Euler systems and Iwasawa theory

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Introduction

The theory of Euler systems is one of the most powerful tools available for studying the arithmetic of global Galois representations. However, constructing Euler systems is a difficult problem, and the list of known constructions was until recently accordingly rather short. In these lecture notes, we outline a general strategy for constructing new Euler systems in the cohomology of Shimura varieties: these Euler systems arise via pushforward of certain units on subvarieties.

We study in detail two special cases of this construction: the Euler system of Beilinson–Flach elements, where the underlying Shimura variety is the fibre product of two modular curves; and the Euler system of Lemma–Flach elements, arising in the cohomology of Siegel modular threefolds.

The lecture notes are structured as follows.

- In Chapter 1, we will review the definition of Euler systems for Galois representations, and their arithmetic application to the Bloch–Kato conjecture.
- In Chapter 2, we introduce some general tools for constructing global cohomology classes for Galois representations arising in geometry, assuming the existence of a supply of subvarieties of appropriate codimension and units on them. We also introduce Siegel units, which are the key for all the Euler system constructions to follow.
- Chapter 3 is largely motivational (and can be skipped at a first reading): it explains how one can use Rankin–Selberg-type integral formulas for L-functions as a guide to where to look for Euler systems.
- Chapter 4 is devoted to the construction of the Beilinson–Flach Euler system for pairs of modular forms of weight 2; and in Chapter 5, we discuss how to adapt this construction to pairs of modular forms of higher weight, using cohomology with coefficients.
- In Chapter 6 we explain the construction of the Lemma–Flach Euler system for genus 2 Siegel modular forms of parallel weight 3.
- In the concluding Chapter 7 we outline some projects.

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Contents

CHAPTER 1

Galois representations and Galois cohomology

References: for §§1.1—1.2, an excellent source is Bellaiche's CMI notes on the Bloch–Kato conjecture.

1.1. Galois representations

Definition. Let K be a number field or a finite extension of $\mathbf{Q}_\ell$ for some finite prime ℓ, $\overline{K}$ its algebraic closure, $G_K = \mathrm{Gal}(\overline{K}/K)$; and let p be a prime, and E a finite extension of $\mathbf{Q}_p$.

DEFINITION. *A* p-adic representation *of G_K is a finite-dimensional E-vector space V with a continuous action of G_K. Here, G_K is equipped with the profinite topology and V with the p-adic topology.*

REMARK. If V is a p-adic representation, then so is its dual $V^* = \mathrm{Hom}_{\mathbf{Q}_p}(V, \mathbf{Q}_p)$. $\diamond$

Examples.

The representation $\mathbf{Z}_p(1)$*.* Let $\mu_{p^n} = \{x \in \overline{K}^\times : x^{p^n} = 1\}$. Then μ_{p^n} is finite cyclic of order p^n and G_K acts on it.

The p-power map sends $\mu_{p^{n+1}} \to \mu_{p^n}$ and we define

$$\mathbf{Z}_p(1) \coloneqq \varprojlim_n \mu_{p^n}, \quad \mathbf{Q}_p(1) \coloneqq \mathbf{Z}_p(1) \otimes \mathbf{Q}_p.$$

This is a 1-dimensional continuous $\mathbf{Q}_p$-linear representation, unramified outside the primes dividing p; G_K acts by "cyclotomic character" $\chi_{\mathrm{cyc}} : G_K \to \mathbf{Z}_p^\times$.

NOTATION. *If V is a p-adic representation and $n \in \mathbf{Z}$, let $V(n) = V \otimes \mathbf{Q}_p(1)^{\otimes n}$.*

Tate modules of elliptic curves. A/K elliptic curve $\Rightarrow A(\overline{K})$ abelian group with G_K-action. Let $A(\overline{K})[p^n]$ subgroup of p^n-torsion points.

Define the *p-adic Tate module of A*

$$T_p(A) \coloneqq \varprojlim_n A(\overline{K})[p^n] \text{ (w.r.t. multiplication-by-}p\text{ maps)}, \qquad V_p(A) \coloneqq T_p(A) \otimes \mathbf{Q}_p.$$

This is a 2-dimensional p-adic representation of G_K.

Etale cohomology. Let X/K be a smooth algebraic variety. We can define vector spaces

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \quad \text{for } 0 \le i \le 2 \dim X,$$

which are finite-dimensional p-adic Galois representations.

Representations coming from geometry. Our second example is a special case of the third: for an elliptic curve A, it turns out that we have $V_p(A) \cong H^1_{\text{ét}}(A_{\overline{K}}, \mathbf{Q}_p)(1)$.

DEFINITION. *We say an E-linear Galois rep V* comes from geometry *if it is a subquotient of $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)(j) \otimes_{\mathbf{Q}_p} E$, for some variety X/K and some integers i, j.*

So all my examples come from geometry. In these lectures we're only ever going to be interested in representations coming from geometry.

1.2. Galois cohomology

1.2.1. Setup. A good reference for Galois cohomology is [**NSW08**].

Again we assume that K is either a number field of a finite extension of $\mathbf{Q}_\ell$ for some finite prime ℓ. There is a cohomology theory for Galois representations[1]: for V an E-linear G_K-rep, we get E-vector spaces $H^i(G_K, V)$ (where we will usually shorten the notation to $H^i(K, V)$), zero unless $i = 0, 1, 2$. Mostly we care about H^0 and H^1, which are given as follows

$$H^0(K,V) = V^{G_K}$$
$$H^1(K,V) = \frac{\{\text{cts functions } s : G_K \to V \text{ such that } s(gh) = s(g) + gs(h)\}}{\{\text{functions of the form } s(g) = gv - v \text{ for some } v \in V\}}.$$

Remark. The group $H^1(K, V)$ classifies extensions V' of V by the trivial representation, i.e. it classifies isomorphism classes of short exact sequences of G_K-representations

$$0 \longrightarrow V \longrightarrow V' \longrightarrow E \longrightarrow 0,$$

where E is equipped with the trivial action of G_K. ◇

These are well-behaved:

(1) if $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ is a short exact sequence of G_K-modules, then we obtain a long exact sequence
$$\dots \longrightarrow H^{i-1}(K,W) \longrightarrow H^i(K,U) \longrightarrow H^i(K,V) \longrightarrow H^i(K,W) \longrightarrow \dots$$
(2) if $L \subset \overline{K}$ is a finite Galois extension of K, then we have a corestriction map
$$\mathrm{cores} : H^i(L,V) \longrightarrow H^i(K,V).$$
(3) for any subgroup $H \subset G_K$, we have a restriction map
$$\mathrm{res} : H^i(G_K,V) \longrightarrow H^i(H,V).$$

1.2.2. The Kummer map. For $V = \mathbf{Q}_p(1)$ the Galois cohomology is related to the multiplicative group K^*. To see this, we have to first think a bit about cohomology with *finite* coefficients.

For any n, we have a short exact sequence

$$0 \longrightarrow \mu_{p^n} \longrightarrow \overline{K}^\times \xrightarrow{[p^n]} \overline{K}^\times \longrightarrow 0$$

which leads to a long exact sequence

$$0 \longrightarrow \mu_{p^n}^{G_K} \longrightarrow K^\times \xrightarrow{[p^n]} K^\times \longrightarrow H^1(K, \mu_{p^n})$$

and thus an injection[2]

$$K^\times \otimes \mathbf{Z}/p^n\mathbf{Z} \hookrightarrow H^1(K, \mu_{p^n}).$$

Passing to the inverse limit we get a map (**Kummer map**)

$$\kappa_p : K^\times \otimes \mathbf{Z}_p \hookrightarrow H^1(K, \mathbf{Z}_p(1)) \quad \text{or} \quad K^\times \otimes \mathbf{Q}_p \hookrightarrow H^1(K, \mathbf{Q}_p(1)).$$

Remark. This already shows that if K is a number field, then $H^1(K, \mathbf{Q}_p(1))$ can't be finite-dimensional, because $K^\times$ has countably infinite rank. ◇

The same argument works for elliptic curves: we get an embedding

$$E(K) \otimes \mathbf{Q}_p \overset{\kappa}{\hookrightarrow} H^1(K, V_p(E)).$$

[1] Technical point: our representations are all continuous, so we shall work with cohomology defined by continuous cochains, which is slightly different from the cohomology of G_K as an abstract group.

[2] In fact this is an isomorphism, because $H^1(K, \overline{K}^\times)$ is zero ("Hilbert's theorem 90")

1.2.3. Local Galois cohomology. Assume now that K is a finite extension of $\mathbf{Q}_\ell$, and let V be a p-adic representation of G_K. We assume that V is defined over $\mathbf{Q}_p$; the case of general coefficients is obtained by $\otimes_{\mathbf{Q}_p} E$.

The following result is due to Tate:

THEOREM 1. *(1) Let $i \geq 0$. Then $H^i(K, V)$ is a finite-dimensional $\mathbf{Q}_\ell$-vector space, and it is zero for $i > 2$.*

(2) (Tate local duality) For $i \in \{0, 1, 2\}$, the cup product induces a perfect duality

$$H^i(K, V) \times H^{2-i}(K, V^*(1)) \longrightarrow H^2(K, \mathbf{Q}_p(1)) \cong \mathbf{Q}_p.$$

1.2.4. Subspaces of $H^1(\mathbf{Q}_\ell, V)$: the case $\ell \neq p$. Reference: [**Rub00a**]

Assume now that K is a finite extension of $\mathbf{Q}_\ell$ with $\ell \neq p$. Denote by I the inertia subgroup of $G_{\mathbf{Q}_\ell}$.

DEFINITION. *Say that V* is unramified *if I acts trivially on V. Define the subgroup of unramified cohomology classes $H^1_{\mathrm{ur}}(\mathbf{Q}_\ell, V) \subset H^1(\mathbf{Q}_\ell, V)$ by*

$$H^1_{\mathrm{ur}}(K, V) = \ker\left(H^1(K, V) \longrightarrow H^1(I, V)\right).$$

PROPOSITION 1. *Under local Tate duality, we have*

$$H^1_{\mathrm{ur}}(K, V)^\perp = H^1_{\mathrm{ur}}(K, V^*(1)).$$

PROPOSITION 2. *We have*

$$\dim_{\mathbf{Q}_p} H^1_{\mathrm{ur}}(K, V) = \dim_{\mathbf{Q}_p}(V^{G_K}),$$
$$\dim_{\mathbf{Q}_p}\left(\frac{H^1(K, V)}{H^1_{\mathrm{ur}}(K, V)}\right) = \dim_{\mathbf{Q}_p} H^2(K, V).$$

1.2.5. Subspaces of $H^1(\mathbf{Q}_p, V)$. References: [**Ber04**] for the definitions and properties of Fontaine's rings of periods, [**BK90**] for the definitions of the subspaces

Let K be a finite extension of $\mathbf{Q}_p$. Defining interesting subspaces of $H^1(K, V)$ is more complicated, since

$$\ker\left(H^1(K, V) \longrightarrow H^1(I, V)\right) = 0$$

for almost all V! In order to define a good analogue of $H^1_{\mathrm{ur}}(K, V)$, we use Fontaine's classification of p-adic representations.

Fontaine's classification. Fontaine has defined subcategories of the category of p-adic representations of G_K using so-called rings of periods: let B be a topological $\mathbf{Q}_p$-algebra with a continuous and linear action of G_K and some additional structures (e.g. filtration, Frobenius, monodromy operator) compactible with the action of G_K. Assume that B is G_K-regular, i.e. if $b \in B$ is such that $\mathbf{Q}_p.b$ is G_K-stable, then $b \in B^\times$. (Exercise: this implies that B^{G_K} is a field.)

DEFINITION. *Let $D_B(V) = (V \otimes B)^{G_K}$. Then $\dim_{B^{G_K}} D_B(V) \leq \dim_{\mathbf{Q}_p} V$ (exercise), and we say that V is B-admissible if equality holds.*

EXAMPLE. (1) $B = \overline{K}$: V is B-adimissible if and only if the action of G_K factors through a finite quotient;

(2) $B = \hat{\overline{K}}$: V is B-admissible if and only if the action of the inertia subgroup of G_K factors through a finite quotient (Ax–Sen–Tate theorem).

$\diamond$

Fonaine has constructed several rings of periods; here are two of them:

- $\mathbf{B}_{\mathrm{dR}}$: this is a field, equipped with a filtration, and $\mathbf{B}_{\mathrm{dR}}^{G_K} = K$
- $\mathbf{B}_{\mathrm{cris}}$: this is a subring of $\mathbf{B}_{\mathrm{dR}}$, equipped with a filtration and a Frobenius operator φ; we have $\mathbf{B}_{\mathrm{cris}}^{G_K} = K_0$, which is the maximal unramified extension of $\mathbf{Q}_p$ in K.

- $\mathbf{B}_{\mathrm{st}}$: this is (non-canonically) a subring of $\mathbf{B}_{\mathrm{dR}}$, equipped with a filtration, a monodromy operator N and a Frobenius operator φ; we have $\mathbf{B}_{\mathrm{st}}^{G_K} = K_0$, which is the maximal unramified extension of $\mathbf{Q}_p$ in K.

The B-admissible representations are respectively called *de Rham*, *crystalline* and *semistable*, and the $D_B(V)$ are denoted $D_{\mathrm{dR}}(V)$, $D_{\mathrm{cris}}(V)$ and $D_{\mathrm{st}}(V)$.

Remark. crystalline $\Rightarrow$ semistable $\Rightarrow$ de Rham $\diamond$

Theorem 2. *(Faltings) Let X be a smooth proper variety over K. Then $V = H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ is de Rham, and $D_{\mathrm{dR}}(V) \cong H^i_{\mathrm{dR}}(X/K)$ as filtered G_K-vector spaces.*

There are similar comparison theorems with crystalline and log-crystalline cohomology (Tsuji, Niziol,...), involving $\mathbf{B}_{\mathrm{cris}}$ and $\mathbf{B}_{\mathrm{st}}$.

Example. Let A be an abelian variety over K, and let $V = V_p(A)$. Then Faltings' theorem implies that V is de Rham. Is it crystalline or semistable? One can show that

- (Iovita) V is crystalline if and only if A has good reduction;
- (Breuil) V is semistable if and only if it has semistable reduction.

This is a p-adic version of the Neron-Ogg-Shafarevich criterion and justifies the claim that crystalline is a good p-adic analogue of unramified. (Recall that the Neron–Ogg-Shafarevich criterion states that when $\ell \neq p$, then T_pA is unramified if and only if A has good reduction $\pmod{\ell}$.) $\diamond$

The two rings are related by the following result:

Proposition 3. *We have a short exact sequence of G_K-modules*

$$0 \longrightarrow \mathbf{Q}_p \longrightarrow \mathbf{B}_{\mathrm{cris}} \xrightarrow{((1-\varphi)x,x)} \mathbf{B}_{\mathrm{cris}} \oplus \mathbf{B}_{\mathrm{dR}}/\mathrm{Fil}^0\mathbf{B}_{\mathrm{dR}}^+ \longrightarrow 0.$$

Definition of the subspaces of $H^1(K,V)$.

Definition. *Denote by $H^1_f(K,V)$ (resp. $H^1_g(K,V)$) the classes of extensions of V by $\mathbf{Q}_p$ which are crystalline (respectively de Rham).*

Remark. The subspace $H^1_f(K,V)$ turns out to be the right analogue of $H^1_{\mathrm{ur}}(K,V)$ when K is a finite extension of $\mathbf{Q}_p$. $\diamond$

Lemma 1. *If V is crystalline, then*

$$H^1_f(K,V) = \ker\left(H^1(K,V) \longrightarrow H^1(K, V \otimes \mathbf{B}_{\mathrm{cris}})\right).$$

Definition. *Define*

$$H^1_e(K,V) = \ker(H^1(K,V) \longrightarrow H^1(V \otimes \mathbf{B}_{\mathrm{cris}}^{\varphi=1}).$$

Theorem 3. *If V is crystalline, then*

$$H^1_f(K,V)^\perp = H^1_f(K,V^*(1)) \qquad \text{and} \qquad H^1_e(K,V)^\perp = H^1_g(K,V^*(1)),$$

where the orthogonal complement is taken with repect to Tate local duality.

We can use Proposition 3 to calculate the dimensions of $H^1_\star(K,V)$:

Proposition 4. *We have*

- $\dim H^1_f(K,V) = [K:\mathbf{Q}_p]\left(\dim_{\mathbf{Q}_p} V - \dim_K \mathrm{Fil}^0 D_{\mathrm{dR}}(V)\right) + \dim_{\mathbf{Q}_p} V^{G_K}$,
- $\dim H^1_e(K,V) = \dim H^1_f(K,V) - \dim_{\mathbf{Q}_p} D_{\mathrm{cris}}(V)^{\varphi=1}$,
- $\dim H^1_g(K,V) = \dim H^1_f(K,V) + \dim_{\mathbf{Q}_p} D_{\mathrm{cris}}(V^*(1))^{\varphi=1}$.

Remark. (1) These subspaces were defined by Bloch–Kato in their formulation of the global Tamagawa number conjecture; see [**BK90**].

(2) It is clear that $H^1_e \subseteq H^1_f \subseteq H^1_g$. It is conjectured that in most cases, these subspaces should be equal; this is known when $V = V_p(A)$ for $A/\mathbf{Q}_p$ an abelian variety. One of the few cases when the subspaces *fail* to be equal is $V = \mathbf{Q}_p(r)$ for $r \in \mathbf{Z}$: see [**BK90**, Example 3.9].

$\diamond$

The Bloch–Kato exponential map. From the exact sequence in Proposition 3, we obtain the short exact sequence

$$0 \longrightarrow \mathbf{Q}_p \longrightarrow \mathbf{B}_{\mathrm{cris}}^{\varphi=1} \longrightarrow \mathbf{B}_{\mathrm{dR}}/\operatorname{Fil}^0\mathbf{B}_{\mathrm{dR}} \longrightarrow 0.$$

Let V be a de Rham representation of G_K; note that

$$\left(V \otimes \mathbf{B}_{\mathrm{dR}}/\operatorname{Fil}^0\mathbf{B}_{\mathrm{dR}}\right)^{G_K} = D_{\mathrm{dR}}(V)/\operatorname{Fil}^0 D_{\mathrm{dR}}(V).$$

Tensoring the exact sequence with V and taking G_K-cohomology, we obtain a connection homomorphism

$$\exp_V : D_{\mathrm{dR}}(V)/\operatorname{Fil}^0 D_{\mathrm{dR}}(V) \longrightarrow H^1(K, V),$$

which is called the Bloch–Kato exponential map.

LEMMA 2. *The image of* $\exp_V$ *is contained in* $H^1_e(K, V)$.

