LTCC course: Euler systems and Iwasawa Theory
COURSE WORK 2

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1. Let $N \geq 1$, and let ℓ be prime. Denote by $\pi : Y_1(N\ell) \longrightarrow Y_1(N)$ the natural quotient map. Show that
$$\pi_*({}_cg_{0,1/N\ell}) = \begin{cases} {}_cg_{0,1/N} & \text{if } \ell | N \\ {}_cg_{0,1/N} \cdot ({}_cg_{0,u/N})^{-1} & \text{if } \ell \nmid N \end{cases}$$
where u is the inverse of $\ell \pmod N$.

2. Let $M, N \geq 1$ such that $M|N$, and let ℓ be a prime not dividing N. Denote by $\pi : Y(M, N\ell) \to Y(M,N)$ the natural quotient map. Calculate $(\pi \times \pi)_*({}_c\mathrm{REis}_{M,N\ell})$.

3. Let M, N, ℓ be integers with ℓ prime, $\ell \nmid M$ and $M\ell|N$. Denote by $\tau_\ell : Y(M\ell, N) \to Y(M,N)$ the map constructed in lectures.
   (a) Let $a \in \mathbb{Z}/M\ell\mathbb{Z}$ be the unique lift of $1 \in \mathbb{Z}/M\mathbb{Z}$ to an element of $\mathbb{Z}/M\ell\mathbb{Z}$ divisible by ℓ.
   (b) Show that the matrix $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ defines an embedding of $Y(M(\ell), N)$ into $Y(M(\ell), N)^2$.
   (c) Show that the following diagram is Cartesian:
$$\begin{array}{ccc} Y(M\ell, N) \sqcup Y(M(\ell), N) & \xrightarrow{(\iota', \gamma)} & Y(M(\ell), N)^2 \\ \downarrow & & \downarrow \\ Y(M,N) & \xrightarrow{\iota_{M,N}} & Y(M,N)^2 \end{array}$$
   Here, the vertical maps are just the natural degeneracy maps.

4. Let f, g be cuspidal modular eigenforms of weight 2 and level $\Gamma_1(N)$; write $f = \sum a_n q^n$ and $g = \sum b_n q^n$. Assume that $a_n, b_n \in \mathbb{Q}$ for all n. Let p be a prime, and denote by $V_p(f)$, $V_p(g)$ the p-adic representations attached to f and g, respectively. Assume[1] that the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in $\mathrm{GL}(V_p(\star))$ is open for $\star \in \{f, g\}$. Show that $V_p(f) \otimes V_p(g)$ is irreducible as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation unless there exists a character χ such that
$$a_\ell = b_\ell \chi(\ell) \qquad \text{for } \ell \nmid N.$$

[1] This is true when f and g are not of CM type.