LGS Notes: Classical Finite Differences

Draft of Chapter 13 of Modelling Financial Derivatives with *Mathematica* 2nd edition

Scope of Discussion

■ PDE methods encourage use of advanced FD methods

■ Explain advantages in context of heat equation, Black-Scholes reducible problems

■ Theoretical - operator approach, truncation and stability

■ Issues with Greeks

13.1 A Reminder from Basic Analysis

When we value derivative financial instruments, we usually need to establish the value of not just a function of many variables, but also several of its first and second derivatives. The complications and additional discipline that the computation of such "Greeks" imposes seem not to be widely appreciated. To appreciate what is happening it is a good idea to temporarily forget all about financial derivatives and remind ourselves of some results from basic analysis.

In the following you may assume that x denotes, for example, an asset price or interest rate, and that n denotes the number of nodes in a numerical grid. The function $f_n(x)$ is the difference between the exact solution and that which is computed numerically on a grid.

In one sense this whole course is about controlling f_n, the error arising from a numerical scheme, and making it tend to zero in a controlled fashion. There is a nasty issue I want to highlight right away. Its significance will emerge during the course.

■ Non-uniform Convergence 101

Suppose we have a sequence of functions $f_n(x)$, $n = 1, 2, 3, \ldots$, with the property that for each x,

$$f_n(x) \to 0 \tag{13.1}$$

as

$$n \to \infty \tag{13.2}$$

In this case a mathematical analyst would say that f converges pointwise to zero. The question arises what happens to the derivative of f_n as n becomes large. Naively one might expect that also becomes small as the function becomes small. Indeed, many functions satisfy this requirement, such as, for x in some finite interval,

$$\frac{x^m}{n} \qquad \frac{\sin(x)}{n} \tag{13.3}$$

Unfortunately, it is not always true. The following classic example makes this clear. We consider the function

$$f_n(x) = \frac{\sin(n\,x)}{n} \tag{13.4}$$

Its first derivative, or "Delta", is then

$$\frac{\partial f_n(x)}{\partial x} = \cos(n\,x) \tag{13.5}$$

Its second derivative, or "gamma", is then

$$\frac{\partial^2 f_n(x)}{\partial x^2} = -n\sin(n\,x) \tag{13.6}$$

So we have the situation where although the function goes to zero, its Delta remains of $O(1)$, and its Gamma is $O(n)$ and tends to infinity. This is important to us when we regard f as the error arising in some numerical scheme. In other words, the following situation is perfectly possible:

Small error in function ⇒ possible moderate error in first derivative δ ⇒ possible huge error in second derivative Γ.

There are other strange things that can happen with sequences of functions, although the issue with differentiation is of the most importance to financial derivatives modelling. There is a type of convergence called uniform convergence, which places stringent limits on the behaviour of functions which ensure that derivatives are as controlled as the function value itself.

For our purposes it is more important to appreciate when we consider the function f to represent the errors in some numerical scheme, where n is related to some grid or tree parameter, there may be substantial errors in our Greeks even when a close inspection of the valuation suggests all is well.

This sort of problem is of considerable importance and I highlight it here as it (a) is routinely ignored "on the desk", and in most textbooks, (b) can results in major errors in risk information (c) comes up remarkably often as a result of the non-smooth initial data in many financial problems causing oscillations.

■ How do we differentiate anyway?

We also have to figure out how to compute "Greeks" at all from a function defined only on a grid, given that partial derivatives are limits. This is a separate matter from the one highlighted above, and also needs to be given special attention.

13.2 From Black Scholes to Diffusion

■ Limits field of application, but there is lots of theoretical work done, a significant fraction of which is not generally known and under-used

We can do some proper numerical analysis on this class of problems. This is an ideal level of sophistication to which one should strive when tackling harder problems.

We consider first the partial differential equation:

$$\frac{\partial V}{\partial t} + S\,(r-q)\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 V}{\partial S^2} - r\,V = 0 \tag{13.7}$$

Let K be any suitable base value for S. It could be the strike of an option, for example. Set

$$S = K\,e^x \tag{13.8}$$

and observe that for any function f

$$S\,\frac{\partial f(S)}{\partial S} = \frac{\partial f(S)}{\partial x} \tag{13.9}$$

the time-dependent Black-Scholes equation becomes

$$\frac{\partial V}{\partial t} + \left(r - q - \frac{\sigma^2}{2}\right) \frac{\partial V}{\partial x} + \frac{1}{2} \frac{\partial^2 V}{\partial x^2} \sigma^2 - r\,V = 0 \tag{13.10}$$

Making one further re-arrangement, this becomes,

$$\frac{\partial^2 V}{\partial x^2} - k_1\, V + (k_2 - 1) \frac{\partial V}{\partial x} = -\frac{2}{\sigma^2} \frac{\partial V}{\partial t} \tag{13.11}$$

$$k_1 = \frac{2\,r}{\sigma^2} \tag{13.12}$$

$$k_2 = \frac{2\,(r - q)}{\sigma^2} \tag{13.13}$$

The next step is to re-scale the time variable. Generally, we are interested in an instrument with an expiry or maturity at a time T in the future. Bearing this in mind, we set, assuming that the volatility is constant in both x and t,

$$\tau = \frac{1}{2} \sigma^2 \,(T - t) \tag{13.14}$$

It is clear that if the volatility depends only on time, we can work more generally with

$$\tau = \frac{1}{2} \int_t^T \sigma^2(t')\, dt' \tag{13.15}$$

Either way, we have arrived at

$$\frac{\partial^2 V}{\partial x^2} - k_1\, V + (k_2 - 1) \frac{\partial V}{\partial x} = \frac{\partial V}{\partial \tau} \tag{13.16}$$

How one proceeds next depends on whether we can regard k_i as constant. If we can, matters are rather trivial, for writing

$$V(S, t) = C\, e^{-\frac{1}{2}(k_2 - 1)\,x - \left(\frac{1}{4}(k_2 - 1)^2 + k_1\right)\tau}\; u(x, \tau) \tag{13.17}$$

eliminates several of the remaining terms, so that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial \tau} \tag{13.18}$$

This is a very important observation, for both analytical and numerical approaches. From the analytical point of view, you should appreciate that many of the vanilla instruments in common use can be priced using rather basic and very old solution techniques for the heat equation:

Separation of Variables - Log and Power contracts;

Green's function methods - Calls, Puts, Binaries;

Method of Images (zero boundary conditions) - Single and Double Barrier options;

Rebates - Duhamel integrals for heat equation

Impedance Boundary conditions - Lookback options (Riemann's solution with a change of variable!).

More complex drifts can be eliminated by grid-skewing - allows for easy management of discrete dividends.

13.3 Simple CB model (time-dependent rates; dividends; coupons)

An example of rather more substance and considerable practical importance is obtained by allowing interest rates to become time-dependent, and allowing the dividends to become time-dependent (in particular discrete), and at the same time admitting coupons. This allows us to start to consider simple models of convertible bonds, where we fold in yield curve information as well as equity volatility. We should also add credit if we want to be completely realistic, going to a full stochastic IR treatment appears to be less important than getting the credit and equity vol aspects rights. So what happens to the heat equation when we have all these complications? Not a lot if you do it right. The following, I believe, originates with Harper:

Harper, J., 1994, Reducing parabolic partial differential equations to canonical form, European Journal of Applied Mathematics, 5, p. 159.

the title of which says it all!

So we consider now

$$\frac{1}{2} S^2 \frac{\partial^2 V}{\partial S^2} \sigma^2(t) - r(t) V + \frac{\partial V}{\partial t} + (r(t) - q(t)) S \frac{\partial V}{\partial S} + K(S, t) = 0 \tag{13.19}$$

Proceeding as before leads to

$$\frac{\partial V}{\partial \tau} + \mathrm{k}_1 V = \frac{\partial^2 V}{\partial x^2} + (\mathrm{k}_2 - 1) \frac{\partial V}{\partial x} + \frac{2}{\sigma^2} K \tag{13.20}$$

where k_i are now the time-dependent functions

$$k_1 = \frac{2 r}{\sigma^2} \tag{13.21}$$

$$k_2 = \frac{2 (r - q)}{\sigma^2} \tag{13.22}$$

Thus far the change of variables has proceeded as for the European options discussed previously. The final part of the previous analysis does not work for the present problem unless $k_2[\tau]$ is constant. To treat the time-dependent case, let

$$z = x + F(\tau) \tag{13.23}$$

and set

$$V(x, \tau) = u(z, \tau) e^{-B(\tau)} \tag{13.24}$$

Making the required change of variable leads to

$$u\left(\mathrm{k}_1 - \frac{\partial B}{\partial \tau}\right) + \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial z^2} + \frac{2\,\mathrm{K} e^{B(\tau)}}{\sigma^2} + \left(-\frac{\partial F}{\partial \tau} + \mathrm{k}_2 - 1\right) \frac{\partial V}{\partial x} \tag{13.25}$$

so we can reduce the problem by making the choices

$$k_1[\tau] = \frac{\partial B}{\partial \tau} \tag{13.26}$$

$$k_2[\tau] - 1 = \frac{\partial F}{\partial \tau} \tag{13.27}$$

whence we get

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial z^2} + Q(\tau) \tag{13.28}$$

where the diffusion equation now has a "source term" Q representing the coupon payments in these coordinates:

$$Q(\tau) = \frac{2\,K\,e^{B(\tau)}}{\sigma^2} \tag{13.29}$$

The quantities B and F are like integrating factors (B is precisely this for the straight bond). The general plan is therefore:
(1) Integrate to obtain F and B;
(2) Solve the PDE given the coupon payments (and for a CB, call/put/exercise conditions), using any suitable analytical or numerical scheme.
In practice we need some supplementary analysis to understand how to complete these steps, and to handle the boundary condition arising from considering the straight bond.
Note how we have made yet another link with the heat equation in a traditional form. The coupon payments in a bond are like a source of body heat for the heat equation.

