

# Small rainbow cliques in randomly perturbed dense graphs

Elad Aigner-Horev <sup>\*</sup>    Oran Danon <sup>§</sup>    Dan Hefetz <sup>‡</sup>    Shoham Letzter <sup>†</sup>

July 3, 2020

## Abstract

For two graphs  $G$  and  $H$ , write  $G \xrightarrow{\text{rbw}} H$  if  $G$  has the property that every *proper* colouring of its edges yields a *rainbow* copy of  $H$ . We study the thresholds for such so-called *anti-Ramsey* properties in randomly perturbed dense graphs, which are unions of the form  $G \cup \mathbb{G}(n, p)$ , where  $G$  is an  $n$ -vertex graph with edge-density at least  $d > 0$ , and  $d$  is independent of  $n$ .

In a companion paper, we proved that the threshold for the property  $G \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_\ell$  is  $n^{-1/m_2(K_{\lceil \ell/2 \rceil})}$ , whenever  $\ell \geq 9$ . For smaller  $\ell$ , the thresholds behave more erratically, and for  $4 \leq \ell \leq 7$  they deviate downwards significantly from the aforementioned aesthetic form capturing the thresholds for *large* cliques.

In particular, we show that the thresholds for  $\ell \in \{4, 5, 7\}$  are  $n^{-5/4}$ ,  $n^{-1}$ , and  $n^{-7/15}$ , respectively. For  $\ell \in \{6, 8\}$  we determine the threshold up to a  $(1 + o(1))$ -factor in the exponent: they are  $n^{-(2/3+o(1))}$  and  $n^{-(2/5+o(1))}$ , respectively. For  $\ell = 3$ , the threshold is  $n^{-2}$ ; this follows from a more general result about odd cycles in our companion paper.

## 1 Introduction

A *random perturbation* of a fixed  $n$ -vertex graph  $G$ , denoted by  $G \cup \mathbb{G}(n, p)$ , is a distribution over the supergraphs of  $G$ . The elements of such a distribution are generated via the addition of randomly sampled edges to  $G$ . These random edges are taken from the binomial random graph on  $n$  vertices with edge-density  $p$ , denoted  $\mathbb{G}(n, p)$ . The fixed graph  $G$  being *perturbed* or *augmented* in this manner is referred to as the *seed* of the *perturbation* (or *augmentation*)  $G \cup \mathbb{G}(n, p)$ . Let  $\mathcal{G}_{d,n}$  denote the family of  $n$ -vertex graphs with edge density at least  $d > 0$ ; the notation  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p)$  then suggests itself to mean the collection of distributions arising from the members of  $\mathcal{G}_{d,n}$ .

The above model of randomly perturbed graphs was introduced by Bohman, Frieze, and Martin [7]. Since then, two prominent strands of study regarding the distribution of randomly perturbed dense graphs have emerged. The first is the generalisation of the results of [7], regarding the Hamiltonicity of perturbed dense graphs, to the study of spanning structures in said graph distributions. Here, one encounters numerous results such as [3, 5, 6, 9, 10, 14, 16, 24, 25, 28].

---

<sup>\*</sup>Department of Computer Science, Ariel University, Ariel 40700, Israel. Email: [horev@ariel.ac.il](mailto:horev@ariel.ac.il).

<sup>§</sup>Department of Computer Science, Ariel University, Ariel 40700, Israel. Email: [oran.danon@msmail.ariel.ac.il](mailto:oran.danon@msmail.ariel.ac.il).

<sup>‡</sup>Department of Computer Science, Ariel University, Ariel 40700, Israel. Email: [danhe@ariel.ac.il](mailto:danhe@ariel.ac.il). Research supported by ISF grant 822/18.

<sup>†</sup>Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, UK. Email: [s.letzter@ucl.ac.uk](mailto:s.letzter@ucl.ac.uk). Research supported by the Royal Society.

The second strand of study, initiated by Krivelevich, Sudakov, and Tetali [26], deals with Ramsey properties of such graph distributions, thus extending the classical results regarding Ramsey properties of random graphs [27, 31, 33, 34, 35]. Das and Treglown [13] and Powierski [32] significantly extended the body of results set by [26] regarding the thresholds of (symmetric and asymmetric) Ramsey properties of the form  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \rightarrow (K_s, K_r)$ . In particular, Das and Treglown [13] also study asymmetric Ramsey properties of  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p)$  involving cliques and cycles. Das, Morris, and Treglown [12] extended the results of Kreuter [23] pertaining to *vertex Ramsey* properties of random graphs to the perturbed model. In the Ramsey-arithmetic scene, the first author and Person [2] established an (asymptotically) optimal Schur-type theorem for randomly perturbed dense sets of integers. Sudakov and Vondrák [37] studied the non-2-colourability of randomly perturbed dense hypergraphs.

The term *Anti-Ramsey* is commonly used in order to refer to a body of problems and results concerning the emergence of non-monochromatic configurations in every (sensible) edge-colouring of a given graph. Here, one encounters a large diversity concerning this theme; the reader is referred to the excellent survey [15] and references therein for more details.

An edge-colouring  $\psi$  of a graph  $G$  is said to be *b-bounded* if no colour is used on more than  $b$  edges. It is said to be *locally-b-bounded* if every colour appears at most  $b$  times at every vertex. In particular, locally-1-bounded edge-colourings are the traditional *proper* edge-colourings. A subgraph  $H \subseteq G$  is said to be *rainbow* with respect to an edge-colouring  $\psi$ , if every two of its edges are assigned different colours under  $\psi$ , that is, if  $|\psi(H)| := |\psi(E(H))| = e(H)$ , where  $\psi(E(H)) := \{\psi(e) : e \in E(H)\}$ . We write  $G \xrightarrow{\text{rbw}} H$ , if  $G$  has the property that every *proper* colouring of its edges admits a rainbow copy of  $H$ .

A fairly complete overview regarding the emergence of small fixed rainbow configurations in random graphs with respect to every  $b$ -bounded colouring, can be found in the work of Bohman, Frieze, Pikhurko, and Smyth [8] and references therein. The first to consider the emergence of fixed rainbow configurations in random graphs with respect to proper edge-colourings were Rödl and Tuza [36]. Subsequently, Kohayakawa, Kostadinidis and Mota [19, 20] launched the systematic study of such rainbow configurations in random graphs with respect to proper edge-colourings. They [19] proved that for every graph  $H$ , there exists a constant  $C > 0$  such that  $\mathbb{G}(n, p) \xrightarrow{\text{rbw}} H$ , whenever  $p \geq Cn^{-1/m_2(H)}$ . Here

$$m_2(H) := \max \left\{ \frac{e(F) - 1}{v(F) - 2} : F \subseteq H, e(F) \geq 2 \right\}$$

is the so-called *maximum 2-density* of  $H$ . For  $H \cong C_\ell$  with  $\ell \geq 7$ , and  $H \cong K_r$  with  $r \geq 19$ , Nenadov, Person, Škorić, and Steger [30] proved, amongst other things, that  $n^{-1/m_2(H)}$  is, in fact, the threshold for the property  $\mathbb{G}(n, p) \xrightarrow{\text{rbw}} H$ .

Barros, Cavalari, Mota, and Parczyk [4] extended the result of [30] for cycles, proving that the threshold of the property  $\mathbb{G}(n, p) \xrightarrow{\text{rbw}} C_\ell$  remains  $n^{-1/m_2(C_\ell)}$  also when  $\ell \geq 5$ . Moreover, Kohayakawa, Mota, Parczyk, and Schnitzer [21] extended the result of [30] for complete graphs, proving that the threshold of the property  $\mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_r$  remains  $n^{-1/m_2(K_r)}$  also when  $r \geq 5$ .

For  $C_4$  and  $K_4$  the situation is different. The threshold for the property  $\mathbb{G}(n, p) \xrightarrow{\text{rbw}} C_4$  is  $n^{-3/4} = o(n^{-1/m_2(C_4)})$ , as proved by Mota [29]. For the property  $\mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_4$ , the threshold

is  $n^{-7/15} = o(n^{-1/m_2(K_4)})$  as proved by Kohayakawa, Mota, Parczyk, and Schnitzer [21]. More generally, Kohayakawa, Kostadinidis and Mota [20] proved that there are infinitely many graphs  $H$  for which the threshold for the property  $\mathbb{G}(n, p) \xrightarrow{\text{rbw}} H$  is significantly smaller than  $n^{-1/m_2(H)}$ .

Note that the threshold for the property  $\mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_3$  coincides with the threshold for the emergence of  $K_3$  in  $\mathbb{G}(n, p)$ , which is  $n^{-1}$ , as every properly-coloured triangle is rainbow.

For a real  $d > 0$ , we say that  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p)$  a.a.s. satisfies a graph property  $\mathcal{P}$ , if

$$\lim_{n \rightarrow \infty} \mathbb{P}[G_n \cup \mathbb{G}(n, p) \in \mathcal{P}] = 1$$

holds for every sequence  $\{G_n\}_{n \in \mathbb{N}}$  satisfying  $G_n \in \mathcal{G}_{d,n}$  for every  $n \in \mathbb{N}$ . We say that  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p)$  a.a.s. does not satisfy  $\mathcal{P}$ , if

$$\lim_{n \rightarrow \infty} \mathbb{P}[G_n \cup \mathbb{G}(n, p) \in \mathcal{P}] = 0$$

holds for at least one sequence  $\{G_n\}_{n \in \mathbb{N}}$  satisfying  $G_n \in \mathcal{G}_{d,n}$  for every  $n \in \mathbb{N}$ .

A sequence  $\hat{p} := \hat{p}(n)$  is said to form a *threshold* for the property  $\mathcal{P}$  in the perturbed model, if  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p)$  a.a.s. satisfies  $\mathcal{P}$  whenever  $p = \omega(\hat{p})$ , and if  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p)$  a.a.s. does not satisfy  $\mathcal{P}$  whenever  $p = o(\hat{p})$ .

Throughout, we suppress this sequence-based terminology and write more concisely that  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p)$  a.a.s. satisfies (or does not) a certain property. In particular, for a fixed graph  $H$ , we say that  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} H$  holds a.a.s. if the aforementioned anti-Ramsey property is upheld a.a.s. by every sequence of graphs in  $\mathcal{G}_{d,n}$ . We say that  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} H$  holds a.a.s. if there exists a sequence of graphs in  $\mathcal{G}_{d,n}$  for which the property fails asymptotically almost surely.

The following is an *abridged* formulation of the main result of the our companion paper [1].

**Theorem 1.1.** [1, Proposition 5.1] *Let a real number  $0 < d \leq 1/2$  and an integer  $\ell \geq 5$  be given. Then, the property  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_\ell$  holds a.a.s. whenever  $p := p(n) = \omega(n^{-1/m_2(K_{\lceil \ell/2 \rceil})})$ .*

Theorem 1.1 in conjunction with the aforementioned results of [21, 30] assert that  $n^{-1/m_2(K_{\lceil \ell/2 \rceil})}$  is the threshold for the property  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_\ell$ , whenever  $\ell \geq 9$ . It follows from another result of [1], regarding odd cycles, that  $n^{-2}$  is the threshold for the property  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_3$ .