PROOF. Immediate from the long exact sequence of Galois cohomology. $\square$

REMARK. When $V = V_p(A)$ for some elliptic curve (or more generally an abelian variety) A, then the quotient $D_{\mathrm{dR}}(V)/\operatorname{Fil}^0 D_{\mathrm{dR}}(V)$ can be identified with the tangent space $\tan(A)$, and the Bloch–Kato exponential map is closely related to the usual exponential map:

$$\begin{array}{ccc} \tan(A) & \xrightarrow{\exp} & \mathbf{Q} \otimes A(O_K) \\ \Big\downarrow{=} & & \Big\downarrow{\kappa_A} \\ D_{\mathrm{dR}}(V_p(A)) & \xrightarrow{\exp_{V_p(A)}} & H^1(K, V_p(A)). \end{array}$$

$\diamond$

1.2.6. Selmer groups. Assume now that K is a number field. The group $H^1(K, V)$ can be infinite-dimensional, so it's useful to "cut down to size" by imposing extra conditions on our H^1 elements. We'll do this by localising at primes of K. Note that we have restriction maps

$$\mathrm{res}_v : H^i(K, V) \to H^i(K_v, V) \text{ for all primes } v,$$

and the local groups $H^i(K_v, V)$ are finite-dimensional.

ASSUMPTION. *We assume that V is unramified outside a finite set of primes, i.e. that for almost all v, I_v acts trivially on $V|_{G_{K_v}}$. This is true for all representations arising from geometry.*

REMARK. Conjecturally the representations coming from geometry should be exactly those which are continuous, unramified almost everywhere, and *potentially semistable* at the primes above p. ('Semistable' means that it is $\mathbf{B}_{\mathrm{st}}$-admissible, where $\mathbf{B}_{\mathrm{st}}$ is another one of Fontaine's rings. 'Potentially semistable' means that the representation becomes semistable over a finite extension.) This is called the **Fontaine–Mazur conjecture**. $\diamond$

DEFINITION. *A* local condition *on V at prime v is an E-linear subspace $\mathcal{F}_v \subseteq H^1(K_v, V)$.*

Examples:

- *strict* local condition $\mathcal{F}_{v,\mathrm{strict}} = \{0\}$
- *relaxed* local condition $\mathcal{F}_{v,\mathrm{relaxed}} = H^1(K_v, V)$
- *unramified* local condition $\mathcal{F}_{v,\mathrm{ur}} = H^1_{\mathrm{ur}}(K_v, V)$ for $v \nmid p$
- *Bloch–Kato "finite"* local condition $\mathcal{F}_{v,\mathrm{BK}} = H^1_f(K_v, V)$ for $v \mid p$

- a *Greenberg* local condition (for $v \mid p$)

$$\mathcal{F}_{v,\mathrm{Gr}} = \mathrm{image}\left(H^1(K_v, V^+) \longrightarrow H^1(K_v, V)\right),$$

for some G_{K_v}-stable subrepresentation V^+ of V (c.f. [**Gre89**])

EXAMPLE. Let A be an elliptic curve over $\mathbf{Q}$, and suppose that A has good ordinary reduction at p. Let $\hat{A}$ (resp. $\tilde{A}$) denote the formal group attached to A over $\mathbf{Q}_p$ (resp. the reduction of A (mod p)). Then we have a short exact sequence of $G_{\mathbf{Q}_p}$-modules

$$0 \longrightarrow V_p\hat{A} \longrightarrow V_pA \longrightarrow V_p\tilde{A} \longrightarrow 0,$$

so we can define a Greenberg local condition by taking $V^+ = V_p\hat{A}$. $\diamond$

DEFINITION. *A* Selmer structure *is a collection $\mathcal{F} = (\mathcal{F}_v)_{v \text{ prime of } K}$, satisfying the following condition: for almost all v we have $\mathcal{F}_v = \mathcal{F}_{v,\mathrm{ur}}$. If $\mathcal{F}$ is a Selmer structure we define the corresponding Selmer group by*

$$\mathrm{Sel}_{\mathcal{F}}(K, V) = \{x \in H^1(K, V) : \mathrm{loc}_v(x) \in \mathcal{F}_v \ \forall v\}.$$

THEOREM 4 (Tate). *For any Selmer structure $\mathcal{F}$, the space* $\mathrm{Sel}_{\mathcal{F}}(K, V)$ *is finite-dimensional over $\mathbf{Q}_p$.*

SKETCH OF PROOF. It's easy to see that if this statement is true for one $\mathcal{F}$, it's true for any $\mathcal{F}$, since the *local* Galois cohomology groups $H^1(K_v, V)$ are all finite-dimensional. We now choose a particular Selmer structure $\mathcal{F}$: let Σ be a finite set of primes containing all infinite places, all places above p, and all places where V is ramified, and consider the Selmer structure

$$\mathcal{F}_v = \begin{cases} H^1_{\mathrm{ur}}(K_v, V) & \text{if } v \notin \Sigma \\ H^1(K_v, V) & \text{if } v \in \Sigma \end{cases}$$

Then Tate has shown that the Selmer group associated to this Selmer structure is finite-dimensional. (It turns out that one can identify this Selmer group with the Galois cohomology group $H^1(\mathrm{Gal}(K^\Sigma/K), V)$, where K^Σ is the maximal extension of K in $\overline{K}$ unramified outside Σ. For the details, see [**Rub00b**, Lemma 5.3].) $\square$

We're mostly interested in three specific choices of Selmer structure, differing only in the choices of the $\mathcal{F}_v$ at primes $v \mid p$: we define the *strict Selmer group*

$$\mathrm{Sel}_{\mathrm{strict}}(K, V) = \begin{cases} \mathcal{F}_{v,\mathrm{ur}} & \text{if } v \nmid p \\ \mathcal{F}_{v,\mathrm{strict}} & \text{if } v \in \Sigma \end{cases}$$

and similarly the *relaxed Selmer group* and *Bloch–Kato Selmer group*.

Hence the strict, relaxed, and Bloch–Kato Selmer groups satisfy

$$\mathrm{Sel}_{\mathrm{strict}}(K, V) \subseteq \mathrm{Sel}_{\mathrm{BK}}(K, V) \subseteq \mathrm{Sel}_{\mathrm{relaxed}}(K, V).$$

REMARK. As will soon become clear, it is $\mathrm{Sel}_{\mathrm{BK}}(K, V)$ which is the most important of all. We care about $\mathrm{Sel}_{\mathrm{strict}}(K, V)$ and $\mathrm{Sel}_{\mathrm{relaxed}}(K, V)$ because they are easier to study, and will give us a stepping-stone towards $\mathrm{Sel}_{\mathrm{BK}}(K, V)$. $\diamond$

EXAMPLE. Recall that for $V = \mathbf{Q}_p(1)$ we have the Kummer map

$$K^\times \otimes \mathbf{Q}_p \hookrightarrow H^1(K, \mathbf{Q}_p(1)).$$

One can check that this induces isomorphisms

$$\mathcal{O}_K[1/p]^\times \otimes \mathbf{Q}_p \xrightarrow{\cong} \mathrm{Sel}_{\mathrm{relaxed}}(K, \mathbf{Q}_p(1)), \tag{1}$$

$$\mathcal{O}_K^\times \otimes \mathbf{Q}_p \xrightarrow{\cong} \mathrm{Sel}_{\mathrm{BK}}(K, \mathbf{Q}_p(1)). \tag{2}$$

The strict Selmer group, on the other hand, should be zero; this is exactly Leopoldt's conjecture for K. $\diamond$

REMARK. There is a global duality (called Poitou–Tate duality) relating Selmer groups for V and for $V^*(1)$ with 'dual' local conditions. A good reference is [**Rub00b**]. $\diamond$

1.3. L-functions of Galois representations

1.3.1. Local Euler factors. Let V as above, v unramified prime. Then $\rho(\mathrm{Frob}_v)$ is well-defined up to conjugacy, where Frob_v is the arithmetic Frobenius.

Definition. *The* local Euler factor *of V at v is the polynomial*

$$P_v(V,t) := \det(1 - t \cdot \rho(\mathrm{Frob}_v^{-1})) \in E[t].$$

Examples:

V	$P_v(V,t)$	
$\mathbf{Q}_p$	$1-t$	
$\mathbf{Q}_p(n)$	$1-\frac{t}{q_v^n}$,	$q_v = \#\mathbf{F}_v$
$H^1(A_{\overline{K}}, \mathbf{Q}_p)$	$1 - a_v(A)t + q_v t^2$,	$a_v(A) := 1 + q_v - \#A(\mathbf{F}_v)$

Remark. Chebotarev density theorem $\Rightarrow$ if V and W are irreducible and have same Euler factors at almost all v, then $V \cong W$. ◇

1.3.2. Global L-functions (sketch). Assume V comes from geometry, and V is semisimple (direct sum of irreducibles). Then $P_v(V,t)$ has coefficients in $\overline{\mathbf{Q}}$ (Deligne); and there is a way of defining $P_v(V,t)$ for bad primes v (case $v \mid p$ is hardest).

Fix an embedding $\iota : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$. Then we consider the product

$$L(V,s) := \prod_{v \text{ prime}} P_v(V, q_v^{-s})^{-1}.$$

Miraculously, this converges for $\Re(s) \gg 0$.

Example.
(1) $V = \mathbf{Q}_p(n)$, $K = \mathbf{Q}$: $L(V,s) = \zeta(s+n)$;
(2) $V = \mathbf{Q}_p(n)$, K a number field: $L(V,s) = \zeta_K(s+n)$, where $\zeta_K(s)$ is the Dedekind zeta function of K;
(3) $V = H^1(A_{\overline{K}}, \mathbf{Q}_p)$, A/K an elliptic curve: $L(V,s)$ is the Hasse–Weil L-function $L(A/K,s)$

◇

Conjecture 1. *For V semisimple and coming from geometry, $L(V,s)$ has meromorphic continuation to $s \in \mathbf{C}$ with finitely many poles, and satisfies a functional equation relating $L(V,s)$ and $L(V^*, 1-s)$.*

Note that if V is semisimple and comes from geometry, the same is true[3] of V^*, so the conjecture is well-posed. This conjecture is of course very hard – the only cases where it is known is where we can relate V to something *automorphic*, e.g. a modular form or a genus 2 Siegel modular form. The latter was proven recently by Boxer–Calegary–Gee–Pilloni.

There are lots of conjectures (and a rather smaller set of theorems) relating properties of arithmetic objects to values of their L-functions; the Birch–Swinnerton-Dyer conjecture is perhaps the best-known of these. As we've just seen, all the information about an elliptic curve you need to define its L-function is encoded in the Galois action on its Tate module; so can we express the BSD conjecture purely in terms of Galois representations? This will be the topic of the next section.[4]

1.4. The Bloch–Kato conjecture

Let K be a number field, and let V be a p-adic representation of G_K coming from geometry.

[3] It is not obvious if this holds without the semisimplicity assumption.

[4] Actually the answer is "no, we can't" – as far as I'm aware, there is no purely Galois-representation-theoretic statement that is precisely equivalent to BSD. But we can get pretty close, as we'll shortly see.

Conjecture 2 (Bloch–Kato). *We have*

$$\dim \mathrm{Sel}_{\mathrm{BK}}(K, V) - \dim H^0(K, V) = \mathrm{ord}_{s=0} L(V^*(1), s).$$

Remark. There are refined versions using $\mathbf{Z}_p$-modules in place of $\mathbf{Q}_p$-vector spaces, which predict the leading term of the L-function up to a unit; but we won't go into these here. ◇

Let's look at what the conjecture says in some example cases.

Example 1: $V = \mathbf{Q}_p$. Here

$$L(V^*(1), s) = L(\mathbf{Q}_p, s+1) = \zeta_K(s+1),$$

so the right-hand side is the order of vanishing of $\zeta_K(s)$ at $s = 1$, which is -1 (there's a simple pole). The left-hand side is $\dim \mathrm{Sel}_{\mathrm{BK}}(K, \mathbf{Q}_p) - 1$, so the conjecture predicts that $\mathrm{Sel}_{\mathrm{BK}}(K, \mathbf{Q}_p) = 0$.

Exercise: Prove this. You'll need to use the finiteness of the ideal class group of K, together with the fact that for this representation the local condition $\mathcal{F}_{v,\mathrm{BK}}$ agrees with $\mathcal{F}_{v,\mathrm{ur}}$ for primes $v \mid p$.

Example 2: $V = \mathbf{Q}_p(1)$. Here $L(V^*(1), s) = \zeta_K(s)$. We recall the functional equation for $\zeta_K(s)$: let

$$\Gamma_{\mathbf{R}}(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right),$$
$$\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s} \Gamma(s).$$

and denote by Δ_K the discriminant of K. Let

$$\Lambda_K(s) = |\Delta_K|^{\frac{s}{2}} \Gamma_{\mathbf{R}}^{r_1} \Gamma_{\mathbf{C}}(s)^{r_2} \zeta_K(s),$$

where r_1 (resp. $2r_2$) denote the number of real (resp. complex) embeddings of K. Then

$$\Lambda_K(s) = \Lambda_K(1-s).$$

From this we can deduce $\mathrm{ord}_{s=0} \zeta_K(s)$: we know that $\Gamma(s)$ has a simple pole at $s = 0$ and is nonzero at $s = \frac{1}{2}$ and $s = 1$, and that ζ_K has a simple pole at $s = 1$. Hence

$$\mathrm{ord}_{s=0} \zeta_K(s) = r_1 + r_2 - 1.$$

Remark. For $K = \mathbf{Q}$, then $\zeta(0) = -\frac{1}{2}$ is finite and non-zero. ◇

On the algebraic side, we have $H^0(K, \mathbf{Q}_p(1)) = 0$ and

$$\dim \mathrm{Sel}_{\mathrm{BK}}(K, V) = \dim_{\mathbf{Q}_p}\left(\mathcal{O}_K^\times \otimes \mathbf{Q}_p\right) = \mathrm{rank}\, \mathcal{O}_K^\times$$

by (2). (For details, see [**Rub00b**, Prop. 6.1].) Hence the Bloch–Kato conjecture here is exactly Dirichlet's unit theorem.

Example 3: Elliptic curves. If V is $V_p(E)$ for an elliptic curve E, then:

- $H^0(K, V) = 0$;
- the Kummer map lands inside the $\mathrm{Sel}_{\mathrm{BK}}(K, V)$ by Lemma 2, and it gives an embedding

  $$E(K) \otimes \mathbf{Q}_p \hookrightarrow \mathrm{Sel}_{\mathrm{BK}}(K, V),$$

  so that $\dim \mathrm{Sel}_{\mathrm{BK}} \geq \mathrm{rank}(E/K)$, with equality iff the p-part of Sha is finite;
- $\mathrm{ord}_{s=0} L(V^*(1), s) = \mathrm{ord}_{s=1} L(E/K, s)$.

Hence the Bloch–Kato conjecture predicts that

$$\mathrm{ord}_{s=1} L(E/K, s) = \mathrm{Sel}_{\mathrm{BK}}(K, V),$$

which is closely related to (but slightly weaker than) the Birch–Swinnerton-Dyer conjecture.

Remark. Notice that $L(V^*(1), s)$ is expected to be related to $L(V, -s)$ via a functional equation; but this functional equation will involve various Γ functions as factors, which can have poles, so the orders of vanishing of the two functions at 0 are not the same in general, as we saw for $\mathbf{Q}_p$ and $\mathbf{Q}_p(1)$. On the Selmer-group side there's a corresponding relation between $\mathrm{Sel}_{\mathrm{BK}}(K, V)$ and $\mathrm{Sel}_{\mathrm{BK}}(K, V^*(1))$ coming from the Poitou–Tate global duality theorem in Galois cohomology. One can check that these factors precisely cancel out: if

$L(V,s)$ has a functional equation of the expected form, then the Bloch–Kato conjecture holds for $V^*(1)$ if and only if it holds for V. This is a wonderful (but rather involved) exercise. $\diamond$

1.5. Euler systems

We'll now introduce the key subject of these lectures: Euler systems, which are tools for studying and controlling Selmer groups. In this section we'll give the abstract definition of an Euler system, and explain (without proofs) why the existence of an Euler system for some Galois representation has powerful consequences for Selmer groups.

References: The standard work on this topic is Karl Rubin's orange book *Euler Systems* [**Rub00b**]. There are also two alternative accounts in Rubin's 2004 Park City lecture notes, and in the book *Kolyvagin Systems* [**MR04**] by Mazur and Rubin.

1.5.1. The definition.

Let:

- V a $G_{\mathbf{Q}}$-representation (for simplicity)
- $T \subset V$ a $G_{\mathbf{Q}}$-stable $\mathbf{Z}_p$-lattice
- Σ a finite set of primes containing p and all ramified primes for V

Since V is a $G_{\mathbf{Q}}$-rep, we can consider it as a G_K-rep for any number field K and form $H^i(K,V)$, and there are **corestriction** or **norm** maps

$$\mathrm{norm}_K^L : H^i(L,V) \to H^i(K,V) \quad \text{if } L \supset K.$$

If K is Galois, $H^i(K,V)$ is a module over $\mathbf{Q}_p[\mathrm{Gal}(K/\mathbf{Q})]$. Similarly for cohomology of lattices $H^i(K,T)$.

Definition. *An* Euler system for (T,Σ) *is a collection* $\mathbf{c} = (c_m)_{m\geq 1}$*, where* $c_m \in H^1(\mathbf{Q}(\mu_m),T)$*, satisfying the following compatibility for any* $m \geq 1$ *and* ℓ *prime:*

$$\mathrm{norm}_{\mathbf{Q}(\mu_m)}^{\mathbf{Q}(\mu_{m\ell})}(c_{m\ell}) = \begin{cases} c_m & \text{if } \ell \in \Sigma \text{ or } \ell \mid m \\ P_\ell(V^*(1),\sigma_\ell^{-1}) \cdot c_m & \text{otherwise} \end{cases}$$

where σ_ℓ *is the image of* Frob_ℓ *in* $\mathrm{Gal}(\mathbf{Q}(\mu_m)/\mathbf{Q})$*. An* Euler system for V *is an Euler system for* (T,Σ)*, for some* $T \subset V$ *and some* Σ*.*

This definition is not very transparent, I admit! Fear not: we'll see an example before too long. Intuitively, each class c_m has "something to do with" the L-function $L(V^*(1),s)$ with its Euler factors at primes dividing $m\Sigma$ missing[5]; so when we compare elements for different m, the Euler factors appear.

The main reason to care about these objects is the following theorem, which is due to Rubin [**Rub00b**], building on earlier work of Kato [**Kat04**], Kolyvagin [**Kol91**], and Thaine [**Tha88**]:

Theorem 5. [**Rub00a**, Theorem 2.3] *Suppose* $\mathbf{c}$ *is an Euler system for* (T,Σ) *with* c_1 *non-zero, and suppose* V *satisfies various technical conditions. Then* $\mathrm{Sel}_{\mathrm{strict}}(\mathbf{Q},V^*(1))$ *is zero.*

For the purposes of these lectures we don't need to know how this theorem is proved – our goal is to understand how to *build* Euler systems, which is a separate problem. If you do want to know about the proof, then see the references listed above.

Remark.

- The technical conditions are to do with the image of $G_{\mathbf{Q}}$ in $\mathrm{GL}(V)$. This needs to be "large enough" in the following sense:
  (1) V is irreducible as a $\mathbf{Q}_p[G_{\mathbf{Q}}]$-module;
  (2) there exists $\tau \in \mathrm{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}(\mu_{p^\infty})\right)$ such that $\dim\left(V/(\tau-1)V\right) = 1$.