■ Exercise

Solve the supplementary conditions and find B and F. What happens when q is discrete, representing dividend payments at particular times?

13.4 Cox-Ingersoll-Ross/Constant Elasticity models

The type of stochastic process under consideration is probably more familiar in the CIR interest-rate model. The CIR world is governed by an interest-rate diffusion of the form

$$d\,r = a(b - r)\,d\,t + \sigma\sqrt{r}\;d\,z \tag{13.30}$$

This describes a mean-reverting process. It has nice analytical properties, such as the closed-form solution for bond options. On the other hand, if we set $b = 0$, and $r \to S,\ a \to (q - r)$, we get a process of the form

$$d\,S = (r - q)\,S\,d\,t + \sigma\sqrt{S}\;d\,z \tag{13.31}$$

We can write (2) in the form

$$\frac{d\,S}{S} = (r - q)\,d\;t + \frac{\sigma}{\sqrt{S}}\,d\,z \tag{13.32}$$

which is an equity-like process with a volatility varying as the inverse square root of the stock price. This is commonly known as an SRCEV model - "Square root constant elasticity of variance". There are more general CEV models, where the process takes the form

$$\frac{d\,S}{S} = (r - q)\,d\;t + \frac{\sigma}{S^{1-\alpha}}\,d\,z \tag{13.33}$$

The Black-Scholes equation with time-dependent interest rates and dividend yield, and volatility $\sigma(S)$, is

$$\frac{\partial V}{\partial t} - r(t)\,V + \frac{1}{2}\,\sigma^2(S)\,S^2\,\frac{\partial^2 V}{\partial S^2} + (r(t) - q(t))\,S\,\frac{\partial V}{\partial S} = 0 \tag{13.34}$$

■ Exercise

Starting with the transformations used for the CB model, investigate how close you can get to the diffusion equation.

13.5 Differentiation on a Grid

We introduce a discrete grid with steps $\Delta\tau = k,\ \Delta\,x = h$, where $\Delta\,x$ is the grid step for the (log) stock price, and $\Delta\tau$ is the grid step for the time, and set

$$u_n^m = u(m\,\Delta\tau, n\,\Delta\,x) \tag{13.35}$$

All the difference schemes involve a parameter α that is given by

$$\alpha = \frac{\Delta\tau}{\Delta x^2} = \frac{k}{h^2} \tag{13.36}$$

What we need to establish first are some relations for approximating derivatives on a grid. We allow ourselves to consider a refined grid, so that e.g. $u^m_{n+\frac{1}{2}}$ makes sense.

■ The Difference Operators

Let's go back to one dimension and consider Taylor's Theorem in the form

$$f(h+x) = f(x) + h\,\frac{\partial f}{\partial x} + \frac{1}{2}\,h^2\,\frac{\partial^2 f}{\partial x^2} + \frac{h^3}{3!}\,\frac{\partial^3 f}{\partial x^3} + \ldots \tag{13.37}$$

Introduce the operator:

$$D\,f = \frac{\partial f}{\partial x} \tag{13.38}$$

Then Taylor's theorem can be written in the compact form:

$$f(h+x) = e^{h\,D}\,f(x) \tag{13.39}$$

The exponential function is used as a convenient encoding of the infinite sum of terms.

■ One-sided differences

First consider

$$\Delta\,f = f(h+x) - f(x) \tag{13.40}$$

Using the operator form of Taylor's theorem we can write

$$\Delta\,f = e^{h\,D}\,f(x) - f(x) = (e^{h\,D} - 1)\,f(x) \tag{13.41}$$

So as operators

$$\Delta = e^{h\,D} - 1 \tag{13.42}$$

We can invert this as (note that log is just used to encode an infinite sum):

$$D = \frac{\log(1+\Delta)}{h} = \frac{1}{h}\left(\Delta - \frac{\Delta^2}{2} + \ldots\right) \tag{13.43}$$

Unpacking this expression, we obtain, first keeping just one term, the Euler approximation to the derivatives

$$D\,f \approx \frac{1}{h}\,\Delta\,f = \frac{1}{h}\,(f(x+h) - f(x)) \tag{13.44}$$

Keeping two terms we obtain instead

$$D\,f \approx \frac{\Delta\,f - \frac{\Delta^2 f}{2}}{h} = \frac{f(h+x) - f(x) - \frac{1}{2}\,(f(x+2\,h) - 2\,f(x+h) + f(x))}{h} \tag{13.45}$$

which simplifies to the approximation:

$$D\,f \approx = \frac{4\,f(x+h) - 3\,f(x) - f(x+2\,h)}{2\,h} \tag{13.46}$$

This last formula is particularly useful for estimating derivatives at the edge of a grid. E.g. Theta, the option time derivative.

■ Central Differences

Now we consider a finite difference centred on a point of interest:

$$\delta f = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) \tag{13.47}$$

In terms of the D operator, proceeding as before, we can write

$$\delta = e^{\frac{hD}{2}} - e^{-\frac{1}{2}hD} = 2\sinh\left(\frac{hD}{2}\right) \tag{13.48}$$

Inverting this, we see that there is an exact relationship:

$$D = \frac{2\sinh^{-1}(\frac{\delta}{2})}{\Delta x} \tag{13.49}$$

We can obtained various orders of approximation by taking various numbers of terms in the series. This series is interesting as it contains only odd powers, in particular no quadratic term arises:

$$D = \frac{1}{\Delta x} 2\sinh^{-1}\left(\frac{\delta}{2}\right) \approx \frac{1}{\Delta x}\left(\delta - \frac{\delta^3}{24} + \frac{3\,\delta^5}{640} + O(\delta^7)\right) \tag{13.50}$$

We are going to need the square of this in the form:

$$D^2 \approx \frac{1}{(\Delta x)^2}\left(\delta^2 - \frac{\delta^4}{12} + \frac{\delta^6}{90} + O(\delta^8)\right) \tag{13.51}$$

Going back to our grid with both a space and time dimension, we need this operator form for the x-direction, that is: we define the central difference operator δ_x by

$$\delta_x u_n^m = u_{n+\frac{1}{2}}^m - u_{n-\frac{1}{2}}^m \tag{13.52}$$

Its square is

$$\delta_x^2 u_n^m = u_{n+1}^m + u_{n-1}^m - 2\,u_n^m \tag{13.53}$$

13.6 Overview of Two-Time-Level Difference Schemes

■ The Operator Approach (Mitchell and Griffiths)

Mitchell, A.R. and Griffiths, D.F., 1980, The Finite Difference Method in Partial Differential Equations, John Wiley (Corrected reprinted edition, 1994).

We introduce the operators L, D, given by

$$L f = \frac{\partial f}{\partial t} \qquad D f = \frac{\partial f}{\partial x} \tag{13.54}$$

So the diffusion equation is just $L f = D^2 f$. Assuming that the Taylor series expansion holds, we can write

$$u(\tau + \Delta\tau, x) = e^{\Delta\tau L}\, u(\tau, x) \tag{13.55}$$

In other words

$$u_n^{m+1} = e^{\Delta\tau L}\, u_n^m = e^{\Delta\tau D^2}\, u_n^m \tag{13.56}$$

More generally, if we consider the value, u_θ of u at $x = n\,\Delta x$, and $\tau = \theta\, n\,\Delta\tau + (1-\theta)(n+1)\,\Delta\tau$, we can write it in two ways. First, by using a forwards Taylor expansion, we have

$$u_\theta = e^{\Delta\tau(1-\theta)L}\, u_n^m = e^{\Delta\tau(1-\theta)D^2}\, u_n^m \tag{13.57}$$

By considering a Taylor series backwards from the next time-level, we can also say that

$$u_\theta = e^{-\Delta\tau\,\theta\,L}\; u_n^{m+1} = e^{-\Delta\tau\,\theta\,D^2}\; u_n^{m+1} \tag{13.58}$$

So on the assumption that we have such Taylor series, we can equate the two to obtain

$$e^{-\Delta\tau\,\theta\,D^2}\; u_n^{m+1} = e^{\Delta\tau\,(1-\theta)\,D^2}\; u_n^m \tag{13.59}$$

Note that no approximations have been made.