## 1.1 Our results

Theorem 1.1 does not apply to  $\ell = 4$ . Moreover, while it does provide an upper bound on the threshold for the property  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_\ell$  for every  $5 \leq \ell \leq 8$ , a matching lower bound is not known to hold. For  $4 \leq \ell \leq 7$ , it turns out that  $n^{-1/m_2(K_{\lceil \ell/2 \rceil})}$  is not the threshold of the corresponding property; indeed the threshold deviates downwards quite significantly from this function. Our first main result determines the threshold for the associated properties when  $\ell \in \{4, 5, 7\}$ .

**Theorem 1.2.** *Let  $0 < d \leq 1/2$  be given.*

1. *The threshold for the property  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_4$  is  $n^{-5/4}$ .*
2. *The threshold for the property  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_5$  is  $n^{-1}$ .*

3. The threshold for the property  $\mathcal{G}_{d,n} \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_7$  is  $n^{-7/15}$ .

For  $K_6$ , we prove the following.

**Theorem 1.3.** *Let  $0 < d \leq 1/2$  be given.*

1. The property  $\mathcal{G}_{d,n} \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_6$  holds a.a.s. whenever  $p = \omega(n^{-2/3})$ .

2. For every  $\varepsilon > 0$ , the property  $\mathcal{G}_{d,n} \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_6$  holds a.a.s., whenever  $p := p(n) = n^{-(2/3+\varepsilon)}$ .

For  $K_8$ , Theorem 1.1 asserts that  $\mathcal{G}_{d,n} \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_8$  holds a.a.s. whenever  $p := p(n) = \omega(n^{-2/5}) = \omega(n^{-1/m_2(K_4)})$ . We prove the following.

**Theorem 1.4.** *For every  $0 < d \leq 1/2$  and  $\varepsilon > 0$ , the property  $\mathcal{G}_{d,n} \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_8$  holds a.a.s., whenever  $p := p(n) = n^{-(2/5+\varepsilon)}$ .*

The proof of Theorem 1.1 in [1] relies heavily on the so-called KLR-theorem [11, Theorem 1.6(i)]; the proofs of all the results stated above, employ entirely different approaches. Indeed, more refinement and control are required in order to handle *small* cliques.

The rest of the paper is organized as follows. In Section 2 we mention some preliminaries and useful observations. We consider the thresholds of the properties  $G \cup \mathcal{G}_{d,n} \xrightarrow{\text{rbw}} K_\ell$  for  $\ell \in \{4, 5, 6, 7, 8\}$  in Sections 3, 4, 5, 6, and 7, respectively.

## 2 Preliminaries

Throughout, in the proofs of the 1-statements we make repeated (standard) appeals to the so-called *dense regularity lemma* [38] (see also [22]). For a bipartite graph  $G := (U \sqcup W, E)$  and two sets  $U' \subseteq U$  and  $W' \subseteq W$ , write  $d_G(U', W') := \frac{e_G(U', W')}{|U'| |W'|}$  for the edge-density of the induced subgraph  $G[U', W']$ . The graph  $G$  is called  $\varepsilon$ -regular if

$$|d_G(U', W') - d_G(U, W)| < \varepsilon$$

holds whenever  $U' \subseteq U$  and  $W' \subseteq W$  satisfy  $|U'| \geq \varepsilon|U|$  and  $|W'| \geq \varepsilon|W|$ .

Given a sequence  $f := f(n)$  and constants  $\varepsilon_1, \dots, \varepsilon_k > 0$  independent of  $n$ , we write  $\Omega_{\varepsilon_1, \dots, \varepsilon_k}(f)$ ,  $\Theta_{\varepsilon_1, \dots, \varepsilon_k}(f)$ , and  $O_{\varepsilon_1, \dots, \varepsilon_k}(f)$  to mean that the constants which are implicit in the asymptotic notation depend on  $\varepsilon_1, \dots, \varepsilon_k$ . We will occasionally replace these constants with fixed graphs, writing  $O_L(f)$  to indicate that the implicit constants in the asymptotic notation depend on the graph  $L$ . In addition, given two constants  $\mu > 0$  and  $\nu > 0$  we write  $\mu \ll \nu$  to mean that, while  $\mu$  and  $\nu$  are fixed, they can be chosen so that  $\mu$  is arbitrarily smaller than  $\nu$ .

### 2.1 Sparse bipartite graphs

For two vertex disjoint graphs  $L$  and  $R$ , let  $K_{L,R}$  denote the graph  $(V(L) \sqcup V(R), F)$ , where

$$F := E(L) \sqcup E(R) \sqcup \{\ell r : \ell \in V(L), r \in V(R)\}.$$

(This is known as the *join of  $L$  and  $R$* .) In the special case that  $e(R) = 0$ , we write  $K_{L,v(R)}$  instead; further still, if in addition  $L$  is complete, then we write  $\widehat{K}_{v(L),v(R)}$ . We denote by  $\tilde{L}$  and  $\tilde{R}$  the realisations of  $L$  and  $R$  in  $K_{L,R}$ .

Let  $G$  be a graph and let  $K \subseteq G$  be a subgraph of  $G$ . Let  $\psi$  be a proper edge-colouring of  $G$ . If  $K$  appears rainbow under  $\psi$ , then  $K$  is said to be  $\psi$ -rainbow. A vertex  $x$  found in the *common neighbourhood* of the members of  $V(K)$ , i.e.,

$$x \in N_G(K) := \left\{ y \in V(G) : y \in \bigcap_{k \in V(K)} N_G(k) \right\},$$

is said to be of *interest to  $K$  with respect to  $\psi$*  if

$$\psi(K) \cap \{\psi(xk) : k \in V(K)\} = \emptyset.$$

If, in addition,  $K$  is  $\psi$ -rainbow, then the above definition stipulates that  $G[V(K) \cup \{x\}]$  is  $\psi$ -rainbow (though, perhaps unintuitively, we make use of the more general definition). A set  $X \subseteq V(G)$  whose members are *all* of interest to  $K$  with respect to  $\psi$ , is said to be *compatible with  $K$  with respect to  $\psi$*  provided that the sets  $\{\psi(xk) : k \in V(K)\}$  are pairwise disjoint. If, in addition,  $K$  is known to be  $\psi$ -rainbow, then the latter definition stipulates that  $G[V(K) \cup G[V(K), X]]$  is  $\psi$ -rainbow.

**Observation 2.1.** *Let  $L$  be a fixed graph and let  $n$  be sufficiently large. Every proper edge-colouring  $\psi$  of  $K_{L,n}$  admits a subset  $I_\psi := I_\psi(\tilde{L}) \subseteq V(\tilde{R})$ , satisfying  $|I_\psi| = n - O_L(1)$ , such that all of its members are of interest to  $\tilde{L}$  with respect to  $\psi$ .*

**Proof.** Being proper, the colour classes of  $\psi$  define (pairwise edge-disjoint) matchings in  $K_{L,n}$ . Hence, the number of common neighbours of  $V(\tilde{L})$  that are incident with at least one of the colours present in the set  $\psi(\tilde{L})$  is at most  $e(L)(v(L) - 2) = O_L(1)$ . The claim follows.  $\square$

**Observation 2.2.** *Let a graph  $L$  be fixed and let  $n$  be sufficiently large. Every proper edge-colouring  $\psi$  of  $H := K_{L,n}$  admits a set  $C_\psi := C_\psi(\tilde{L}) \subseteq V(\tilde{R})$ , satisfying  $|C_\psi| = \Omega_L(n)$ , that is compatible with  $\tilde{L}$  with respect to  $\psi$ .*

**Proof.** Fix a proper edge-colouring  $\psi$  of  $H$ . Let  $I_\psi = I_\psi(\tilde{L})$  be the set whose existence is ensured by Observation 2.1 and let  $\{u_1, \dots, u_t\}$  be an arbitrary ordering of its elements; note that  $t = \Omega_L(n)$  holds by Observation 2.1. The set  $C_\psi$  is constructed recursively as follows. Initially, we set  $C_\psi = \{u_1\}$  and proceed to iterate over  $I_\psi$  according to the ordering of its elements fixed above, making a decision for each member considered whether or not to include it in the set  $C_\psi$ .

Suppose that for some  $1 \leq j \leq t - 1$ , the decision on whether or not to include  $u_i$  in  $C_\psi$  has been made for every  $1 \leq i \leq j$ , and that the current set  $C_\psi$  is compatible with  $\tilde{L}$  with respect to  $\psi$ ; this trivially holds for  $j = 1$ . Add  $u_{j+1}$  to  $C_\psi$  if and only if  $C_\psi \cup \{u_{j+1}\}$  is compatible with  $\tilde{L}$  with respect to  $\psi$ . Since  $\psi$  is proper, each vertex added to  $C_\psi$  *disqualifies* at most  $v(L)(v(L) - 1)$  vertices in  $I_\psi(\tilde{L})$  from being added in subsequent rounds, as each of the  $v(L)$  colours appearing on the edges incident with that vertex forms a matching of size at most  $v(L)$ . Hence, at least  $n/O_L(1)$  vertex-additions are performed throughout the above process and the claim follows.  $\square$

**Remark 2.3.** *Observations 2.1 and 2.2 can be applied to a set of vertices  $X$  in a graph  $G$  and its common neighbourhood  $N_G(X)$ , by taking  $L \cong K_{|X|}$  in these observations. We make use of this fact in Section 5.1.*

### 3 Rainbow copies of $K_4$

In this section we prove the first part of Theorem 1.2 asserting that the threshold for the property  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_4$  is  $n^{-5/4}$ .

#### 3.1 1-statement

In this section we prove that for every  $d > 0$ , the property  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_4$  holds a.a.s. whenever  $p := p(n) = \omega(n^{-5/4})$ . We commence with the following observation.

**Observation 3.1.**  $K_{K_{1,3}, K_{1,4}} \xrightarrow{\text{rbw}} K_4$ .

**Proof.** Fix an arbitrary proper colouring  $\psi$  of the edges of  $K_{L,R}$ , where  $L \cong K_{1,3}$  and  $R \cong K_{1,4}$ . Let  $e \in E(R)$  be an edge for which  $\psi(e) \notin \{\psi(e') : e' \in E(L)\}$  holds. It is now straightforward to verify that the graph  $K_{L,e}$  contains a copy of  $K_4$  which is rainbow under  $\psi$ .  $\square$

Observation 3.1 reduces the 1-statement at hand to that of determining the threshold for the property  $K_{K_{1,3}, K_{1,4}} \subseteq \mathcal{G}_{d,n} \cup \mathbb{G}(n, p)$ . The threshold for the latter property has been determined long ago by Krivelevich, Sudakov, and Tetali [26]. In this regard, we require only the 1-statement associated with their result.