[5] This becomes more precise if you work with the *equivariant* L-function $L(V^*(1),\mathbf{Q}(\mu_m)/\mathbf{Q},s)$ which is a Dirichlet series taking values in the group ring $\mathbf{C}[(\mathbf{Z}/m\mathbf{Z})^\times]$ rather than just in $\mathbf{C}$, encoding the L-values of V twisted by Dirichlet characters modulo m. The definition of this function only makes sense if you drop the Euler factors at primes dividing m.

- For the proof of the theorem, we don't actually need c_m to be defined for all m; it's enough to have c_m for all integers m of the form $p^k m_0$, where $k \geq 0$ and m_0 is a square-free product of primes not in Σ.
- More generally, one can also define Euler systems for G_K-representations, for K a number field. In place of cyclotomic fields, one has to have classes over different ray class fields of K. However, we'll only work with $K = \mathbf{Q}$ here.
- There is also a notion of "anticyclotomic Euler system", which applies when you have a representation V of G_K, a quadratic extension L/K, and cohomology classes for V over the *anticyclotomic extensions* of L, which are the abelian extensions of L such that conjugation by $\mathrm{Gal}(L/K)$ acts on their Galois groups by -1. The most important example of an anticyclotomic Euler system is Kolyvagin's **Euler system of Heegner points** [**Kol91**], where $K = \mathbf{Q}$, $V = V_p(E)$ for E an elliptic curve, and L is an imaginary quadratic field. Other examples of anticyclotomic Euler systems have recently been found by Cornut, and by Jetchev and his collaborators.

$\diamond$

1.5.2. Cyclotomic units. We're going to build an Euler system for $V = \mathbf{Q}_p(1)$. Recall that we have Kummer maps $K^\times \hookrightarrow H^1(K, \mathbf{Z}_p(1))$. Also, for L/K finite, we have a commutative square

$$\begin{array}{ccc} L^\times & \xrightarrow{\kappa_p} & H^1(L, \mathbf{Z}_p(1)) \\ {\scriptstyle \mathrm{norm}_K^L}\downarrow & & \downarrow{\scriptstyle \mathrm{norm}_K^L} \\ K^\times & \xrightarrow{\kappa_p} & H^1(K, \mathbf{Z}_p(1)) \end{array}$$

where the left-hand norm map is the usual field norm, and the right-hand one is the Galois corestriction. So we have to find good elements of the multiplicative groups of cyclotomic fields, satisfying compatibilities under the norm maps.

Fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}^\times$ and let $\zeta_m = \iota^{-1}(e^{2\pi i/m}) \in \mu_m$.

Definition. *For $m > 1$, set $u_m = 1 - \zeta_m \in \mathbf{Q}(\mu_m)^\times$.*

A pleasant computation (exercise!) shows that[6]

$$\mathrm{norm}_{\mathbf{Q}(\mu_m)}^{\mathbf{Q}(\mu_{m\ell})} u_{m\ell} = \begin{cases} u_m & \text{if } \ell \mid m \\ (1 - \sigma_\ell^{-1}) \cdot u_m & \text{if } \ell \nmid m \text{ and } m > 1 \\ \ell & \text{if } m = 1 \end{cases}$$

This is almost what we need for an Euler system, but there are two problems: firstly, there is no sensible way to define u_1; secondly, we are seeing Euler factors at *all* primes, whereas we only want to see them for primes outside Σ (and Σ can't be empty because it has to contain p). We can get around both of these problems by setting

$$v_m = \begin{cases} u_m & \text{if } p \mid m, \\ \mathrm{norm}_{\mathbf{Q}(\mu_m)}^{\mathbf{Q}(\mu_{pm})}(u_{pm}) & \text{if } p \nmid m \text{ (including } m = 1). \end{cases}$$

Theorem 6. *The classes $c_m = \kappa_p(v_m)$ are an Euler system for $(\mathbf{Z}_p(1), \{p\})$.* $\square$

1.5.3. Soulé twists. Rubin's theorem applied directly to the cyclotomic unit Euler system isn't actually very interesting (it follows easily from class field theory that $\mathrm{Sel}_{\mathrm{strict}}(\mathbf{Q}, \mathbf{Q}_p) = 0$). However, there is a notion of *twisting* for Euler systems.

[6] This is stated in Rubin's book "Euler Systems", §3.2, but with a sign error: he sets $u_m = \zeta_m - 1$, which doesn't quite work, since $\mathrm{norm}_{\mathbf{Q}(\mu_4)}^{\mathbf{Q}(\mu_8)}(\zeta_8 - 1) \neq \zeta_4 - 1$.

Theorem 7. *Let $\chi : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$ be a continuous character unramified outside Σ (e.g. any power of the cyclotomic character). Then there is a canonical bijection $\mathbf{c} \mapsto \mathbf{c}^\chi$ between Euler systems for T and for the twist $T(\chi)$.*

Let me sketch how to prove this when χ is the cyclotomic character. Let e be a basis of $\mathbf{Z}_p(1)$, and write e_n for $e \pmod{p^n}$. Suppose we have an Euler system $(c_m)_{m\geq 1}$ with $c_m \in H^1(\mathbf{Q}(\mu_m), T)$, where $T \subset V$ is a $G_{\mathbf{Q}}$-stable $\mathbf{Z}_p$-lattice. We then have natural map

$$\mathrm{tw}_n : H^1(\mathbf{Q}(\mu_{mp^n}), T/p^nT) \xrightarrow{\otimes e_n} H^1(\mathbf{Q}(\mu_{mp^n}), T/p^nT) \otimes e_n \cong H^1(\mathbf{Q}(\mu_{mp^n}), T/p^nT(1));$$

the second equality is true because $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}(\mu_{mp^n}))$ acts trivially on e_n. Note that if $c_{mp^n,n}$ denotes the image of c_{mp^n} in $H^1(\mathbf{Q}(\mu_{mp^n}), T/p^nT)$, then

$$\mathrm{tw}_n(c_{mp^n,n}) \in H^1(\mathbf{Q}(\mu_{mp^n}), T/p^nT(1)).$$

Define

$$c_m^\chi = \varprojlim_n \mathrm{norm}_{\mathbf{Q}(\mu_m)}^{\mathbf{Q}(\mu_{mp^n})} \mathrm{tw}_n(c_{m,n});$$

note that $c_m^\chi \in \varprojlim_n H^1(\mathbf{Q}(\mu_m), T/p^nT(1)) \cong H^1(\mathbf{Q}(\mu_m), T(1))$. On can show that these classes form an Euler system for $V(1)$.

Remark. Note that the "bottom class" c_1^χ in the twisted Euler system depends on the collection of classes $\{c_{p^n}\}_{n\geq 1}$, not just on c_1. So even if $c_1 \neq 0$ we might have $c_1^\chi = 0$, and we have to check carefully that the twisted Euler system satisfies the conditions for Rubin's theorem. $\diamond$

The twists of the cyclotomic unit Euler system have many applications in number theory; see e.g. §3.2 of [**Rub00b**]. For instance, they play a major role in Huber and Kings' proof of the Bloch–Kato conjecture for $\mathbf{Q}_p(n)$ for all $n \in \mathbf{Z}$, an account of which can be found in [**CRSS15**].

CHAPTER 2

A toolkit for building Euler systems

2.1. Etale cohomology and the Hochschild–Serre spectral sequence

(*References*: not as many as there should be. Jannsen's article "Continuous étale cohomology" [**Jan88**] has the details, but it is not an easy read.)

We saw before that, for a variety X/K, the étale cohomology groups $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ were an interesting source of Galois representations.

But this isn't the only thing we can do with étale cohomology. Rather than base-extending to $\overline{K}$, we can also take étale cohomology of X/K directly[1]; there are groups $H^i_{\text{ét}}(X, \mathbf{Q}_p(m))$ for all i and m. These "absolute" étale cohomology groups are *not* themselves Galois representations, but it turns out that these are related to the Galois cohomology of the étale cohomology over $\overline{K}$:

Theorem 8 (Jannsen). *For any variety X/K, and any n, there is a convergent "Hochschild–Serre" spectral sequence*

$$E_2^{ij} = H^i\left(K, H^j_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)(n)\right) \Rightarrow H^{i+j}_{\text{ét}}(X, \mathbf{Q}_p(n)).$$

In particular, we get edge maps $H^i(X, \mathbf{Q}_p(n)) \to H^i(X_{\overline{K}}, \mathbf{Q}_p(n))^{G_K}$, and if $F^1 H^i$ denotes the kernel of this map (the "homologically trivial" classes), there is a map

$$F^1 H^i(X, \mathbf{Q}_p(n)) \to H^1\left(K, H^{i-1}(X_{\overline{K}}, \mathbf{Q}_p)(n)\right).$$

So, if X is defined over $\mathbf{Q}$ and V is the Galois representation $H^{i-1}(X_{\overline{\mathbf{Q}}})$ (or a direct summand of it), we can try to construct an Euler system for V by building classes in $F^1 H^i(X_{\mathbf{Q}(\mu_m)})$ for varying m.

How will we do this? We'll use geometry! To be precise, we'll rely on the following rather simple bag of tricks:

- **Cup products**: étale cohomology has cup-product maps
  $$H^i(X, \mathbf{Q}_p(m)) \times H^j(X, \mathbf{Q}_p(n)) \to H^{i+j}(X, \mathbf{Q}_p(m+n)).$$
- **Kummer maps**: if $f \in \mathcal{O}(X)^\times$ is a unit in the ring of rational functions on X, then there is a class $\kappa_p(f) \in H^1(X, \mathbf{Q}_p(1))$.
- **Pushforward maps**: if $Z \subset X$ is a closed subvariety of codimension d (and X and Z are both smooth), then there are pushforward maps
  $$H^i(Z, \mathbf{Q}_p(n)) \to H^{i+2d}(X, \mathbf{Q}_p(n+d)).$$
  In particular, the pushforward of the identity class $1_Z \in H^0(Z, \mathbf{Q}_p(0))$ is a class in $H^{2d}(X, \mathbf{Q}_p(d))$, the *cycle class* of Z.

So if we have a good supply of units on X, or of subvarieties of X (or of subvarieties of X with units on them, etc) then we have some objects to play with; and we can try to write down classes landing in the "right" cohomological degree to map into H^1 of our target Galois representation.

[1] Technical point: what we actually want here is "continuous étale cohomology" in the sense of Jannsen. This is consistent with our use of continuous cochains to define cohomology of Galois representations.

If you have a random variety, it's not clear how to find lots of subvarieties, or lots of units, on it; but we're going to home in on the case where X is a *Shimura variety* – a variety coming from automorphic theory, such as a modular curve. Then we can try and write down units and subvarieties using automorphic ideas.

2.2. Modular curves and modular forms

(*References*: Diamond–Shurman [**DS05**], Darmon–Diamond–Taylor [**DDT97**].)

We're particularly interested in the Galois representations associated to modular forms, which come from geometry via modular curves. We'll consider weight 2 modular forms first, as these are the simplest to handle.

2.2.1. Modular curves. For $N \geq 1$ let

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : c = 0, d = 1 \bmod N \right\}.$$

This acts on the upper half-plane $\mathcal{H}$ via $\tau \mapsto \frac{a\tau+b}{c\tau+d}$. It turns out that the quotient is naturally an algebraic variety:

THEOREM 9. *For $N \geq 4$ there is an algebraic variety $Y_1(N)$ over $\mathbf{Q}$ with the following properties:*

- *$Y_1(N)$ is a smooth geometrically connected affine curve.*
- *For any field extension[2] $F/\mathbf{Q}$, the F-points of $Y_1(N)$ biject with isomorphism classes of pairs (E, P), where E/F is an elliptic curve and $P \in E(F)$ is a point of order N on E.*
- *$Y_1(N)(\mathbf{C}) \cong \Gamma_1(N)\backslash\mathcal{H}$, via the map sending $\tau \in \mathcal{H}$ to (E_τ, P_τ) where $E_\tau = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$ and $P_\tau = 1/N \bmod \mathbf{Z} + \mathbf{Z}\tau$.*

(Much stronger theorems are known – for instance, $Y_1(N)$ has a canonical smooth model over $\mathbf{Z}[1/N]$ – but we won't need this just now.)

REMARK. There are two different choices of conventions for $\mathbf{Q}$-models for $Y_1(N)$; everyone agrees what $Y_1(N)$ means over $\mathbf{C}$, but there are two different ways to descend it to $\mathbf{Q}$, classifying elliptic curves with either a point of order N (our convention) or an embedding of the group scheme μ_N (the alternative convention). $\diamond$

2.2.2. Galois representations. We can use these rational models of modular curves to attach Galois representations to modular forms. Let $f = \sum a_n q^n$ be a cuspidal modular eigenform of weight 2 and level $\Gamma_1(N)$, normalised so that $a_1 = 1$. By a theorem of Shimura, there is a number field L such that all $a_n \in L$. We shall fix an embedding $\iota : L \hookrightarrow \overline{\mathbf{Q}}_p$, and assume that our p-adic coefficient field $E/\mathbf{Q}_p$ contains the image of ι.

DEFINITION. *We let $V_p(f)$ be the largest subspace of $H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes E$ on which the Hecke operators $T(\ell)$, for $\ell \nmid N$, act as multiplication by $a_\ell(f)$.*

By construction, $V_p(f)$ is an E-linear Galois representation coming from geometry. However, one can also show that

1. $V_p(f)$ is 2-dimensional and irreducible.
2. $V_p(f)$ is a direct summand of $H^1_{\text{ét}}$ (not just a subspace).
3. For $\ell \nmid pN$, $V_p(f)$ is unramified at ℓ and the trace of Frob_ℓ^{-1} on $V_p(f)$ is $a_\ell(f)$. More precisely, the local Euler factor is given by
$$P_\ell(V_p(f), t) = 1 - a_\ell(f)t + \ell\chi(\ell)t^2,$$
where χ is the character of f.
4. $V_p(f)^* = V_p(f \otimes \chi^{-1})(1)$.

[2] Any $\mathbf{Q}$-algebra, in fact; this is important if you want to make precise the idea that $Y_1(N)$ represents a functor.

It follows from (3) that (up to finitely many bad Euler factors at primes $\ell \mid pN$)[3] the global L-series $L(V_p(f), s)$ is just the L-series of f,

$$L(f,s) = \sum a_n(f) n^{-s}.$$

In particular if $L = \mathbf{Q}$, so that f corresponds to an elliptic curve A, then we have $V_p(f) \cong H^1_{\text{ét}}\left(A_{\overline{\mathbf{Q}}}, \mathbf{Q}_p\right) \cong V_p(A)(-1)$.

Remark. Warning: in Diamond–Shurman chapter 9, the representation they denote by $\rho_{f,p}$ is the *dual* of our $V_p(f)$, which is compensated for by the fact that they use arithmetic Frobenius Frob_p rather than geometric Frobenius Frob_p^{-1} to define the Euler factor. The same applies to Romyar Sharifi's notes at this Arizona Winter School: the representation (ρ_f, V_f) defined in §3.5 of his notes is the dual of our $V_p(f)$. $\diamond$

2.2.3. Tensor products. Later on, we'll be interested in *tensor products* of Galois representations associated to modular forms. If you take two newforms f, g (both with coefficients in E) and let V be the four-dimensional Galois representation $V = V_p(f) \otimes V_p(g)$, then using the Kunneth formula for étale cohomology you can show that V is a direct summand of $H^2_{\text{ét}}\left(Y_1(N)^2_{\overline{\mathbf{Q}}}, \mathbf{Q}_p\right) \otimes E$, for any N divisible by N_f and N_g.

The L-function attached to this tensor product representation is a rather classical object: it's the so-called *Rankin–Selberg convolution L-function* of f and g, denoted by $L(f \otimes g, s)$. Up to finitely many bad Euler factors, this agrees with the Dirichlet series

$$L(\chi_f \chi_g, 2s-2) \sum_{n \geq 1} a_n(f) a_n(g) n^{-s}. \tag{‡}$$

2.3. Numerology

For instance, let's suppose we want to build an Euler system for $V_p(f)$, where f is a modular form of weight 2. Since we can twist Euler systems, we can choose to work with $V_p(f)(n)$ for any integer n.

Because $Y = Y_1(N)_{\mathbf{Q}}$ is affine, we have $H^2_{\text{ét}}(Y_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) = 0$, and $H^1_{\text{ét}}(Y_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)(n)$ contains $V_p(f)(n)$ as a direct summand. So the Hochschild–Serre spectral sequence gives us a map

$$H^2_{\text{ét}}(Y, \mathbf{Q}_p(n)) \to H^1\left(\mathbf{Q}, H^1_{\text{ét}}(Y_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)(n)\right) \to H^1(\mathbf{Q}, V_p(f)(n)).$$

How can we get at the groups $H^2_{\text{ét}}(Y, \mathbf{Q}_p(n))$ using our geometric toolkit?

- For $n \leq 0$ this is hopeless, because our toolkit will only ever give classes in $H^i(-, \mathbf{Q}_p(n))$ for $n \geq \frac{i}{2}$ (check this!)
- For $n = 1$, you can use cycle classes of codimension 1 subvarieties of Y – i.e., points. This is Kolyvagin's original approach [**Kol91**]: to build an Euler system using cycle classes of Heegner points. However, this gives an anticyclotomic Euler system (relative to some choice of imaginary quadratic field), not a full Euler system in the sense of §1.5.1.[4].
- For $n = 2$, you can use cup-products of units: the Kummer map gives you classes in $H^1_{\text{ét}}(Y, \mathbf{Q}_p(1))$, and the cup-product of two such classes lands in $H^2_{\text{ét}}(Y, \mathbf{Q}_p(2))$. This is Kato's approach [**Kat04**].
- $n \geq 3$ can also be made to work similarly (but gives no more information than for $n = 2$).

We can also ask the same question for $V_p(f) \otimes V_p(g)$, using the geometry of $Y \times Y$. Again, different twists n give very different geometric setups; and taking n too small is hopeless – you want $n \geq 2$ at least. The sensible choices are:

[3] In fact, if f is a newform, then $L(f,s)$ and $L(V_p(f), s)$ have the same Euler factors at the bad primes too, although this is much harder to check. This doesn't work for the Rankin–Selberg L-function; the "naive" Rankin–Selberg L-series (‡) frequently has the wrong local factors at the bad primes, even if f and g are newforms.

[4] This is an instance of a general phenomenon. We have seen that one needs $i \leq 2n$ for geometric techniques to work. It turns out that in the boundary case $i = 2n$, one can only work with cycle classes of subvarieties (not with units); and these cannot give a full Euler system, only an anticyclotomic one.

- $n = 3$: we can get classes here as cup-products $\kappa_p(f_1) \cup \kappa_p(f_2) \cup \kappa_p(f_3)$, where f_1, f_2, f_3 are units on $Y \times Y$.
- $n = 2$: we can get classes by taking a curve $Z \subset Y \times Y$ and a unit $f \in \mathcal{O}(Z)^\times$, and pushing forward $\kappa_p(f) \in H^1_{\text{ét}}(Z, \mathbf{Q}_p(1))$ along the embedding $Z \hookrightarrow Y \times Y$.

The $n = 3$ approach has, I believe, never been carried out (and people have tried very hard to make it work without success). The $n = 2$ approach leads to the Euler system of Beilinson–Flach elements, which we'll discuss later in these lectures.

