EULER SYSTEMS AND THE BIRCH–SWINNERTON-DYER CONJECTURE

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These are the notes from my talk in Essen in June 2014.

1. The BSD conjecture

Let E be an elliptic curve over $\mathbb{Q}$, so $E(\mathbb{Q}) \cong \Delta \times \mathbb{Z}^{r_E}$, where Δ is a finite torsion group and $r_E \geq 0$. Let $L(E, s)$ be the L-function of E, which is defined as an infinite product of local terms (one for each prime p). The product is known to converge for $\Re(s) > \frac{3}{2}$.

Theorem 1.1. *(Wiles, Breuil-Conrad-Diamond-Taylor) $L(E, s)$ has analytic continuation to $\mathbb{C}$.*

The BSD conjecture predicts that $L(E, s)$ contains global information of E:

Conjecture 1.2.
- *$\mathrm{ord}_{s=1} L(E, s) = r_E$;*
- *There is an explicit formula for the leading term at $s = 1$ in terms of the global arithmetic invariants of E, e.g. the order of the group $Ш(E/\mathbb{Q})$ (which is conjectured to be finite).*

We are interested in the following generalisation: let ρ be an Artin representation of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ factoring through a finite extension F of $\mathbb{Q}$, and assume that ρ is odd and 2-dimensional. One can then define $L(E, \rho, s)$, and it is known that this L-function has analytic continuation to $\mathbb{C}$.

Conjecture 1.3. *(BSD_ρ)* $\mathrm{ord}_{s=1} L(E, \rho, s) = \mathrm{rank}\, E(F)[\rho]$

Back to the original BSD conjecture. One of the strongest results in this direction is due to Kolyvagin and Kato [Kat04]:

Theorem 1.4. *If $L(E, 1) \neq 0$, then $r_E = 0$ and $Ш(E/\mathbb{Q})[p^\infty]$ is finite for almost all p.*

Kato's strategy consists of three parts:

(1) make the problem p-adic: let $T_pE = \varprojlim E(\overline{\mathbb{Q}})[p^n]$, and define $V_pE = T_pE \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. This is a p-adic representation of $G_{\mathbb{Q}}$: it is a finite-dimensional $\mathbb{Q}_p$-vector space with a continuous action of $G_{\mathbb{Q}}$. One can hence consider the frst Galois cohomology group $H^1(\mathbb{Q}, V_pE) = H^1(G_{\mathbb{Q}}, V_pE)$. This group contains the Selmer group $\mathrm{Sel}(\mathbb{Q}, V_pE)$ which has two important properties:
   - $E(\mathbb{Q}) \otimes \mathbb{Q}_p \hookrightarrow \mathrm{Sel}(\mathbb{Q}, V_pE)$;
   - the quotient is related to $Ш(E/\mathbb{Q})[p^\infty]$.

   To prove the theorem, it is hence sufficient to show that if $L(E, 1) \neq 0$, then $\mathrm{Sel}(\mathbb{Q}, V_pE) = 0$.

(2) construct an Euler system (ES) for V_pE, which is a collection of classes $(z_m)_{m\geq 1}$,
$$z_m \in H^1(\mathbb{Q}(\mu_m), V_pE),$$
satisfying certain compatibility relations (the Euler system norm relations) under the Galois corestriction maps. This ES is related to the value $L(E, 1)$: there exists a linear functional (the Bloch-Kato dual exponential map)
$$\exp^* : H^1(\mathbb{Q}_p, V_pE) \longrightarrow \mathbb{Q}_p,$$
and one can show that $\exp^*(z_1) = \frac{L(E,1)}{\Omega}$ for some period Ω.

(3) use duality theorem from global Galois cohomology to show
$$\exp^*(z_1) \neq 0 \qquad \Rightarrow \qquad \mathrm{Sel}(\mathbb{Q}, V_pE) = 0.$$
(For this implication to hold, one needs to existence of the whole Euler system, and not just the existence of the class Z_1.)

The aim of this talk is to prove the analogue of this result for BSD_ρ, following Kato's strategy.

