EULER SYSTEMS

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These are the notes from my talk at the Harris conference at MSRI in December 2014.

1. A conjecture of Perrin-Riou

Let M be a motive over $\mathbb{Q}$ which is not the trivial motive. Denote by $L(M, s)$ the L-function of M.

Assumption 1.1. *$L(M, s)$ has analytic continuation to $\mathbb{C}$ and satisfies a functional equation.*

Denote by M_p the p-adic realisation of M, so M_p is a finite-dimensional $\mathbb{Q}_p$-vector space with a continuous action of $G_{\mathbb{Q}} = \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Denote by $H^1_f(\mathbb{Q}, M_p)$ the Bloch-Kato Selmer group of M_p, which is a subgroup of $H^1(\mathbb{Q}, M_p)$, cut out by local conditions

- the unramified local condition at $\ell \neq p$;
- the H^1_f-local condition at p.

The Bloch-Kato conjecture predicts the size of $H^1_f(\mathbb{Q}, M_p^*(1))$ in terms of the L-function of M:

Conjecture 1.2. *(Bloch-Kato)*

$$\operatorname{ord}_{s=0} L(M, s) = \dim H^1_f(\mathbb{Q}, M_p^*(1)) - \dim H^0(\mathbb{Q}, M_p^*(1)),$$

together with an explicit formula for the leading term up to a p-adic unit.

Definition. The motive M is effective if all Hodge numbers (= steps in the Hodge filtration of M_{dR}) are ≥ 0.

Example. The trivial motive $M = \mathbb{Q}$ and the motive $M(f)$ attached to a modular form of weight ≥ 2 are effective, but $M = \mathbb{Q}(1)$ is not effective.

Fact 1.3. *If M is effective and $M \neq \mathbb{Q}$, then*

- *the H^1_f=local condition at p is the relaxed local condition unless the local L-factor vanishes;*
- $\operatorname{ord}_{s=0} L(M, s) = \dim(M_{\mathrm{Betti}})^{c=1}$, *where c denotes complex conjugation*
  $\Rightarrow$ *the Bloch-Kato conjecture predicts that*
  $$\dim H^1_f(\mathbb{Q}, M_p^*(1)) = \dim(M_{\mathrm{Betti}})^{c=1} =: d^+.$$

Euler systems are a tool for proving special cases of the Bloch-Kato conjecture for effective motives:

Conjecture 1.4. *(Perrin-Riou* [PR98]*) If M is effective, there exists a non-trivial system of elements $(z_m)_{m\geq 1}$,*

$$z_m \in \bigwedge^{d^+} H^1\left(\mathbb{Q}(\mu_m)^+, M_p^*(1)\right),$$

such that

$$\operatorname{cores}^{\mathbb{Q}(\mu_{m\ell})^+}_{\mathbb{Q}(\mu_m)^+}(z_{m\ell}) = \begin{cases} z_m & \text{if } \ell | m \text{ or } M_p^*(1) \text{ is ramified at } \ell \\ P_\ell(\sigma_\ell^{-1}) z_m & \text{otherwise,} \end{cases}$$

where $P_\ell(X) = \det(1 - X\sigma_\ell^{-1} | M_p)$ and σ_ℓ^{-1} denotes the arithmetic Frobenius at ℓ.

Definition. Such a system of elements $(z_m)_{m\geq 1}$ is called a *rank d^+ Euler system* for M_p.

Remark. If M is defined over a number field K, then an Euler system for M should have classes over all the abelian extensions of K.

Theorem 1.5. *(Kolyvagin, Rubin* [Rub00], *Perrin-Riou* [PR98]*) If such a rank d^+ Euler system exists and $z_1 \neq 0$, then (under some technical hypotheses) the Selmer group $H^1_f(\mathbb{Q}, M_p^*(1))$ is d^+-dimensional, as prediced by the Bloch-Kato conjecture.*

Remark. In proving this theorem, it is very important that the H^1_f-local condition at p is the relaxed one.

Examples. (rank $d^+ = 1$ Euler systems)

- $M = \mathbb{Q}$ (cyclotomic units)
- $M = M(f)$, where f is a modular form of weight ≥ 2 (Kato's Euler system, see [Kat04])
- $M = \mathbb{Q}$ over an imaginary quadratic field (eliptic units)

Remark. There exist other Euler systems attached to self-dual motives (i.e. $M \cong M^*(1)$)) satisfying a sign condition, but these don't give access to non-central L-values.

Problem. There are no non-trivial examples for $d^+ > 1$! (Unless one assumes the 'prectic conjecture' of Nekovar-Scholl, which gves access to certian settings related to totally real fields.)