■ General High Order Difference Versions of the Diffusion Equation

Now our diffusion equation involves $\Delta\tau\, D^2$, which, after some algebra, we can expand out as

$$\alpha\left(\delta_x^2 - \frac{\delta_x^4}{12} + \frac{\delta_x^6}{90} + \ldots\right) \tag{13.60}$$

We can combine our exact diffusion equation with the series expansion of the operators contained within it to obtain a description of the problem to any desired order. Keeping all terms up to order δ_x^6, and performing some tedious simplifications, the combination of the last two equations becomes, neglecting eighth and higher order differences

$$\begin{aligned}
&-\alpha\,\theta\left(\frac{\alpha^2\,\theta^2}{6} + \frac{\alpha\,\theta}{12} + \frac{1}{90}\right)\delta_x^6\, u_n^{m+1} + \frac{1}{2}\left(\alpha\,\theta + \frac{1}{6}\right)\alpha\,\theta\,\delta_x^4\, u_n^{m+1} - \alpha\,\theta\,\delta_x^2\, u_n^{m+1} + u_n^{m+1} = \\
&\left(\frac{1}{2}\,(1-\theta)^2\,\alpha^2 - \frac{1}{12}\,(1-\theta)\,\alpha\right)\delta_x^4\, u_n^m + \alpha\,(1-\theta)\,\delta_x^2\, u_n^m + u_n^m \\
&+\left(\frac{1}{6}\,(\alpha^2\,(1-\theta))^2 - \frac{1}{12}\,\alpha\,(1-\theta) + \frac{1}{90}\right)\alpha\,(1-\theta)\,\delta_x^6\, u_n^m
\end{aligned} \tag{13.61}$$

This in general is a matrix equation, and can be represented in terms of "difference matrices", A, B that govern the mapping from one time level to the next. All such schemes can be written in the form $B\,u^{m+1} = A\,u^m$ for suitable difference matrices.

■ Explicit Schemes

These are obtained by setting $\theta = 0$, thereby obtaining, to sixth order,

$$u_n^{m+1} = u_n^m + \alpha\;\delta_x^2\, u_n^m + \frac{\alpha}{2}\left(\alpha - \frac{1}{6}\right)\delta_x^4\, u_n^m + \frac{\alpha}{6}\left(\alpha^2 - \frac{1}{2}\,\alpha + \frac{1}{15}\right)\;\delta_x^6\, u_n^m \tag{13.62}$$

■ Second Order Explicit and Binomial Schemes

If we keep terms to second order we obtain

$$u_n^{m+1} = u_n^m + \alpha\;\delta_x^2\, u_n^m = (1 - 2\,\alpha)\, u_n^m + \alpha\;(u_{n-1}^m + u_{n+1}^m) \tag{13.63}$$

The choice $\alpha = 1/2$ in fact gives the scheme embodied by the binomial model (if a tree-shaped grid is used instead of a rectangular grid).

■ Fourth Order Explicit and Pentanomial/Trinomial Schemes

If we keep terms to fourth order we obtain the family of pentanomial schemes

$$u_n^{m+1} = u_n^m + \alpha\;\delta_x^2\, u_n^m + \frac{\alpha}{2}\left(\alpha - \frac{1}{6}\right)\delta_x^4\, u_n^m \tag{13.64}$$

The choice $\alpha = 1/6$ gives the scheme embodied by the trinomial model (if a tree-shaped grid is used instead of a rectangular grid) - the fourth order terms then vanish identically, and we have a simple scheme but with high order accuracy.

■ The θ-method Family

The so-called θ-method is obtained by considering the general system and keeping terms to second order in the difference operator δ.

$$u_n^{m+1} - \alpha\,\theta\,\delta_x^2\,u_n^{m+1} = u_n^m + \alpha\,(1-\theta)\,\delta_x^2\,u_n^m \tag{13.65}$$

If we expand this out we obtain:

$$(1 + 2\,\alpha\,\theta)\,u_n^{m+1} - \alpha\,\theta\,(u_{n-1}^{m+1} + u_{n+1}^{m+1}) = (1 - 2\,\alpha\,(1-\theta))\,u_n^m + \alpha\,(1-\theta)\,(u_{n-1}^m + u_{n+1}^m) \tag{13.66}$$

From this we obtain the four important special cases:

■ Explicit

When $\theta = 0$ we obtain

$$u_n^{m+1} = (1 - 2\,\alpha)\,u_n^m + \alpha\,(u_{n-1}^m + u_{n+1}^m) \tag{13.67}$$

■ Fully Implicit

When $\theta = 1$ we obtain

$$(1 + 2\,\alpha)\,u_n^{m+1} - \alpha\,(u_{n-1}^{m+1} + u_{n+1}^{m+1}) = u_n^m \tag{13.68}$$

■ Crank-Nicolson

When $\theta = 1/2$ we obtain

$$(1 + \alpha)\,u_n^{m+1} - \alpha/2\,(u_{n-1}^{m+1} + u_{n+1}^{m+1}) = (1 - \alpha)\,u_n^m + \alpha/2\,(u_{n-1}^m + u_{n+1}^m) \tag{13.69}$$

■ Douglas (2-time level)

$$\theta = \frac{1}{2} - \frac{1}{12\,\alpha} \tag{13.70}$$

n.b. this is obtained by manipulating the fourth-order difference equation to eliminate the fourth-order terms.

Let's look at this one from a different viewpoint. Consider the general high order scheme with $\theta = 1/2$, and multiplying both sides by (exercise for LGS students!)

$$1 + \mu\,\delta_x^2 - \lambda\,\delta_x^4 \tag{13.71}$$

The choice $\mu = 1/12$, $\lambda = \alpha^2/8$ leads to a very interesting equation where the fourth order terms disappear:

$$u_n^{m+1} - (1/12 - \alpha/2)\,\delta_x^2\,u_n^{m+1} + O(\delta_x^6\,u_n^{m+1}) = u_n^m + (1/12 + \alpha/2)\,\delta_x^2\,u_n^m + O(\delta_x^6\,u_n^m) \tag{13.72}$$

The second order truncated form, i.e.

$$u_n^{m+1} - (1/12 - \alpha/2)\,\delta_x^2\,u_n^{m+1} = u_n^m + (1/12 + \alpha/2)\,\delta_x^2\,u_n^m \tag{13.73}$$

is called the Douglas scheme. It is very important due to the fact that it is exact to order δ_x^4, even though it contains terms only of order δ_x^2.

13.7 Link to Binomial and Trinomial Schemes

The following links may be helpful in appreciating the importance of the Douglas scheme, and its relationship to the explicit schemes that are common to FD models and binomial/trinomial tree models.

(1) When $\theta = 0$ and $\alpha = 1/2$, our schemes, as noted already, become the same difference rule as in a binomial tree, but on a rectangular grid in standard coordinates.

(2) When we use a Douglas scheme with $\alpha = 1/6$, so that equation (29) gives $\theta = 0$, we obtain a highly accurate explicit scheme. This is none other than the trinomial tree model, but on a rectangular grid. So the Douglas scheme is the natural implicit form of the trinomial model. We have already seen that this case also corresponds to a special case of a high order explicit scheme based on five points (the pentanomial scheme).

The relationship to standard tree models may be made clearer if we state up front the simplest form of the change of variables used to reduce an option-pricing problem to the diffusion equation. For a problem with a flat term-structure parametrized by a variable K (e.g. strike or barrier) the change of coordinates being used here is, for an underlying S, time variable T, volatility σ,

$$x = \log\left(\frac{S}{K}\right) \tag{13.74}$$

$$\tau = \frac{\sigma^2\, T}{2} \tag{13.75}$$

so that

$$\Delta S = S\,\sigma \sqrt{\frac{\Delta T}{2\,\alpha}} \tag{13.76}$$

In these coordinates the relationship between FD schemes with $\alpha = 1/2$, 1/6 and binomial, trinomial trees becomes clear. To practitioners, we emphasize the following:
The two-time-level Douglas scheme is the natural implicit generalization of the trinomial tree.