For a graph  $J$ , the quantity

$$m_1(J) := \max\{e(J')/v(J') : J' \subseteq J, v(J') > 0\}$$

is referred to as the *maximum density* of  $J$ . The *maximum bipartition density*<sup>1</sup> of  $J$  is given by

$$m_{(2)}(J) := \min_{V(J)=V_1 \cup V_2} \max\{m_1(J[V_1]), m_1(J[V_2])\}.$$

**Theorem 3.2.** ([26, Theorem 2.1] – abridged) *For every real  $d > 0$  and every graph  $J$ , the perturbed graph  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p)$  a.a.s. contains a copy of  $J$ , whenever  $p := p(n) = \omega(n^{-1/m_{(2)}(J)})$ .*

We are now ready to prove the 1-statement associated with the emergence of rainbow copies of  $K_4$ .

**Proof of the 1-statement for  $K_4$ .** Let  $d > 0$  and  $p := p(n) = \omega(n^{-5/4})$  be given. Observe that

$$m_{(2)}(K_{K_{1,3}, K_{1,4}}) = \max\{m_1(K_{1,3}), m_1(K_{1,4})\} = \max\{3/4, 4/5\} = 4/5.$$

Hence,  $K_{K_{1,3}, K_{1,4}} \subseteq \mathcal{G}_{d,n} \cup \mathbb{G}(n, p)$  holds a.a.s. by Theorem 3.2. The claim now follows by Observation 3.1.  $\square$

---

<sup>1</sup>Note that a more general quantity, namely  $m_{(r)}(J)$  for some prescribed  $r \geq 2$ , is defined in [26]. We only require the quantity corresponding to the case  $r = 2$ .

### 3.2 0-statement

Let  $G := (U \cup W, E) \cong K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$  and let  $p := p(n) = o(n^{-5/4})$ . We prove that a.a.s.  $G \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_4$  holds, by describing a proper colouring of the edges of  $G \cup G_{n,p}$  admitting no rainbow  $K_4$ . With  $p$  squarely below the threshold for the emergence of  $K_3$  in  $\mathbb{G}(n, p)$  (see, e.g., [18, Theorem 3.4]), the random perturbation  $\mathbb{G}(n, p)$  itself is a.a.s. triangle-free. Consequently,  $G \cup \mathbb{G}(n, p)$  a.a.s. has the property that all its copies of  $K_4$  are comprised of a copy of  $C_4$ , present in  $G$ , and two additional edges brought on by the perturbation  $\mathbb{G}(n, p)$  such that one is spanned by  $U$  and the other by  $W$ .

With  $p$  being below the threshold for the emergence of any connected graph on five vertices in  $\mathbb{G}(n, p)$  (see, e.g., [18, Theorem 3.4]), it follows that a.a.s. the edges of the perturbation are captured through a collection of vertex-disjoint copies of  $K_2$ ,  $P_3$ ,  $K_{1,3}$ , and  $P_4$ , where  $P_i$  is the path on  $i$  vertices. Let  $G' \sim \mathbb{G}(n, p)$  having this component structure be fixed and let  $\Gamma = G \cup G'$ . Then, every copy of  $K_4$  in  $\Gamma$  is found within some copy of  $K_{L,R}$ , with  $L, R \in \{K_2, P_3, K_{1,3}, P_4\}$  and such that  $V(L) \subseteq U$  and  $V(R) \subseteq W$ .

Let  $L_1, \dots, L_s$  and  $R_1, \dots, R_t$  be arbitrary enumerations of the connected components of  $\Gamma[U]$  and  $\Gamma[W]$ , respectively; so  $L_i$  and  $R_j$  are copies of one of  $K_1, K_2, P_3, K_{1,3}, P_4$  for every  $i \in [s]$  and  $j \in [t]$ . In what follows, we define a colouring of  $\Gamma$  in which all of the aforementioned components appearing in  $\Gamma[U]$  and  $\Gamma[W]$  are coloured using the colours 1, 2, and 3. For each pair  $(i, j)$  we assign a set  $A_{i,j}$  of  $|L_i| \cdot |R_j|$  colours to be used on the edges from  $R_i$  to  $L_j$  (we may repeat colours, thus not using all of the colours in  $A_{i,j}$ ), such that the sets  $A_{i,j}$  are pairwise disjoint and do not intersect  $\{1, 2, 3\}$ . We obtain a proper edge-colouring of  $\Gamma$  as follows.

**(A1)** Colour the edges of each connected component  $C \in \{L_1, \dots, L_s, R_1, \dots, R_t\}$  as follows.

- (a) If  $C$  is a single vertex, there is nothing to colour.
- (b) If  $C \cong K_2$ , colour its edge using the colour 1.
- (c) If  $C \cong P_3$ , colour it properly using the colours 1, 2.
- (d) If  $C \cong K_{1,3}$ , colour it properly using the colours 1, 2, 3.
- (e) If  $C \cong P_4$ , colour it properly using the colours 1, 2, 3 such that all three colours are used and the colour 2 is used for the middle edge.

**(A2)** Given any  $1 \leq i \leq s$  and  $1 \leq j \leq t$ , colour the edges of  $G$  connecting  $L_i$  and  $R_j$  properly, using colours from the set  $A_{i,j}$ , such that the corresponding copy of  $K_{L_i, R_j}$  admits no rainbow copy of  $K_4$ . (The validity of this step is verified below.)

It is evident that the proposed colouring, if exists, is proper and admits no rainbow copy of  $K_4$ . Proving that the desired colouring exists, can be done by a fairly straightforward yet somewhat tedious case analysis. It suffices to describe a colouring  $\psi_{ij} : V(L_i) \times V(R_j) \rightarrow A_{i,j}$  for every  $1 \leq i \leq s$  and  $1 \leq j \leq t$ , such that the following holds. Let  $\varphi_{ij}$  be the colouring of the edges of  $K_{L_i, R_j}$  under which the edges of  $V(L_i) \times V(R_j)$  are coloured as in  $\psi_{ij}$  and the edges of  $V(L_i) \times V(R_j)$  are coloured as in Item **(A1)** above. Then, for every  $1 \leq i \leq s$  and  $1 \leq j \leq t$ , the colouring  $\varphi_{ij}$  is proper and no copy of  $K_4$  in  $K_{L_i, R_j}$  is rainbow under  $\varphi_{ij}$ .

It thus suffices to describe such a colouring of  $V(L) \times V(R)$  for any  $L, R \in \{K_1, K_2, P_3, K_{1,3}, P_4\}$ . Observe that  $K_{1,3}$  and  $P_4$  contain  $K_1$ ,  $K_2$  and  $P_3$ , where the edges of all five graphs are coloured

per articles **(A1)** and **(A2)** specified above. Therefore, the desired colouring for  $K_{L,R}$ , where  $L, R \in \{K_{1,3}, P_4\}$ , would yield the desired colouring for  $K_{L',R'}$  for every  $L', R' \in \{K_1, K_2, P_3, K_{1,3}, P_4\}$ . Up to symmetry, we are thus left with only three cases to consider.

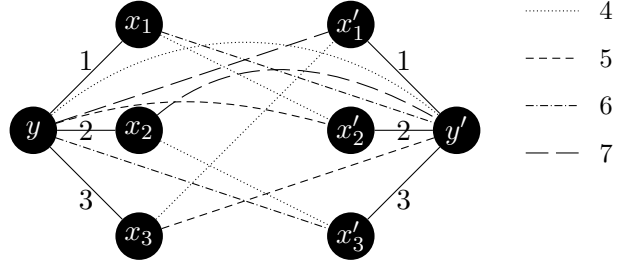
1.  $L \cong R \cong K_{1,3}$ . Let  $\{y, x_1, x_2, x_3\}$  and  $\{y', x'_1, x'_2, x'_3\}$  be the vertices of  $L$  and  $R$ , respectively, with  $y$  and  $y'$  being the vertices of degree 3, and  $yx_i$  and  $y'x'_i$  being the edges of colour  $i$ . Define the colouring  $\psi$  of  $V(L) \times V(R)$  as follows.

$$\psi(\{x_1x'_2, yy', x_2x'_3, x_3x'_1\}) = \{4\}$$

$$\psi(\{yx'_2, x_3y'\}) = \{5\}$$

$$\psi(\{x_1y', yx'_3\}) = \{6\}$$

$$\psi(\{x_2y', yx'_1\}) = \{7\}.$$



To complete the definition of  $\psi$ , colour each remaining edge using a new unique colour.

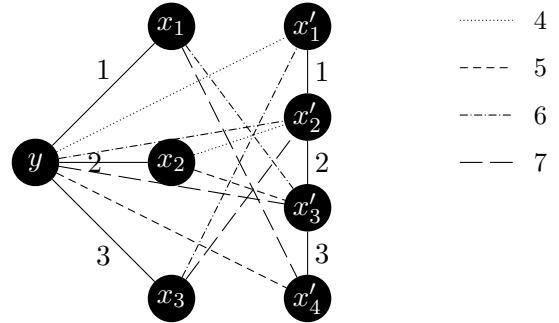
2.  $L \cong K_{1,3}$  and  $R \cong P_4$ . Let  $\{y, x_1, x_2, x_3\}$  be the vertices of  $L$ , with  $y$  being the vertex of degree 3 and  $yx_i$  having colour  $i$  for  $i \in [3]$ ; and let  $\{x'_1, x'_2, x'_3, x'_4\}$  be the vertices of  $R$  giving rise to the path  $x'_1x'_2x'_3x'_4$  with  $x'_1x'_2$  coloured 1. Define the colouring  $\psi$  of  $V(L) \times V(R)$  as follows.

$$\psi(\{yx'_1, x_2x'_2\}) = \{4\}$$

$$\psi(\{yx'_4, x_2x'_3\}) = \{5\}$$

$$\psi(\{x_1x'_3, yx'_2, x_3x'_1\}) = \{6\}$$

$$\psi(\{x_1x'_4, yx'_3, x_3x'_2\}) = \{7\}$$

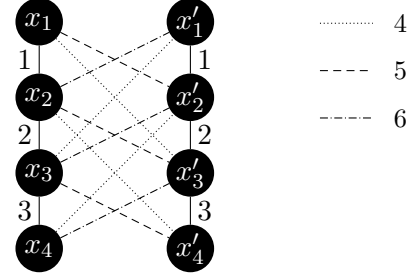


To complete the definition of  $\psi$ , colour each remaining edge using a new unique colour.

3.  $L \cong R \cong P_4$ . Let  $\{x_1, x_2, x_3, x_4\}$  and  $\{x'_1, x'_2, x'_3, x'_4\}$  be the vertices of  $L$  and  $R$ , respectively, giving rise to the paths  $x_1x_2x_3x_4$  and  $x'_1x'_2x'_3x'_4$ , with  $x_1x_2$  and  $x'_1x'_2$  coloured 1. Define the colouring  $\psi$  of  $V(L) \times V(R)$  as follows.



$$\begin{aligned}\psi(\{x_1x'_3, x_2x'_4, x_3x'_1, x_4x'_2\}) &= \{4\} \\ \psi(\{x_1x'_2, x_2x'_3, x_3x'_4\}) &= \{5\} \\ \psi(\{x_2x'_1, x_3x'_2, x_4x'_3\}) &= \{6\}.\end{aligned}$$



To complete the definition of  $\psi$ , colour each remaining edge using a new unique colour.