2.4. Changing the field and changing the level

To build an Euler system using the Hochschild–Serre spectral sequence, we need to build classes in $H^i(X_K, \mathbf{Q}_p(n))$ as K varies over cyclotomic fields. It turns out that, for modular curves, we can "sneak up" on this field extension by varying the level of our modular curves instead.

Definition. *We write μ_m° for the $\mathbf{Q}$-variety of primitive m-th roots of unity.*

Concretely, this is the 0-dimensional subvariety of the affine line cut out by $\Phi_m(X) = 0$, where Φ_m is the m-th cyclotomic polynomial. This variety is connected (since the cyclotomic polynomials are irreducible over $\mathbf{Q}$) but not, of course, geometrically connected once $m > 2$.

Hence, for any variety $X/\mathbf{Q}$, we can consider the product variety $X \times \mu_m^\circ$, which is also a variety over $\mathbf{Q}$.

Proposition 5. *For any i, m, n, we have isomorphisms of $G_{\mathbf{Q}}$-representations*

$$H^i_{\text{ét}}\left((X \times \mu_m^\circ)_{\overline{\mathbf{Q}}}, \mathbf{Q}_p\right) \cong \operatorname{Ind}_{G_{\mathbf{Q}(\mu_m)}}^{G_{\mathbf{Q}}} H^i_{\text{ét}}\left(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_p\right)$$

and isomorphisms of $\mathbf{Q}_p$-vector spaces

$$H^i_{\text{ét}}\left(X_{\mathbf{Q}(\mu_m)}, \mathbf{Q}_p(n)\right) \cong H^i_{\text{ét}}\left(X \times \mu_m^\circ, \mathbf{Q}_p(n)\right).$$

(This is a form of Shapiro's lemma; it corresponds to the fact that $\mu_m^\circ = \operatorname{Spec} \mathbf{Q}(\mu_m)$, and hence $X_{\mathbf{Q}} \times \mu_m^\circ$ is the image of $X_{\mathbf{Q}(\mu_m)}$ under the forgetful functor from $\mathbf{Q}(\mu_m)$-varieties to $\mathbf{Q}$-varieties.)

This is useful to us because, if $X = Y_1(N)$, the base-extension $Y_1(N) \times \mu_m^\circ$ is *also* a modular curve. More precisely, for any open compact subgroup $U \subset \mathrm{GL}_2(\mathbf{A}_{\mathrm{f}})$, there is an algebraic curve $Y(U)$ defined over $\mathbf{Q}$, whose $\mathbf{C}$-points are the quotient

$$Y(U)(\mathbf{C}) = \mathrm{GL}_2^+(\mathbf{Q}) \backslash \Big[\mathrm{GL}_2(\mathbf{A}_{\mathrm{f}}) \times \mathcal{H} \Big] / U. \tag{3}$$

(Here the left action of $\mathrm{GL}_2^+(\mathbf{Q})$ is on both factors of $\mathrm{GL}_2(\mathbf{A}_{\mathrm{f}}) \times \mathcal{H}$, while U acts only on $\mathrm{GL}_2(\mathbf{A}_{\mathrm{f}})$.)

If U is the subgroup

$$U_1(N) \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbf{Z}}) : c = 0, d = 1 \bmod N\widehat{\mathbf{Z}} \right\}$$

then $Y(U)$ is just $Y_1(N)$. However, if we set $U' = \{u \in U_1(N) : \det(u) = 1 \bmod m\}$, then $Y(U')$ is canonically isomorphic to $Y_1(N) \times \mu_m^\circ$, and the action of the quotient U/U' on $Y(U')$ matches up with the Galois action via the usual isomorphism $\mathrm{Gal}(\mathbf{Q}(\mu_m)/\mathbf{Q}) \cong (\mathbf{Z}/m\mathbf{Z})^*$.

This transports our problem – constructing cohomology classes for $Y_1(N)$ over varying cyclotomic fields – into a more "automorphic" problem: constructing cohomology classes for modular curves over $\mathbf{Q}$ of varying levels.

Remark. To some extent this is just a superficial change of language. However, it seems to be a helpful one, as will be clear from our proofs of norm relations later in these lectures. $\diamond$

2.5. Siegel units

As we saw above, we can get potentially useful cohomology classes if we have a source of units in the coordinate rings of our varieties. Fortunately, for modular curves, we have lots of nice units at our disposal. (References: §§1–2 of [**Kat04**] are the definitive source; [**Lan87**] is also useful.)

2.5.1. The construction. Let U be an open compact subgroup of $\mathrm{GL}_2(\mathbf{A}_\mathrm{f})$ (such as the group $U_1(N)$ from the previous section).

Definition. *A **modular unit** of level U is a unit in the coordinate ring of the algebraic variety $Y(U)$.*

This definition is very clean, but hard to work with concretely. So we'll unwrap it a bit. Recall that $Y(U)(\mathbf{C})$ is defined as a quotient of $\mathcal{H} \times \mathrm{GL}_2(\mathbf{A}_\mathrm{f})$, so the image of $\mathcal{H} \times \{1\}$ in this quotient is a connected component of $Y(U)(\mathbf{C})$. It turns out that this image is exactly $\Gamma\backslash\mathcal{H}$, where Γ is the discrete group $U \cap \mathrm{GL}_2^+(\mathbf{Q})$ (which is commensurable with $\mathrm{SL}_2(\mathbf{Z})$). So we get a map

$$\begin{pmatrix} \text{modular units} \\ \text{of level } U \end{pmatrix} \longrightarrow \begin{pmatrix} \text{nowhere-zero holomorphic fcns} \\ \text{on } \Gamma\backslash\mathcal{H} \text{ with finite-order poles at cusps} \end{pmatrix}.$$

Fact: This map is *injective*, because the Galois group acts transitively on the components of $Y(U)$. □

For a general subgroup U the image is a little fiddly to describe. However, for some nice subgroups we can make it very concrete:

Proposition 6. *Let $U(N) \subset \mathrm{GL}_2(\widehat{\mathbf{Z}})$ be the kernel of the reduction map $\mathrm{GL}_2(\hat{\mathbf{Z}}) \to \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$, and $\Gamma(N) = U(N) \cap \mathrm{SL}_2(\mathbf{Z})$. Then the modular units of level $U(N)$ are precisely the functions on $\Gamma(N)\backslash\mathcal{H}$ which are holomorphic and nonzero away from the cusps, are meromorphic at the cusps, and have q-expansion coefficients in $\mathbf{Q}(\mu_N)$.* □

We're going to construct some "special" modular units of level $U(N)$, using nothing but classical 19th-century elliptic function theory. These functions are called **Siegel units** and they are really amazingly powerful gadgets. In fact, you can recover virtually every known example of an Euler system by starting from Siegel units!

Definition. *Let $\alpha, \beta \in \mathbf{Q}/\mathbf{Z}$, not both zero. Define the function $g_{\alpha,\beta} : \mathcal{H} \to \mathbf{C}$ as follows: write $(\alpha, \beta) = (a/N, b/N)$ for some $N \geq 1$ and $a, b \in \mathbf{Z}$, with $0 \leq a < N$ without loss of generality. Then*

$$g_{\alpha,\beta}(\tau) = q^w \prod_{n\geq 0} \left(1 - q^{n+a/N}\zeta_N^b\right) \prod_{n \geq 1} \left(1 - q^{n-a/N}\zeta_N^{-b}\right),$$

where $q = e^{2\pi i \tau}$ and $w = \frac{1}{12} - \frac{a}{N} + \frac{a^2}{2N^2}$.

This is well-defined (independent of the choice of common denominator N). We'd like to say it's modular of level N, but this doesn't quite work: acting on it by an element of $\Gamma(N)$ multiplies it by a root of unity. These error terms can be killed by a very simple modification:

Definition (Siegel units). *For $c > 1$ coprime to 6 and to the order of α, β in $\mathbf{Q}/\mathbf{Z}$, let*

$${}_c g_{\alpha,\beta} = \frac{(g_{\alpha,\beta})^{c^2}}{g_{c\alpha,c\beta}}.$$

Proposition 7. *The functions ${}_c g_{\alpha,\beta}$, for $(\alpha, \beta) \in (\frac{1}{N}\mathbf{Z}/\mathbf{Z})^{\oplus 2} - \{(0,0)\}$, are modular units of level $U(N)$. The left action of $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ on $Y(U(N))$ transforms these units via the rule*

$${}_c g_{\alpha,\beta} \mid \sigma = {}_c g_{\alpha',\beta'}, \quad \text{where} \quad (\alpha', \beta') = (\alpha, \beta)\sigma. \quad \square$$

In particular, because $(0, \frac{1}{N})$ is preserved by right-multiplication by matrices of the form $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$, which give the action of the quotient $U_1(N)/U(N)$, we see that:

Proposition 8. *The function ${}_c g_{0,1/N}$ is a modular unit of level $U_1(N)$.* □

2.5.2. Changing the level: the basic norm relation.

THEOREM 10. *Let $\alpha, \beta \in \mathbf{Q}/\mathbf{Z}$, not both zero, and let $A \geq 1$. Then we have the three relations*

$$\prod_{\alpha' : A\alpha' = \alpha} {}_c g_{\alpha',\beta}(\tau) = {}_c g_{\alpha,\beta}(A^{-1}\tau), \tag{4}$$

$$\prod_{\beta' : A\beta' = \beta} {}_c g_{\alpha,\beta'}(\tau) = {}_c g_{\alpha,\beta}(A\tau), \tag{5}$$

$$\prod_{\substack{\alpha',\beta' \\ A(\alpha',\beta')=(\alpha,\beta)}} {}_c g_{\alpha',\beta'}(\tau) = {}_c g_{\alpha,\beta}(\tau). \tag{6}$$

SKETCH OF PROOF. Note that (1) and (2) imply (3), and (2) follows from (1) via the action of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; so it suffices to prove (1). This can be bashed out directly from the infinite product formula, but there is a much slicker argument in[**Kat04**], involving a 2-variable theta function ${}_c\theta(\tau, z)$ such that ${}_c\theta(\tau, \alpha\tau + \beta) = {}_c g_{\alpha\beta}$. □

The most important relation is (3), which can be written in a more conceptual way using push-forward maps between modular curves. Suppose ℓ is a prime; then there's a quotient map $\pi : Y_1(N\ell) \to Y_1(N)$, and associated to this is a norm map $\pi_* : \mathcal{O}(Y_1(N\ell))^\times \to \mathcal{O}(Y_1(N))^\times$, characterised by

$$(\pi_* f)(x) = \prod_{y \in \pi^{-1}(x)} f(y) \quad \text{for } x \in \Gamma_1(N)\backslash\mathcal{H}.$$

COROLLARY 1. *The Siegel units satisfy*

$$\pi_*({}_c g_{0,1/N\ell}) = \begin{cases} {}_c g_{0,1/N} & \text{if } \ell \mid N, \\ {}_c g_{0,1/N} \cdot \left({}_c g_{0,u/N}\right)^{-1} & \text{if } \ell \nmid N. \end{cases}$$

where u is the inverse of ℓ modulo N. □

PROOF. Exercise. □

This is hugely important, because it's the underlying input for all of the Euler systems we will build out of Siegel units.

CHAPTER 3

The Beilinson–Flach Euler system

In this section we're going to write down the classes, and prove the "p-direction" norm relations, for one important example of an Euler system: the Euler system of Beilinson–Flach elements. That is, we'll define classes over the fields $\mathbf{Q}(\mu_m)$ for all integers m, and we'll show that if m is of the form p^r, then these classes are compatible under the norm maps for varying r.

References for this lecture: here there is really no alternative to the original papers [**LLZ14**], [**KLZ15**] and [**KLZ17**].

3.1. Beilinson–Flach elements

As we've seen in Sections 2.2 and 4.1, we can find this Galois representations attached to Rankin–Selberg convolutions of pairs of weight 2 modular forms in the geometry of $Y_1(N)^2$, for a suitable integer N. Suppose now that both modular forms have weight 2. Then we want to construct classes in the cohomology groups

$$H^3_{\text{ét}}\left(Y_1(N) \times Y_1(N) \times \mu_m^\circ, \mathbf{Z}_p(2)\right)$$

for $m \geq 1$. Notice that we have only one copy of μ_m° here, not two; so this is best interpreted not as a Shimura variety for $\mathrm{GL}_2 \times \mathrm{GL}_2$, but for the fibre product

$$\mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2 = \{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 : \det(g_1) = \det(g_2)\}.$$

3.1.1. Strategy. In the "Numerology" section above, we saw that one natural line of attack is to find curves $C \subset Y \times Y$, where $Y = Y_1(N)$, and units on C. This approach goes back to Beilinson in 1984 (and was further refined by Flach in 1992, hence the name).

An obvious first guess is to take C to be the diagonally-embedded copy of Y in $Y \times Y$, and then put modular units on C. This is exactly what we'll do for $m = 1$: we define

$${}_c\,\mathrm{BF}_{1,N} = \iota_*\left({}_c g_{0,1/N}\right),$$

where ι is the diagonal embedding, and $c > 1$ is some integer coprime to everything in sight.

However, how will we get classes over $\mathbf{Q}(\mu_m)$ for $m > 1$? If we had modular units on the curves $Y_1(N) \times \mu_m^\circ$ which were norm-compatible in m, then we could just push these forward in the same way. However, units with this kind of norm-compatibility seem to be hard to find; the Siegel units have very good compatibility properties in the "N-direction", but no interesting compatibility in the "m-direction".

So we have to make the curve C vary too, and get some contribution to our norm-compatibility this way instead. This is the first hint at a rather powerful general machine that can turn easy norm relations on a small group into "hard" norm relations on a larger group.

We'll have a lot of use for the following basic lemma relating pushforward and pullback maps in étale cohomology:

Proposition 9 (Push-pull lemma). *Suppose we have a commutative diagram of morphisms of smooth varieties*

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\beta"'] & Y \arrow[d, "\gamma"] \\ Z \arrow[r, "\delta"'] & W, \end{tikzcd}$$

in which the horizontal maps α and δ are closed embeddings of codimension c, and the vertical maps β and γ are unramified coverings of equal degrees. Then the morphisms $H^i_{\text{ét}}(Z, \mathbf{Z}_p(n)) \to H^{i+2c}_{\text{ét}}(Y, \mathbf{Z}_p(n+c))$ given by $\alpha_* \circ \beta^*$ and $\gamma^* \circ \delta_*$ coincide. □

This is a simple instance of a much more general result: the hypotheses imply that the diagram is *Cartesian*, identifying X with the fibre product $Y \times_W Z$. The identity of push-pull and pull-push maps holds for any Cartesian diagram, although we'll only use diagrams of this simple kind.

3.1.2. Rankin–Eisenstein classes.

Definition. *For integers $M \mid N$, let*

$$U(M,N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbf{Z}}) : \begin{array}{c} a = 1, b = 0 \bmod M, \\ c = 0, d = 1 \bmod N \end{array} \right\}.$$

A more compact notation for the same thing, which I'll use henceforth, is that $U(M,N)$ is the subgroup of level $\begin{pmatrix} M & M \\ N & N \end{pmatrix}$. The definition makes perfect sense without assuming $M \mid N$, of course, but we will only use it in this case. We write $Y(M,N)$ for the corresponding modular curve. Notice that we've already seen two special cases: we have $U(1,N) = U_1(N)$, and $U(N,N) = U(N)$.

The following is an easy check:

Proposition 10. *If $M \mid N$, the group $U(M,N)$ is normalised by the element* $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\hat{\mathbf{Z}})$. □

So we can make the following definition:

Definition. *Let $\iota_{M,N}$ be the embedding $Y(M,N) \hookrightarrow Y(M,N)^2$ given by*

$$P \mapsto \left(P, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot P \right).$$

Notice that this corresponds to $\tau \mapsto (\tau, \tau+1)$ on the upper half-plane.

Definition. *The* Rankin–Eisenstein *class ${}_c\mathrm{REis}_{M,N}$ is the image of ${}_c g_{0,1/N}$ under $(\iota_{M,N})_*$.*

3.1.3. Beilinson–Flach elements. The final piece of the puzzle is to descend from the higher-level modular curves where the Rankin–Eisenstein classes live to $Y_1(N) \times \mu_M^\circ$. As above, we're identifying $Y_1(N) \times \mu_M^\circ$ with the Shimura variety of level $U' = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : c = 0, d = 1 \bmod N, ad - bc = 0 \bmod M \}$.

One checks easily that

$$\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} U(M, MN) \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}^{-1} \subseteq U',$$

so there is a map $s_M : Y(M, MN) \to Y_1(N) \times \mu_M^\circ$ corresponding to $\tau \mapsto \tau/M$ on $\mathcal{H}$. This gives us a pushforward map in cohomology, $(s_M \times s_M)_*$.

Definition. *We define the* Beilinson–Flach class *as the class*

$${}_c\mathrm{BF}_{M,N} = (s_M \times s_M)_* \left({}_c\mathrm{REis}_{M,MN} \right) \in H^3_{\text{ét}} \left(Y_1(N)^2 \times \mu_M^\circ, \mathbf{Q}_p(2) \right).$$

These are the classes we really want to study. However, it turns out that proving the norm-compatibility relations for the Beilinson–Flach elements directly is difficult; it's easiest to investigate the norm-compatibility of the auxiliary classes ${}_c\mathrm{REis}_{M,N}$ first, and deduce norm-compatibility relations for the classes ${}_c\mathrm{BF}_{M,N}$ as a consequence. This is what we'll do in the next section.

3.2. Norm-compatibility for the Rankin–Eisenstein classes

It's easy to see that Rankin–Eisenstein classes "inherit" from the Siegel units good norm-compatibility properties in the N-aspect.

Proposition 11. *Let ℓ be a prime dividing N, and let π_ℓ denote the natural quotient map $Y(M, N\ell) \to Y(M, N)$. Then we have*

$$(\pi_\ell \times \pi_\ell)_* \left({}_c\mathrm{REis}_{M,\ell N}\right) = {}_c\mathrm{REis}_{M,N}$$

when $\ell \mid N$.

Proof. We have a commutative diagram

$$\begin{array}{ccc} Y(M,N\ell) & \xrightarrow{\iota_{M,N\ell}} & Y(M,N\ell)^2 \\ \Big\downarrow{\scriptstyle \pi_\ell} & & \Big\downarrow{\scriptstyle \pi_\ell \times \pi_\ell} \\ Y(M,N) & \xrightarrow{\iota_{M,N}} & Y(M,N)^2 \end{array}$$

from which we deduce that

$$(\iota_{M,N} \circ \pi_\ell)_* = ((\pi_\ell \times \pi_\ell) \circ \iota_{M,N\ell})_*.$$

By the functoriality of pushforward in etale cohomology, we deduce that

$$(\iota_{M,N})_* \circ (\pi_\ell)_* = (\pi_\ell \times \pi_\ell)_* \circ (\iota_{M,N\ell})_*.$$

We saw in Corollary 1 that $(\pi_\ell)_*({}_cg_{0,1/N\ell}) = {}_cg_{0,1/N}$. Hence

$$(\iota_{M,N})_*({}_cg_{0,1/N}) = (\pi_\ell^2)_* \circ (\iota_{M,N\ell})_*({}_cg_{0,1/N\ell}) \tag{7}$$

$$\Leftrightarrow {}_c\mathrm{REis}_{M,N} = (\pi_\ell \times \pi_\ell)_* \left({}_c\mathrm{REis}_{M,\ell N}\right). \tag{8}$$

□

Remark. There is a similar formula when $\ell \nmid N$. (exercise) ◇

The Rankin–Eisenstein classes have also, miraculously, acquired an extra norm-compatibility in the M-aspect, which the Siegel units do not have. We define a twisted degeneracy map $\tau_\ell : Y(M\ell, N) \to Y(M, N)$ as follows.