13.8 The Concept of Truncation Error

Books and papers vary in their normalization of "the truncation error", the variations being in the form of an overall multiplicative factor. Informally speaking, any definition of truncation error gives a measure of the extent to which an exact solution of the differential equation fails to satisfy the difference equation. Suppose that we write the difference equation in a form with zero on the right hand side of the equation:

$$T(u_n^m) = 0 \tag{13.77}$$

where T is an operator that takes linear combinations of the u_n^m with the indices raised and lowered according to the particular difference scheme. For example, in the explicit method, we can define the operator T to be given by

$$T(u_n^m) = u_n^{m+1} - (1 - 2\,\alpha)\, u_n^m - \alpha\, (u_{n-1}^m + u_{n+1}^m) = 0 \tag{13.78}$$

Similarly in the fully implicit method, an obvious choice for T is:

$$T(u_n^m) = (1 + 2\,\alpha)\, u_n^{m+1} - \alpha\, (u_{n-1}^{m+1} + u_{n+1}^{m+1}) - u_n^m = 0 \tag{13.79}$$

It is clear in each case that another choice for T could be obtained by scaling T by any number, including the grid parameters $k,\ h$. Now let $v(\tau,\ x)$ be an exact solution of the diffusion equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} \tag{13.80}$$

which evaluates at the grid points to:

$$v_n^m = v(m\,\Delta\tau,\, n\,\Delta\, x) \tag{13.81}$$

The raw truncation error is given by:

$$\text{TE} = T(\,v_n^m) \tag{13.82}$$

This definition may be multiplied by some constant, as we have noted. The scaling varies depending on whose book you read!

Most authors prefer to scale the difference equation so that as $k \to 0$, a factor like

$$\frac{\partial v}{\partial \tau} \tag{13.83}$$

emerges. This is what we shall take to give the normalized truncation error, which we shall denote as $\hat{\text{TE}}$. So for example, in the explicit method:

$$\hat{\text{TE}} = \hat{T}(v_n^m) = 1/k(\,v_n^{m+1} - (1 - 2\,\alpha\,)\,v_n^m - \alpha\,\,(v_{n-1}^m + v_{n+1}^m)) \tag{13.84}$$

This is my personal preferred form, and is used by Smith, Richtmyer-Morton, Wilmott *et al*.

Finally, note what is being defined here: it is a measure of the extent to which the EXACT solution to the DIFFERENTIAL equation fails to satisfy the DIFFERENCE equation. This does not immediately tell us how good a solution to the DIFFERENCE equation is in satisfying the DIFFERENTIAL equation.

13.9 Truncation Error Calculations

The calculation of truncation errors is a straightforward but rather tedious process requiring care with a bunch of Taylor series expansions. We take the exact solution, pick a base point on the grid, which is just (m, n), except for the fully implicit method, where it is better to pick $(m + 1,\ n)$, and expand the exact solution as a Taylor series around the base point. One does Taylor expansions in (a) time, (b) space, and (c) (for more complicated schemes), both time and space, and inserts these into the definition of TE or the normalized form. The truncation error then emerges as a collection of powers of h and k multiplying various high-order derivatives.

The complexity of the calculation depends strongly on what scheme is being considered, and how many terms one wants to keep in the series that emerges for the truncation error. The best approach is to first consider the two simplest cases, explicit and fully implicit, which, with a good choice of base point, do not require any double space and time Taylor series. Then we shall look at the general θ-method situation.

In the LGS lectures I will just do the details for the explicit case. The notes have fully-implicit and θ-method worked out in glorious detail.

■ 13.9.1 Truncation Error for Explicit Method

In this case the standard raw and normalized truncation errors are given by:

$$\text{TE} = T(v_n^m) = v_n^{m+1} - (1 - 2\,\alpha\,)\,v_n^m - \alpha\,\,(v_{n-1}^m + v_{n+1}^m) \tag{13.85}$$

$$\hat{\text{TE}} = \hat{T}(v_n^m) = 1/k(\,v_n^{m+1} - (1 - 2\,\alpha\,)\,v_n^m - \alpha\,\,(v_{n-1}^m + v_{n+1}^m)) \tag{13.86}$$

We shall calculate the raw form and finally divide by k to get the normalized form. First we need the expansion for v_n^{m+1}. This requires a Taylor expansion in the time variable:

$$v_n^{m+1} = v_n^m + k\left(\frac{\partial v}{\partial \tau}\right)_n^m + \frac{k^2}{2!}\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^m + O(k^3) \tag{13.87}$$

Next we need two spatial Taylor series to deal with the terms v_{n-1}^m, v_{n+1}^m.

$$v_{n+1}^m = v_n^m + h\left(\frac{\partial v}{\partial x}\right)_n^m + \frac{h^2}{2!}\left(\frac{\partial^2 v}{\partial x^2}\right)_n^m + \frac{h^3}{3!}\left(\frac{\partial^3 v}{\partial x^3}\right)_n^m + \frac{h^4}{4!}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^m + O(h^5) \quad (13.88)$$

$$v_{n-1}^m = v_n^m - h\left(\frac{\partial v}{\partial x}\right)_n^m + \frac{h^2}{2!}\left(\frac{\partial^2 v}{\partial x^2}\right)_n^m - \frac{h^3}{3!}\left(\frac{\partial^3 v}{\partial x^3}\right)_n^m + \frac{h^4}{4!}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^m + O(h^5) \quad (13.89)$$

The sum of these is, bearing in mind that the fifth powers will also cancel given that they have opposite sign:

$$v_{n+1}^m + v_{n-1}^m = 2\,v_n^m + h^2\left(\frac{\partial^2 v}{\partial x^2}\right)_n^m + \frac{h^4}{12}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^m + O(h^6) \quad (13.90)$$

We now insert all these results into equation (17). Without doing any simplification, we obtain:

$$\begin{aligned} \text{TE} = {} & v_n^m + k\left(\frac{\partial v}{\partial \tau}\right)_n^m + \frac{k^2}{2!}\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^m + O(k^3) \\ & -(1-2\,\alpha)\,v_n^m \\ & -\alpha\left(2\,v_n^m + h^2\left(\frac{\partial^2 v}{\partial x^2}\right)_n^m + \frac{h^4}{12}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^m + O(h^6)\right) \end{aligned} \quad (13.91)$$

Clearly all the terms involving just v_n^m cancel, leaving us with (doing some re-ordering)

$$\text{TE} = k\left(\frac{\partial v}{\partial \tau}\right)_n^m - \alpha\,h^2\left(\frac{\partial^2 v}{\partial x^2}\right)_n^m + \frac{k^2}{2!}\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^m - \alpha\left(+\frac{h^4}{12}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^m + O(h^6)\right) + O(k^3) \quad (13.92)$$

Now we use the fact that $\alpha = k/h^2$ to simplify this to:

$$\text{TE} = k\left(\left(\frac{\partial v}{\partial \tau}\right)_n^m - \left(\frac{\partial^2 v}{\partial x^2}\right)_n^m\right) + \frac{k}{2}\left(k\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^m - \frac{h^2}{6}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^m\right) + O(k\,h^4) + O(k^3) \quad (13.93)$$

Given that v satisfies the diffusion equation the first pair of terms cancel, so that the raw truncation error is given, finally, by

$$\text{TE} = \frac{k}{2}\left(k\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^m - \frac{h^2}{6}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^m\right) + O(k\,h^4) + O(k^3) \quad (13.94)$$

The normalized form is therefore

$$\hat{\text{TE}} = \frac{1}{2}\left(k\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^m - \frac{h^2}{6}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^m\right) + O(h^4) + O(k^2) \quad (13.95)$$

The "principal part" consists of just the first two terms, and this result is often summarized by saying that the (normalized) truncation error for the explicit method is

$$O(k) + O(h^2) \quad (13.96)$$

There is an important special case to consider. Note that if

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} \quad (13.97)$$

$$\frac{\partial^2 v}{\partial \tau^2} = \frac{\partial}{\partial \tau}\frac{\partial v}{\partial \tau} = \frac{\partial}{\partial \tau}\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x^2}\frac{\partial v}{\partial \tau} = \frac{\partial}{\partial x^2}\frac{\partial^2 v}{\partial x^2} = \frac{\partial^4 v}{\partial x^4} \quad (13.98)$$

assuming that the partial derivatives can be swapped around. So the normalized truncation error can be written as

$$\hat{\text{TE}} = \frac{1}{2}\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^m\left(k - \frac{h^2}{6}\right) + O(h^4) + O(k^2) \quad (13.99)$$

and the principal part vanishes when

$$k = \frac{h^2}{6} \tag{13.100}$$

That is, when

$$\alpha = \frac{1}{6} \tag{13.101}$$

In this case the normalized truncation error is of the form

$$\hat{\text{TE}} = O(k^2) + O(h^4) \tag{13.102}$$

This is another view of the importance of the special case $\alpha = 1/6$. Recall that this also corresponds to the case where the grid parameters are equivalent to those used in a standard trinomial tree, and that the Douglas scheme, that is usually implicit, collapses to this same explicit case when we set $\alpha = 1/6$ in the general Douglas scheme.