## 4 Rainbow copies of $K_5$

In this section we prove the second part of Theorem 1.2 asserting that the threshold for the property  $\mathcal{G}_{d,n} \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_5$  is  $n^{-1}$ . To see the 0-statement, fix some  $d \leq 1/2$  and let  $G$  be a bipartite graph on  $n$  vertices with density  $d$ , and let  $p = o(1/n)$ . Since  $G$  is bipartite, any copy of  $K_5$  in  $\Gamma \sim G \cup \mathbb{G}(n,p)$  must contain some triangle of  $\mathbb{G}(n,p)$ . However,  $\mathbb{G}(n,p)$  is a.a.s. triangle-free whenever  $p = o(1/n)$ . In particular, a.a.s. no edge-colouring of  $\Gamma$  can yield a rainbow  $K_5$ .

Proceeding to the 1-statement, let  $d > 0$  be fixed, let  $G \in \mathcal{G}_{d,n}$  be given, and let  $p := p(n) = \omega(1/n)$  be set, where throughout this section we assume that  $n$  is a sufficiently large integer. By a standard application of the (dense) regularity lemma [38] (see also [22]), we may assume that  $G = (U \cup W, E)$  is an  $\varepsilon$ -regular bipartite graph of edge-density  $d' \gg \varepsilon > 0$  satisfying  $|U| = |W| = m = \Theta_{d',\varepsilon}(n)$ , where  $\varepsilon$  and  $d'$  are some fixed constants.

Let  $\tilde{K}_{3,5} = K_{K_3, K_{1,4}}$  and let  $\hat{K}_{3,5}$  be the join of a triangle and an independent set of size 5. The following claim captures the principal property we require  $G \cup \mathbb{G}(n,p)$  to satisfy.

**Claim 4.1.** *Asymptotically almost surely any proper edge-colouring  $\psi$  of  $G \cup \mathbb{G}(n,p)$  admits a copy of  $\tilde{K}_{3,5}$  with its unique subgraph isomorphic to  $\hat{K}_{3,5}$  being rainbow under  $\psi$ .*

Prior to proving Claim 4.1, we use it to derive the 1-statement for  $K_5$ . Fix a graph  $\Gamma \sim G \cup \mathbb{G}(n,p)$  satisfying the property described in Claim 4.1, and fix a proper colouring  $\psi$  of its edges. Let  $K \subseteq \Gamma$  be a copy of  $\tilde{K}_{3,5}$  with its unique subgraph isomorphic to  $\hat{K}_{3,5}$  being rainbow under  $\psi$ ; such a copy exists by Claim 4.1. Let  $V(\tilde{L}) = \{x_1, x_2, x_3\}$  and  $V(\tilde{R}) = \{y, z_1, z_2, z_3, z_4\}$ , where  $y$  is the central vertex of the star  $\tilde{R}$ . Since  $\psi$  is proper, there exists  $t \in [4]$  such that  $\psi(yz_t) \notin \{\psi(x_1x_2), \psi(x_1x_3), \psi(x_2x_3)\}$ . Using yet again the fact that  $\psi$  is proper, it follows that  $\psi(yz_t) \notin \psi(\{x_1, x_2, x_3\} \times \{y, z_t\})$ . We conclude that  $x_1, x_2, x_3, y, z_t$  induce a rainbow copy of  $K_5$ . It remains to prove Claim 4.1.

**Proof of Claim 4.1.** Write  $G \cup \mathbb{G}(n,p)$  as the union  $G \cup G_1 \cup G_2$ , where  $G_1 \sim (\mathbb{G}(n,p))[W] = \mathbb{G}(m,p)$  and  $G_2 \sim (\mathbb{G}(n,p))[U] = \mathbb{G}(m,p)$  (formally, there may also be random edges between  $U$  and  $W$ , but these are ignored). Let

$$\mathcal{T} := \left\{ \{x, y, z\} \in \binom{W}{3} : |N_G(\{x, y, z\})| = \Omega_{d',\varepsilon}(m) \right\}.$$

Then,  $|\mathcal{T}| = \Theta_{d',\varepsilon}(m^3)$ , owing to  $G$  being  $\varepsilon$ -regular with density  $d'$ .

By Claim A.4 (see Appendix A), a.a.s.  $G_1 \sim \mathbb{G}(m, p)$  admits a triangle which is supported on a member of  $\mathcal{T}$ . Let such a graph  $G_1$  be fixed and let  $K \subseteq G \cup G_1$  be the aforementioned copy of  $K_3$ ; set  $N = N_G(V(K))$ , so  $|N| = \Omega(m)$ . By Claim A.5 (see Appendix A),  $G_2 \sim \mathbb{G}(m, p)$  a.a.s. has the property that every linear subset of its vertices spans a copy of  $K_{1,4}$ . Fix a graph  $G_2$  satisfying this property, and let  $\psi$  be a proper edge-colouring of the now fully determined graph  $\Gamma := G \cup G_1 \cup G_2$ . By Observation 2.2, there is a set  $C_\psi \subseteq N$  of size  $\Omega(|N|) = \Omega(m)$  which is compatible with  $K$  with respect to  $\psi$ . Then, owing to the property specified above for  $G_2$ , the graph  $\Gamma[C_\psi]$  spans a copy of  $K_{1,4}$ , thus concluding the proof of Claim 4.1.  $\square$

## 5 Rainbow copies of $K_6$

In this section we prove Theorem 1.3.

### 5.1 1-statement

In this section we prove the first part of Theorem 1.3, which asserts that for every (fixed)  $d > 0$ , the property  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_6$  holds a.a.s., whenever  $p = \omega(n^{-2/3})$ . Let  $d > 0$  be fixed, let  $G \in \mathcal{G}_{d,n}$  be given, and let  $p := p(n) = \omega(n^{-2/3})$  be set. Throughout this section, we assume that  $n$  is a sufficiently large integer.

Let  $\xi := \xi(d) > 0$  be an arbitrarily small yet fixed constant. By a standard application of the (dense) regularity lemma [38] (see also [22]), we may assume that  $G = (U \cup W, E)$  is an  $\varepsilon$ -regular bipartite graph of edge-density  $d'$  satisfying  $|U| = |W| = m = \Theta_{d', \varepsilon}(n)$ , for some fixed constants  $\varepsilon > 0$  and  $d' > 0$  such that  $\varepsilon \ll \xi$  and  $\varepsilon \ll d'$ . Let

$$\mathcal{Z} := \left\{ X \in \binom{W}{217} : |N_G(X)| = \Omega_{d', \varepsilon}(m) \right\}.$$

Then, owing to  $G$  being  $\varepsilon$ -regular with edge-density  $d'$  and to our assumption that  $\varepsilon \ll \xi$ , the size of  $\mathcal{Z}$  is at least  $(1 - \xi) \binom{m}{217}$ .

Write  $G \cup \mathbb{G}(n, p)$  as the union  $G \cup G_1 \cup G_2$ , where  $G_1 \sim (\mathbb{G}(n, p)) [W] = \mathbb{G}(m, p)$  and  $G_2 \sim (\mathbb{G}(n, p)) [U] = \mathbb{G}(m, p)$  (we ignore random edges with one endpoint in  $U$  and the other in  $W$ ). Let  $R_7$  denote the graph obtained from  $K_{1,2}$  by attaching two triangles to each of its edges, that is,  $V(R_7) = \{u_1, u_2, u_3, w_1, w_2, w_3, w_4\}$  and

$$E(R_7) = \{u_1u_2, u_2u_3, u_1w_1, u_1w_2, u_2w_1, u_2w_2, u_2w_3, u_2w_4, u_3w_3, u_3w_4\}; \quad (1)$$

(see Figure 1a). Let  $R$  be the vertex-disjoint union of 31 copies of  $R_7$ . By Claim A.6 (see Appendix A), a.a.s.  $G_1 \sim \mathbb{G}(m, p)$  admits a copy of  $R$  supported on a member of  $\mathcal{Z}$ . Fix such a graph  $G_1$  and let  $X \in \mathcal{Z}$  be the vertex set of a copy of  $R$ .

Let

$$T_k := (\{x, v_1, \dots, v_{2k}\}, \{xv_i : 1 \leq i \leq 2k\}) \cup \{v_{2i-1}v_{2i} : 1 \leq i \leq k\} \quad (2)$$

obtained by *gluing*, so to speak,  $k$  edge-disjoint triangles along a single (central) vertex (see Figure 1b). By Claim A.8 (see Appendix A), a.a.s.  $G_2 \sim \mathbb{G}(m, p)$  satisfies the property that every linear subset of its vertices spans a copy of  $T_{10}$ . Let  $G_2 \sim \mathbb{G}(m, p)$  satisfying this property be fixed.

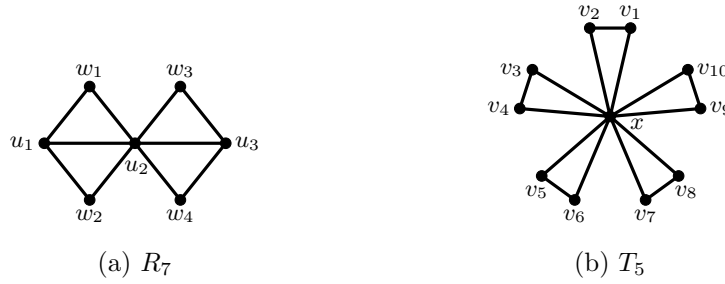


Figure 1:  $R_7$  and  $T_5$

It remains to prove that the fully determined graph  $\Gamma := G \cup G_1 \cup G_2$  satisfies the property  $\Gamma \xrightarrow{\text{rbw}} K_6$ . Let then a proper edge-colouring  $\psi$  of  $\Gamma$  be fixed. Observation 2.2 (applied with  $L = K_{|X|}$ ), asserts that every proper edge-colouring  $\psi$  of  $\Gamma[X] \cup \Gamma[X, N_G(X)]$  admits a set  $C_\psi$  of size  $\Omega(m)$  which is compatible with  $\Gamma[X]$  with respect to  $\psi$ . Owing to the aforementioned property satisfied by  $G_2$ , the graph  $\Gamma[C_\psi]$  spans a copy of  $T_{10}$ . The following claim aids in the construction of a  $\psi$ -rainbow copy of  $K_6$  in  $\Gamma[X \cup C_\psi]$ .