Let $U(M(\ell), N)$ be the group of level $\begin{pmatrix} M & M\ell \\ N & N \end{pmatrix}$. Then there is a natural quotient map $Y(M\ell, N) \to Y(M(\ell), N)$; and there are two maps

$$\hat{\pi}_{1,\ell}, \hat{\pi}_{2,\ell} : Y(M(\ell), N) \to Y(M, N),$$

where $\hat{\pi}_{1,\ell}$ is the natural quotient map, and $\hat{\pi}_{2,\ell}$ corresponds to $\tau \mapsto \tau/\ell$ on $\mathcal{H}$.

Remark. The map $\hat{\pi}_{2,\ell}$ is well-defined, since

$$\begin{pmatrix} 1 & \\ & \ell \end{pmatrix} U(M(\ell), N) \begin{pmatrix} 1 & \\ & \ell \end{pmatrix}^{-1} \subseteq U(M, N).$$

◇

Definition. *We write τ_ℓ for the composite*

$$Y(M\ell, N) \to Y(M(\ell), N) \xrightarrow{\hat{\pi}_{2,\ell}} Y(M, N).$$

Theorem 11. *If M, N, ℓ are integers with ℓ prime, $\ell \mid M$ and $M\ell \mid N$, then the Rankin–Eisenstein classes satisfy*

$$(\tau_\ell \times \tau_\ell)_* \left({}_c\mathrm{REis}_{\ell M,N}\right) = (U'(\ell) \times U'(\ell)) \cdot {}_c\mathrm{REis}_{M,N}.$$

Here $U'(\ell)$ is the transpose of the usual Hecke operator $U(\ell)$. The proof of this involves a very important commutative diagram of maps of algebraic varieties over $\mathbf{Q}$:

$$\begin{tikzcd}
Y(\ell M, N) \arrow[r, hook, "\iota_{\ell M,N}"] \arrow[d, equal] & Y(\ell M, N)^2 \arrow[d] \arrow[ddr, "\tau_\ell \times \tau_\ell"] & \\
Y(\ell M, N) \arrow[r, "\iota'"] \arrow[d] \arrow[dr, phantom, "\diamondsuit"] & Y(M(\ell), N)^2 \arrow[d, "\hat{\pi}_{1,\ell} \times \hat{\pi}_{1,\ell}"] \arrow[dr, "\hat{\pi}_{2,\ell} \times \hat{\pi}_{2,\ell}"] & \\
Y(M, N) \arrow[r, hook, "\iota_{M,N}"] & Y(M, N)^2 & Y(M, N)^2
\end{tikzcd}$$

Here the two diagonal maps are the ones introduced in the previous section, and the vertical maps are the natural quotient maps. The commutativity of the diagram is obvious by construction; the two really important and nonobvious properties are the following:

Proposition 12. *Under the hypotheses of the theorem, the map ι' is a closed embedding, and the lower left square marked $\diamondsuit$ is a Cartesian diagram of the kind described in Proposition 9.*

Proof. It's easy to see that the image of ι' is precisely the modular curve associated to the group

$$U(M(\ell), N) \cap \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} U(M(\ell), N) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

However, a straightforward matrix computation shows that this intersection is nothing but $U(\ell M, N)$ itself. So ι' is a closed embedding.

Since both horizontal maps in the square $\diamondsuit$ are closed embeddings, and the vertical maps are automatically finite coverings, it suffices to check that the degrees of the vertical maps agree. These degrees are equal to the indices of corresponding inclusions of level groups: on the left-hand side we have $[U(M, N) : U(\ell M, N)] = \ell^2$, and on the right-hand side $[U(M, N)^2 : U(M(\ell), N)^2] = \ell^2$. □

Corollary 2. *The following two classes in $H^3_{\text{ét}}(Y(M(\ell), N)^2, \mathbf{Z}_p(2))$ coincide:*

- *the pushforward of ${}_c\mathrm{REis}_{\ell M,N}$ along the upper vertical arrow $Y(\ell M, N)^2 \to Y(M(\ell), N)^2$;*
- *the pullback of ${}_c\mathrm{REis}_{M,N}$ along the lower vertical arrow $Y(M(\ell), N)^2 \to Y(M, N)^2$.*

Proof. This is exactly the "push-pull" lemma applied to the square $\diamond$ (since the unit ${}_c g_{0,1/N}$ on $Y(\ell M, N)$ is, by definition, the pullback of the unit with the same name on $Y(M, N)$.) □

Since these two classes are equal on $Y(M(\ell), N)^2$, they certainly must have the same pushforward along the diagonal map to $Y(M, N)^2$. So we obtain an equality between $(\tau_\ell \times \tau_\ell)_* \left({}_c\mathrm{REis}_{\ell M,N}\right)$ and the image of ${}_c\mathrm{REis}_{\ell M,N}$ under pullback and pushforward around the triangle. This composite of pushforward and pullback maps is exactly the Hecke operator $U'(\ell) \times U'(\ell)$, so we have proved the theorem. $\square$

Exercise. Show that if M, N, ℓ are integers with ℓ prime, $\ell \nmid M$ and $M\ell \mid N$, then the Rankin–Eisenstein classes satisfy

$$(\tau_\ell \times \tau_\ell)_* \left({}_c\mathrm{REis}_{\ell M,N}\right) = \left(U'(\ell) \times U'(\ell) - \Delta_\ell^*\right) \cdot {}_c\mathrm{REis}_{M,N}, \tag{9}$$

where Δ_ℓ denotes any element of $\mathrm{GL}_2(\mathbf{Z}/MN\mathbf{Z})^2$ of the form $\left(\begin{pmatrix} x & \\ & 1\end{pmatrix}, \begin{pmatrix} x & \\ & 1\end{pmatrix}\right)$ with $x \equiv \ell \pmod{M}$.

Remark. Note that the assumption $\ell M \mid N$ is essential, since otherwise the definition of the Rankin–Eisenstein element doesn't even make sense. $\diamond$

3.3. Norm-compatibility for the Beilinson–Flach classes

We can now state and prove the main theorem:

Theorem 12. *If ℓ is prime with $\ell \mid M$ and $\ell \mid N$, then we have*

$$\mathrm{norm}_{\mathbf{Q}(\mu_M)}^{\mathbf{Q}(\mu_{\ell M})}\left({}_c\mathrm{BF}_{\ell M,N}\right) = [U'(\ell) \times U'(\ell)] \cdot {}_c\mathrm{BF}_{M,N}.$$

Proof. This follows from the commutativity of the diagram

$$\begin{tikzcd} Y(\ell M, \ell MN) \arrow[r, "\tau_\ell"] \arrow[d, "s_{\ell M}"'] & Y(M, \ell MN) \arrow[r, "\pi_\ell"] & Y(M, MN) \arrow[d, "s_M"] \\ Y_1(N) \times \mu_{\ell M}^\circ \arrow[rr] & & Y_1(N) \times \mu_M^\circ, \end{tikzcd}$$

and the following compatibilities:

- Theorem 11, which we use to compare $\mathrm{REis}_{\ell M,\ell MN}$ with $\mathrm{REis}_{M,M\ell N}$;
- Proposition 11, which allows us to compare $\mathrm{REis}_{M,M\ell N}$ with $\mathrm{REis}_{M,MN}$;
- the fact that $U'(\ell)$ commutes with the pushforward along the maps π_ℓ and s_M. $\square$

Exercise. Using (9), formulate and prove the analogous statement in the case when $\ell \nmid M$ and $\ell \mid N$.

Remarks.

(i) It is also possible to describe the class ${}_c\mathrm{BF}_{M,N}$ directly at level N (rather than going via the higher-level curves $Y(M, MN)$ as we have done). The curve $\mathrm{image}(\iota_{M,MN}) \subset Y(M, MN)^2$ maps down via $s_M \times s_M$ to a curve $C_{M,N} \subset Y_1(N)^2 \times \mu_M^\circ$, and our class can be characterised as the pushforward of a unit on $C_{M,N}$. However, the curve $C_{M,N}$ is rather messy (it can have many self-intersections, for instance), which makes it more difficult to prove the norm relation by this approach.

(ii) The compatibility of $U'(\ell)$ with pushforwards may seem like a minor point, but I want to emphasise it here, because this is the point where the proof breaks down in the case $\ell \nmid MN$. In this case, there is an operator $U'(\ell)$ on $Y(M, \ell MN)$, and an operator $T'(\ell)$ on $Y(M, MN)$, but these aren't compatible under π_*. So to complete the argument we would need to relate

$$(\pi \times \pi)_* \Big[(U'(\ell) \times U'(\ell)) \cdot {}_c\mathrm{REis}_{M,N\ell} \Big]$$

to the objects we know about on $Y(M, MN)$. This can be done – in fact there are at least three separate approaches – but it isn't easy. The eventual outcome is that for $\ell \nmid MN$ we have a formula

$$\mathrm{norm}_{\mathbf{Q}(\mu_M)}^{\mathbf{Q}(\mu_{\ell M})}\left({}_c\mathrm{BF}_{\ell M,N}\right) = Q_\ell(\sigma_\ell^{-1}) \cdot {}_c\mathrm{BF}_{M,N},$$

where $Q_\ell(X)$ is a degree 4 polynomial with coefficients in the Hecke algebra. $\diamond$

3.4. Projection to the (f,g) component

We now bring the eigenforms f and g into the picture. It's important to impose some local conditions at p. We take f and g to be eigenforms of some level N, with $p \mid N$, whose $U(p)$-eigenvalues α_f, α_g are p-adic units (we say f and g are *ordinary* at p).

Remark. If we start with some form f of level N_0 with $p \nmid N_0$, then we replace f with one of the two $U(p)$-eigenforms of level $N = pN_0$ which have the same Hecke eigenvalues away from p. This process is called ***p*-stabilisation**. This doesn't change the Galois representations: the Galois representations attached to the p-stabilisations of f are isomorphic to that of the original form f, although they live on a different modular curve. ◇

The quotient

$$H^1_{\text{ét}}\left(Y_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Q}_p(1)\right) / \left\langle \begin{array}{c} T'(\ell) - a_\ell(f)\ \forall \ell \nmid N, \\ U'(\ell) - a_\ell(f)\ \forall \ell \mid N \end{array} \right\rangle$$

turns out to be isomorphic to the dual[1] V_f^* of V_f. The image of the cohomology with $\mathbf{Z}_p$-coefficients gives a lattice T_f^* in V_f^*. Doing this for both f and g, and combining this with the Hochschild–Serre spectral sequence, we get a projection map

$$\mathrm{Pr}_{f,g} : H^3_{\text{ét}}(Y_1(N) \times \mu_M^\circ, \mathbf{Z}_p(2)) \to H^1(\mathbf{Q}(\mu_M), T_f^* \otimes T_g^*).$$

By construction, the Hecke operator $U'(\ell) \times U'(\ell)$ on the source corresponds to multiplication by $\alpha_f \alpha_g$ on the target. This gives us the following theorem:

Proposition 13. *The classes*

$$(\alpha_f \alpha_g)^{-r} \,\mathrm{Pr}_{f,g}\left({}_c\mathrm{BF}_{p^r,N,1}\right) \in H^1\left(\mathbf{Q}(\mu_{p^r}), T_f^* \otimes T_g^*\right)$$

are norm-compatible for $r \geq 1$. □

Notice that it's crucial that α_f, α_g are p-adic units, since otherwise these renormalised classes wouldn't land in $T_f^* \otimes T_g^*$ any more.

Exercise. Show that if $p \mid N$, then

$$\mathrm{cores}_{\mathbf{Q}}^{\mathbf{Q}(\mu_p)} \mathrm{Pr}_{f,g}\left({}_c\mathrm{BF}_{p,N}\right) = (\alpha_f \alpha_g - 1)\, \mathrm{Pr}_{f,g}\left({}_c\mathrm{BF}_{1,N}\right).$$

The exercise shows that the case $r = 0$ doesn't quite work; there is an unwanted Euler factor appearing, just as in the case of cyclotomic units. Exactly as in that case, we can get rid of this error term by re-defining the $r = 0$ class to be the norm of the $r = 1$ class. This gives an element of the module

$$H^1_{\mathrm{Iw}}\left(\mathbf{Q}(\mu_{p^\infty}), T_f^* \otimes T_g^*\right) := \varprojlim_{r \geq 0} H^1\left(\mathbf{Q}(\mu_{p^r}), T_f^* \otimes T_g^*\right),$$

which is the *Iwasawa cohomology* of $T_f^* \otimes T_g^*$.

3.4.1. Euler factors. Having got this far, we can ask what happens if ℓ doesn't divide M and N. If $\ell \mid N$ but $\ell \nmid M$ then a slight modification of the argument gives

$$\mathrm{norm}_{\mathbf{Q}(\mu_M)}^{\mathbf{Q}(\mu_{\ell M})}\left({}_c\xi_{\ell M,N}\right) = (U(\ell)' - \sigma_\ell) \cdot {}_c\xi_{M,N}.$$

If $\ell \nmid N$ things get quite a lot more difficult, because one has to keep track of the difference between the Hecke operators $U(\ell)'$ at level ℓN and $T(\ell)'$ at level N. The eventual result is that

$$\mathrm{norm}_{\mathbf{Q}(\mu_M)}^{\mathbf{Q}(\mu_{\ell M})}\left({}_c\xi_{\ell M,N}\right) = -\sigma_\ell Q_\ell(\sigma_\ell^{-1}) \cdot {}_c\xi_{M,N}$$

where Q_ℓ is a degree 4 polynomial with coefficients in the Hecke algebra.

[1]The dual appears here because the $T'(\ell)$ are the adjoints of the $T(\ell)$ under Poincaré duality.

Remark. We expected that Q_ℓ would act on the (f, g)-eigenspace as the Euler factor of $V^*(1)$, where $V = V_p(f) \otimes V_p(g)(2)$ is our Galois representation. But this isn't the polynomial we get: if P_ℓ is the polynomial giving the Euler factor, then $Q_\ell(X) = P_\ell(X) + (\ell - 1)R_\ell(X^2)$, where R_ℓ is the Euler factor of $\chi_f \chi_g$.

David and I spent a week in the summer of 2012 repeatedly checking and re-checking the calculations thinking that this error term was a mistake. We eventually concluded that it really is there. We later understood that this is again a consequence of our Euler system being a "shadow" of a conjectural rank 2 Euler system. For applications, the error term is not an issue: fortunately, it is zero modulo $\ell - 1$, and one always chooses the primes ℓ in Euler system arguments to be congruent to 1 modulo high powers of p so the error term can be neglected. (A similar phenomenon occurs for the Euler system of Stickelberger elements, cf. §3.4 of Rubin's book.) $\diamond$

CHAPTER 4

Modular forms of higher weight

4.1. Galois representations

When we defined Galois representations attached to modular forms, we assumed that the modular forms has weight 2. Let's now see how this extends to other weights.

Assume now that f is a cuspidal modular eigenform of level $\Gamma_1(N)$ and weight $k+2$ for some $k \geq 0$. (We say that f has *cohomological weight.*) It turns out that we can still attach a Galois representation to f, but if $k > 0$, then we have to consider *étale cohomology with coefficients.*

It follows from Theorem 9 that there is a universal elliptic curve over $Y_1(N)$, say $\pi : \mathcal{E} \to Y_1(N)$. Denote by $\mathscr{H}$ the étale sheaf $V_p(\mathcal{E})$ on $Y_1(N)$; this is a locally constant sheaf of $\mathbf{Q}_p$-vector spaces of dimension 2, whose fibre at any geometric point x is canonically identified with the Tate module $V_p(\mathcal{E}_x)$ of the elliptic curve $\mathcal{E}_x/\overline{K}$.

Remark. We have a functor

$$(\star) \qquad \{\text{algebraic representations of } \mathrm{GL}_2 / \mathbf{Q}_p\} \to \{\text{étale } \mathbf{Q}_p\text{-sheaves on } Y_1(N)\}.$$

The sheaf $\mathscr{H}$ is the image under this functor of the defining 2-dimensional representation of GL_2. ◇

Definition. *We let $V_p(f)$ be the largest subspace of $H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}, \operatorname{Sym}^k \mathscr{H}(-k))$ on which the Hecke operators $T(\ell)$, for $\ell \nmid N$, act as multiplication by $a_\ell(f)$.*

Then $V_p(f)$ has the expected properties (generalising those we had above for $k = 0$):

1. $V_p(f)$ is 2-dimensional and irreducible.
2. $V_p(f)$ is a direct summand of $H^1_{\text{ét}}$ (not just a subspace).
3. For $\ell \nmid pN$, $V_p(f)$ is unramified at ℓ and the the local Euler factor is
$$P_\ell(V_p(f), t) = 1 - a_\ell(f)t + \ell^{k+1}\chi(\ell)t^2.$$
4. $V_p(f)^* = V_p(f \otimes \chi^{-1})(k+1)$.

Remark. There are also Galois representations attached to weight 1 modular forms, but these are harder to construct – they don't show up in étale cohomology with coefficients in any reasonable sheaf. ◇

In much the same way, if we have a pair of integers $(k, k') \geq 0$ we can form a sheaf $\operatorname{Sym}^k \mathscr{H} \boxtimes \operatorname{Sym}^{k'} \mathscr{H}$ on $Y_1(N)^2$, and the tensor product $V(f) \otimes V(g)$, for f and g eigenforms of weight $k+2$ and $k'+2$, appears as a direct summand of the space

$$H^2_{\text{ét}}\left(Y_1(N)^2_{\overline{\mathbf{Q}}}, (\operatorname{Sym}^k \mathscr{H} \boxtimes \operatorname{Sym}^{k'} \mathscr{H})(-k-k')\right).$$

4.2. Eisenstein classes

The Kummer images of Siegel units give us classes in $H^1_{\text{ét}}(Y_1(N), \mathbf{Q}_p(1))$. What should our higher-weight analogues of this be?

It turns out that for any $k \geq 0$ and $N \geq 4$, and $c > 1$ coprime to $6pN$, there exists an *étale Eisenstein class*

$${}_c\operatorname{Eis}^k_{0,1/N} \in H^1_{\text{ét}}\left(Y_1(N), \operatorname{Sym}^k \mathscr{H}(1)\right),$$

which in the case $k = 0$ agrees with the Kummer-map image of the Siegel unit. These étale Eisenstein symbols satisfy similar basic relations (Theorem 10) to those of the Siegel units.