■ 13.9.2 **Truncation Error for Fully Implicit Method

In this case the standard raw and normalized truncation errors are given by:

$$\text{TE} = T(v_n^m) = (1 + 2\alpha)\, v_n^{m+1} - \alpha\, (v_{n-1}^{m+1} + v_{n+1}^{m+1}) - v_n^m \tag{13.103}$$

$$\hat{\text{TE}} = \hat{T}(v_n^m) = 1/k((1 + 2\alpha)\, v_n^{m+1} - \alpha\, (v_{n-1}^{m+1} + v_{n+1}^{m+1}) - v_n^m) \tag{13.104}$$

We shall calculate the raw form and finally divide by k to get the normalized form. Note that in this case it is not so useful to use v_n^m as the base point, as the terms of the form v_{n-1}^{m+1}, v_{n+1}^{m+1} would require a double Taylor series. Matters become much more straightforward if we base the argument around the value of v_n^{m+1}.
First we need the expansion for v_n^m. This requires a *backwards* Taylor expansion in the time variable:

$$v_n^m = v_n^{m+1} - k\left(\frac{\partial v}{\partial \tau}\right)_n^{m+1} + \frac{k^2}{2!}\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^{m+1} + O(k^3) \tag{13.105}$$

Next we need two spatial Taylor series to deal with the terms v_{n-1}^{m+1}, v_{n+1}^{m+1}. These are just as in the explicit case but with $m$incremented by one:

$$v_{n+1}^{m+1} = v_n^{m+1} + h\left(\frac{\partial v}{\partial x}\right)_n^{m+1} + \frac{h^2}{2!}\left(\frac{\partial^2 v}{\partial x^2}\right)_n^{m+1} + \frac{h^3}{3!}\left(\frac{\partial^3 v}{\partial x^3}\right)_n^{m+1} + \frac{h^4}{4!}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^{m+1} + O(h^5) \tag{13.106}$$

$$v_{n-1}^{m+1} = v_n^{m+1} - h\left(\frac{\partial v}{\partial x}\right)_n^{m+1} + \frac{h^2}{2!}\left(\frac{\partial^2 v}{\partial x^2}\right)_n^{m+1} - \frac{h^3}{3!}\left(\frac{\partial^3 v}{\partial x^3}\right)_n^{m+1} + \frac{h^4}{4!}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^{m+1} + O(h^5) \tag{13.107}$$

The sum of these is, bearing in mind that the fifth powers will also cancel given that they have opposite sign:

$$v_{n+1}^{m+1} + v_{n-1}^{m+1} = 2\, v_n^{m+1} + h^2\left(\frac{\partial^2 v}{\partial x^2}\right)_n^{m+1} + \frac{h^4}{12}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^{m+1} + O(h^6) \tag{13.108}$$

We now insert all these results into equation (35). Without doing any simplification, we obtain:

$$\begin{aligned}\text{TE} = {} & (1 + 2\alpha)\, v_n^{m+1} - \left(v_n^{m+1} - k\left(\frac{\partial v}{\partial \tau}\right)_n^{m+1} + \frac{k^2}{2!}\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^{m+1} + O(k^3)\right) \\ & -\alpha\left(2\, v_n^{m+1} + h^2\left(\frac{\partial^2 v}{\partial x^2}\right)_n^{m+1} + \frac{h^4}{12}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^{m+1} + O(h^6)\right)\end{aligned} \tag{13.109}$$

Clearly all the terms involving just v_n^mcancel, leaving us with (doing some re-ordering)

$$\text{TE} = k\left(\frac{\partial v}{\partial \tau}\right)_n^{m+1} - \alpha\, h^2\left(\frac{\partial^2 v}{\partial x^2}\right)_n^{m+1} - \frac{k^2}{2!}\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^{m+1} - \alpha\left(\frac{h^4}{12}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^{m+1} + O(h^6)\right) + O(k^3) \tag{13.110}$$

Now we use the fact that $\alpha = k/h^2$ to simplify this to:

$$\text{TE} = k\left(\left(\frac{\partial v}{\partial \tau}\right)_n^{m+1} - \left(\frac{\partial^2 v}{\partial x^2}\right)_n^{m+1}\right) - \frac{k}{2}\left(k\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^{m+1} + \frac{h^2}{6}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^{m+1}\right) + O(k\,h^4) + O(k^3) \tag{13.111}$$

Given that v satisfies the diffusion equation the first pair of terms cancel, so that the raw truncation error is given, finally, by

$$\text{TE} = -\frac{k}{2}\left(k\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^{m+1} + \frac{h^2}{6}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^{m+1}\right) + O(k\,h^4) + O(k^3) \tag{13.112}$$

The normalized form is therefore

$$\hat{\text{TE}} = -\frac{1}{2}\left(k\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^{m+1} + \frac{h^2}{6}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^{m+1}\right) + O(h^4) + O(k^2) \tag{13.113}$$

The "principal part" consists of just the first two terms, and this result is often summarized by saying that the (normalized) truncation error for the fully implicit method is, as with the explicit method:

$$O(k) + O(h^2) \tag{13.114}$$

However, in this case there are clearly no cancellations to be obtained by picking particular values for α.

■ 13.9.3 **High-Order Truncation Error for More General Θ-Method

In this case the standard raw and normalized truncation errors are given by:

$$\text{TE} = T(v_n^m) = (1 + 2\,\alpha\,\theta)\,v_n^{m+1} - \alpha\,\theta\,(v_{n-1}^{m+1} + v_{n+1}^{m+1}) - [1 - 2\,\alpha(1-\theta)]\,v_n^m - \alpha\,(1-\theta)\,(v_{n-1}^m + v_{n+1}^m) \tag{13.115}$$

$$\hat{\text{TE}} = \hat{T}(v_n^m) = T(v_n^m)/k \tag{13.116}$$

We shall calculate the raw form and finally divide by k to get the normalized form. Note that in this case we cannot avoid making a double Taylor series so we shall take v_n^m as the base value. We shall do this and just work through the series. (One should note that if one sets $\theta = 1$ in this argument, the derivation of the result for the fully implicit method is then a bit more involved than the one we gave in the previous sub-section.)
First we need the expansion for v_n^{m+1}. This requires a *forwards* Taylor expansion in the time variable, as for the explicit method. This time, as we are going to investigate some higher-order cancellation effects, we take more terms.

$$v_n^{m+1} = v_n^m + k\left(\frac{\partial v}{\partial \tau}\right)_n^m + \frac{k^2}{2!}\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^m + \frac{k^3}{3!}\left(\frac{\partial^3 v}{\partial \tau^3}\right)_n^m + O(k^4) \tag{13.117}$$

Next we need two spatial Taylor series to deal with the terms v_{n-1}^m, v_{n+1}^m.

$$v_{n+1}^m = v_n^m + h\left(\frac{\partial v}{\partial x}\right)_n^m + \frac{h^2}{2!}\left(\frac{\partial^2 v}{\partial x^2}\right)_n^m + \frac{h^3}{3!}\left(\frac{\partial^3 v}{\partial x^3}\right)_n^m + \frac{h^4}{4!}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^m + \frac{h^5}{5!}\left(\frac{\partial^5 v}{\partial x^5}\right)_n^m + \frac{h^6}{6!}\left(\frac{\partial^6 v}{\partial x^6}\right)_n^m + O(h^7) \tag{13.118}$$

$$v_{n-1}^m = v_n^m - h\left(\frac{\partial v}{\partial x}\right)_n^m + \frac{h^2}{2!}\left(\frac{\partial^2 v}{\partial x^2}\right)_n^m - \frac{h^3}{3!}\left(\frac{\partial^3 v}{\partial x^3}\right)_n^m + \frac{h^4}{4!}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^m - \frac{h^5}{5!}\left(\frac{\partial^5 v}{\partial x^5}\right)_n^m + \frac{h^6}{6!}\left(\frac{\partial^6 v}{\partial x^6}\right)_n^m + O(h^7) \tag{13.119}$$

The sum of these is, bearing in mind that the seventh powers will also cancel given that they have opposite sign:

$$v_{n+1}^m + v_{n-1}^m = 2\,v_n^m + h^2\left(\frac{\partial^2 v}{\partial x^2}\right)_n^m + \frac{h^4}{12}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^m + \frac{2\,h^6}{6!}\left(\frac{\partial^6 v}{\partial x^6}\right)_n^m + O(h^8) \tag{13.120}$$

We abbreviate this as

$$v_{n+1}^m + v_{n-1}^m = 2\, v_n^m + \Gamma_n^m \tag{13.121}$$

where

$$\Gamma_n^m = h^2 \left(\frac{\partial^2 v}{\partial x^2}\right)_n^m + \frac{h^4}{12}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^m + \frac{2\, h^6}{6!}\left(\frac{\partial^6 v}{\partial x^6}\right)_n^m + O(h^8) \tag{13.122}$$

We have a similar expression at the later time step:

$$v_{n+1}^{m+1} + v_{n-1}^{m+1} = 2\, v_n^{m+1} + \Gamma_n^{m+1} \tag{13.123}$$

At this stage we can simplify the raw truncation error somewhat - so far we have

$$\text{TE} = (1 + 2\,\alpha\,\theta)\, v_n^{m+1} - \alpha\,\theta\,(2\, v_n^{m+1} + \Gamma_n^{m+1}) - [1 - 2\,\alpha(1-\theta)]\, v_n^m - \alpha\,(1-\theta)\,(2\, v_n^m + \Gamma_n^m) \tag{13.124}$$