**Claim 5.1.** *Let  $\psi$  be a proper edge colouring of a vertex-disjoint union of  $R_7$  and  $T_{10}$ . Then there exist triangles  $Q_1 \subseteq T_{10}$  and  $Q_2 \subseteq R_7$  such that  $\psi(Q_1) \cap \psi(Q_2) = \emptyset$ .*

**Proof.** Let  $V(R_7) = \{u_1, u_2, u_3, w_1, w_2, w_3, w_4\}$  and let  $E(R_7)$  be defined as in (1). Let  $V(T_{10}) = \{x, v_1, \dots, v_{10}\}$  and let  $E(T_{10})$  be defined as in (2). Since  $\psi$  is proper, every colour appears at most once on an edge incident with  $u_2$ . Hence, we may assume without loss of generality that

$$\psi(u_1u_2) = 1, \psi(u_2w_1) = 2, \psi(u_2w_2) = 3, \psi(u_2w_3) = 4, \psi(u_2w_4) = 5, \psi(u_2u_3) = 6.$$

Write

$$\psi(u_1w_1) = \alpha, \psi(u_1w_2) = \beta,$$

and note that  $\alpha \neq \beta$ . Similarly, every colour appears at most once on an edge incident with  $x$ , and thus, without loss of generality, we may assume that

$$\{\psi(xv_{2i-1}), \psi(xv_{2i})\} \cap \{1, 2, 3, 4, 5, 6\} = \emptyset$$

holds for every  $i \in [4]$ . Write

$$\psi(v_{2i-1}v_{2i}) = \gamma_i$$

for every  $i \in [4]$ .

Suppose first that  $\gamma_i = \gamma_j$  for some distinct  $i, j \in [4]$ . Assume without loss of generality that  $\gamma_1 = \gamma_2 =: \gamma$ , and  $\gamma \notin [3]$  (the complementary case  $\gamma \in [3] \Rightarrow \gamma \notin \{4, 5, 6\}$  can be treated similarly). Moreover, as  $\alpha \neq \beta$ , without loss of generality  $\gamma \neq \alpha$ . Since  $\psi(xv_i)$  are distinct for  $i \in [4]$ , without loss of generality  $\alpha \notin \{\psi(xv_1), \psi(xv_2)\}$ . We may thus pick  $Q_1 = xv_1v_2$  and  $Q_2 = u_1u_2w_1$ .

Next, we may assume that  $\gamma_1, \gamma_2, \gamma_3, \gamma_4$  are distinct. Without loss of generality,  $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$  contains at most one of 1 and 2 (otherwise, it contains at most one of 1 and 3 or at most one of 4 and 6 and these cases can be treated similarly). It follows that at most one of the triangles  $xv_1v_2, xv_3v_4, xv_5v_6, xv_7v_8$  has an edge coloured 1 or 2. Moreover, at most two of these triangles contain an edge coloured  $\alpha$ . Thus, one of these triangles does not have edges coloured 1, 2, or  $\alpha$ ; take  $Q_1$  to be such a triangle and let  $Q_2 = u_1u_2w_1$ .  $\square$

With Claim 5.1 at hand, we prove that  $\Gamma[X \cup C_\psi]$  admits a copy of  $K_6$  which is rainbow under  $\psi$  as follows. Let  $Y_1, Y_2, \dots, Y_{31}$  be pairwise vertex-disjoint copies of  $R_7$  in  $\Gamma[X]$ . By Claim 5.1, for every  $i \in [31]$  there is a triangle  $Q'_i \subseteq Y_i$  and a triangle  $Q''_i \subseteq T_{10} \subseteq \Gamma[C_\psi]$  such that  $\psi(Q'_i) \cap \psi(Q''_i) = \emptyset$ . Hence, there are four pairwise vertex-disjoint triangles  $Q_1, Q_2, Q_3, Q_4 \subseteq \Gamma[X]$  and a triangle  $Q \subseteq \Gamma[C_\psi]$  and such that

$$\psi(Q_i) \cap \psi(Q) = \emptyset \text{ for every } i \in [4]. \quad (3)$$

Since  $\psi$  is proper, there exists an  $i \in [4]$  such that  $\psi(V(Q_i) \times V(Q)) \cap \psi(Q) = \emptyset$ . Since, by assumption,  $\Gamma[V(Q_i), V(Q)]$  is rainbow under  $\psi$  and  $\psi(Q_i) \cap \psi(V(Q_i) \times V(Q)) = \emptyset$  (as  $C_\psi$  is compatible with  $X$ ), it follows by (3) that  $\Gamma[V(Q_i) \cup V(Q)] \cong K_6$  is rainbow under  $\psi$ .

## 5.2 0-statement

In this section we prove the second part of Theorem 1.3 asserting that for every  $0 < d \leq 1/2$  and every  $\varepsilon > 0$ , a.a.s.  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_6$ , whenever  $p := p(n) = n^{-(2/3+\varepsilon)}$ . We deduce this from the following lemma which is the main result of this section.

**Lemma 5.2.** *For every  $\varepsilon > 0$  and  $p := p(n) = n^{-(2/3+\varepsilon)}$ , a.a.s.  $R \sim \mathbb{G}(n, p)$  contains four pairwise edge-disjoint matchings, namely  $M_0, M_1, M_2, M_3$ , such that the following holds.*

1.  $M_0$  and  $M_i$  are vertex-disjoint for every  $i \in [3]$ ; and
2. every triangle in  $R$  either contains an edge of  $M_0$  or contains edges from at least two of the matchings  $M_1, M_2, M_3$ .

Prior to proving Lemma 5.2, we use it to derive the aforementioned 0-statement for the emergence of rainbow copies of  $K_6$  in the perturbed model, i.e., the second part of Theorem 1.3.

**Proof of the 0-statement for  $K_6$  using Lemma 5.2.** Let  $\varepsilon > 0$  be fixed and let  $p := p(n) = n^{-(2/3+\varepsilon)}$ . Then,  $R \sim \mathbb{G}(n, p)$  is a.a.s.  $K_4$ -free (as the expected number of copies of  $K_4$  is  $O(n^4 p^6) = o(1)$ ). Let a  $K_4$ -free graph  $R$ , satisfying the assertion of Lemma 5.2, be fixed. We prove that  $\Gamma := G \cup R$  satisfies  $\Gamma \xrightarrow{\text{rbw}} K_6$ , where  $G \cong K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$  is a balanced complete bipartite graph with bipartition  $V(G) = A \cup B$ .

Define an assignment of colours to the edges of  $\Gamma$  as follows.

- (C1) Colour the edges of the matchings  $M_0$  and  $M_1$  (found in  $R$ ) red. Colour the edges of  $M_2$  blue; and colour the edges of  $M_3$  green.
- (C2) Given an unordered pair of edges  $xy \in M_0$  and  $zw \in M_2$  such that either  $\{x, y\} \subseteq A$  and  $\{z, w\} \subseteq B$ , or  $\{x, y\} \subseteq B$  and  $\{z, w\} \subseteq A$ , the members of  $E_G(\{x, y\}, \{z, w\})$  define a copy of  $C_4$  in  $G$ . Colour the members of  $E_G(\{x, y\}, \{z, w\})$  using two colours that are unique to the pair  $\{xy, zw\}$  (i.e., the colours have never been used before on any other edge coloured thus far) and in such a way that a proper edge colouring is defined over the copy of  $C_4$  arising from  $E_G(\{x, y\}, \{z, w\})$ .
- (C3) Colour the remaining uncoloured edges of  $\Gamma$  distinctively; each with its unique new colour.

Let  $\psi$  be the resulting colour assignment. First, observe that  $\psi$  is a well-defined edge-colouring of  $\Gamma$ . It is clear that each edge of  $\Gamma$  is assigned at least one colour. Owing to  $M_0, M_1, M_2, M_3$  being pairwise edge-disjoint and owing to  $M_0$  and  $M_2$  being vertex disjoint, no edge of  $\Gamma$  is assigned more than one colour. Next, note that  $\psi$  is a proper edge-colouring of  $\Gamma$ . For the edges coloured red, this holds as  $M_0$  and  $M_1$  are vertex-disjoint. For all other colours this is self-evident.

It remains to prove that no  $\psi$ -rainbow copy of  $K_6$  exists in  $\Gamma$ . To this end, let a copy of  $K_6$  in  $\Gamma$ , denoted  $K$ , be fixed. As  $R$  is  $K_4$ -free, the set  $V(K)$  is comprised of three vertices from  $A$  and the other three from  $B$ ; each such triple forming a triangle in  $R$ . Let  $T \subseteq R[A]$  and  $S \subseteq R[B]$  denote these two triangles. Since  $R$  satisfies the property described in Lemma 5.2, at least one of the following alternatives holds.

- (A1) Both  $T$  and  $S$  contain an edge from  $M_0$ .
- (A2) Both  $T$  and  $S$  contain edges from two of  $M_1, M_2, M_3$ .
- (A3)  $T$  contains an edge from  $M_0$  and  $S$  contains an edge from  $M_1$  (or vice versa).
- (A4)  $T$  contains an edge from  $M_0$  and  $S$  contains an edge from  $M_2$  and an edge from  $M_3$  (or vice versa).

If one of (A1), (A2), (A3) holds, then the triangles  $T$  and  $S$  have a colour (red, blue or green) in common. If (A4) holds, then there are two edges of the same colour between  $T$  and  $S$ . Either way, the  $K_6$ -copy  $K$  is not  $\psi$ -rainbow, as required.  $\square$

It remains to prove Lemma 5.2.

**Proof of Lemma 5.2.** Fix  $\varepsilon > 0$ . Given  $R \sim \mathbb{G}(n, p)$ , let  $R'$  be the subgraph of  $R$  which is the union of all triangles in  $R$ . It suffices to prove that a.a.s. the required matchings exist for every connected component of  $R'$ .

Given a connected component  $F$  of  $R'$ , let  $F_0, F_1, \dots, F_\ell$  be a (nested) sequence of connected subgraphs of  $F$  defined (recursively) as follows. The *starting* graph, namely  $F_0$ , is an arbitrary copy of  $K_3$  in  $F$ . Suppose that  $F_0, \dots, F_{i-1}$  have already been defined. If  $F_{i-1} = F$  or if  $i - 1 > 1/\varepsilon$ , stop and set  $\ell := i - 1$ . Otherwise, since  $F$  is connected, there is an edge  $e_i = x_i y_i \in E(F) \setminus E(F_{i-1})$  such that  $x_i \in V(F_{i-1})$ . Let  $z_i \in V(F)$  be a vertex such that the set  $\{x_i, y_i, z_i\}$  forms a triangle in  $F$  (such a  $z_i$  exists by the definition of  $R'$ ). Then, one of the following alternatives holds (up to relabelling).

- (a)  $x_i \in V(F_{i-1}), y_i, z_i \notin V(F_{i-1})$ .
- (b)  $x_i, z_i \in V(F_{i-1}), y_i \notin V(F_{i-1})$ , and  $x_i z_i \in E(F_{i-1})$ .
- (c)  $x_i, z_i \in V(F_{i-1}), y_i \notin V(F_{i-1})$ , and  $x_i z_i \notin E(F_{i-1})$ .
- (d)  $x_i, y_i, z_i \in V(F_{i-1})$ , and  $y_i z_i, x_i z_i \in E(F_{i-1})$ .
- (e)  $x_i, y_i, z_i \in V(F_{i-1})$ , and at least one of  $y_i z_i, x_i z_i$  is not in  $E(F_{i-1})$ .

Define  $F_i$  to be the subgraph of  $F$  with vertex set  $V(F_{i-1}) \cup \{y_i, z_i\}$  and edge set  $E(F_{i-1}) \cup \{x_i y_i, x_i z_i, y_i z_i\}$ .