Remark. One can make sense of "motivic cohomology with coefficients in $\mathscr{H}$", and then one finds that these Eisenstein classes are the étale images of motivic Eisenstein classes, whose images under the Beilinson regulator are non-holomorphic Eisenstein series of weight $-k$. This is a higher-weight generalisation of the Kronecker limit formula, since for $k = 0$ the Beilinson regulator map on $H^1_{\mathrm{mot}}(Y_1(N), \mathbf{Z}(1)) \cong \mathcal{O}(Y_1(N))^\times$ maps a unit u to the function $\log|u| : Y_1(N)(\mathbf{C}) \to \mathbf{R}$. ◇

4.3. The Euler system for higher weight modular forms

We can adapt the above construction for pairs of modular forms of higher weight. Suppose that f and g have weights $k+2$ and $k'+2$ with $k, k' \geq 2$. Then it follows from Section 4.1, we need to construct classes in the cohomology groups

$$H^3_{\text{ét}}\left(Y_1(N) \times Y_1(N) \times \mu_m^\circ, \operatorname{Sym}^k \mathscr{H} \boxtimes \operatorname{Sym}^{k'} \mathscr{H}(n)\right),$$

for some appropriate $n \in \mathbf{Z}$, and these classes should arise via pushforward from the cohomology of $Y_1(N)$. Assume that $m = 1$, so we want to pushforward along the diagonal embedding $\iota : Y_1(N) \to Y_1(N)^2$. (Once we have understood this case, we can construct classes for $m > 1$ using the methods from the previous sections.)

4.3.1. Pushforward with coefficients. It turns out that pushforward maps "work" with coefficients: there's a natural map

$$\iota_* : H^1_{\text{ét}}\left(Y_1(N), \iota^*(\mathscr{L})(1)\right) \to H^3_{\text{ét}}\left(Y_1(N)^2, \mathscr{L}(2)\right)$$

for any étale sheaf $\mathscr{L}$, with the case above being $\mathscr{L}$ the constant sheaf $\mathbf{Q}_p$. Here $\iota^*\mathscr{L}$ is just the pullback of $\mathscr{L}$ to $Y_1(N)$. So what does the sheaf $\iota^*\left(\operatorname{Sym}^k \mathscr{H} \boxtimes \operatorname{Sym}^{k'} \mathscr{H}\right)$ look like?

Since $\mathscr{H}$ and its symmetric powers arise from irreducible algebraic representations of GL_2, we can use group theory to answer this question. Let V denote the standard 2-dimensional $\mathbf{Q}_p$-representation of GL_2. Then the sheaf $\operatorname{Sym}^k \mathscr{H} \boxtimes \operatorname{Sym}^{k'} \mathscr{H}$ on $Y_1(N)^2$ arises from the irreducible representation $\operatorname{Sym}^k V \boxtimes \operatorname{Sym}^{k'} V$ of $G \coloneqq \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$. If $H \subset G$ denotes the diagonally-embedded copy of GL_2, then the restriction of this G-representation to H breaks up as a sum of irreducible H-representations; and we have a corresponding decomposition of the pullback $\iota^*\left(\operatorname{Sym}^k \mathscr{H} \boxtimes \operatorname{Sym}^{k'} \mathscr{H}\right)$ in the category of sheaves on $Y_1(N)$.

Remark. A posh way of stating this compatibility is that we have a commutative diagram of functors

$$\begin{array}{ccc} \mathrm{Rep}_G & \longrightarrow & \text{sheaves}\,/Y_1(N)^2 \\ {\scriptstyle \mathrm{res}^G_H}\big\downarrow & & \big\downarrow{\scriptstyle \iota^*} \\ \mathrm{Rep}_H & \longrightarrow & \text{sheaves}\,/Y_1(N) \end{array}$$

where Rep_G and Rep_H are the categories of representations of G and its subgroup H, res^G_H is restriction of representations, and the horizontal arrows are the functors ($\star$) for G and H. An analogue of this naturality property has been established for motivic cohomology in recent works of Ancona and Torzewski. ◇

The decomposition of $\mathrm{res}^G_H\left(\operatorname{Sym}^k V \boxtimes \operatorname{Sym}^{k'} V\right) = \operatorname{Sym}^k V \otimes \operatorname{Sym}^{k'} V$ into irreducible representations of H is described by the *Clebsch–Gordan formula*:

$$\operatorname{Sym}^k V \otimes \operatorname{Sym}^{k'} V = \bigoplus_{j=0}^{\min\{k,k'\}} \operatorname{Sym}^{k+k'-2j} V \otimes \det{}^j,$$

so for every $0 \le j \le \min\{k, k'\}$, we have a GL_2-equivariant map

$$\mathrm{Sym}^{k+k'-2j} V \otimes \det{}^j \longrightarrow \iota^*\left(\mathrm{Sym}^k V \boxtimes \mathrm{Sym}^{k'} V\right).$$

The representation $\det^j$ of GL_2 corresponds to the sheaf $\mathbf{Q}_p(j)$, so this means that for every $0 \le j \le \min\{k, k'\}$ we get a map of sheaves on $Y_1(N)$,

$$\mathrm{Sym}^{k+k'-2j} \mathscr{H} \longrightarrow \iota^*\left(\mathrm{Sym}^k \mathscr{H} \boxtimes \mathrm{Sym}^{k'} \mathscr{H}(-j)\right),$$

which induces a map in étale cohomology

$$\iota_* : H^1_{\text{ét}}\left(Y_1(N), \mathrm{Sym}^{k+k'-2j} \mathscr{H}(1)\right) \longrightarrow H^3_{\text{ét}}\left(Y_1(N)^2, \mathrm{Sym}^k \mathscr{H} \boxtimes \mathrm{Sym}^{k'} \mathscr{H}(2-j)\right).$$

4.3.2. Definition of the classes. As we saw above, there is a special element in $H^1_{\text{ét}}\left(Y_1(N), \mathrm{Sym}^{k+k'-2j} \mathscr{H}(1)\right)$, the étale Eisenstein class ${}_c\,\mathrm{Eis}^{(k+k'-2j)}_{0,1/N} \in H^1_{\text{ét}}\left(Y_1(N), \mathrm{Sym}^{k+k'-2j} \mathscr{H}(1)\right)$.

Definition. *Let $0 \le j \le \min\{k, k'\}$. We define the Rankin–Eisenstein class*

$${}_c\,\mathrm{REis}^{(k,k',j)}_{1,N} = \iota_*\left({}_c\,\mathrm{Eis}^{(k+k'-2j)}_{0,1/N}\right),$$

which is an element of $H^3_{\text{ét}}\left(Y_1(N)^2, \mathrm{Sym}^k \mathscr{H} \boxtimes \mathrm{Sym}^{k'} \mathscr{H}(2-j)\right)$.

Using the same methods as in Sections 3.1.2 and 3.1.3, we more generally define Rankin–Eisenstein classes ${}_c\,\mathrm{REis}^{(k,k',j)}_{M,N}$ for $M|N$; and (finally) Beilinson–Flach classes

$${}_c\,\mathrm{BF}^{(k,k',j)}_{m,N} \in H^3_{\text{ét}}\left(Y_1(N)^2 \times \mu^\circ_m, \mathrm{Sym}^k \mathscr{H} \otimes \mathrm{Sym}^{k'} \mathscr{H}(2-j)\right)$$

as the image of ${}_c\,\mathrm{REis}^{(k,k',j)}_{m,mN}$ under the map[1] $(s_m \times s_m)_*$.

Now let f and g be eigenforms of weights $k+2, k'+2 \ge 2$ with $k, k' \ge 0$ and level N, where $p \mid N$, both of which are ordinary at p. It then follows from Section 4.1 and the arguments in Section 3.4 that we have a projection map

$$\mathrm{Pr}_{f,g} : H^3_{\text{ét}}\left(Y_1(N)^2 \times \mu^\circ_m, \mathrm{Sym}^k \mathscr{H} \otimes \mathrm{Sym}^{k'} \mathscr{H}(2-j)\right) \longrightarrow H^1(\mathbf{Q}(\mu_m), (V_f \otimes V_g)^*(-j)),$$

and as in the parallel weight 2 case one can show that the images of the Beilinson–Flach classes $\left({}_c\,\mathrm{BF}^{(k,k',j)}_{m,N}\right)_{m \ge 1}$ under this projection map have the same properties under corestriction maps as in Proposition 13.

Remark. We also have to check that these objects land in a $\mathbf{Z}_p$-lattice independent of m. To do this, we need to find good integral lattices in $\mathrm{Sym}^k \mathscr{H}$. There is a natural $\mathbf{Z}_p$-lattice subsheaf $\mathscr{H}_{\mathbf{Z}_p} \subset \mathscr{H}$, but a small complication arises because with $\mathbf{Z}_p$-coefficients the Sym^k functor is not compatible with duality unless $p > k$. To repair this one has to introduce a slightly different sheaf, the sheaf $\mathrm{TSym}^k \mathscr{H}_{\mathbf{Z}_p}$ of symmetric tensors, which is only isomorphic to $\mathrm{Sym}^k \mathscr{H}_{\mathbf{Z}_p}$ if $p > k$. ◇

4.4. Twist-compatibility

The upshot of this construction is that for a fixed pair of forms f and g, we have not one but $1 + \min\{k, k'\}$ *different* Euler systems, which live in different cyclotomic twists of the representation $V_f^* \otimes V_g^*$. However, as we've seen above, Soulé's twisting construction gives an isomorphism between the space of Euler systems for V and for $V(m)$ for any $m \in \mathbf{Z}$, so it makes sense to compare these Euler systems to each other.

[1] One needs to be a bit careful here with extending s_m to a map on cohomology with coefficients, but we don't discuss this issue here. For reference, see [**KLZ17**, §6.1].

Theorem 13. *The Beilinson–Flach Euler systems associated to different values of j in the range $0 \le j \le \min\{k, k'\}$ are all compatible under the Soulé twist.*

This simple-looking statement turns out to be deceptively hard. See [**KLZ17**, §6].

Remark. A similar issue arises for Kato's Euler system associated to a single modular form, and in this case one even has *infinitely many* potentially different Euler systems! More precisely, for a weight 2 form f, one can use cup-products of two weight n Eisenstein classes, for any $n \ge 0$, to construct an Euler system with values in $V_p(f)^*(1+n)$.

Naturally, one expects that the Euler system thus constructed for any $n \ge 1$ should coincide with the n-th Soulé twist of the $n = 0$ Euler system. This was checked in the PhD thesis of Matthew Gealy [**Gea06**]. $\diamond$

4.5. An adelic modification

Just in order to motivate some of the constructions we'll use in later chapters, it's worth pointing out that one can make a slight modification to the construction. Since we have defined our modular curves $Y(U)$ as quotients of $\mathrm{GL}_2(\mathbf{A}_f) \times \mathcal{H}$ (c.f. (3)), where $\mathbf{A}_f$ are the finite adéles, we have a (right) action of the normaliser of $\mathrm{GL}_2(\mathbf{A}_f)$ on the tower of curves $Y(U)$ for varying U. This is compatible with the action of $\mathrm{GL}_2^+(\mathbf{Q}) \subset \mathrm{GL}_2(\mathbf{A}_f)$ via Möbius transformations on $\mathcal{H}$, after modifying by an inverse to interchange left and right actions.

With these conventions, we can define our Hecke operators, and our degeneracy maps τ_ℓ, s_M etc, using elements of $\mathrm{GL}_2(\mathbf{A}_f)$ which are the identity outside the place ℓ. This does not change anything major (the difference between the "old" and "new" elements is given by the action of an element of $\mathrm{Gal}(\mathbf{Q}(\mu_m)/\mathbf{Q})$) but the adelic presentation makes it a little easier to leverage results from representation theory.

CHAPTER 5

Finding Euler systems: motivic cohomology and period integrals

Note: This chapter is provided only for motivation, and involves some very deep and advanced concepts; these will not be needed in the following sections, so you may wish to skip this part at a first reading.

We've seen in the last section that:

- Interesting Galois representations often appear in the étale cohomology (over $\overline{\mathbf{Q}}$) of Shimura varieties.
- One can build classes in H^1 of these Galois representations via Hochschild–Serre, using cup-products, pushforwards from subvarieties, and the Kummer map.
- We have a supply of interesting units on modular curves to use as input to the Kummer map.

For example, if we want to build Euler systems for tensor products $V_f \otimes V_g$, we want classes in $H^3_{\text{ét}}(Y_1(N) \times Y_1(N), \mathbf{Z}_p(2))$; and we can get these by choosing curves $Z \subset Y_1(N)^2$, and pushing forward $\kappa_p(u)$ for some $u \in \mathcal{O}(Z)^\times$. A natural thing to try, of course, is to take Z to be the diagonal copy of $Y_1(N)$, and $u = {}_c g_{0,1/N}$ the Siegel unit.

However, why should this construction give *interesting* classes? How are we going to relate them to the special values of L-functions?

5.1. The Rankin–Selberg integral formula

Here's a very classical result, discovered independently by Rankin and by Selberg in the 1930s.

THEOREM 14. *Let $N \geq 1$, and for $s \in \mathbf{C}$ with $\Re(s) \gg 0$, let E_s be the (non-holomorphic) function on the upper half-plane $\mathcal{H}$ defined by*

$$E_s(\tau) = \pi^{-s}\Gamma(s) \sum_{(c,d)\in \mathbf{Z}^2} \frac{\Im(\tau)^s}{|c\tau + d + 1/N|^{2s}}.$$

Then, for any two newforms f, g of level N and weight 2, we have

$$\langle \bar{f}, gE_s\rangle = \int_{\Gamma_1(N)\backslash\mathcal{H}} f(-\bar{\tau})g(\tau)E_s(\tau)\, \mathrm{d}\tau \wedge \mathrm{d}\bar{\tau} = (*) \cdot L(V_f \otimes V_g, s+1),$$

where $(*)$ is an explicit factor.

This is surprisingly simple to prove: after interchanging summation and integration, you get the integral of $f(-\bar{\tau})g(\tau)\Im(\tau)^{-s}$ over the region $\{x + iy : 0 \leq x \leq 1, 0 \leq y \leq \infty\}$, and substituting in the q-expansions of f and g and integrating term-by-term gives the result. However, it has a lot of important consequences; for instance, it follows easily from this formula and the properties of E_s that $L(V_f \otimes V_g, s)$ has meromorphic continuation to all $s \in \mathbf{C}$ (with well-understood poles) and satisfies a functional equation relating s and $3 - s$.

However, the reason I want to consider it here is the following classical result ("Kronecker's second limit formula")[1]

THEOREM 15 (Kronecker). *We have $E_0(\tau) = -\log|g_{0,1/N}|$.*

[1] I did say this was 19th-century stuff; Kronecker died in 1891.

So there's some connection between $E_0(\tau)$ and Siegel units, and on the other hand between $E_s(\tau)$ and Rankin–Selberg convolutions. In order to state this properly, we need to introduce another cohomology theory.

5.2. Motivic cohomology

References: Mazza–Voevodsky–Weibel, *Lecture notes on motivic cohomology* [**MVW06**]; Beilinson, *Higher regulators and values of L-functions* [**Beĭ84**].

There is a cohomology theory for algebraic varieties called *motivic cohomology*, introduced by Beilinson and greatly refined by the late Vladimir Voevodsky. It gives groups $H^i_{\mathrm{mot}}(X, \mathbf{Z}(n))$, and for each prime p, there are maps (étale regulators)

$$r_{\text{ét}} : H^i_{\mathrm{mot}}(X, \mathbf{Z}(n)) \otimes \mathbf{Z}_p \to H^i_{\text{ét}}(X, \mathbf{Z}_p(n)).$$

For small i and n the motivic cohomology groups have explicit descriptions. $H^1_{\mathrm{mot}}(X, \mathbf{Z}(1))$ is literally equal to $\mathcal{O}(X)^\times$, and the étale regulator on this group is the Kummer map κ_p.

Remark. The étale regulator is compatible with pushforward and cup-products, so in fact our entire toolkit for building elements of étale cohomology factors through motivic cohomology. This also explains why our tools can't get at $H^i_{\text{ét}}(X, \mathbf{Z}_p(n))$ when $i > 2n$: in this range the group $H^i_{\mathrm{mot}}(X, \mathbf{Z}(n))$ is zero. $\diamond$

Theorem 16 (Landsburg [**Lan91**]). *If S is an algebraic surface over a field k, $H^3_{\mathrm{mot}}(S, \mathbf{Z}(2))$ is isomorphic to the quotient*

$$\left\{ \begin{array}{c} \text{formal sums} \sum_i (Z_i, u_i),\ Z_i \subset S \text{ irreducible curve,} \\ u_i \in k(Z_i)^\times, \text{ with} \sum_i \operatorname{div} u_i = 0 \end{array} \right\} / \sim$$

where $\sim$ is some equivalence relation.

In particular, if we have a curve $Z \subset S$ and an element $u \in \mathcal{O}(Z)^\times$, then $\operatorname{div} u$ is trivial, so (Z, u) defines a class in $H^3_{\mathrm{mot}}(S, \mathbf{Z}(2))$; and (unsurprisingly) the image of this class in $H^3_{\text{ét}}(X, \mathbf{Z}_p(2))$ is just $\iota_*(\kappa_p(u))$, where $\iota : Z \hookrightarrow S$ is the inclusion morphism.

However, as well as the étale regulator $r_{\text{ét}}$, there's a second regulator map defined on $H^3_{\mathrm{mot}}(S, \mathbf{Z}(2)) \otimes \mathbf{R}$, the *Beilinson regulator* $r_{\mathbf{C}}$: if ω is a (sufficiently nice) differential 2-form on $S(\mathbf{C})$, we can map an element $\mathfrak{z} = \sum_i (Z_i, u_i)$ to

$$(\dagger) \qquad \sum_i \int_{Z_i} \omega \log |u_i|.$$

This is clearly linear in ω, so we get a map from $H^3_{\mathrm{mot}}(S, \mathbf{Z}(2))$ to the dual space of a space of differential forms – more precisely, to $\left(\mathrm{Fil}^1 H^2_{\mathrm{dR}}(S_{\mathbf{C}})\right)^*$.

Combining this with what we know about logs of Siegel units, something magical happens: if $S = Y_1(N) \times Y_1(N)$ and $\mathfrak{z}$ is the class of (diagonal, ${}_c g_{0,1/N}$), and we take $\omega = (f(-\bar{\tau})\, \mathrm{d}\bar{\tau}) \wedge (g(\tau)\, \mathrm{d}\tau)$, then *the integral* ($\dagger$) *is exactly the Rankin–Selberg integral at $s = 1$!* So, to sum up,

- the class we've built in $H^3_{\text{ét}}(Y_1(N)^2, \mathbf{Z}_p(2))$ is naturally the image of something in $H^3_{\mathrm{mot}}(Y_1(N)^2, \mathbf{Z}(2))$,
- the Beilinson regulator of this class, paired with a differential coming from f and g, computes a value of the L-function $L(f \otimes g, s)$.

This is pretty strong evidence that the Galois cohomology class we're building (the *Beilinson–Flach class*) is the right class to consider: it's the image under the étale regulator of a motivic class which is a "motivic incarnation" of the Rankin–Selberg integral.