This simplifies immediately to:

$$\text{TE} = v_n^{m+1} - v_n^m - \alpha\,\theta\,\Gamma_n^{m+1} - \alpha\,(1-\theta)\,\Gamma_n^m \tag{13.125}$$

We now work on this. Substituting the time Taylor series and the definitions of the Γgives us the raw tracking error as:

$$\begin{aligned}
\text{TE} = \\
& k\left(\frac{\partial v}{\partial \tau}\right)_n^m + \frac{k^2}{2!}\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^m + \frac{k^3}{3!}\left(\frac{\partial^3 v}{\partial \tau^3}\right)_n^m + O(k^4) \\
& -\alpha\,\theta\left(h^2\left(\frac{\partial^2 v}{\partial x^2}\right)_n^{m+1} + \frac{h^4}{12}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^{m+1} + \frac{2\, h^6}{6!}\left(\frac{\partial^6 v}{\partial x^6}\right)_n^{m+1} + O(h^8)\right) \\
& -\alpha\,(1-\theta)\left(h^2\left(\frac{\partial^2 v}{\partial x^2}\right)_n^m + \frac{h^4}{12}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^m + \frac{2\, h^6}{6!}\left(\frac{\partial^6 v}{\partial x^6}\right)_n^m + O(h^8)\right)
\end{aligned} \tag{13.126}$$

What we do next is to gather together the terms involving $\alpha\,\theta$, and group the terms in the third line with just α with those in the first line:

$$\begin{aligned}
\text{TE} = \\
& k\left(\frac{\partial v}{\partial \tau}\right)_n^m - \alpha\, h^2\left(\frac{\partial^2 v}{\partial x^2}\right)_n^m \\
& + \frac{k^2}{2!}\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^m - \alpha\,\frac{h^4}{12}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^m \\
& + \frac{k^3}{3!}\left(\frac{\partial^3 v}{\partial \tau^3}\right)_n^m - \alpha\,\frac{2\, h^6}{6!}\left(\frac{\partial^6 v}{\partial x^6}\right)_n^m \\
& + O(k^4) + O(\alpha\, h^8) \\
& -\alpha\,\theta\left(h^2\left(\left(\frac{\partial^2 v}{\partial x^2}\right)_n^{m+1} - \left(\frac{\partial^2 v}{\partial x^2}\right)_n^m\right) + \frac{h^4}{12}\left(\left(\frac{\partial^4 v}{\partial x^4}\right)_n^{m+1} - \left(\frac{\partial^4 v}{\partial x^4}\right)_n^m\right)\right. \\
& \qquad \left. + \frac{2\, h^6}{6!}\left(\left(\frac{\partial^6 v}{\partial x^6}\right)_n^{m+1} - \left(\frac{\partial^6 v}{\partial x^6}\right)_n^m\right) + O(h^8)\right)
\end{aligned} \tag{13.127}$$

Now we use the fact that v satisfies the diffusion equation, and that $\alpha = k/h^2$. The first two terms then cancel and we can make some simplifications in rows two and three. For the rest, we now must make a time Taylor series expansion to simplify the differences between the even spatial derivatives at the two time-levels.

$$\left(\frac{\partial^2 v}{\partial x^2}\right)_n^{m+1} - \left(\frac{\partial^2 v}{\partial x^2}\right)_n^m = k\left(\frac{\partial}{\partial \tau}\,\frac{\partial^2 v}{\partial x^2}\right)_n^m + \frac{k^2}{2!}\left(\frac{\partial^2}{\partial \tau^2}\,\frac{\partial^2 v}{\partial x^2}\right)_n^m + O(k^3) \tag{13.128}$$

$$\left(\frac{\partial^4 v}{\partial x^4}\right)_n^{m+1} - \left(\frac{\partial^4 v}{\partial x^4}\right)_n^m = k\left(\frac{\partial}{\partial \tau}\frac{\partial^4 v}{\partial x^4}\right)_n^m + \frac{k^2}{2!}\left(\frac{\partial^2}{\partial \tau^2}\frac{\partial^4 v}{\partial x^4}\right)_n^m + O(k^3) \tag{13.129}$$

$$\left(\frac{\partial^6 v}{\partial x^6}\right)_n^{m+1} - \left(\frac{\partial^6 v}{\partial x^6}\right)_n^m = k\left(\frac{\partial}{\partial \tau}\frac{\partial^6 v}{\partial x^6}\right)_n^m + \frac{k^2}{2!}\left(\frac{\partial^2}{\partial \tau^2}\frac{\partial^6 v}{\partial x^6}\right)_n^m + O(k^3) \tag{13.130}$$

and so on. Implementing these observations, we arrive at the intermediate result:

$$\begin{aligned}
\text{TE} =\ & \\
& k\left(\frac{k}{2}\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^m - \frac{h^2}{12}\left(\frac{\partial^4 v}{\partial x^4}\right)_n^m\right) \\
& + k\left(\frac{k^2}{6}\left(\frac{\partial^3 v}{\partial \tau^3}\right)_n^m - \frac{2h^4}{6!}\left(\frac{\partial^6 v}{\partial x^6}\right)_n^m\right) \\
& + O(k^4) + O(k\,h^6) \\
& - k\,\theta\left(\left(k\left(\frac{\partial}{\partial \tau}\frac{\partial^2 v}{\partial x^2}\right)_n^m + \frac{k^2}{2!}\left(\frac{\partial^2}{\partial \tau^2}\frac{\partial^2 v}{\partial x^2}\right)_n^m + O(k^3)\right)\right. \\
& \quad + \frac{h^2}{12}\left(k\left(\frac{\partial}{\partial \tau}\frac{\partial^4 v}{\partial x^4}\right)_n^m + \frac{k^2}{2!}\left(\frac{\partial^2}{\partial \tau^2}\frac{\partial^4 v}{\partial x^4}\right)_n^m + O(k^3)\right) \\
& \quad \left. + \frac{2h^4}{6!}\left(k\left(\frac{\partial}{\partial \tau}\frac{\partial^6 v}{\partial x^6}\right)_n^m + \frac{k^2}{2!}\left(\frac{\partial^2}{\partial \tau^2}\frac{\partial^6 v}{\partial x^6}\right)_n^m + O(k^3)\right) + O(h^6)\right)
\end{aligned} \tag{13.131}$$

To analyze this, we reduce everything to time derivatives using the diffusion equation and assuming that space and time derivatives commute. This allows us to simplify with the relation

$$\left(\frac{\partial^k v}{\partial \tau^k}\right)_n^m = \left(\frac{\partial^{2k} v}{\partial x^{2k}}\right)_n^m \tag{13.132}$$

Using this, we obtain:

$$\begin{aligned}
\text{TE} =\ & k\left(\frac{k}{2} - \frac{h^2}{12}\right)\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^m + k\left(\frac{k^2}{6} - \frac{2h^4}{6!}\right)\left(\frac{\partial^3 v}{\partial \tau^3}\right)_n^m + O(k^4) + O(k\,h^6) \\
& - k\,\theta\left(\left(k\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^m + \frac{k^2}{2!}\left(\frac{\partial^3 v}{\partial \tau^3}\right)_n^m + O(k^3)\right)\right. \\
& + \frac{h^2}{12}\left(k\left(\frac{\partial^3 v}{\partial \tau^3}\right)_n^m + \frac{k^2}{2!}\left(\frac{\partial^4 v}{\partial \tau^4}\right)_n^m + O(k^3)\right) \\
& \left. + \frac{2h^4}{6!}\left(k\left(\frac{\partial^4 v}{\partial \tau^4}\right)_n^m + \frac{k^2}{2!}\left(\frac{\partial^5 v}{\partial \tau^5}\right)_n^m + O(k^3)\right) + O(h^6)\right)
\end{aligned} \tag{13.133}$$

Finally we gather together the various time orders of the time derivative.

$$\begin{aligned}
\text{TE} =\ & \\
& k\left(\frac{k}{2} - \frac{h^2}{12} - k\,\theta\right)\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^m + k\left(\frac{k^2}{6} - \frac{2h^4}{6!} - \theta\,\frac{k^2}{2} - \theta\,k\,\frac{h^2}{12}\right)\left(\frac{\partial^3 v}{\partial \tau^3}\right)_n^m + O(k^4) + O(k\,h^6) \\
& + k\,\theta\left(\frac{h^2}{12}\left(\frac{k^2}{2!}\left(\frac{\partial^4 v}{\partial \tau^4}\right)_n^m\right) + \frac{2h^4}{6!}\left(k\left(\frac{\partial^4 v}{\partial \tau^4}\right)_n^m + \frac{k^2}{2!}\left(\frac{\partial^5 v}{\partial \tau^5}\right)_n^m + O(k^3)\right) + O(h^6)\right)
\end{aligned} \tag{13.134}$$