2. Euler systems

Definition. (Rubin, [Rub00]) Let K be a number field, and let V be a p-adic representation of G_K. Assume that V is unramified outside a finite set of primes Σ which contains all the primes above p. An ES for (K, V) is a collection of classes $(z_\mathfrak{m})$, $z_\mathfrak{m} \in H^1(K(\mathfrak{m}), V^*(1))$, indexed by the integral ideals of K ($K(\mathfrak{m})$ denotes the ray class field $\pmod{\mathfrak{m}}$), satisfying the following conditions:

- $z_\mathfrak{m}$ lands in a fixed lattice of $V^*(1)$, independent of $\mathfrak{m}$;
- (Euler system norm relations) $\mathrm{cores}^{K(\ell\mathfrak{m})}_{K(\mathfrak{m})} = \begin{cases} z_\mathfrak{m} & \text{if } \ell \mid \mathfrak{m} \text{ or } \ell \in \Sigma \\ P_\ell(\sigma_\ell^{-1}) z_\mathfrak{m} & \text{otherwise} \end{cases}$ where σ_ℓ is the arithmetic Frobenius at ℓ and $P_\ell(X) = \det(1 - \sigma_\ell^{-1} X | V)$ is the Euler factor at ℓ.

Theorem 2.1. *(Rubin) If $\exp^*(z_1) \neq 0$, then we get a bound for $\mathrm{Sel}(K, V)$ (or related Selmer groups with slightly different local conditions at p).*

Remark.
- We need to consider $V^*(1)$ because of global duality theorems.
- In the case $V = V_p E$, we have $V \cong V^*(1)$.

Conjecture 2.2. *A non-zero Euler system should exist whenever V comes from geometry.*

Despite this conjecture, the list of known Euler systems is rather short:

- cyclotomic units: $K = \mathbb{Q}$, $V = \mathbb{Q}$;
- elliptic units: K imag. quad., $V = \mathbb{Q}$;
- Kato's Euler system: $K = \mathbb{Q}$, $V = V_p E$ or $V_p f$, where f is a modular form of weight ≥ 2;
- Heegner points/Heegner cycles: K imag. quad. or CM, $V = V_p E$

Here is a new one:

Theorem 2.3. *(Lei-Loeffler-Zerbes [LLZ14], Kings-Loeffler-Zerbes [KLZ14]) Let f, g be modular forms of weights $k+2, k'+2 \geq 2$, levels N_f, N_g. Let $0 \leq j \leq \min\{k, k'\}$, and define $V = V_p f \otimes V_p g(1+j)$. Then there exist classes*

$$\mathrm{BF}_n^{(f,g,j)} \in H^1(\mathbb{Q}(\mu_n), V^*(1))$$

satisfying Euler system like relations under the corestriction maps maps.

This Euler system is related to L-values: one can show that $\mathrm{BF}_1^{(f,g,j)} \in \ker(\exp^*_{\mathrm{BK}})$, but if $p \nmid m N_f N_g$, then

$$\log_{\mathrm{BK}}(\mathrm{BF}_1^{(f,g,j)}) = (\star) L_p(f, g, 1+j),$$

where $L_p(f, g)$ is Hida's Rankin-Selberg p-adic L-function.

Remark.
- This formula was proved by Bertolini-Darmon-Rotger in the case when $k = k' = j = 0$.
- We have a similar formula for the image of $\mathrm{BF}_1^{(f,g,j)}$ under the complex regulator, which was proven independently by Brunault-Chida.

3. Idea of construction

Suppose that $k = k' = j = 0$. The geometric input is the Siegel unit

$$g_{\frac{1}{m^2 N}} \in O(Y_1(m^2 N))^\times = H^1_{\mathrm{Mot}}(Y_1(m^2 N, \mathbb{Q}(1))).$$

Let $\iota_{m,N} : Y_1(M^2 N) \to Y_1(N)^2$ denote the map given on $\mathcal{H}$ by $z \mapsto \left(z, z + \frac{1}{m}\right)$; observe that this map is defined over $\mathbb{Q}(\mu_m)$. Pushforward of $g_{\frac{1}{m^2 N}}$ along $\iota_{m,N}$ gives a class in $H^3_{\mathrm{Mot}}(Y_1(N)^2 \times \mu_m, \mathbb{Q}(2))$.

Via the p-adic étale regulator and the Hochschild-Serre spectral sequence we obtain an element in $H^1\left(\mathbb{Q}(\mu_m), H^2_{\mathrm{et}}\left(\overline{Y_1(N)}^2, \mathbb{Q}_p(2)\right)\right)$. If f and g are eigenforms of weight 2, levels N_f, N_g dividing N, project from $Y_1(N)^2$ into $Y_1(N_f) \times Y_1(N_g)$ and then into the $(\bar{f}, \bar{g})$-isotypical component to obtain the element $\mathrm{BF}_m^{(f,g,0)} \in H^1(\mathbb{Q}(\mu_m), V_f^* \times V_g^*)$.

