STUDY GROUP ON LUBIN-TATE (φ,Γ)-MODULES

1. Introduction

The theory of (φ,Γ)-modules has been a crucial tool in Number Theory during the last years, with the stablishment of the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ as one of its culminating applications. This study group will be focused on Lubin-Tate (φ,Γ)-modules, which are natural analogues to deal with towers other than the cyclotomic one. One of its main interests being the hope of extending the p-adic Langlands correspondence to the case of $\mathrm{GL}_2(F)$ for a finite extension F of $\mathbf{Q}_p$.

1.1. Cyclotomic (φ,Γ)-modules.

Let's recall quickly the classical (cyclotomic) theory. Let $\mathbf{Q}_p$ denote the p-adic numbers, $F_n = \mathbf{Q}_p(\mu_{p^n})$ the p^n-cyclotomic extension, $F_\infty = \cup_{n\geq 0} F_n$. Call $\Gamma = \mathrm{Gal}(F_\infty/\mathbf{Q}_p)$, $\mathscr{G}_{\mathbf{Q}_p} = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Call $\chi\colon \Gamma \xrightarrow{\sim} \mathbf{Z}_p^\times$ the cyclotomic character, defined by $\sigma(\zeta_{p^n}) = \zeta_{p^n}^{\chi(\sigma)}$ for any primitive p^n-th root of unity ζ_{p^n} and $\sigma\in\Gamma$. Let $\mathscr{O}_{\mathscr{E}_{\mathbf{Q}_p}}$ denote the set of Laurent series $\sum_{n\in\mathbf{Z}} a_n T^n$ such that $a_n\in\mathbf{Z}_p$ and $a_n\to 0$ as $n\to-\infty$ (or, in other words, the p-adic completion of $\mathcal{O}_F[[T]][1/T]$). It is a ring equipped with a valuation, and we note $\mathscr{E}_{\mathbf{Q}_p} = \mathscr{O}_{\mathscr{E}_{\mathbf{Q}_p}}[1/p]$ (it is the set of Laurent series such that $a_n\in\mathbf{Q}_p$, $v_p(a_n)$ are uniformly bounded below and $a_n\to 0$ as $n\to-\infty$), which is a two dimensional field. These two objects are equipped with a continuous action of Γ and an operator φ given by

$$\sigma(T) = (1+T)^{\chi(\sigma)} - 1, \quad \varphi(T) = (1+T)^p - 1.$$

A (φ,Γ)-module over $\mathscr{E}_{\mathbf{Q}_p}$ of rank d is a free $\mathscr{E}_{\mathbf{Q}_p}$-module (of rank d) equipped with continuous semi-linear actions of Γ and φ, commuting with each other. We say that D is etale if it admits a basis for which the matrices of every $\sigma\in\Gamma$ and φ belong to $\mathrm{GL}_d(\mathscr{O}_{\mathscr{E}_{\mathbf{Q}_p}})$. Call $\Phi\Gamma(\mathscr{E}_{\mathbf{Q}_p})$ (resp. $\Phi\Gamma_{\mathrm{et}}(\mathscr{E}_{\mathbf{Q}_p})$) the category of (resp. etale) (φ,Γ)-modules over $\mathscr{E}_{\mathbf{Q}_p}$. Their main interest resides in the following result:

Theorem 1.1 (Fontaine)**.** *There is an equivalence of categories between the category* $\mathrm{Rep}_{\mathbf{Q}_p}\mathscr{G}_{\mathbf{Q}_p}$ *of finite dimensional local p-adic Galois representations and* $\Phi\Gamma_{et}(\mathscr{E}_{\mathbf{Q}_p})$.

At the base of this construction live the so-called Fontaine's rings and the theory of the field of norms of Wintenberger.

1.2. Overconvergent (φ,Γ)-modules.

Call $\mathscr{E}^\dagger_{\mathbf{Q}_p}$ the set of elements of $\mathscr{E}_{\mathbf{Q}_p}$ that actually converge on some non-empty annulus. We say that a (φ,Γ)-module D over $\mathscr{E}_{\mathbf{Q}_p}$ is overconvergent if admits a basis for which the matrices of any $\sigma\in\Gamma$ and φ belong to $\mathrm{GL}_d(\mathscr{E}^\dagger_{\mathbf{Q}_p})$. We have the following result, which is absolutely crucial to make a link between the theory of (φ,Γ)-modules to p-adic Hodge theory and Iwasawa theory:

Theorem 1.2 (Cherbonnier-Colmez)**.** *Every etale (φ,Γ)-module D over $\mathscr{E}_{\mathbf{Q}_p}$ is overconvergent.*