Finally reducing the "leftovers" to *O* notation we obtain:

$$\text{TE} = k\left(\frac{k}{2} - \frac{h^2}{12} - k\,\theta\right)\left(\frac{\partial^2 v}{\partial\tau^2}\right)_n^m + k\left(\frac{k^2}{6} - \frac{2\,h^4}{6!} - \theta\,\frac{k^2}{2} - \theta\,k\,\frac{h^2}{12}\right)\left(\frac{\partial^3 v}{\partial\tau^3}\right)_n^m + O(k^4) + O(k\,h^6) + O(k^3\,h^2) + O(k^2\,h^4) \tag{13.135}$$

The normalized tracking errors is then:

$$\hat{\text{TE}} = \left(\frac{k}{2} - \frac{h^2}{12} - k\,\theta\right)\left(\frac{\partial^2 v}{\partial\tau^2}\right)_n^m + \left(\frac{k^2}{6} - \frac{2\,h^4}{6!} - \theta\,\frac{k^2}{2} - \theta\,k\,\frac{h^2}{12}\right)\left(\frac{\partial^3 v}{\partial\tau^3}\right)_n^m + O(k^3) + O(h^6) + O(k^2\,h^2) + O(k\,h^4) \tag{13.136}$$

13.10 θ-Method Trunction Summary

The normalized truncation error for the θ-method family is:

$$\hat{\text{TE}} = \left(\frac{k}{2} - \frac{h^2}{12} - k\,\theta\right)\left(\frac{\partial^2 v}{\partial\tau^2}\right)_n^m + \left(\frac{k^2}{6} - \frac{2\,h^4}{6!} - \theta\,\frac{k^2}{2} - \theta\,k\,\frac{h^2}{12}\right)\left(\frac{\partial^3 v}{\partial\tau^3}\right)_n^m + O(k^3) + O(h^6) + O(k^2\,h^2) + O(k\,h^4) \tag{13.137}$$

■ Analysis of Special Cases

It is quite clear that for a general value of θ the leading order term is of order

$$O(k) + O(h^2) \tag{13.138}$$

In particular, when $\theta = 0$, which is the explicit method, the leading order behaviour is

$$\hat{\text{TE}} = \left(\frac{k}{2} - \frac{h^2}{12}\right)\left(\frac{\partial^2 v}{\partial\tau^2}\right)_n^m \tag{13.139}$$

This is killed when $\alpha = k/h^2 = 1/6$, which corresponds to the "standard trinomial" method.

Also, when $\theta = 1$, which is the fully implicit method, the leading order behaviour is

$$\hat{\text{TE}} = -\left(\frac{k}{2} + \frac{h^2}{12}\right)\left(\frac{\partial^2 v}{\partial\tau^2}\right)_n^m \tag{13.140}$$

Now consider the case when $\theta = 1/2$, which is the Crank-Nicolson case. The terms of $O(k)$ in the first term then cancel, giving a term of $O(h^2)$ from that factor. The first power of k that appears is k^2 in the second group of terms. So we can see that the principal part of the truncation error in the CN approach is

$$O(k^2) + O(h^2) \tag{13.141}$$

But what is now clear is that the optimal choice is not to take $\theta = 1/2$ but to pick it so that the first term vanishes identically! This is the choice leading to the Douglas scheme:

$$\theta = \frac{1}{2} - \frac{1}{12\,\alpha} \tag{13.142}$$

The leading order term in the truncation error is then

$$\hat{\text{TE}} = \left(\frac{k^2}{6} - \frac{2\,h^4}{6!} - \theta\,\frac{k^2}{2} - \theta\,k\,\frac{h^2}{12}\right)\left(\frac{\partial^3 v}{\partial\tau^3}\right)_n^m + O(k^3) + O(h^6) + O(k^2\,h^2) + O(k\,h^4) \tag{13.143}$$

With the given choice of θ, this reduces to a truncation error

$$\hat{\text{TE}} = -\frac{1}{12}\left(k^2 - \frac{h^4}{20}\right)\left(\frac{\partial^3 v}{\partial\tau^3}\right)_n^m + O(k^3) + O(h^6) + O(k^2\,h^2) + O(k\,h^4) \tag{13.144}$$

So in general we obtain an error for the Douglas scheme with order

$$O(k^2) + O(h^4) \tag{13.145}$$

Note that the principal part of the tracking error can be written as

$$\hat{\text{TE}} = \frac{k^2}{12}\left(\frac{1}{20\,\alpha^2} - 1\right)\left(\frac{\partial^3 v}{\partial \tau^3}\right)_n^m \tag{13.146}$$

and that this itself can be made to vanish when

$$\alpha = \frac{1}{\sqrt{20}} \tag{13.147}$$

This I think was first derived in about 1958 by Saulev. This gives a rather small time step, but slightly larger than that used in a standard trinomial tree. Used in conjunction with the implicit approach through the Douglas method, it can in practice give substantial error reduction - see the example in Mitchell and Griffiths.

13.11 von Neumann Stability

There are many approaches to analyzing whether a difference scheme is "stable". First we should explain the importance of stability. In general a difference equation may possess solutions that are growing, decaying, or remaining approximately the same in magnitude as the system is evolved in time. What we wish to prevent is the appearance of growing solutions to the difference equation that are unrelated to solutions to the exact differential equation. In the case of the diffusion equation, an analytical study of that equation shows that the Green's function decays with time, so that in this case we wish to prevent growing solutions from appearing at all. More practically, if our numerical initial data contains some rounding, and bearing in mind that all numerical representations on a computer are approximate, we do not wish the errors that are implicit in such representations to be able to grow to completely dominate the numerically computed answer. With an unstable scheme, it is possible to put the same algorithms on different computers, or use different software systems on the same computer, and get totally different answers!

The simplest approach to stability is to consider the possibility for growth of errors terms that have a trigonometric dependence on the spatial variable x. Given that Fourier analysis allows us to resolve any function into Fourier modes, this is actually a very general method. But we should also point out that if we can find any function that grows un-naturally, we have demonstrated instability - trig functions will do for this purpose.

We suppose that we have a solution u_n^m to our difference equation and that the neighbouring solution $v_n^m = u_n^m + \varepsilon_n^m$ also satisfies the same difference equation. By subtraction we deduce that the perturbation ε_n^m also satisfies the difference equation.

$$(1 + 2\,\alpha\,\theta)\,\varepsilon_n^{m+1} - \alpha\,\theta\,(\varepsilon_{n-1}^{m+1} + \varepsilon_{n+1}^{m+1}) = (1 - 2\,\alpha\,(1-\theta))\,\varepsilon_n^m + \alpha\,(1-\theta)\,(\varepsilon_{n-1}^m + \varepsilon_{n+1}^m) \tag{13.148}$$

We consider the solutions of this difference equation for a perturbation of the form

$$\varepsilon_n^m = \lambda^m \sin(n\,\omega) \tag{13.149}$$

The detailed possibilities for ω will not be considered here. What we are interested in are the possible values for the amplification factor λ. If it is the case that

$$|\lambda| > 1 \tag{13.150}$$

then the system is unstable. We need

$$|\lambda| \leq 1 \tag{13.151}$$

if the system is to be stable. This is the standard view. However, we note further that even in the stable case

(i) if λ is close to $+1$or -1 there may be highly persistent errors in solutions. It is better if $|\,\lambda\,|$ is rather less than unity.

(ii) if λ is negative then the errors will oscillate in time. This will potentially corrupt the time derivative and, through the diffusion equation, the spatial second derivative. What this means for financial calculations is that θ

(i.e. the time rate of change of the value of the financial instrument - this is an annoying case of the same symbol standing for two completely different things in two parts of the same subject!) and Γ can be severely corrupted.

■ Sufficient Conditions for stability

One λ has $|\lambda| \leq 1$, all others have $|\lambda| < 1$. Matters if consider three or more time-level schemes.

Stability for Θ-Method and Special Cases

We substitute our trigonometric error term into the θ-method difference equation and obtain:

$$\begin{aligned}&(1+2\alpha\theta)\lambda^{m+1}\sin(n\omega)-\alpha\theta(\lambda^{m+1}\sin((n+1)\omega)+\lambda^{m+1}\sin((n-1)\omega)) = \\ &\quad (1-2\alpha(1-\theta))\lambda^m\sin(n\omega)+\alpha(1-\theta)(\lambda^m\sin((n+1)\omega)+\lambda^m\sin((n-1)\omega))\end{aligned} \tag{13.152}$$

This simplifies immediately to:

$$\begin{aligned}&\lambda[(1+2\alpha\theta)\ \sin(n\omega)-\alpha\theta(\sin((n+1)\omega)+\sin((n-1)\omega))] = \\ &\quad (1-2\alpha(1-\theta))\ \sin(n\omega)+\alpha(1-\theta)(\ \sin((n+1)\omega)+\sin((n-1)\omega))\end{aligned} \tag{13.153}$$

Now using a trigonmetric identity we write

$$\sin((n+1)\omega)+\sin((n-1)\omega)=2\sin(n\omega)\cos(\omega) \tag{13.154}$$

If we substitute this into (13.153), and then cancel the common factor $\sin(n\,\omega)$ we get

$$\lambda(1+2\alpha\theta-2\alpha\theta\cos(\omega)) = 1-2\alpha(1-\theta)+2\alpha(1-\theta)\cos(\omega) \tag{13.155}$$

Now recall the identity:

$$\cos(\omega)=1-2\sin^2\left(\frac{\omega}{2}\right) \tag{13.156}$$

If we now substitute this into (13,155) we get

$$\lambda\left(1+4\alpha\theta\sin^2\left(\frac{\omega}{2}\right)\right)=1-4\alpha(1-\theta)\sin^2\left(\frac{\omega}{2}\right) \tag{13.157}$$

We can now divide through to express the amplification factor explicitly as:

$$\lambda=\frac{1-4\alpha(1-\theta)\sin^2(\frac{\omega}{2})}{1+4\alpha\theta\sin^2(\frac{\omega}{2})} \tag{13.158}$$

At this stage it is convenient to analyze several special cases.