For mudular forms of heigher weight, we replace $g_{\frac{1}{m^2 N}}$ by a motivic Eisenstein symbol (c.f. [Kin13])

$$\mathrm{Eis}^k_{m^2 N} \in H^1_{\mathrm{Mot}}\left(Y_1(m^2 N), \mathrm{Sym}^k \mathcal{H}_\mathbb{Q}(1)\right),$$

whee $\mathcal{H}$ is the relative cohomology sheaf of the universal elliptic curve over $Y_1(m^2N)$. If $k,k' \geq 0$, $0 \leq j \leq \min\{k,k'\}$, then there exists a map to take $\mathrm{Eis}^{k+k'-2j}_{m^2N}$ into

$$H^3_{\mathrm{Mot}}\left(Y_1(N)^2 \times \mu_m, \mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}} \boxtimes \mathrm{Sym}^{k'} \mathcal{H}_{\mathbb{Q}}(2-j)\right).$$

Roughly, this map is composition of $\iota_{m,N}$ with the Clebsch-Gordan decomposition.

4. A 3-variable Euler system

Recall that we want to construct an Euler system for $V = V_pE(\rho)$, where ρ is a 2-dimensional odd Artin representation of $G_{\mathbb{Q}}$. However:

- E corresponds to a modular form of weight 2;
- ρ corresponds to a modular form of weight 1.

Hence Theorem 2.3 does not apply! In order to get around this, we use p-adic deformation.

Let $\mathcal{F}$, $\mathcal{G}$ be Hida families of tame levels $N_{\mathcal{F}}$, $N_{\mathcal{G}}$, so they are maximal ideals of ordinary Hecke algebras. Let $\Lambda_{\mathcal{F}}$, $\Lambda_{\mathcal{G}}$ be the corresponding localisations of the Hecke algebras, and let $M_{\mathcal{F}}$ and $M_{\mathcal{G}}$ denote the Λ-adic representations attached to $\mathcal{F}$ and $\mathcal{G}$. Also, let $\Gamma = \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$.

Theorem 4.1. *(Kings-Loeffler-Zerbes* [KLZ14]*) For all $m \geq 1$, not divisible by p, there exists a class*

$$\mathrm{BF}^{(\mathcal{F},\mathcal{G})}_m \in H^1\left(\mathbb{Q}(\mu_m), M^*_{\mathcal{F}} \hat{\otimes} M^*_{\mathcal{G}} \hat{\otimes} \Lambda(\Gamma)\right)$$

satisfying the following conditions:

(1) *they satisfy Euler system like relations under the corestriction maps;*
(2) *if f, g in $\mathcal{F}, \mathcal{G}$ are of weights $k+2, k'+2 \geq 2$ and $0 \leq j \leq \min\{k,k'\}$, then the specialisation of $\mathrm{BF}^{(\mathcal{F},\mathcal{G})}_m$ at (f,g,j) recovers the element $\mathrm{BF}^{(f,g,j)}_m$ constructed in Theorem 2.3 up to some Euler factors;*
(3) *these points are dense in the Hida families, so we get a relation to L-values* everywhere *in the families, even at critical specialisations.*

Remark. (2) is proven directly in étale cohomology. The variation in k,k' reduces to a compatibility on $Y_1(N)$ for Eisenstein classes, which was proven by Kings in [Kin13]. The variation in j reduces to variation in k,k' by a geometric argument.

5. Application to BSD_ρ

Let E be an elliptic curve corresponding to a modular form f, and let ρ be an odd 2-dimensional Artin representation corresponding to a mdoular form g. Assume that f and g are ordinary at p (which is automatic for g). Let $\mathcal{F}$ and $\mathcal{G}$ be Hida families through f and g. Consider the Euler system $\left(\mathrm{BF}^{(\mathcal{F},\mathcal{G})}_m\right)_m$ and specialize it at $(f,g,0)$. We obtain an Euler system for $V_pE(\rho)$ related to the critical L-value $L(E,\rho,1)$. Applying Rubin's Euler system machine, we obtain the following result:

Theorem 5.1. *(Kings-Loeffler-Zerbes* [KLZ14]*) Let $p \geq 5$, assume that E does not have complex multiplication and that E is ordinary at p. Suppose that ρ factors through F. If some technical hypotheses are hold (one can show that they are satisfied for infinitely many p) and $L(E,\rho,1) \neq 0$, then* $\mathrm{rank}\, E(F)[\rho] = 0$ *and the p-primary part of $\text{Ш}(E/\mathbb{Q})[\rho]$ is finite.*

Remark.
- The fact that $L(E,\rho,1) \neq 0$ implies $\mathrm{rank}\, E(F)[\rho] = 0$ was first proven by Bertolini-Darmon-Rotger in [BDR14] via a different method.
- It is work in progress (jointly with Kings and Loeffler) to show that we can remove the hypothesis that E be ordinary at p, i.e. we can replace Hida families by Coleman families.

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