2. A new Euler system

Theorem 2.1. *(Lei-LZ* [LLZ14], *Kings-LZ* [KLZ14]*) Let f, g be modular forms of weights $k+2, k'+2 \geq 2$, and let $0 \leq j \leq \min\{k, k'\}$. Then there exists a rank 1 Euler system for $M = M(f) \otimes M(g)(1+j)$, related to the p-adic L-value $L_p(f, g, 1+j)$.*

The existence of this Euler system does not fit the setting of Perrin-Riou's conjecture:

- $d^+(M) = 2$, but M is *not* effective: its Hodge numbers are
$$-1-j, \quad ,k'-j, \quad k-j, \quad k+k'+1-j;$$
- $\mathrm{ord}_{s=0} L(M, s) = \mathrm{ord}_{s=1+j} L(f, g, s) = 1$;
- the H^1_f-local condition at p is not the relaxed one;
- the Euler system classes take values in the Bloch-Kato Selmer group.

We therefore need a generalisation of Perrin-Riou's conjecture for motives which are not effective. We first generalize the notion of *effective*:

Definition. Let $r \geq 0$. Then M is *r-critical* if the Archimedean Γ-factor $L_\infty(M, s)$ has a pole at $s = 0$ of order r, and $L_\infty(M^*(1), 0) \neq \infty$.

Remarks. (1) 0-critical is precisely Deligne's definition of critical;
(2) if M is r-critical, then $\mathrm{ord}_{s=0} L(M, s) \geq r$ (and this lower bound shold be sharp);
(3) M is r-critical if and only if $d^+ - r$ Hodge numbers are < 0
$\Rightarrow M$ is effective if and only if it is d^+-critical.

Conjecture 2.2. *If M is r-critical, there exists a non-trivial rank r Euler system $(z_m)_{m \geq 1}$,*
$$z_m \in \bigwedge^r H^1_f\left(\mathbb{Q}(\mu_m)^+, M_p^*(1)\right).$$

Note. The Euler system in Theorem 2.1 is an example of an Euler system for a 1-critical motive.

Remarks.
- Conjecture 2.2 reduces to Perrin-Riou's conjecture when $r = d^+$. (Note that in this case the H^1_f-local condition is relaxed, so the condition that the Euler system take values in $H^1_f\left(\mathbb{Q}(\mu_m)^+, M_p^*(1)\right)$ is pretty much automatic.)
- A rank 0 Euler system should be thought of as a p-adic L-function. I will elaborate on this later.

Theorem 2.3. *(LZ) If M is 1-critical and $(z_m)_{m \geq 1}$ is a rank 1 Euler system with $z_1 \neq 0$, then (under some technical hypotheses) $H^1_f(\mathbb{Q}, M_p^*(1))$ is 1-dimensional, as predicted by the Bloch-Kato conjecture.*

Remark. For proving this theorem, we need to adapt the Euler system machine to take into account the non-relaxed local condition at p.

Here are some examples of 1-critical motives:

(1) $M = M(f) \otimes M(g)(1+j)$, where f, g are modular forms of weights $k+2, k'+2 \geq 2$ and $0 \leq j \leq \min\{k, k'\}$;
(2) $M = M_{\mathrm{A}sai}(f)(1+j)$, where f is a quadratic Hilbert modular form of weights $(k+2, k'+2)$ and $0 \leq j \leq \min\{k, k'\}$;
(3) $M = M_{\mathrm{Spin}}(F)(1+j)$, where F is a genus 2 Siegel modular form of weights $(k+3, k'+3)$, $k \geq k' \geq 0$ and $0 \leq j \leq k'$;
(4) $M(1)$, where $M \subset h^2\left(\mathrm{S}h(U(2,1))\right)$ is a rank 3 motive over an imaginary quadratic field.

Remark. There are lots more examples: if M is any motive with distinct Hodge numbers, then some twist of it will be 1-critical.

For the examples above, Conjecture 2.2 predicts the existence of a rank 1 Euler system: case (1) is Theorem 2.1, and cases (2), (3) and (4) are joint work in progress with $\begin{cases}\text{Antonio Lei}\\ \text{Francesco Lemma}\\ \text{Chris Skinner}\end{cases}$.

In other words, we can construct some exmaples of Euler systems for 1-critical motives. However, the interesting cases are those twists of the motives which are 0-critical. We can get at those using *p-adic deformation.*

3. Euler systems in p-adic families

Definition. Let A be a complete local $\mathbb{Z}_p$-algebra, and let $X = \mathrm{Spf}(A)$. A family of motivic Galois representations is a finite free A-module V with an A-linear action of $G_{\mathbb{Q}}$ such that for a Zariski dense set $X_{\mathrm{cl}} \subset X$, we have

$$V_x \cong M_{x,p} \qquad \text{for some motive } M_x.$$

The key example is that of the *cyclotomic deformation* of a motivic Galois representation:

Example. Let $A = \Lambda(\mathbb{Z}_p^\times)$ and $V = M_p \otimes A$, where $G_{\mathbb{Q}}$ acts on A via multiplication by the canonical character $G_{\mathbb{Q}} \to \mathbb{Z}_p^\times \hookrightarrow A^\times$. Note that V specializes to $M_p(j)$ for all j.