1.3. The Robba ring.

Let $\mathscr{R}_{\mathbf{Q}_p}$ be the set of Laurent series $\sum_{n\in\mathbf{Z}} a_n T^n$ (not necessarily bounded) which converge on some annulus $0 < v_p(T) < r$. We also have commuting actions of Γ and φ on $\mathscr{R}_{\mathbf{Q}_p}$. A (φ,Γ)-module over $\mathscr{R}_{\mathbf{Q}_p}$ is a free $\mathscr{R}_{\mathbf{Q}_p}$-module of finite rank endowed with commuting continuous semi-linear actions of Γ and φ. We can as well define what does it mean for a (φ,Γ)-module over $\mathscr{R}_{\mathbf{Q}_p}$ to be etale. Define $\Phi\Gamma(\mathscr{R}_{\mathbf{Q}_p})$ and $\Phi\Gamma_{\mathrm{et}}(\mathscr{R}_{\mathbf{Q}_p})$ analogously.

Theorem 1.3 (Kedlaya)**.** *There is an equivalence of categories between* $\Phi\Gamma_{et}(\mathscr{E}^\dagger_{\mathbf{Q}_p})$ *and* $\Phi\Gamma_{et}(\mathscr{R}_{\mathbf{Q}_p})$.

Working with (φ,Γ)-modules over the Robba ring turns out to be very convenient since we can apply differential methods.

1.4. **p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$.** If $D \in \Phi\Gamma(\mathscr{E}_{\mathbf{Q}_p})$, then we can package the action of Γ, the operator φ and the multiplication by $(1+T)$ into an action of the mirabolic semi-group $\left(\begin{smallmatrix} \mathbf{Z}_p-\{0\} & \mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$, the action of $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ corresponding to φ, that of $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $a \in \mathbf{Z}_p^\times$, corresponding to $\sigma_a \in \Gamma$, and $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ to multiplication by $(1+T)^b$. Colmez then manages to construct a sheaf $U \mapsto D \boxtimes U$ over $\mathbf{P}^1 = \mathbf{P}^1(\mathbf{Q}_p)$, whose sections over $\mathbf{Z}_p$ are D, and such that its global sections $D \boxtimes \mathbf{P}^1$ are equipped with an action of $\mathrm{GL}_2(\mathbf{Q}_p)$. If V is a two dimensional Galois representation and $D = \mathbf{D}(V) \in \Phi\Gamma_{\mathrm{et}}(\mathscr{E}_{\mathbf{Q}_p})$ is the (φ,Γ)-module associated by Fontaine's equivalence, it is from the module $D \boxtimes \mathbf{P}^1$ that one cuts out the (dual of the) $\mathrm{GL}_2(\mathbf{Q}_p)$-representation associated to V by the p-adic local Langlands correspondence.

1.5. **p-adic Langlands correspondence for $\mathrm{GL}_2(F)$?** If one expects to generalize the above results for a finite extension F of $\mathbf{Q}_p$, one should replace $\Gamma = \mathrm{Gal}(F_\infty/\mathbf{Q}_p)$ by some $\Gamma_F \cong \mathcal{O}_F^\times$. This suggests to consider (φ,Γ)-modules using Lubin-Tate towers instead of the cyclotomic one.

2. Lubin-Tate (φ,Γ)-modules

2.1. **Lubin-Tate theory.** Let F be a finite extension of $\mathbf{Q}_p$ of degree d, ring of integers $\mathcal{O}_F$, uniformizer π and residue field k_F of cardinality $q = p^h$, so that $d = eh$, where e is the ramification index of F. Let LT be a Lubin-Tate formal $\mathcal{O}_F$-module attached to π and choose a variable T for the formal group law. We then have, for every $a \in \mathcal{O}_F$, a power series

$$[a](T) = aT + \{\deg \geq 2\}.$$

Call $F_n = F(\mathrm{LT}[\pi^n])$ the field generated by the $[\pi^n]$-torsion points of LT acting on $\mathfrak{m}_{\overline{\mathbf{Q}}_p}$, $F_\infty = \cup_n F_n$. We note $\chi_F \colon \mathscr{G}_F \to \mathcal{O}_F^\times$ the Lubin-Tate character given by the action of $\mathscr{G}_F$ on the Tate module of LT, inducing an isomorphism $\mathrm{Gal}(F_\infty/F) \xrightarrow{\sim} \mathcal{O}_F^\times$. Let $\mathscr{H}_F = \mathrm{Gal}(\overline{\mathbf{Q}}_p/F_\infty)$, $\Gamma_n := \mathrm{Gal}(F_\infty/F_n) \subseteq \Gamma_0 = \mathrm{Gal}(F_\infty/F) =: \Gamma_F$, so that $\Gamma_n \xrightarrow{\sim} 1 + \pi^n\mathcal{O}_F$ via the Lubin-Tate character.

