the function

$$f(t) = \begin{cases} 0; & t < 0 \\ e^{at}; & t > 0 \end{cases} \tag{17.64}$$

where we make no particular requirement on the sign of the real parameter a. The Fourier transform is just

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{(a+i\omega)t}\, dt \tag{17.65}$$

This integral is given by evaluating the difference in the values at the limits of the indefinite integral

$$\frac{1}{\sqrt{2\pi}} \frac{e^{t(a+i\omega)}}{a+i\omega} \tag{17.66}$$

Under what circumstances does the limit of this, for $t \to \infty$, exist and equal zero? We need the real part of

$$a + i\omega \tag{17.67}$$

to be less than zero, i.e.

$$a < \mathrm{Im}(\omega) \tag{17.68}$$

Under these circumstances the transform exists and equals

$$-\frac{1}{\sqrt{2\pi}} \frac{1}{(a+i\omega)} \tag{17.69}$$

So it is convenient to allow the transform variable ω to be complex. The transform exists provided the imaginary part is large enough to kill the exponential growth in the function being transformed.

The inversion theorem must be adjusted accordingly - we need to do the inversion integration by integating along any horizontal contour *above* $\mathrm{Im}[\omega] = a$. We shall not give a proof of the following remark, but it turns out that this type of behaviour is absolutely typical for the transforms of functions that are zero for $t < 0$. More specifically, let $f(t)$ be zero for $t < 0$ and satisfy a condition that

$$|f(t)| \leq K e^{at} \tag{17.70}$$

for $t > 0$ and some real $K > 0$ and real a. Then the Fourier transform exists and is a holomorphic function of ω for $\mathrm{Im}(\omega) > a$ - the upper half-plane above a - see Dettman (1984) for a discussion of this, and other related properties. The inversion takes place along a horizontal contour in the half-plane above a. More generally still, for functions that are not zero for $t < 0$, there will typically be a strip in which the transform is holomorphic. See Exercise 17.7 for an example.