In order to vary a rank r Euler system in a p-adic family, we need to make an auxiliary choice called an *r-refinement.*

Definition. An r-refinement of V is an A-direct summand $W \subset V$ which is $G_{\mathbb{Q}_p}$-stable together with a Zariski-dense set of points $X_{\mathrm{cl},W} \subset X_{\mathrm{cl}}$, such that $\forall x \in X_{\mathrm{cl},W}$,

- M_x is r-critical,
- W_x has all Hodge-Tate weights ≥ 1, and V_x/W_x has all Hodge-Tate weights ≤ 0 *(Panchishkin condition).*

Conjecture 3.1. *Let V be a family of motivic Galois representations, and let $W \subset V$ be an r-refinement. Then there exists a non-trivial rank r Euler system $(z_m)_{m\geq 1}$,*

$$z_m \in \bigwedge^r H^1\left(\mathbb{Q}(\mu_m)^+, V\right)$$

such that

- $\mathrm{loc}_p(z_m) \in \mathrm{im}\left(H^1\left(\mathbb{Q}(\mu_m)^+, W\right) \to H^1\left(\mathbb{Q}(\mu_m)^+, V\right)\right)$,
- *$\forall x \in X_{\mathrm{cl},W}$, the specialisation of the Euler system at x agrees with the Euler system from Conjecture 2.2.*

Remark. $\forall x \in X_{\mathrm{cl},W}$ (generically), one has

$$H^1_f\left(\mathbb{Q}(\mu_m)^+, V_x\right) = \mathrm{im}\left(H^1\left(\mathbb{Q}(\mu_m)^+, W_x\right) \to H^1\left(\mathbb{Q}(\mu_m)^+, V_x\right)\right),$$

so $z_{m,x} \in \bigwedge^r H^1_f$ as required.

Example: Rankin-Selberg convolutions

Let $\mathbf{f}$, $\mathbf{g}$ be Hida families. Then there exists a 3-parameter of $G_{\mathbb{Q}}$-representations

$$V(\mathbf{f})^* \hat{\otimes} V(\mathbf{g})^* \hat{\otimes} \Lambda(\mathbb{Z}_p^\times)$$

(two Hida parameters and one cyclotomic parameter) interpolating $M_p(f_k)^* \otimes M_p(g_{k'})^*(1+j)$ for varying k, k', j. We have the following refinements:

2-refinement:	$W_2 = \{0\}$,	$X_{\mathrm{cl},W_2} = \{k, k' \geq 0, j \leq -1\}$
1-refinement:	$W_1 = \mathcal{F}^+ V(\mathbf{f})^* \hat{\otimes} \mathcal{F}^+ V(\mathbf{g})^* \hat{\otimes} \Lambda(\mathbb{Z}_p^\times)$,	$X_{\mathrm{cl},W_1} = \{0 \leq j \leq \min\{k, k'\}\}$
0-refinements:	$W_{0a} = \mathcal{F}^+ V(\mathbf{f})^* \hat{\otimes} V(\mathbf{g})^* \hat{\otimes} \Lambda(\mathbb{Z}_p^\times)$,	$X_{\mathrm{cl},W_{0a}} = \{k' + 1 \leq j \leq k\}$
	$W_{0b} = V(\mathbf{f})^* \hat{\otimes} \mathcal{F}^+ V(\mathbf{g})^* \hat{\otimes} \Lambda(\mathbb{Z}_p^\times)$,	$X_{\mathrm{cl},W_{0b}} = \{k + 1 \leq j \leq k'\}$

Here, $\mathcal{F}^+ V(\mathbf{f})$ denotes the rank 1 submodule as constructed by Wiles in [Wil88].

Theorem 3.2. *(Kings-LZ,* [KLZ14]*)*

(1) *Our Euler systems from Theorem 2.1 for $M(f_k)^* \otimes M(g_{k'})^*(1+j)$ interpolate along W_1;*
(2) *(Explicit reciprocity law) There exist rank-lowering operators corresponding to the inclusions $W_1 \hookrightarrow \begin{cases} W_{0a} \\ W_{0b} \end{cases}$ mapping the rank 1 Euler system to Hida's two 2-variable p-adic L-functions;*
(3) *if a rank 2 Euler system exist, then it maps to our Euler system under the rank-lowering operator.*

Remarks.

- The rank-lowering operators are multi-variable analogues of Perrin-Riou's regulator map, as constructed in [LZ14].
- (KLZ, in progress) The Hida families can be replaced by Coleman families. In this case, the subrepresentations are replaced by sub-(φ, Γ)-modules over the Roba ring.

In general, we expect there to be a *hierachy of Euler systems*, governed by inclusion of refinements, and related to each other via rank-lowering operators: if W' and W are r'-, resp. r-refinements of V, then Conjecture 3.1 predicts the existence of rank r', resp. rank r Euler systems. If $W' \subset W$, then these Euler systems should be related to each other via a rank-lowering operator

$$\bigwedge^r H^1\left(\mathbb{Q}(\mu_m)^+, V\right) \longrightarrow \bigwedge^{r'} H^1\left(\mathbb{Q}(\mu_m)^+, V\right).$$

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