Write  $\alpha, \beta, \gamma, \delta, \zeta$  to denote the number of values  $i$  for which the first, second, third, fourth, and fifth alternative held throughout the construction of the sequence, respectively. Then,

$$v := v(F_\ell) = 3 + 2\alpha + \beta + \gamma \text{ and } e := e(F_\ell) \geq 3 + 3\alpha + 2\beta + 3\gamma + \delta + 2\zeta.$$


Given values of  $\alpha, \beta, \gamma, \delta, \zeta$  whose sum is at most  $1/\varepsilon + 1$ , there are  $O_\varepsilon(1)$  possible configurations for the terminating graph  $F_\ell$ . For any single such configuration  $C$ , the expected number of copies of  $C$  in  $\mathbb{G}(n, p)$  is at most

$$O(n^v p^e) = O\left(n^{3+2\alpha+\beta+\gamma-(2/3+\varepsilon)\cdot(3+3\alpha+2\beta+3\gamma+\delta+2\zeta)}\right) = O\left(n^{1-\beta/3-\gamma-2\delta/3-4\zeta/3-\ell\varepsilon}\right),$$

where in the last equality we use the fact that  $e \geq \ell$ , entailing the term  $\varepsilon\ell$  appearing in the exponent. We may assume that  $1 - \beta/3 - \gamma - 2\delta/3 - 4\zeta/3 - \ell\varepsilon \geq 0$ , for otherwise there are no copies of  $C$  in  $\mathbb{G}$  a.a.s. across all of its possible configurations  $C$  with values  $\alpha, \beta, \gamma, \delta, \zeta$ , owing to Markov's inequality and the fact that the number of possible configurations is  $O_\varepsilon(1)$ . It follows that


$$\gamma = \zeta = 0, \quad \ell \leq 1/\varepsilon, \quad 0 \leq \beta \leq 2, \quad 0 \leq \delta \leq 1, \quad \text{and } \beta + 2\delta \leq 2.$$




The fact that  $\ell \leq 1/\varepsilon$  implies that, by definition, the sequence terminated due to  $F_\ell$  coinciding with  $F$  so that  $F_\ell = F$  holds.

We may assume, without loss of generality, that  $\delta = 0$ . Indeed, otherwise  $\delta = 1$  and thus  $\beta = 0$ . It then follows that the graph  is a subgraph of  $F$ . This in turn means that the sequence could have started with two steps of type (b), i.e., that  $\beta \geq 2$  and thus  $\delta = 0$ .

In what follows we construct the required matchings via a case analysis ranging over the three possible values of  $\beta$ .

**Case I:**  $\beta = 0$ . In this case all steps are of type (a). Take  $M_0$  to be a matching that consists of some edge in  $F_0$ , and the edges  $\{y_i z_i : i \in [\ell]\}$  and let  $M_1 = M_2 = M_3 = \emptyset$ . It is self-evident that, in this case,  $M_0$  is a matching meeting all triangles of  $F$ .

**Case II:**  $\beta = 1$ . In this case  is a subgraph of  $F$ . We may thus assume that the first step is of type (b), and all other steps are of type (a). Let  $M_0$  be the matching consisting of the edges  $x_1 z_1$  and  $\{y_i z_i : 2 \leq i \leq \ell\}$  and let  $M_1 = M_2 = M_3 = \emptyset$ . Then, again,  $M_0$  is a matching meeting all triangles of  $F$ .

**Case III:**  $\beta = 2$ . In this case  $F$  can be formed by making steps of type (a), starting with one of the graphs  or , or there are two edge-disjoint copies of . In the former case, one can verify that there exists a matching  $M_0$  meeting all triangles of  $F$ , by finding such a matching in the starting graph and extending it by adding the edges  $y_i z_i$ . In the latter case,  $F$  can be formed by making steps of type (a), starting with one of the families of graphs depicted in Figure 2.

For each of the first two families of starting graphs, namely of type I (see Figure 2a) and of type II (see Figure 2b), there is a matching  $M'_0$  meeting all of their triangles (see Figure 3);

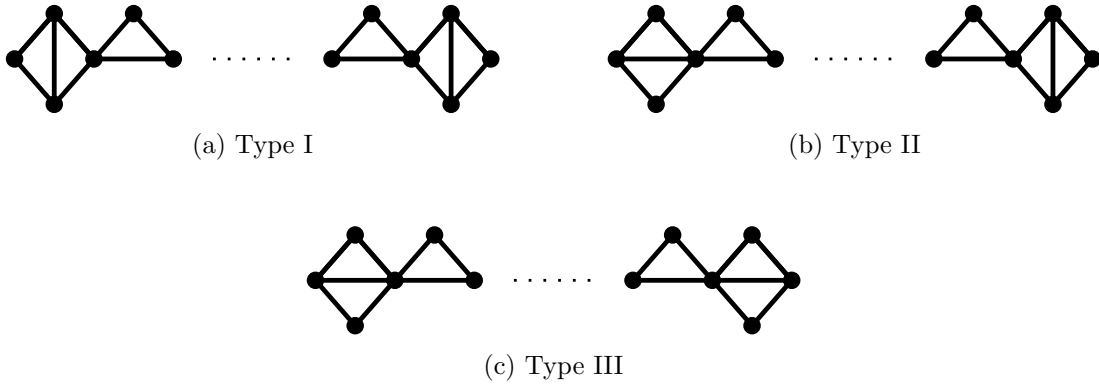


Figure 2: Three families of starting graphs.

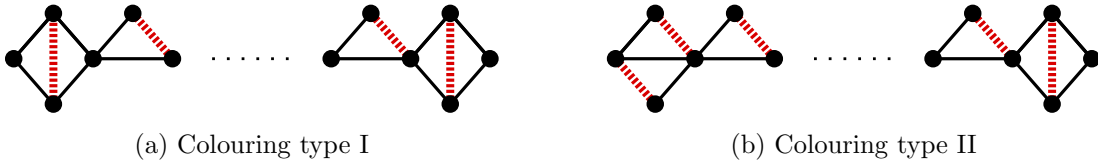


Figure 3: Colouring the first two starting graphs

and this matching can be extended into a matching  $M_0$  in  $F$  meeting all triangles of  $F$ , by adding the edges of the form  $y_i z_i$  defined in subsequent steps. As in previous cases, we set  $M_1 = M_2 = M_3 = \emptyset$ .

For the third family of starting graph, of type III (see Figure 2c), there are three edge-disjoint matchings  $M_1, M_2, M_3$  such that every triangle of the starting graph contains edges from at least two of these matchings (see Figure 4). In this case, set  $M_0$  to consist of the edges of the form  $y_i z_i$  defined in subsequent steps.

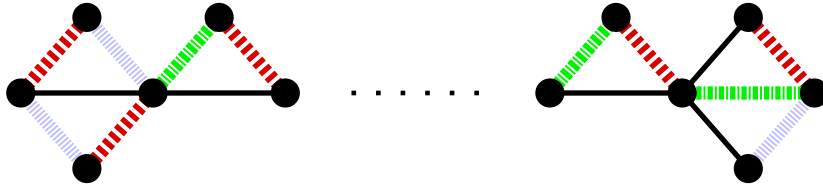


Figure 4: Colouring type III graphs

One may readily check that the matchings  $M_0, M_1, M_2, M_3$ , defined above, satisfy the properties stipulated in Lemma 5.2.  $\square$

## 6 Rainbow copies of $K_7$

In this section we prove the third part of Theorem 1.2. That is, we prove that the threshold for the property  $\mathcal{G}_{d,n} \cup \mathbb{G}(n,p) \xrightarrow{\text{rbw}} K_7$  is  $n^{-7/15}$ . To see the 0-statement, fix some  $d \leq 1/2$  and let  $G$  be a bipartite graph on  $n$  vertices with density  $d$ , and let  $p = o(n^{-7/15})$ . Since  $G$  is bipartite, any

rainbow copy of  $K_7$  in  $\Gamma \sim G \cup \mathbb{G}(n, p)$  must contain a rainbow copy of  $K_4$  in  $\mathbb{G}(n, p)$ . However, as proved in [21], a.a.s. the property  $\mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_4$  does not hold whenever  $p = o(n^{-7/15})$ .

Proceeding to the 1-statement, let  $d > 0$  be fixed, let  $G \in \mathcal{G}_{d,n}$  be given, let  $p := p(n) = \omega(n^{-7/15})$  be set, and let  $\xi := \xi(d) > 0$  be an arbitrarily small yet fixed constant. Throughout this section, we assume  $n$  is a sufficiently large integer. By a standard application of the (dense) regularity lemma [38] (see also [22]), we may assume that  $G = (U \cup W, E)$  is an  $\varepsilon$ -regular bipartite graph of edge-density  $d'$  satisfying  $|U| = |W| = m = \Theta_{d',\varepsilon}(n)$ , for some fixed constants  $\varepsilon > 0$  and  $d' > 0$  such that  $\varepsilon \ll \xi$  and  $\varepsilon \ll d'$ .

Let

$$\mathcal{Z} := \left\{ X \in \binom{W}{28} : |N_G(X)| = \Omega_{d',\varepsilon}(m) \right\}.$$

Then, owing to  $G$  being  $\varepsilon$ -regular with edge-density  $d'$  and to our assumption that  $\varepsilon \ll \xi$ , the size of  $\mathcal{Z}$  is at least  $(1-\xi)\binom{m}{28}$ . Write  $G \cup \mathbb{G}(n, p)$  as the union  $G \cup G_1 \cup G_2$ , where  $G_1 \sim (\mathbb{G}(n, p)) [W] = \mathbb{G}(m, p)$  and  $G_2 \sim (\mathbb{G}(n, p)) [U] = \mathbb{G}(m, p)$  (we ignore random edges with one endpoint in  $U$  and the other in  $W$ ).

Let  $H$  be the disjoint union of four copies of  $\widehat{K}_{3,4}$  (i.e., the join of a triangle and an independent set of size four; recall such definitions in Section 2.1), and let  $F$  be the graph obtained from  $K_{1,25}$  by attaching 49 triangles to each of its edges, where the vertex not in  $K_{1,25}$  is unique to each triangle. The copy of  $K_{1,25}$  giving rise to  $F$  is referred to as its *skeleton*.

By Claim A.9 (see Appendix A), a.a.s.  $G_1 \sim \mathbb{G}(m, p)$  admits a copy of  $H$  supported on a member of  $\mathcal{Z}$ . Fix such a graph  $G_1$ , let  $X \in \mathcal{Z}$  be the vertex set of a copy of  $H$  in  $G_1$ , and let  $N := N_G(X)$ ; note that  $|N| = \Omega_{d',\varepsilon}(m)$  holds by the definition of  $\mathcal{Z}$ . By Claim A.10 (see Appendix A), a.a.s.  $G_2 \sim \mathbb{G}(m, p)$  has the property that every subset of its vertices of linear size spans a copy of  $F$ . Let  $G_2 \sim \mathbb{G}(m, p)$  satisfying this property be fixed.