2.2. **Fontaine's rings revisited and Lubin-Tate (φ,Γ)-modules.** Let $\mathscr{O}_{\mathscr{E}_F}$ be the set of Laurent series $\sum_{n\in\mathbf{Z}} a_n T^n$ such that $a_n \in \mathcal{O}_F$ and $a_n \to 0$ as $n \to -\infty$ and call, as before, $\mathscr{E}_F = \mathscr{O}_{\mathscr{E}_F}[1/\pi]$, which is a two dimensional local field. Endow these objects with a relative Frobenius φ_q and an action of Γ_F given by

$$(\varphi_q f)(T) = f([\pi]T), \quad (\sigma f)(T) = f([\chi_F(\sigma)]T).$$

A (φ,Γ)-module over $\mathscr{E}_F$ is a finite dimensional $\mathscr{E}_F$-module equipped with commuting continuous semi-linear actions of Γ_F and φ_q. Define analogously what it means to a (φ,Γ)-module over $\mathscr{E}_F$ to be etale. We then have:

Theorem 2.1 (Kisin-Ren). *There is an equivalence of categories between* $\mathrm{Rep}_F\,\mathscr{G}_F$ *and* $\Phi\Gamma_{et}(\mathscr{E}_F)$.

2.3. **Overconvergence.** The first problem appears once we take a look at analogues of Cherbonnier-Colmez's theorem:

Theorem 2.2 (Fourquaux-Xie). *If $d > 1$, there are two-dimensional (φ,Γ)-modules over $\mathscr{E}_F$ which are not overconvergent.*

2.4. **F-analytic (φ,Γ)-modules and multi variable (φ,Γ)-modules.** In fact, thanks to the work of Berger, we can describe exactly which two-dimensional (φ,Γ)-modules over $\mathscr{E}_F$ are overconvergent. Call $\mathscr{I}$ the set of embeddings of F into $\overline{\mathbf{Q}}_p$. We say that a representation $V \in \mathrm{Rep}_F\,\mathscr{G}_F$ is F-analytic if V has zero Hodge-Tate weights at every direction $\mathrm{id} \neq \tau \in \mathscr{I}$, i.e., if, for every such τ, the representation $V \otimes_{F,\tau} \mathbf{C}_p$ is the trivial $\mathbf{C}_p$-semilinear representation of $\mathscr{G}_F$.

Theorem 2.3 (Berger). *If V is absolutely irreducible and overconvergent, then there exists a character $\delta \colon \Gamma_F \to \mathcal{O}_F^\times$ such that $V(\delta)$ is F-analytic. Conversely, if V is F-analytic, then it is overconvergent.*

In order to prove the last statement of the theorem above, Berger introduced a series of 'multivariable rings' so as to avoid the problem of the non-existence of normalised Tate trace maps when $F \neq \mathbf{Q}_p$, which will play an important role.

2.5. **Character variety and (φ,Γ)-modules.** There is yet another approach to the theory, developed in [9], which we explain now. Let $\mathfrak{X}$ denote the rigid analytic group variety over F, whose closed points in an extension K/F parametrize locally F-analytic characters $\mathcal{O}_F \to K^\times$, as constructed in [19]. The ring $\mathcal{O}_F^b(\mathfrak{X})$ of bounded analytic functions on $\mathfrak{X}$ is equipped with an action of $\mathcal{O}_F - \{0\}$ and can be localised and completed (in the same way as the ring $\mathcal{O}_{\mathscr{E}}$) so as to get rings $\mathscr{E}_F(\mathfrak{X})$, $\mathscr{E}_F^\dagger(\mathfrak{X})$ and $\mathscr{R}_F(\mathfrak{X})$ over which we can define what a (φ,Γ)-module is.

Call $\mathfrak{B}_{/L}$ the rigid analytic open unit ball over L. It is shown in [19] that $\mathfrak{B}_{/L}$ and $\mathfrak{X}_{\mathbf{C}_p}$ become isomorphic over $\mathbf{C}_p$, but that they are not isomorphic over any base extension over a finite extension of F. So that we have $\mathscr{R}_{\mathbf{C}_p}(\mathfrak{B}) = \mathscr{R}_{\mathbf{C}_p}(\mathfrak{X})$. On the other hand, there is an action of $\mathscr{G}_F$ over those rings such that $\mathscr{R}_F(\mathfrak{B}) = \mathscr{R}_{\mathbf{C}_p}(\mathfrak{B})^{\mathscr{G}_F}$ and a 'twisted' action of $\mathscr{G}_F$ over them such that $\mathscr{R}_F(\mathfrak{X}) = \mathscr{R}_{\mathbf{C}_p}(\mathfrak{B})^{\mathscr{G}_F}$. In this way, one can comparing the theory of (φ,Γ)-modules over both rings, and we have

