Modular Curves: EXAM

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1. Let p be a prime.
   (a) [2 points] Show that the natural projection $p : X_1(p) \to X_0(p)$ is unramified at the cusps.
   (b) [3 points] Show that $X_1(p)$ has $p-1$ cusps when $p > 2$ and 2 cusps when $p = 2$.
   (c) [2 points] Calculate the width of the cusps, and determine which cusps are regular.
   (d) [2 points] Show that $X_1(p)$ has no elliptic points when $p > 3$, $X_1(3)$ has no elliptic points of order 2 and one elliptic point of order 3, and $X_1(2)$ has one elliptic point of order 2 and no elliptic point of order 3.
   (e) [2 points] Compute $g(X_1(p))$.

2. [3 points] Let X be a compact Riemann surface, $\mathcal{L}$ an invertible sheaf on X, and $P_1, \ldots, P_n$ any finite set of points on X. Show that there exists a meromorphic section of $\mathcal{L}$ which is holomorphic and non-vanishing at all the P_j.

3. [3 points] Let $f : X \to Y$ be a non-constant morphism of Riemann surfaces. Let $P \in X$ and $Q = f(P) \in Y$. Show that if ω is a differential on Y which is holomorphic and nonvanishing at Q, then the pullback $f^*\omega$ vanishes to order $e_P(f) - 1$ at P.

   Hence deduce the Riemann-Hurwitz formula, assuming that the sheaf of holomorphic differentials on a compact Riemann surface of genus g has degree $2g-2$.

4. Check the the following assertions from the lectures:
   (a) [2 points] The map $\tau \mapsto (\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau), \frac{1}{N}\mathbb{Z})$ gives a bijection between $\Gamma_0(N)\backslash\mathcal{H}$ and the set of equivalence classes of pairs (E, C), where E is an elliptic curve over $\mathbb{C}$ and C is a cyclic subgroup of E of order N.
   (b) [2 points] The map $\tau \mapsto (\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau), \frac{1}{N})$ gives a bijection between $\Gamma_1(N)\backslash\mathcal{H}$ and the set of equivalence classes of pairs (E, P), where E is an elliptic curve over $\mathbb{C}$ and P is a point of E of exact order N.

5. [3 points] Let E be an elliptic curve over $\mathbb{C}$, and $N > 1$. The *Weil pairing* is a perfect pairing $E[N] \times E[N] \to \mu_N$ (the exact definition is not relevant for this question, but it is given in Silverman's elliptic curves book). You may assume the following fact: if $E = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$, then $\langle \tau/N, 1/N\rangle_{E[N]} = e^{2\pi i/N}$.

   Using this, prove that the map $\tau \mapsto (\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau), \frac{\tau}{N}, \frac{1}{N})$ gives a bijection between $\Gamma(N)\backslash\mathcal{H}$ and the set of equivalence classes of triples (E, P, Q) with E an elliptic curve over $\mathbb{C}$ and P, Q two points of order N on E with $\langle P, Q\rangle_{E[N]} = e^{2\pi i/N}$.

6. For each of the following functors $\mathcal{F} : \mathcal{C} \to \underline{Set}$, either write down an object X of $\mathcal{C}$ and an element of $\mathcal{F}(X)$ which represent $\mathcal{F}$, or prove that $\mathcal{F}$ is not representable.
   (a) [2 points] The functor $\underline{Ring} \to \underline{Set}$ mapping a ring R to the set of cube roots of 1 in R.
   (b) [2 points] The restriction of the functor from (a) to the subcategory $\underline{\mathbf{F}_5 - Alg}$ of $\mathbf{F}_5$-algebras.

(c) [2 points] The functor $\underline{Ring} \to \underline{Set}$ mapping R to the set of cubes in R.

(d) [2 points] The functor $\underline{\mathbb{R} - Alg} \to \underline{Set}$ mapping R to the set of all vector space homomorphisms $\mathbb{R}^2 \to R$.

7. [3 points] Give an example of a scheme S, an elliptic curve E/S, an integer $N > 1$, and a section $P \in E(S)$ such that $nP \neq 0$ but $nP_x = 0$ as a point on E_x for every $x \in S$.

8. [3 points] Let E be the elliptic curve over $\mathbb{Z}[1/(2 \times 37)]$ defined by $y^2 = x^3 - 16x + 16$, and P the point $(0, 4)$. Find $\alpha, \beta \in \mathbb{Q}$ and an isomorphism between E and the Tate-normal-form elliptic curve $E(\alpha, \beta)$ that maps P to $(0, 0)$.

9. [3 points] Find an equation for $Y_1(6)$ (as a $\mathbb{Z}[1/6]$-scheme), and the universal pair (E, P) over it.

10. Recall that, for S a scheme, $\underline{Sch/S}$ denotes the "slice category" whose objects are pairs consisting of a scheme T and a morphism of schemes $T \to S$ (and whose morphisms are the obvious ones: morphisms of schemes $T \to U$ commuting with the morphism to S). There is a natural "forgetful functor" $\underline{Sch/S} \to \underline{Sch}$.

    (a) [3 points] Show that for any two schemes S, T, there is a canonical bijection between the following two sets:

    - morphisms of schemes $S \to S'$;
    - functors $\underline{Sch/S} \to \underline{Sch/S'}$ commuting with the forgetful map to $\underline{Sch}$.

    (b) [3 points] Show that for any scheme S there is a canonical bijection between

    - elliptic curves over S,
    - functors $\underline{Sch/S} \to \underline{Ell/\mathbb{Z}}$ commuting with the forgetful map to $\underline{Sch}$.

11. [3 points] Let $\mathcal{P}$ be a moduli problem on $\underline{Ell/R}$, for some ring R, and let $\widetilde{\mathcal{P}}$ be the associated contravariant functor $\underline{Sch/R} \to \underline{Sets}$. Show that

$$(\mathcal{P} \text{ is representable}) \Leftrightarrow (\mathcal{P} \text{ is rigid and } \widetilde{\mathcal{P}} \text{ is representable}).$$