It remains to prove that the fully determined graph  $\Gamma := G \cup G_1 \cup G_2$  satisfies  $\Gamma \xrightarrow{\text{rbw}} K_7$ . To this end, fix a proper edge-colouring  $\psi$  of  $\Gamma$ . By Observation 2.2, there is a set  $C_\psi \subseteq N$  of size  $\Omega(|N|)$  which is compatible with  $X$  with respect to  $\psi$ , i.e.,  $\Gamma[X, C_\psi]$  is  $\psi$ -rainbow. Owing to the aforementioned property satisfied by  $G_2$ , there is a copy of  $F$  in  $\Gamma[C_\psi]$ ; denote this copy by  $F_\psi$  and write  $Y_\psi := V(F_\psi)$ .

It is easy to verify that  $\widehat{K}_{3,4} \xrightarrow{\text{rbw}} K_4$  (this was also observed in [21]). Consequently,  $X$  admits four pairwise vertex-disjoint  $\psi$ -rainbow copies of  $K_4$ ; denote their vertex sets by  $X_\psi^1, X_\psi^2, X_\psi^3, X_\psi^4$  and write  $X_\psi := X_\psi^1 \cup \dots \cup X_\psi^4$  and  $H_\psi = \Gamma[X_\psi^1] \cup \dots \cup \Gamma[X_\psi^4]$ .

In the remainder of the proof, we find a  $\psi$ -rainbow  $K_7$  in  $\Gamma[X_\psi \cup Y_\psi]$ . Observe that  $\Gamma[A \cup B] \cong K_7$  for every  $A \in \{X_\psi^1, \dots, X_\psi^4\}$  and  $B \subseteq V(Y_\psi)$  such that  $\Gamma[B] \cong K_3$ . Since  $\Gamma[A] \cup \Gamma[A, B]$  is  $\psi$ -rainbow for all such choices of  $A$  and  $B$ , if such a copy of  $K_7$  is not rainbow under  $\psi$ , then there exist edges  $e_A \in E_\Gamma(A) \cup E_\Gamma(A, B)$  and  $e_B \in E_\Gamma(B)$  such that  $\psi(e_A) = \psi(e_B)$ . Dealing with the case  $e_A \in E_\Gamma(A)$  first, we delete from  $F_\psi$  every edge whose colour under  $\psi$  appears in  $\psi(E(H_\psi))$ . Owing to  $\psi$  being proper, this entails the removal of at most 24 matchings from  $F_\psi$ . We claim that this does not destroy all of the triangles of  $F_\psi$ .

**Observation 6.1.** *The removal of any 24 matchings from  $F$  yields a graph which is not triangle-free.*

**Proof.** Let  $M_1, \dots, M_{24}$  be any 24 matchings in  $F$  and let  $F' = F \setminus (M_1 \cup \dots \cup M_{24})$ . At least one of



the edges of the skeleton of  $F$ , say  $e$ , is retained in  $F'$ . Observe that, for every  $i \in [24]$ , the matching  $M_i$  meets the edges of at most two of the triangles of  $F$  associated with  $e$ . Therefore, at least one of the 49 triangles associated with  $e$  in  $F$  remains intact in  $F'$ .  $\square$

Following Observation 6.1, let  $T \subseteq F_\psi$  be a triangle that has persisted the removal of all edges of  $F_\psi$  that were assigned a colour which appears in  $\psi(E(H_\psi))$ . It thus remains to take care of colour clashes between the edges of  $T$  and the edges connecting it to  $X_\psi$ . For every  $i \in [4]$ , let  $E_i = E_\Gamma(X_\psi^i, V(T))$ . Since  $\psi$  is proper, if  $\psi(E(T)) \cap \psi(\cup_{i=1}^4 E_i) \neq \emptyset$ , then there are two independent edges  $e \in E(T)$  and  $e' \in \cup_{i=1}^4 E_i$  such that  $\psi(e) = \psi(e')$ . Since  $T$  is a triangle and  $\psi$  is proper, there are at most three such pairs of edges. Consequently, there exists an index  $i^* \in [4]$  such that  $\psi(E(T)) \cap \psi(E_{i^*}) = \emptyset$ . Then,  $\Gamma[X_\psi^{i^*} \cup V(T)] \cong K_7$  is rainbow under  $\psi$ .

## 7 Rainbow copies of $K_8$

In this section, we prove Theorem 1.4. That is, we prove that given  $0 < d \leq 1/2$  and  $\varepsilon > 0$ , the property  $\mathcal{G}_{d,n} \cup \mathbb{G}(n, p) \xrightarrow{\text{rbw}} K_8$  holds a.a.s., whenever  $p := p(n) = n^{-(2/5+\varepsilon)}$ . The following implies Theorem 1.4.

**Proposition 7.1.** *Let  $\varepsilon > 0$  and let  $p = n^{-2/5-\varepsilon}$ . Then, a.a.s. the edges of  $\mathbb{G}(n, p)$  can be properly coloured so that all rainbow copies of  $K_4$  share at least one common colour.*

Prior to proving Proposition 7.1, we use it to deduce Theorem 1.4.

**Proof of Theorem 1.4 using Proposition 7.1.** Fix  $G \sim \mathbb{G}(n, p)$  satisfying the property specified in Proposition 7.1. Then  $G$  admits a proper edge-colouring  $\psi$  such that all copies of  $K_4$  in  $G$  which are rainbow under  $\psi$  contain an edge coloured, say, red. This further implies that  $\psi$  gives rise to no rainbow copy of  $K_5$ . Indeed, suppose the vertex set  $\{a, b, c, d, e\}$  induces a rainbow copy of  $K_5$ , then there is an edge of that copy, say  $ab$ , which is coloured red. Then, the vertex set  $\{b, c, d, e\}$  induces a rainbow copy of  $K_4$  without a red edge, a contradiction.

Given an  $n$ -vertex bipartite graph  $B$ , extend the edge-colouring  $\psi$  into a proper edge-colouring of  $G \cup B$  arbitrarily, and let  $\psi'$  denote the resulting colouring. Let  $K$  be a copy of  $K_8$  in  $G \cup B$ , and let  $K'$  and  $K''$  denote the intersections of  $K$  with the two parts of the bipartition of  $B$ . We may assume that both  $K'$  and  $K''$  are  $\psi$ -rainbow, for otherwise  $K$  is clearly not  $\psi'$ -rainbow. As  $\psi$  does not give rise to any rainbow copies of  $K_5$  in  $G$ , it follows that  $K', K'' \cong K_4$ . Then, while  $\psi$ -rainbow on their own,  $K'$  and  $K''$  have a colour in common and the proof follows.  $\square$

The remainder of this section is dedicated to the proof of Proposition 7.1. In Section 7.1 we introduce some useful terminology. In Section 7.2 we deduce Proposition 7.1 from the main result of this section, namely Lemma 7.3, stated below. In Section 7.3, we prove Lemma 7.3.

### 7.1 Stretched generating sequences and their properties

For a graph  $H$ , let  $\mathcal{K}_4(H)$  be the auxiliary graph whose vertices are the copies of  $K_4$  in  $H$ , with two such copies being adjacent if and only if they are not edge-disjoint. We say that  $H$  is  $K_4$ -connected if  $\mathcal{K}_4(H)$  is connected. Moreover, we say that  $H$  is  $K_4$ -covered if every edge of  $H$  lies in some copy

of  $K_4$ . Graphs  $H$  that are both  $K_4$ -connected and  $K_4$ -covered are called  $K_4$ -tilled. Such graphs can be *generated* through a (nested) sequence of connected subgraphs of  $H$ , namely

$$H_0 \cong K_4, H_1, \dots, H_r = H,$$

such that for every  $i \in [r]$ , the graph  $H_i$  can be obtained from  $H_{i-1}$  using one of the following steps.

**Standard steps.** Let  $z_i w_i \in E(H_{i-1})$  and let  $x_i, y_i \in V(H) \setminus V(H_{i-1})$  be distinct. Define  $H_i$  by setting

$$V(H_i) := V(H_{i-1}) \cup \{x_i, y_i\} \text{ and } E(H_i) := E(H_{i-1}) \cup \{x_i y_i, x_i z_i, x_i w_i, y_i z_i, y_i w_i\}.$$

**Vertex-steps.** Let  $y_i, z_i, w_i \in V(H_{i-1})$  be distinct vertices that span at least one edge of  $H_{i-1}$ . Let  $x_i \in V(H) \setminus V(H_{i-1})$ . Define  $H_i$  by setting

$$V(H_i) := V(H_{i-1}) \cup \{x_i\} \text{ and } E(H_i) := E(H_{i-1}) \cup \{x_i y_i, x_i z_i, x_i w_i, y_i z_i, y_i w_i, w_i z_i\}.$$

Such vertex-steps are further distinguished and are said to be *with* or *without* missing edges, according to whether or not one of the pairs  $\{y_i, z_i\}$ ,  $\{y_i, w_i\}$ , and  $\{z_i, w_i\}$  forms a non-edge of  $H_{i-1}$ , respectively.

**Edge-steps.** Let  $x_i, y_i, z_i, w_i \in V(H_{i-1})$  be distinct vertices that span between one and five edges. Define  $H_i$  by setting

$$V(H_i) = V(H_{i-1}) \text{ and } E(H_i) := E(H_{i-1}) \cup \{x_i y_i, x_i z_i, x_i w_i, y_i z_i, y_i w_i, w_i z_i\}.$$

Edge-steps adding  $m$  new edges are called  $m$ -*edge-steps*.

Given a sequence generating  $H$ , let  $\gamma$  denote the number of edges added throughout along edge-steps, and between existing vertices in vertex-steps with missing edges.

A  $K_4$ -tilled graph  $H$  may admit numerous generating sequences. Sequences generating  $H$  that

**(T1)** minimise  $\gamma$ , and

**(T2)** amongst generating sequences satisfying **(T1)**, maximise the length of the sequence  $r$ ,

are said to be *stretched*. Such sequences have the property that the addition of the missing edges alone in vertex-steps (with missing edges) does not yield a, so-called, new copy of  $K_4$ . For otherwise, one may split such a vertex-step into an edge-step followed by a vertex-step keeping  $\gamma$  unchanged, yet increasing the length of the sequence; contrary to its maximality stated in **(T2)**. Similarly, adding any proper subset of the set of edges added in some edge-step does not give rise to a new copy of  $K_4$ ; this would again contradict the maximality stated in **(T2)**.

The following claim facilitates our proof of Proposition 7.1. Its proof can be found in Appendix B.

**Claim 7.2.** *Let  $H$  be a  $K_4$ -tilled  $K_5$ -free graph, and let  $H_0 \cong K_4, H_1, \dots, H_r = H$  be a stretched sequence generating  $H$ . Suppose that the first edge-step in the sequence is a 1-edge-step that introduces the new edge  $xy$ , resulting in  $\{x, y, z, w\}$  forming a copy of  $K_4$ . Then,*

- (a)  $\{x, y, z, w\}$  is the sole new copy of  $K_4$  incurred through the addition of the edge  $xy$ .
- (b) The step introducing  $xy$  is preceded by at least one vertex-step with missing edges, or at least two vertex-steps with no missing edges.