Remark. (1) It follows from the Beilinson regulator formula that the motivic class $\mathfrak{z} = (\text{diagonal}, {}_c g_{0,1/N}) \in H^3_{\mathrm{mot}}(S, \mathbf{Z}(2))$ is non-zero. If the étale regulator from here to $H^3_{\text{ét}}(S, \mathbf{Z}_p(2))$ were injective, then we could actually deduce that our class in $H^3_{\text{ét}}(S, \mathbf{Z}_p(2))$ was non-zero, and we'd be in a good position to apply Rubin's theorem.

Sadly, we don't know this. We can replace S with an integral model $\mathcal{S}$ defined over $\mathbf{Z}[1/pN]$. It's known that $H^3_{\mathrm{mot}}(\mathcal{S}, \mathbf{Z}(2))/p^k$ maps injectively to $H^3_{\text{ét}}(\mathcal{S}, \mathbf{Z}_p(2))/p^k$ for every k; but unfortunately we don't know any finite generation properties for $H^3_{\mathrm{mot}}(\mathcal{S}, \mathbf{Z}(2))$, so $\mathfrak{z}$ might potentially be infinitely p-divisible, and hence zero in $H^3_{\mathrm{mot}}(\mathcal{S}, \mathbf{Z}(2))/p^k$ for every k. It's conjectured that the motivic cohomology groups of a scheme of finite type over $\mathbf{Z}$ should always be finitely generated, which would rule out this pathology, but unfortunately this conjecture is wide open.

(2) In order to show that our étale class is non-zero, one uses another kind of regulator, the so-called *syntomic regulator* from motivic cohomology. I will discuss this in the next section.

$\diamond$

5.3. P-adic regulators

So we have a strategy for building Galois cohomology classes which "really ought to be" non-zero, in the sense that they are the étale images of non-zero motivic cohomology classes. However, since we can't prove that the map from motivic to étale cohomology is injective, how can we be sure these Galois cohomology classes aren't all zero?

To do this, we introduce yet another regulator map defined[2] on $H^3(S, \mathbf{Z}(2))$, besides the étale and Beilinson regulators: the *p-adic syntomic regulator* [**Bes00**], which is defined using p-adic rigid geometry, assuming $p \nmid N$. The two key properties of this regulator are that

- like Beilinson's, it can be made explicit enough to compute with: there is a formula for the p-adic regulator map for a surface, due to Besser [**Bes12**], which is very closely analogous to (†), with the integral understood via Coleman's p-adic integration theory.
- unlike Beilinson's, it can be compared to the étale regulator: a very deep theorem in p-adic Hodge theory, due (independently[3]) to Nizioł and Nekovář [**Niz97, Nek98**], shows that there is a commutative diagram relating the étale and syntomic regulators via the Bloch–Kato logarithm map of p-adic Hodge theory.

$$\begin{tikzcd}
& \boxed{\text{motivic coh}} \arrow[dl, "r_{\mathbf{C}}"'] \arrow[d, "r_{\mathrm{syn}}"] \arrow[dr, "r_{\text{ét}}"] & \\
\boxed{\text{de Rham coh / } \mathbf{C}} \arrow[r, leftrightarrow, dotted, "\text{(no relation)}"] & \boxed{\text{de Rham coh / } \mathbf{Q}_p} & \boxed{\text{étale coh}} \arrow[l, "\log_{\mathrm{BK}}"']
\end{tikzcd}$$

Putting these pieces together, if we can build a class $\mathfrak{z} \in H^3(S, \mathbf{Z}(2))$ and show that the syntomic regulator of $\mathfrak{z}$ is non-zero, then its etale regulator must also be non-zero. This programme was carried out in the Rankin–Selberg setting by Bertolini, Darmon and Rotger [**BDR15**], using Besser's formula [**Bes12**] to prove that the syntomic regulators of the Beilinson–Flach classes were p-adic L-values.

[2]This is not quite true: it is defined on the part of $H^3(S, \mathbf{Z}(2))$ coming from a smooth model $\mathcal{S}$ over $\mathbf{Z}_p$. This is a non-trivial restriction; the work of Flach on adjoint Selmer groups of modular forms relies strongly on the existence of motivic cohomology classes for S which don't extend to $\mathcal{S}$.

[3]Stronger results have subsequently been proved by these two authors jointly, in [**NN16**], which treats the case of varieties with bad reduction at p.

5.4. Other Rankin–Selberg formulae

The Rankin–Selberg integral is only the first of a very wide class of formulae, which express the L-values of an automorphic form for some reductive group G in terms of its integral against an Eisenstein series on some subgroup H (a "period integral"). There is a survey article by Bump [**Bum05**]) which catalogues dozens of constructions of this kind.

So we can play the following game: if we want to build an Euler system for some class of automorphic Galois representations, then we can look for known formulae expressing the L-function of our representation in terms of periods of automorphic forms. Then we can stare at the resulting integrals and try to recognise them as Beilinson regulators of motivic cohomology classes. If we can do this, then the étale versions of these classes should be non-zero (although we can't prove this), and they are clearly the right building blocks for an Euler system for our representation.

Remark. This won't always work, sadly. Firstly, in many of the known Rankin–Selberg formulae the groups G and H do not have Shimura varieties, so they lie outside the world of algebraic geometry; there is a perfectly good Rankin–Selberg integral for $\mathrm{GL}_m \times \mathrm{GL}_n$ for any integers (m, n), but it doesn't correspond to anything motivic unless $m = n = 2$.

Even if G corresponds to a Shimura variety (and H to a Shimura subvariety), then there can be more subtle obstacles. One major stumbling block is the Eisenstein series appearing in the formulae; these are often not just Eisenstein series for GL_2 but for more general reductive groups, and we need a way to relate these to motivic cohomology, generalising the way that GL_2 Eisenstein series are related to units via Kronecker's limit formula. This seems to be a difficult problem in general.

Despite these apparently gloomy remarks, all is not lost: there are surprisingly many Rankin–Selberg formulae in which only GL_2 Eisenstein series appear! There's now an ongoing project, being pursued by several research groups, to build Euler systems for each such integral formula. Some examples are

- an Euler system for the Asai representation attached to quadratic Hilbert modular forms, with $H = \mathrm{GL}_2$ and $G = \mathrm{Res}^F_{\mathbf{Q}} \mathrm{GL}_2$, where F is a real quadratic field [**LLZ16**];
- an Euler system for the spin representations attached to genus 2 Siegel modular forms, with $H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$ and $G = \mathrm{GSp}_4$ ([**LSZ17**]; I will discuss this briefly in Chapter 6);
- an Euler system for the spin representation of genus 3 Siegel modular forms, with
$$H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$$
and $G = \mathrm{GSp}_6$, which is studied by Antonio Cauchi and Joaquin Rodrigues [**CR18**];
- an Euler system for Picard modular forms (work in progress with David Loeffler and Chris Skinner), with $H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{Res}^K_{\mathbf{Q}} \mathrm{GL}_1$ and $G = \mathrm{GU}(2, 1)$, where K is an imaginary quadratic field and $\mathrm{GU}(2, 1)$ a unitary group split over K. In this case, we get an *Euler system over* K: in other words, we construct cohomology classes over all the finite abelian extensions of K.

$\diamond$

CHAPTER 6

An Euler system for Siegel modular forms

We will describe the construction of the Euler system using the adelic approach, as described in Section 4.5. This is consistent with the approach taken in the main reference for this chapter [**LSZ17**].

6.1. Siegel modular 3-folds

Definition. *Let J be the skew-symmetric 4×4-matrix $\begin{pmatrix} & & & 1\\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$. Define GSp_4 to be the group scheme over $\mathbf{Z}$ such that for any $\mathbf{Z}$-algebra R, we have*

$$\mathrm{GSp}_4(R) = \{(g,\nu)\in \mathrm{GL}_4(R)\times R^\times : gJg^t = \nu J\}.$$

We let Sp_4 be the subgroup of elements with $\nu = 1$.

The group $\mathrm{GSp}_4^+(\mathbf{R})$ (the elements of $\mathrm{GSp}_4(\mathbf{R})$ with $\nu > 0$) acts on the genus 2 Siegel upper half space

$$\mathcal{H}_2 = \{Z\in M_2(\mathbf{C}) : Z = \left(\begin{smallmatrix} y & z\\ x & y\end{smallmatrix}\right), \quad \Im(\left(\begin{smallmatrix} x & y\\ y & z\end{smallmatrix}\right)) \text{ is positive definite}\}$$

via $\left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right)\cdot Z = (AZ+B)(CZ+D)^{-1}$.

Remark. If we use a slightly different model of GSp_4, as matrices satisfying $gJ'g^t = \nu J'$ where $J' = \left(\begin{smallmatrix} & I_2\\ -I_2 & \end{smallmatrix}\right)$, then we can define $\mathcal{H}_2$ more tidily, as the space of symmetric complex matrices with positive-definite imaginary part. However, defining GSp_4 using the anti-diagonal matrix J, as we have done, is more convenient for representation theory (as the intersection of GSp_4 with the upper-triangular matrices in GL_4 is a Borel subgroup). ◇

If U is an open compact subgroup of $\mathrm{GSp}_4(\mathbf{A}_f)$, then we can define the double quotient analogous to (3),

$$\tilde{Y}(U) = \mathrm{GSp}_4^+(\mathbf{Q})\backslash\Big[\mathrm{GSp}_4(\mathbf{A}_\mathrm{f})\times\mathcal{H}_2\Big]/U.$$

This is a 3-dimensional complex manifold, with finitely many components, each of which looks like $\Gamma\backslash\mathcal{H}_2$ for some discrete subgroup $\Gamma\subset\mathrm{Sp}_4(\mathbf{Q})$.

Theorem 17. *If U is sufficiently small, then $\tilde{Y}(U)$ is the $\mathbf{C}$-points of a smooth algebraic variety $\tilde{Y}(N)$ defined over $\mathbf{Q}$ (a* Siegel 3-fold*), which is a moduli space for principally polarised abelian surfaces with some level structure.*

Of course, the kind of level structure that emerges depends on the group U we choose. A particularly important case is when

$$U = U_1(N) \coloneqq \{(g,\nu)\in\mathrm{GSp}_4(\hat{\mathbf{Z}}) : g = \left(\begin{smallmatrix} * & *\\ 0_2 & I_2\end{smallmatrix}\right) \bmod N\}$$

(where 0_2 and I_2 are the 2×2 zero and identity matrices respectively). The corresponding threefold $\tilde{Y}_1(N)$ parametrises triples (A,λ,P,Q) where A is an abelian surface, λ is a principal polarisation on A, and $P,Q\in A[N]$ are two points of exact order N satisfying $\langle P,Q\rangle = 0$ (where $\langle,\rangle$ is the Weil pairing induced by the polarisation λ).

As in the case of modular curves, we can identify the basechange $\tilde{Y}_1(N)\times\mu_m^\circ$ with a Shimura variety $\tilde{Y}(U)$ for some modified level m. More precisely, if we let

$$\mathcal{U} = \{(g,\nu)\in U_1(N) : \nu = 1 \bmod m\}$$

then $\tilde{Y}(\mathcal{U})$ is canonically isomorphic to $\tilde{Y}_1(N) \times \mu_m^\circ$ as a $\mathbf{Q}$-variety.

Remark. In terms of moduli spaces, the projection to μ_m° is given by the Weil pairing. ◇

6.2. Genus 2 Siegel modular forms

References: van der Geer's article [**vdG08**] is an excellent introduction; more details (particularly on Hecke operators) can be found in Andrianov's book [**And87**].

6.2.1. Definitions.

Definition. *Let $\tilde{\Gamma}_1(N) = \mathrm{Sp}_4(\mathbf{Z}) \cap \tilde{U}_1(N)$. A Siegel modular form of genus 2, level N and weight (k,k) is a holomorphic function $\mathcal{F} : \mathcal{H}_2 \to \mathbf{C}$ such that $\mathcal{F}(g \cdot Z) = \det(CZ+D)^k \mathcal{F}(Z)$ for all $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \tilde{\Gamma}_1(N)$ and $Z \in \mathcal{H}_2$. We write $M^{(2)}_{k,k}(\tilde{\Gamma}_1(N))$ for the space of such functions.*

Note the similarity to the familiar definition of modular forms (which are automorphic forms for $\mathrm{GSp}_2 \cong \mathrm{GL}_2$).

Remark. There is a more general notion of Siegel modular forms of weight (k_1, k_2) for integers $k_1 \ge k_2$; these are holomorphic functions on $\mathcal{H}_2$, taking values the space $\mathbf{C}^{k_1-k_2+1}$, and the transformation law involves the action of $CZ + D$ via the representation $\mathrm{Sym}^{k_1-k_2} \otimes \det^{k_2}$ of $\mathrm{GL}_2(\mathbf{C})$. When $k_1 > k_2$ these are sometimes called *vector-valued Siegel modular forms*, and the forms for $k_1 = k_2$ are called *scalar-valued.* ◇

As for usual modular forms, the space $M^{(2)}_{k_1,k_2}(\tilde{\Gamma}_1(N))$ is finite-dimensional over $\mathbf{C}$, and has a subspace $S^{(2)}_{k_1,k_2}(\tilde{\Gamma}_1(N))$ of *cuspidal* forms.

6.2.2. Hecke operators. We can also describe $M^{(2)}_{k_1,k_2}(\tilde{\Gamma}_1(N))$ and $S^{(2)}_{k_1,k_2}(\tilde{\Gamma}_1(N))$ adelically, using the isomorphism

$$\tilde{\Gamma}_1(N)\backslash\mathcal{H}_2 \cong \mathrm{GSp}_4^+(\mathbf{Q})\backslash \left(\mathrm{GSp}_4(\mathbf{A}_f) \times \mathcal{H}_2\right)/U_1(N).$$

From this interpretation, we get an action on these spaces of the *Hecke algebra* of double cosets $U_1(N)gU_1(N)$, $g \in \mathrm{GSp}_4(\mathbf{A}_f)$. This decomposes as a product of local Hecke algebras for each prime ℓ.

For $\ell \nmid N$, the local Hecke algebra is generated by three operators $T(\ell)$, $T_1(\ell^2)$, and $R(\ell)$, corresponding to the double cosets of $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ell & \\ & & & \ell \end{pmatrix}$, $\begin{pmatrix} 1 & & & \\ & \ell & & \\ & & \ell & \\ & & & \ell^2 \end{pmatrix}$, and $\begin{pmatrix} \ell & & & \\ & \ell & & \\ & & \ell & \\ & & & \ell \end{pmatrix}$ (considered as elements of $\mathrm{GSp}_4(\mathbf{Q}_\ell) \subset \mathrm{GSp}_4(\mathbf{A}_f)$, with components at all places other than ℓ being the identity).

Definition. *If $\mathcal{F} \in S^{(2)}_{k_1,k_2}(\tilde{\Gamma}_1(N))$ is an eigenform for the above three operators, with eigenvalues $t(\ell), t_1(\ell^2), r(\ell)$ respectively, then the* spin L-factor *of $\mathcal{F}$ at ℓ is the degree 4 polynomial*

$$P_{\mathrm{spin},\ell}(\mathcal{F}, X) = 1 - t(\ell)X + \ell\Big(t_1(\ell^2) + (\ell^2+1)r(\ell)\Big)X^2 - \ell^3 t(\ell)r(\ell)X^3 + \ell^6 r(\ell)^2 X^4.$$

(Often we work with the renormalised polynomial $P_{\mathrm{spin},\ell}(\mathcal{F}, \ell^{-3/2}X)$, which has the advantage that its roots all have absolute value 1.)

This is, of course, crying out to be made into an Euler product

$$L_{\mathrm{spin}}(\mathcal{F}, s) = \prod_{\ell \text{ prime}} P_{\mathrm{spin},\ell}\left(\mathcal{F}, \ell^{-s-\frac{3}{2}}\right)^{-1},$$

the *spin L-function* of $\mathcal{F}$, although this only makes sense if we have a good definition of the local factors at primes $\ell \mid N$. Under some mild hypotheses on $\mathcal{F}$, a suitable recipe for these factors was found by Piatetski-Shapiro and Novodvorsky in the 1970s (although not published until 1997 [**PS97**]), and they showed that the resulting function has meromorphic continuation with a functional equation of the form $s \mapsto 1 - s$.

Remark. As well as the spin L-function, there is another L-function associated to $\mathcal{F}$, confusingly called the *standard L-function*, given by a different Euler product in which the local factors at the good primes are reciprocals of polynomials in ℓ^{-s} of degree 5. The terminology "standard" is unfortunate for GSp_4, since the spin L-function is a much more fundamental object than the standard one, but it reflects the fact that the standard L-function generalises more easily to GSp_{2n} for general n. $\diamond$

6.3. Galois representations

Let $\mathcal{F}$ be a genus 2 cuspidal Siegel modular form of weight $(3,3)$ and level N which is an eigenform for the Hecke operators away from N. The following result shows that one can associate to $\mathcal{F}$ a Galois representation.

Theorem 18 (Weissauer, [**Wei05**]). *There exists a finite extension E of $\mathbf{Q}_p$ and a 4-dimensional Galois representation $V_{\mathcal{F}}$ over E, such that for all primes ℓ coprime to pN we have*

$$\det\left(1 - X\,\mathrm{Frob}_\ell^{-1} \mid V_{\mathcal{F}}\right) = P_{\mathrm{spin},\ell}(\mathcal{F}, X).$$

Perhaps surprisingly, these representations aren't always irreducible, even if $\mathcal{F}$ is cuspidal. This is because there are certain special types of cuspidal Siegel eigenforms that are "lifts" of automorphic forms on smaller groups; these are said to be *endoscopic*. There are several types of these, but only two which can occur in weight $(3,3)$, namely *Yoshida lifts* and *Saito–Kurokawa lifts*.

If $\mathcal{F}$ is non-endoscopic, and p is large enough[1] then the representation $V_{\mathcal{F}}$ is irreducible.

Theorem 19. *If $\mathcal{F}$ is non-endoscopic, then $V_{\mathcal{F}}$ appears in the etale cohomology of the level N Siegel 3-fold. More precisely, we have a projection map*

$$\mathrm{Pr}_{\mathcal{F}} : H^3_{\text{ét}}\left(\tilde{Y}_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Q}_p(3)\right) \otimes E \twoheadrightarrow V_{\mathcal{F}}^*.$$

We can similarly construct Galois representations for Siegel modular forms of weight (k_1, k_2) whenever $k_1 \geq k_2 \geq 3$, using étale cohomology with coefficients in sheaves coming from algebraic representations of the group GSp_4.

Remark. Note that weight $(2,2)$ forms are not cohomological – they still have spin Galois representations, but these can't be seen in the cohomology of the Siegel threefold. This is unfortunate, since there is a conjecture due to Brumer and Kramer, the *Paramodular Conjecture*, predicting that (certain) abelian surfaces over $\mathbf{Q}$ correspond to Siegel eigenforms of weight $(2,2)$. It would be very interesting to try to build Euler systems for these non-cohomological eigenforms, by deforming the constructions of this chapter in a p-adic family. $\diamond$

6.4. Lemma-Flach elements

References: [**Lem15, Lem17, LSZ17**].

6.4.1. Strategy. As we have seen above, the spin Galois representation of a genus 2 Siegel modular form can be found in the étale cohomology of the Siegel 3-fold $\tilde{Y}_1(N)$, for a suitable N. We therefore want to construct cohomology classes in $H^4_{\text{ét}}(\tilde{Y}_1(N) \times \mu_m^\circ, \mathbf{Z}_p(3))$ for $m \geq 1$, satisfying norm-compatibility relations as m changes (for a fixed N).