Theorem 2.4 (Berger - Schneider - Xie)**.** *There is a degree-preserving equivalence of categories between the category of F-analytic (resp. etale) (φ,Γ)-modules over $\mathscr{R}_F(\mathfrak{B})$ and the category of F-analytic (resp. etale) (φ,Γ)-modules over $\mathscr{R}_F(\mathfrak{X})$.*

2.6. **Further topics I: Triangulable Lubin-Tate (φ,Γ)-modules.** The classical construction of the p-adic Langlands correspondence relies heavily on the fact that the (φ,Γ)-module is etale. By a clever reverse-engineering of the calculation of the locally analytic vectors of principal series, Colmez was able to bypass this difficult and hence to associate, to any trianguline (φ,Γ)-module over the Robba ring (not necessarily etale), a locally analytic representation of $\mathrm{GL}_2(\mathbf{Q}_p)$. His construction work as well for Lubin-Tate (φ,Γ)-modules.

Theorem 2.5 (Colmez)**.** *Let $D \in \Phi\Gamma(\mathscr{R})$ be idecomposable of rank 2, and let $\omega = \det_D \chi^{-1}$ be its central character. Then there exists a unique $\mathrm{GL}_2(F)$-equivariant sheaf $U \mapsto D \boxtimes U$ over $\mathbf{P}^1 = \mathbf{P}^1(F)$, of analytic type, such that $D \boxtimes \mathcal{O}_F = D$. Moreover, there exists a unique locally analytic representation $\Pi(D)$ of $\mathrm{GL}_2(F)$ such that $D \boxtimes \mathbf{P}^1$ is an extension of $\Pi(D)$ by $\Pi(D)^* \otimes \omega$.*

2.7. **Further topics II: Iwasawa theory.** Recall that if $V \in \mathrm{Rep}_{\mathbf{Q}_p}\mathscr{G}_{\mathbf{Q}_p}$ and T is a $\mathscr{G}_{\mathbf{Q}_p}$-stable $\mathbf{Z}_p$-lattice, its Iwasawa cohomology is defined as

$$H^1_{\mathrm{Iw}}(\mathbf{Q}_p, V) = \varprojlim_{n\geq 0} H^1(F_n, T) \otimes L,$$

where the inverse limit is taken with respect to the corestriction maps. By Shapiro's lemma, this can be interpreted as one single Galois cohomology group with values in measure spaces. This module is where Euler systems's live and it's crucial in the study of Iwasawa theory (explicit reciprocity laws, p-adic L-functions, ...). If $D = \mathbf{D}(V)$ is the cyclotomic (φ,Γ)-module over $\mathscr{E}_{\mathbf{Q}_p}$ associated to V, then a result of Fontaine says that

$$H^1_{\mathrm{Iw}}(\mathbf{Q}_p, V) = D^{\psi=1}.$$

If $V \in \mathrm{Rep}_F\mathscr{G}_F$, then a similar result has been established by Schneider an Venjakob replacing D by its Lubin-Tate version.

On the other hand, Fourquaux and Berger-Fourquaux have similar results concerning Iwasawa theory in the context of F-analytic Lubin-Tate (φ,Γ)-modules and, as they point out, it would be interesting to compare their results with those of Schneider and Venjakob.

3. Talks

January 11
Introductory talk.

January 18 (Tibor)
Fontaine's rings. Field of norms. Cyclotomic (φ,Γ)-modules. Fontaine's correspondence.
References: [15], [2].

January 25 (Kevin)
Cherbonnier-Colmez's theorem. Statement of the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$.
References: [2], [7], [4], [12], [13].

February 1 (Nethan)

p-adic differential equations. Berger's comparison theorems. Construction of (φ,Γ)-modules.
References: [1], [3].

February 8 (Chris)
p-divisible groups. Lubin-Tate formal groups. Lubin-Tate (φ,Γ)-modules and their relation to Galois representations.
References: [18].

February 15 (Joe)
Berger work: 'Multivariable (φ,Γ)-modules and locally analytic vectors.
References: [6].

February 22 (Antonio)
Berger work: 'Multivariable (φ,Γ)-modules and locally analytic vectors.
References: [6].

March 1 (Stephane)
p-adic Fourier theory.
References: [19].

March 8 (Sarah)
Outline of Berger-Schneider-Xie work: 'Rigid character groups , Lubin-Tate theory, and (φ,Γ)-modules'.
References: [9].

March 15 (Joaquin)
Locally analytic representations and (φ,Γ)-modules

March 22 (David)
Iwasawa theory

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