LTCC course: Euler systems and Iwasawa Theory: COURSE WORK 1

Submission date: November 5

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1. Let K be a finite extension of $\mathbf{Q}_p$, and let B be a topological $\mathbf{Q}_p$-algebra with a continuous, semi-linear action of $G_K = \mathrm{Gal}(\bar{K}/K)$. Assume that B is G_K-regular, i.e. if $b \in B$ is non-zero and $\mathbf{Q}_p.b$ is G_K-stable, then $b \in B^\times$.
   (a) Show that B^{G_K} is a field.
   (b) Assume that B is a field, and let $F = B^{G_K}$. Let V be a finite-dimensional $\mathbf{Q}_p$-vector space with a continuous action of G_K. Show that
   $$\dim_F \left(V \otimes_{\mathbf{Q}_p} B\right)^{G_K} \leq \dim_{\mathbf{Q}_p} V.$$

2. Let K be a field, and let V be a p-adic representation of $G_K = \mathrm{Gal}(\bar{K}/K)$ with coefficient field $\mathbf{Q}_p$. Recall that
   $$H^1(K,V) = \frac{\text{cts. functions } s : G_K \to V \text{ such that } s(gh) = s(g) + hs(h) \text{ for all } g,h \in G_K}{\text{functions } s : G_K \to V \text{ of the form } s(g) = gv - v}.$$
   (a) Show that there is a bijection between $H^1(K,V)$ and equivalence classes of short exact sequences of G_K-representations
   $$0 \to V \to V' \to \mathbf{Q}_p \to 0,$$
   where the term $\mathbf{Q}_p$ is the trivial representation.
   (b) Show that if V is crystalline, then $H^1_f(K,V)$ classifies those equivalence classes of short exact sequences where V' is crystalline.

3. Let K be a number field, and let $V = \mathbf{Q}_p$, quipped with the trivial action of G_K. Show that $\mathrm{Sel}_{\mathrm{BK}}(K,V) = 0$. You may use without proof that $\mathcal{F}_{v,\mathrm{BK}} = \mathcal{F}_{v,\mathrm{ur}}$ for all $v|p$.

4. For $m \geq 1$, let $\zeta_m = e^{2\pi i/m}$, and let $u_m = 1 - \zeta_m$. Write μ_m for the group of mth roots of unity.
   (a) Let ℓ be a prime. Show that
   $$\mathrm{norm}^{\mathbf{Q}(\mu_{\ell m})}_{\mathbf{Q}(\mu_m)} u_{\ell m} = \begin{cases} u_m & \text{if } \ell | m \\ (1 - \sigma_\ell^{-1}) u_m & \text{if } \ell \nmid m \text{ and } m > 1 \\ l & \text{if } m = 1 \end{cases}$$
   (b) Let p be a prime, and let
   $$v_m \begin{cases} u_m & \text{if } p|m \\ \mathrm{norm}^{\mathbf{Q}(\mu_{pm})}_{\mathbf{Q}(\mu_m)} u_{pm} & \text{if } p \nmid m \text{ (including if } m = 1 \end{cases}$$
   Denote by κ_p the Kummer map. Show that the classes $\kappa_p(v_m)$ are an Euler system for the representation $\mathbf{Q}_p(1)$.