To do this, we note that we have a natural embedding

$$\iota : \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2 \longrightarrow \mathrm{GSp}_4,$$

which is given explicitly by

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right] \mapsto \begin{pmatrix} a & & & b \\ & a' & b' & \\ & c' & d' & \\ c & & & d \end{pmatrix}.$$

[1] It's expected to be irreducible for all p, but this is only known if we assume that $p \geq 5$ and $p \nmid N$.

This embedding induces a map from the product of two modular curves into a Siegel 3-fold with compatible level structures; for instance, we get maps

$$Y_1(N) \times Y_1(N) \to \tilde{Y}_1(N),$$

which are injective if N is large enough. This in turn induces a pushforward map on the étale cohomology groups

$$\iota_* : H^i_{\text{ét}}(Y_1(N)^2, \mathbf{Z}_p(j)) \longrightarrow H^{i+2}(\tilde{Y}_1(N), \mathbf{Z}_p(j+1)).$$

Consider the case when $i = j = 2$: Then the exterior cup product of two Siegel units ${}_c g_{0,1/N} \sqcup {}_d g_{0,1/N}$ defines an element of $H^2_{\text{ét}}(Y_1(N)^2, \mathbf{Z}_p(2))$, and we define the *Lemma–Flach element* for $m = 1$ to be

$${}_{c,d}\,\mathrm{LF}_{1,N} = \iota_*({}_c g_{0,1/N} \sqcup {}_d g_{0,1/N}).$$

6.4.2. Relation to an integral formula. As in Chapter 5 above, there is a good motivation for why this class ${}_{c,d}\,\mathrm{LF}_{1,N}$ should be interesting.

There is an integral formula for the spin L-function of GSp_4, which is due to Piatetskii-Shapiro. If $\mathcal{F}$ is a non-endoscopic, holomorphic eigenform of weight $(3, 3)$ (or of any cohomological weight), then we can consider the integral

$$\int_{(\Gamma_1(N)\backslash\mathcal{H})^2} E(\tau_1, s)E(\tau_2, s)\mathcal{F}(\tau_1, \tau_2)\mathrm{d}A$$

where $E(\tau, s)$ is a suitably-chosen family of Eisenstein series, and $\mathrm{d}A = \frac{\mathrm{d}\tau_1\wedge\mathrm{d}\bar{\tau}_1\wedge\mathrm{d}\tau_2\wedge\mathrm{d}\bar{\tau}_2}{\Im(\tau_1)\Im(\tau_2)}$ is the invariant measure on $(\Gamma_1(N)\backslash\mathcal{H})^2$. The general theory tells us that this unfolds into a product of local integrals, and the local integral at a finite place computes the spin L-factor.

The problem is that the local integral at ∞ is always zero! This can be fixed by replacing the holomorphic eigenform $\mathcal{F}$ with an "evil twin" $\mathcal{F}^g$, which is a real-analytic but non-holomorphic function on $\tilde{Y}_1(N)(\mathbf{C})$ with the same Hecke eigenvalues as $\mathcal{F}$; this doesn't change the local integrals at the finite places, but gives us a non-vanishing archimedean integral.

As in the Rankin–Selberg setting, the Lemma–Flach class we've defined is naturally the image under the étale regulator of a motivic cohomology class. The main result of [**Lem17**] shows that the Beilinson regulator of this motivic class, paired with an appropriate differential on $\tilde{Y}_1(N)(\mathbf{C})$ coming from $\mathcal{F}^g$, gives Piatetskii-Shapiro's integral for $L_{\mathrm{spin}}(\mathcal{F}, s)$ at $s = -\frac{1}{2}$.

REMARK. The g stands for "generic". Representation-theoretically, the problem is that the discrete-series representations of $\mathrm{GSp}_4(\mathbf{R})$ come in pairs ("local L-packets"), consisting of a holomorphic representation and a non-holomorphic one, and it is the non-holomorphic one which is generic (admits a Whittaker model) and thus can contribute to the integral formula.

One can also replace $Y_1(N) \times Y_1(N)$ with a symmetric space associated to GL_2/K, where K is an imaginary quadratic field; this gives an alternative integral representation which does involve the holomorphic eigenform $\mathcal{F}$. However, it seems to be impossible to interpret this integral as the regulator of a motivic cohomology class, since the symmetric space for GL_2/K is not an algebraic variety. $\diamond$

6.4.3. Lemma–Eisenstein classes. Our task is now to extend this to an Euler system: that is, to define classes ${}_{c,d}\,\mathrm{LF}_{m,N}$ for $m > 1$ satisfying good norm-compatibility properties. As before, we'll start by defining classes on higher-level modular varities, which are easier to work with, and proving norm-compatibility relations for these auxiliary classes.

Let us define $\mathcal{U}(M, N) = \left\{\gamma \in \mathrm{GSp}_4(\hat{\mathbf{Z}}) : \gamma = \left(\begin{smallmatrix} I_2 & 0_2 \\ 0_2 & I_2\end{smallmatrix}\right) \bmod \left(\begin{smallmatrix} M & M \\ N & N\end{smallmatrix}\right)\right\}$, and $\tilde{Y}(M, N)$ the corresponding Siegel threefold.

LEMMA 3. *If $M|N$, the group $\mathcal{U}(M, N)$ is normalised by the element* $u = \begin{pmatrix} 1 & & 1 & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1\end{pmatrix}$.

Define $\iota_{M,N} : Y(M, N)^2 \to \tilde{Y}(M, N)$ to be the composite

$$Y(M, N)^2 \xhookrightarrow{\iota} \tilde{Y}(M, N) \xrightarrow{u} \tilde{Y}(M, N).$$

Here, $Y(M,N)^2$ denotes as above the fibre product of two copies of $Y(M,N)$ over μ_M°.

Definition. *The Lemma-Eisenstein class ${}_{c,d}\,\mathrm{LEis}_{M,N}$ is the image of ${}_c g_{0,1/N} \sqcup {}_d g_{0,1/N}$ under $(\iota_{M,N})_*$. Here, we regard ${}_c g_{0,1/N} \sqcup {}_d g_{0,1/N}$ as an element of $H^2_{\text{ét}}(Y(M,N)^2, \mathbf{Z}_p(2))$ via pullback.*

6.4.4. Norm relations. Exactly as before, one sees straightforwardly that the Lemma–Eisenstein classes satisfy norm relations as N changes.

Proposition 14. *Suppose that $M|N$ and ℓ is a prime with $\ell \mid N$. Then*

$$(\mathrm{pr}_1)_*({}_{c,d}\,\mathrm{LEis}_{M,\ell N}) = {}_{c,d}\,\mathrm{LEis}_{M,N},$$

where pr_1 is the natural quotient map $\tilde{Y}(M,N\ell) \to \tilde{Y}(M,N)$. □

(*Exercise*: formulate and prove a similar formula for $\ell \nmid N$.)

The second norm relation: changing M. Let us write τ_ℓ for the "non-standard" degeneracy map $\tilde{Y}(\ell M,N)$ to $\tilde{Y}(M,N)$, given by the right-translation action of $\begin{pmatrix} \ell &&& \\ & \ell && \\ && 1 & \\ &&& 1 \end{pmatrix} \in \mathrm{GSp}_4(\mathbf{Q}_\ell) \subset \mathrm{GSp}_4(\mathbf{A}_f)$. Note that τ_ℓ factors as

$$\tilde{Y}(\ell M,N) \longrightarrow \tilde{Y}(M(\ell),N) \xrightarrow{\tilde{\pi}_{2,\ell}} \tilde{Y}(M,N),$$

where the first map is the natural degeneracy map.

Theorem 20. *Suppose $\ell|M$ and $\ell M \mid N$. Then we have*

$$(\tau_\ell)_*\left({}_{c,d}\,\mathrm{LEis}_{\ell M,N}\right) = \mathcal{U}'_\ell \cdot {}_{c,d}\,\mathrm{LEis}_{M,N}\,.$$

Here, $\mathcal{U}'_\ell$ is the Hecke correspondence on $\tilde{Y}(M,N)$ given by the element of $\mathrm{GSp}_4(\mathbf{A}_f)$ which is $\begin{pmatrix} \ell &&& \\ & \ell && \\ && 1 & \\ &&& 1 \end{pmatrix}$ at ℓ, and the identity elsewhere.

Remark. Again, there is a similar but slightly more complicated formula in the case when $\ell \nmid M$ (but still $\ell M \mid N$). ◇

Proof. We erect the following commutative diagram, in which all vertical arrows are the natural degeneracy maps:

$$\begin{tikzcd}
Y(\ell M,N)^2 \arrow[r, hook, "\iota_{\ell M,N}"] \arrow[d, equal] & \tilde{Y}(\ell M,N) \arrow[d] \arrow[ddr, "\tau_\ell"] & \\
Y(\ell M,N)^2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamondsuit"] & \tilde{Y}(M(\ell),N)) \arrow[d, "\tilde{\pi}_{1,\ell}"'] \arrow[dr, "\tilde{\pi}_{2,\ell}"] & \\
Y(M,N)^2 \arrow[r, hook, "\iota_{M,N}"] & \tilde{Y}(M,N) & \tilde{Y}(M,N)
\end{tikzcd}$$

We claim that the middle arrow is actually injective. This is equivalent to the claim that

$$H(\mathbf{A}_f) \cap u\tilde{U}(M(\ell),N)u^{-1} = U(M\ell,N)^2,$$

which is an easy matrix computation.

When $\ell \mid M$, we see that the square marked $\diamondsuit$ has both horizontal arrows closed immersions, and both vertical arrows of degree ℓ^3. So it is Cartesian, and we may conclude that the image of ${}_{c,d}\,\mathrm{LEis}_{\ell M,N}$ under pushforward to $\tilde{Y}(M(\ell),N)$ coincides with the pullback of ${}_{c,d}\,\mathrm{LEis}_{M,N}$. The result now follows by pushing both of these elements forward along the diagonal arrow and observing that $\mathcal{U}'_\ell = (\tilde{\pi}_{2,\ell})_* \circ (\tilde{\pi}_{1,\ell})^*$. □

6.4.5. Lemma-Flach classes and their norm relations. Let $m \geq 1$. We let ϖ_m denote the element of $\mathbf{A}_f^\times$ whose ℓ-th component is $\ell^{v_\ell(m)}$. Then right translation by the element $\begin{pmatrix} \varpi_m & & & \\ & \varpi_m & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \in \mathrm{GSp}_4(\mathbf{A}_f)$ induces a map

$$s_m : \tilde{Y}(m, mN) \to \tilde{Y}_1(N) \times \mu_m^\circ.$$

Definition. *We define the Lemma-Flach element* ${}_{c,d}\,\mathrm{LF}_{m,N}$ *to be the image of* ${}_{c,d}\,\mathrm{LEis}_{m,mN}$ *under* $(s_m)_*$.

Theorem 21. *Let ℓ be prime such that $\ell|M$ and $\ell|N$. Then we have*

$$\mathrm{norm}_{\mathbf{Q}(\mu_m)}^{\mathbf{Q}(\mu_{\ell m})}\left({}_{c,d}\,\mathrm{LF}_{\ell m,N}\right) = \mathcal{U}'_\ell \cdot {}_{c,d}\,\mathrm{LF}_{m,N}\,.$$

Proof. Analogous to the proof of the corresponding statement for Beilinson–Flach elements. □

Bibliography

[And87] Anatolij N. Andrianov, *Quadratic forms and Hecke operators*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 286, Springer-Verlag, Berlin, 1987. MR 884891 ↑ 42

[BDR15] Massimo Bertolini, Henri Darmon, and Victor Rotger, *Beilinson–Flach elements and Euler systems I: syntomic regulators and p-adic Rankin L-series*, J. Algebraic Geom. **24** (2015), no. 2, 355–378. MR 3311587 ↑ 39

[Beĭ84] A. A. Beĭlinson, *Higher regulators and values of L-functions*, Current problems in mathematics, Vol. 24, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984, pp. 181–238. MR 760999 ↑ 38

[Ber04] Laurent Berger, *An introduction to the theory of p-adic representations*, Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter, Berlin, 2004, pp. 255–292. MR 2023292 ↑ 9

[Bes00] Amnon Besser, *Syntomic regulators and p-adic integration. I. Rigid syntomic regulators*, Israel J. Math. **120** (2000), no. 1, 291–334. MR 1809626 ↑ 39

[Bes12] ———, *On the syntomic regulator for K_1 of a surface*, Israel J. Math. **190** (2012), no. 1, 29–66. MR 2956231 ↑ 39

[BK90] Spencer Bloch and Kazuya Kato, *L-functions and Tamagawa numbers of motives*, The Grothendieck Festschrift, Vol. I, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, pp. 333–400. MR 1086888 ↑ 9, 10, 11

[Bum05] Daniel Bump, *The Rankin-Selberg method: an introduction and survey*, Automorphic representations, L-functions and applications: progress and prospects, Ohio State Univ. Math. Res. Inst. Publ., vol. 11, de Gruyter, Berlin, 2005, pp. 41–73. MR 2192819 ↑ 40

[CR18] Antonio Cauchi and Joaquin Rodrigues, *Euler systems for GSp(6)*, in preparation, 2018. ↑ 40

[CRSS15] John Coates, A. Raghuram, Anupam Saikia, and R. Sujatha (eds.), *The Bloch-Kato conjecture for the Riemann zeta function*, London Mathematical Society Lecture Note Series, vol. 418, Cambridge University Press, Cambridge, 2015. MR 3410209 ↑ 17

[DDT97] Henri Darmon, Fred Diamond, and Richard Taylor, *Fermat's last theorem*, Elliptic curves, modular forms & Fermat's last theorem (Hong Kong, 1993), Int. Press, Cambridge, MA, 1997, pp. 2–140. MR 1605752 ↑ 20

[DS05] Fred Diamond and Jerry Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, vol. 228, Springer-Verlag, New York, 2005. MR 2112196 ↑ 20

[Gea06] Matthew Thomas Gealy, *On the Tamagawa Number Conjecture for motives attached to modular forms*, Caltech PhD thesis, 2006. ↑ 36

[Gre89] Ralph Greenberg, *Iwasawa theory for p-adic representations*, Algebraic number theory, Adv. Stud. Pure Math., vol. 17, Academic Press, Boston, MA, 1989, pp. 97–137. MR 1097613 ↑ 12

[Jan88] Uwe Jannsen, *Continuous étale cohomology*, Math. Ann. **280** (1988), no. 2, 207–245. MR 929536 ↑ 19

[Kat04] Kazuya Kato, *P-adic Hodge theory and values of zeta functions of modular forms*, Astérisque **295** (2004), ix, 117–290, Cohomologies p-adiques et applications arithmétiques. III. MR 2104361 ↑ 15, 21, 23, 24

[KLZ15] Guido Kings, David Loeffler, and Sarah Livia Zerbes, *Rankin–Eisenstein classes for modular forms*, to appear, 2015. ↑ 25

[KLZ17] ———, *Rankin–Eisenstein classes and explicit reciprocity laws*, Cambridge J Math **5** (2017), no. 1, 1–122. ↑ 25, 35, 36

[Kol91] V. A. Kolyvagin, *On the structure of Shafarevich-Tate groups*, Algebraic geometry (Chicago, IL, 1989), Lecture Notes in Math., vol. 1479, Springer, Berlin, 1991, pp. 94–121. MR 1181210 ↑ 15, 16, 21

[Lan87] Serge Lang, *Elliptic functions*, second ed., Graduate Texts in Mathematics, vol. 112, Springer-Verlag, New York, 1987, With an appendix by J. Tate. MR 890960 ↑ 23

[Lan91] Steven E. Landsburg, *Relative Chow groups*, Illinois J. Math. **35** (1991), no. 4, 618–641. MR 1115990 ↑ 38

[Lem15] Francesco Lemma, *On higher regulators of Siegel threefolds I: The vanishing on the boundary*, Asian J. Math. **19** (2015), no. 1, 83–120. MR 3318014 ↑ 43

[Lem17] ———, *On higher regulators of Siegel threefolds II: the connection to the special value*, Compos. Math. **153** (2017), no. 5, 889–946. MR 3705247 ↑ 43, 44

[LLZ14] Antonio Lei, David Loeffler, and Sarah Livia Zerbes, *Euler systems for Rankin–Selberg convolutions of modular forms*, Ann. of Math. (2) **180** (2014), no. 2, 653–771. MR 3224721 ↑ 25

[LLZ16] ———, *Euler systems for Hilbert modular surfaces*, preprint, 2016. ↑ 40

[LSZ17] David Loeffler, Christopher Skinner, and Sarah Livia Zerbes, *Euler systems for GSp(4)*, preprint, 2017. ↑ 40, 41, 43

[MR04] Barry Mazur and Karl Rubin, *Introduction to Kolyvagin systems*, Stark's conjectures: recent work and new directions, Contemp. Math., vol. 358, Amer. Math. Soc., Providence, RI, 2004, pp. 207–221. MR 2088718 ↑ 15

[MVW06] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs, vol. 2, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006. MR 2242284 ↑ 38

[Nek98] Jan Nekovář, *Syntomic cohomology and p-adic regulators*, preprint, 1998. ↑ 39

[Niz97] Wiesława Nizioł, *On the image of p-adic regulators*, Invent. Math. **127** (1997), no. 2, 375–400. MR 1427624 ↑ 39

[NN16] Jan Nekovář and Wiesława Nizioł, *Syntomic cohomology and p-adic regulators for varieties over p-adic fields*, Algebra Number Theory **10** (2016), no. 8, 1695–1790, With appendices by Laurent Berger and Frédéric Déglise. MR 3556797 ↑ 39

[NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg, *Cohomology of number fields*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, Springer-Verlag, Berlin, 2008. MR 2392026 ↑ 8

[PS97] I. I. Piatetski-Shapiro, *L-functions for* GSp_4, Pacific J. Math. (1997), no. Special Issue, 259–275, Olga Taussky-Todd: in memoriam. MR 1610879 ↑ 42

[Rub00a] Karl Rubin, *Euler systems*, Annals of Mathematics Studies, vol. 147, Princeton University Press, Princeton, NJ, 2000, Hermann Weyl Lectures. The Institute for Advanced Study. MR 1749177 ↑ 9, 15

[Rub00b] ———, *Euler systems*, Annals of Mathematics Studies, vol. 147, Princeton Univ. Press, 2000. MR 1749177 ↑ 12, 14, 15, 17

[Tha88] Francisco Thaine, *On the ideal class groups of real abelian number fields*, Ann. of Math. (2) **128** (1988), no. 1, 1–18. MR 951505 ↑ 15

[vdG08] Gerard van der Geer, *Siegel modular forms and their applications*, The 1-2-3 of modular forms, Universitext, Springer, Berlin, 2008, pp. 181–245. MR 2409679 ↑ 42

[Wei05] Rainer Weissauer, *Four dimensional Galois representations*, Astérisque **302** (2005), 67–150, Formes automorphes. II. Le cas du groupe GSp(4). MR 2234860 ↑ 43