■ Stability of Explicit Method and Related Tree Models

When $\theta = 0$ we obtain

$$\lambda=1-4\alpha\ \sin^2\left(\frac{\omega}{2}\right) \tag{13.159}$$

Note first that if $\alpha < 0$, then $\lambda > 1$ and the system is unstable. Backwards diffusion is analytically unstable in any case. Second, if $\alpha > 1/2$, there are values of ω for which $\lambda < -1$ and the system is unstable.
So stability needs

$$0 \leq \alpha \leq \frac{1}{2} \tag{13.160}$$

Note also that the binomial scheme is an explicit method with $\alpha = 1/2$ and is on the stability limit. Furthermore there are values of ω such that $\lambda = -1$ or is close to -1 so we expect there to be persistent spurious oscillations. The standard trinomial explicit scheme has $\alpha = 1/6$, with an amplification factor satisfying

$$+\frac{1}{3} \leq \lambda \leq +1 \tag{13.161}$$

and is much better behaved.

■ Stability of Fully Implicit Method

When $\theta = 1$ we obtain

$$\lambda = \frac{1}{1 + 4\,\alpha\,\sin^2(\frac{\omega}{2})} \tag{13.162}$$

With $\alpha > 0$, we see that

$$0 < \lambda \leq 1 \tag{13.163}$$

and that $\lambda < 1$ unless $\sin(\omega/2) = 0$. So this system is stable. Furthermore, λ is positive, so the errors should decay in time in a non-oscillatory fashion. This leads us to expect less corruption of the time derivative, and better values for the time-derivative (θ) and Γ in the solution to the financial problem.

■ Stability of Crank-Nicolson Method

If we do the same calculation setting $\theta = 1/2$ everywhere we obtain

$$\lambda = \frac{1 - 2\,\alpha\,\sin^2(\frac{\omega}{2})}{1 + 2\,\alpha\,\sin^2(\frac{\omega}{2})} \tag{13.164}$$

so we deduce that provided $\alpha \geq 0$ we have

$$-1 \leq \lambda \leq +1 \tag{13.165}$$

and that the system is stable. Note that this system can have negative λ and so there may be persistent error oscillations.

■ Stability of θ-Method in General, and Douglas

Now we return to the general formula for the amplification factor:

$$\lambda = \frac{1 - 4\,\alpha\,(1-\theta)\sin^2(\frac{\omega}{2})}{1 + 4\,\alpha\,\theta\sin^2(\frac{\omega}{2})} \tag{13.166}$$

Let's make a substitution to simplify this. We set

$$y = 4\,\alpha\sin^2\left(\frac{\omega}{2}\right) \tag{13.167}$$

so that considered as a function of y:

$$\lambda(y) = \frac{1 - (1-\theta)\,y}{1 + y\,\theta} \tag{13.168}$$

The derivative of this with respect to y is

$$\frac{\theta - 1}{y\,\theta + 1} - \frac{(1 - y\,(1-\theta))\,\theta}{(y\,\theta + 1)^2} \tag{13.169}$$

which simplifies to

$$-\frac{1}{(y\,\theta + 1)^2} \tag{13.170}$$

Note that this is strictly negative, so that λ is a strictly decreasing function of y. Furthermore, some algebra shows that

$$\lambda(y) - \left(1 - \frac{1}{\theta}\right) = \frac{1 - y\,(1-\theta)}{y\,\theta + 1} + \frac{1}{\theta} - 1 = \frac{1}{y\,\theta^2 + \theta} \tag{13.171}$$

which tends to zero as $y \to \infty$, so that

$$\lambda(y) \to 1 - \frac{1}{\theta} \tag{13.172}$$

as $y \to \infty$. Note also that when $y = 0$ then $\lambda = 1$.

Observation 1.
Suppose first that $\theta \geq 1/2$, so that $1/\theta \leq 2$. Then as y becomes large and positive λ approaches a value which is greater than or equal to -1. Because λ is a strictly decreasing function of y and was 1when $y = 0$, it follows that $-1 \leq \lambda \leq +1$ and hence the method is stable.
Observation 2.
If instead $0 \leq \theta < 1/2$, the large y limit is less than -1 and λ attains the value -1 when

$$y = \frac{2}{1 - 2\theta}$$

This can occur only if

$$4\alpha \sin^2\left(\frac{\omega}{2}\right) = y = \frac{2}{1 - 2\theta} \tag{13.173}$$

that is, if

$$\alpha \sin^2\left(\frac{\omega}{2}\right) = \frac{1}{2(1 - 2\theta)} \tag{13.174}$$

As the maximum value of the $\sin^2$ function is 1, this situation cannot apply if

$$\alpha \leq \frac{1}{2(1 - 2\theta)} \tag{13.175}$$

So the θ-method is stable if this condition applies. It might be thought that this suggests that values of θ that are less than $1/2$ are less useful. This is not so - in the case of the Douglas method, where $\theta = \frac{1}{2} - \frac{1}{12\alpha}$

the inequality (13,175) is satisfied, since it is then equivalent to the following, which is satisfied for $\alpha > 0$:

$$\alpha \leq \frac{1}{2(1 - 2\theta)} = \frac{1}{2(1 - 2(\frac{1}{2} - \frac{1}{12\alpha}))} = 3\alpha \tag{13.176}$$

■ Douglas Three time-level

$$\begin{aligned}&\left(\frac{1}{8} - \alpha\right)(u_{n-1}^{m+1} + u_{n+1}^{m+1}) + \left(\frac{5}{4} + 2\alpha\right)u_n^{m+1} \\ &= \frac{1}{6}(u_{n-1}^m + u_{n+1}^m + 10\,u_n^m) - \frac{1}{24}(u_{n-1}^{m-1} + u_{n+1}^{m-1} + 10\,u_n^{m-1})\end{aligned} \tag{13.177}$$

Some calculations (R. Cantwell, KCL MSc, 2002)

$$\hat{\text{TE}} = \frac{5}{6}\left(\frac{h^4}{200} - k^2\right)\left(\frac{\partial^3 v}{\partial \tau^3}\right)_n^m + \ldots\ldots \tag{13.178}$$

Cancellations when $\alpha = 1/\sqrt{200}$! Stability equation is, with $y = 5 + \cos(\omega)$:

$$\left(48\alpha \sin^2\left(\frac{\omega}{2}\right) + 3y\right)\lambda^2 - 4y\lambda + y = 0 \tag{13.179}$$

Check real and complex cases (both exist), to find

$$0 < \text{Re}(\lambda) \leq 1 \tag{13.180}$$

Summary Of Properties

```
TableForm[tabdata = {{Scheme, θ, PPTE, "α_KILL?", VNS, Min[Re[λ]]},
   {Explicit, 0, "O[k]+O[h^2]", 1 / 6, "α≤1/2", "-1(α=2/3),1/3(α=1/6)"},
   {Fully Implicit, 1, "O[k]+O[h^2]", "NO", "OK", 0},
   {CN, 1 / 2, "O[k^2]+O[h^2]", "NO", "OK", -1},
   {Doug2, 1 / 2 - 1 / 12 / α, "O[k^2]+O[h^4]", 1 / Sqrt[20], "OK", -1},
   {Doug3, "NA", "O[k^2]+O[h^4]", 1 / Sqrt[200], "OK", 0}}]
```

Scheme	θ	PPTE	α_KILL?	VNS	Re(λ)
Explicit	0	O[k]+O[h^2]	$\frac{1}{6}$	α≤1/2	−1(α=2/3),1/3(α=1/6)
Fully Implicit	1	O[k]+O[h^2]	NO	OK	0
CN	$\frac{1}{2}$	O[k^2]+O[h^2]	NO	OK	−1
Doug2	$\frac{1}{2} - \frac{1}{12\,\alpha}$	O[k^2]+O[h^4]	$\frac{1}{2\sqrt{5}}$	OK	−1
Doug3	NA	O[k^2]+O[h^4]	$\frac{1}{10\sqrt{2}}$	OK	0