## 7.2 Proof of Proposition 7.1

For a  $K_4$ -tiled graph  $H$  and a stretched sequence  $H_0 \cong K_4, H_1, \dots, H_r = H$  generating  $H$ , write  $\alpha$ ,  $\beta$ , and  $\gamma$  to denote the number of standard steps, vertex-steps, and edges added throughout the sequence along vertex or edge-steps connecting two existing non-adjacent vertices, respectively. Then,

$$v(H) = 4 + 2\alpha + \beta, \quad \text{and} \quad e(H) = 6 + 5\alpha + 3\beta + \gamma. \quad (4)$$

In particular,

$$e(H) \geq (5/2)v(H) - 4. \quad (5)$$

The parameter

$$\varphi(H) := 8 - 5v(H) + 2e(H) = 2\gamma + \beta$$

will arise naturally in various calculations, (see e.g. (6)). Note that, by (5),  $\varphi(H) \geq 0$  holds for every  $K_4$ -tiled graph; we will see below that a.a.s.  $\varphi(H) \leq 7$  holds for every  $K_4$ -tiled graph  $H$  in  $G \sim \mathbb{G}(n, p)$  with  $p = n^{-(2/5+\varepsilon)}$  (see Claim 7.5). A central ingredient in the proof of Proposition 7.1 is the following lemma, asserting the existence of certain proper edge-colourings of  $K_4$ -tiled graphs.

**Lemma 7.3.** *Let  $H$  be a  $K_4$ -tiled graph.*

- (i) *If  $\varphi(H) \leq 2$ , then  $H$  has a proper edge-colouring admitting no rainbow copies of  $K_4$ .*
- (ii) *If  $\varphi(H) \in \{3, 4, 5\}$ , then  $H$  admits a triangle  $T$  and a proper edge-colouring  $\psi$  such that all rainbow copies of  $K_4$  arising from  $\psi$  contain  $T$ .*
- (iii) *If  $\varphi(H) \in \{6, 7\}$ , then  $H$  admits a matching  $M$  of size at most 3 and a proper edge-colouring  $\psi$  such that all rainbow copies of  $K_4$  arising from  $\psi$  meet  $M$ .*

The proof of Lemma 7.3 is postponed to Section 7.3. The remainder of the current section is dedicated to the derivation of Proposition 7.1 from this lemma.

By a  $K_4$ -component of a graph  $G$ , we mean a maximal  $K_4$ -tiled subgraph of  $G$ . Observe that such components are by definition pairwise edge-disjoint (recall the definition of the auxiliary graph  $\mathcal{K}_4(G)$ ); yet they may have vertices in common.

The edges of a graph  $G$  can be decomposed into a collection  $\mathcal{H} := \mathcal{H}(G)$  of (pairwise edge-disjoint)  $K_4$ -components, and a set  $E$  of edges of  $G$  contained in no copy of  $K_4$  in  $G$ . The members of  $E$  will be of no interest to us. Owing to alternative (i) of Lemma 7.3,  $K_4$ -components  $H$  satisfying  $\varphi(H) \leq 2$  are of no threat to us, so to speak. It thus suffices to analyse the union of  $K_4$ -components  $H$  satisfying  $\varphi(H) \geq 3$ . Given a graph  $G$ , consider the graph  $G'$  which is the union of  $K_4$ -components  $H$  of  $G$ , satisfying  $\varphi(H) \geq 3$ , and let  $\mathcal{C} := \mathcal{C}(G)$  be the collection of connected components in  $G'$ .

Let  $\varepsilon > 0$  be given; note that we may assume that  $\varepsilon$  is arbitrarily small yet fixed. Set  $p := p(n) = n^{-(2/5+\varepsilon)}$ . Claims 7.4 to 7.9, stated below, collectively capture properties that are a.a.s. satisfied simultaneously by  $G \sim \mathbb{G}(n, p)$ . Roughly speaking, these properties collectively assert that  $K_4$ -components  $H$  of  $G$ , satisfying  $\varphi(H) \geq 3$ , admit a tree-like structure.

**Claim 7.4.** *Asymptotically almost surely  $G \sim \mathbb{G}(n, p)$  does not have  $K_4$ -tiled subgraphs on more than  $\lceil 1/\varepsilon \rceil$  vertices.*

**Proof.** Owing to (5), the expected number of  $k$ -vertex  $K_4$ -tiled subgraphs of  $G \sim \mathbb{G}(n, p)$  is at most

$$2^{k^2} \cdot n^k p^{(5/2)k-4} = 2^{k^2} \cdot n^{k-(2/5+\varepsilon)((5/2)k-4)} = 2^{k^2} \cdot n^{8/5+4\varepsilon-(5/2)\varepsilon k} \leq 2^{k^2} \cdot n^{2-(5/2)\varepsilon k}, \quad (6)$$

where for the sole inequality above we use the fact that  $\varepsilon$  is arbitrarily small yet fixed. Consequently, by Markov's inequality,  $G \sim \mathbb{G}(n, p)$  a.a.s. admits no  $k$ -vertex  $K_4$ -tiled subgraph with  $\lceil 1/\varepsilon \rceil \leq k \leq \lceil 1/\varepsilon \rceil + 1$ . As every  $K_4$ -tiled graph on at least  $\lceil 1/\varepsilon \rceil + 1$  vertices contains a  $K_4$ -tiled subgraph on either  $\lceil 1/\varepsilon \rceil$  or  $\lceil 1/\varepsilon \rceil + 1$  vertices, the claim follows.  $\square$

**Claim 7.5.** *Asymptotically almost surely  $\varphi(H) \leq 7$  (equivalently,  $5v(H) - 2e(H) \geq 1$ ) holds for every  $H$  which is a  $K_4$ -tiled subgraph of  $G \sim \mathbb{G}(n, p)$ .*

**Proof.** Let  $H$  be a  $K_4$ -tiled subgraph of  $G \sim \mathbb{G}(n, p)$ , satisfying  $5v(H) - 2e(H) \leq 0$ . Then, the expected number of copies of  $H$  in  $G$  is at most

$$n^{v(H)} p^{e(H)} = n^{v(H)-(2/5+\varepsilon)e(H)} = n^{\frac{1}{5} \cdot (5v(H)-2e(H))-\varepsilon \cdot e(H)} \leq n^{-6\varepsilon} = o(1), \quad (7)$$

where the above inequality holds since  $H$  contains a copy of  $K_4$  and thus  $e(H) \geq 6$ . This estimate, along with the fact that the number of graphs on at most  $\lceil 1/\varepsilon \rceil$  vertices has order of magnitude  $O_\varepsilon(1)$ , collectively imply that  $G \sim \mathbb{G}(n, p)$  a.a.s. has the property that all  $K_4$ -tiled subgraphs  $H$  of  $G$  on at most  $\lceil 1/\varepsilon \rceil$  vertices satisfy  $5v(H) - 2e(H) \geq 1$ . This property, together with Claim 7.4, completes the proof.  $\square$

**Claim 7.6.** *Asymptotically almost surely  $G \sim \mathbb{G}(n, p)$  does not have two edge-disjoint  $K_4$ -tiled subgraphs,  $H_1$  and  $H_2$ , that satisfy  $\varphi(H_i) \geq 3$  (equivalently,  $5v(H_i) - 2e(H_i) \leq 5$ ) for  $i \in [2]$ , and that have at least two vertices in common.*

**Proof.** Suppose that  $k := |V(H_1) \cap V(H_2)| \geq 2$  and set  $H := H_1 \cup H_2$ . Then,

$$v(H) = v(H_1) + v(H_2) - k \text{ and } e(H) = e(H_1) + e(H_2),$$

implying that

$$5v(H) - 2e(H) = (5v(H_1) - 2e(H_1)) + (5v(H_2) - 2e(H_2)) - 5k \leq 5 + 5 - 5k \stackrel{k \geq 2}{\leq} 0.$$

Following (7), the expected number of copies of  $H$  in  $G \sim \mathbb{G}(n, p)$  is at most

$$n^{\frac{1}{5} \cdot (5v(H)-2e(H))-\varepsilon \cdot e(H)} \leq n^{-\varepsilon \cdot e(H)} \leq n^{-6\varepsilon} = o(1).$$

The claim now follows by a similar argument to that seen after (7).  $\square$

The following claim precludes long path compositions of  $K_4$ -tiled graphs in  $\mathcal{C}$ .

**Claim 7.7.** *Asymptotically almost surely  $G \sim \mathbb{G}(n, p)$  does not have a collection of (pairwise) edge-disjoint  $K_4$ -tiled subgraphs,  $H_1, \dots, H_k$ , with  $k \geq \lceil 1/\varepsilon \rceil$ , such that  $\varphi(H_i) \geq 3$  (equivalently,  $5v(H_i) - 2e(H_i) \leq 5$ ) for every  $i \in [k]$ , and  $|V(H_i) \cap V(H_{i+1})| = 1$  for every  $i \in [k-1]$ .*

**Proof.** It suffices to prove the claim for  $k = \lceil 1/\varepsilon \rceil$ . Suppose that  $H_1, \dots, H_k$  is such a collection with  $k = \lceil 1/\varepsilon \rceil$ , and let  $H = \bigcup_{i=1}^k H_i$ . Then

$$5v(H) - 2e(H) = \sum_{i=1}^k (5v(H_i) - 2e(H_i)) - 5(k-1) \leq 5.$$

Following (7), the expected number of copies of  $H$  in  $\mathbb{G}(n, p)$  is at most

$$n^{\frac{1}{5} \cdot (5v(H) - 2e(H)) - \varepsilon \cdot e(H)} \leq n^{1 - \varepsilon \cdot e(H)} \leq n^{1 - 6\varepsilon k} = o(1).$$

Since the number of possible such graphs  $H$  is  $O_\varepsilon(1)$  (using Claim 7.4), the claim follows.  $\square$

The following claim precludes *cyclic* compositions of  $K_4$ -tiled subgraphs in  $\mathbb{G}(n, p)$ .

**Claim 7.8.** *Asymptotically almost surely  $G \sim \mathbb{G}(n, p)$  does not have a collection of (pairwise) edge-disjoint  $K_4$ -tiled subgraphs,  $H_1, \dots, H_k$ , such that  $\varphi(H_i) \geq 3$  (equivalently,  $5v(H_i) - 2e(H_i) \leq 5$ ) for every  $i \in [k]$ , and  $|V(H_i) \cap V(H_{i+1})| = 1$  for every  $i \in [k]$  (with indices taken modulo  $k$ , i.e.,  $H_k$  and  $H_1$  share a vertex).*

**Proof.** Suppose that  $H_1, \dots, H_k$  is such a collection, and let  $H = \bigcup_{i=1}^k H_i$ . Then

$$5v(H) - 2e(H) = \sum_{i=1}^k (5v(H_i) - 2e(H_i)) - 5k \leq 0. \quad (8)$$

As in previous claims, it follows that the expected number of copies of  $H$  is  $o(1)$ . Since, by Claims 7.7 and 7.4, the number of possible such graphs  $H$  is  $O_\varepsilon(1)$ , the claim follows.  $\square$

The following claim further restricts the paths of  $K_4$ -tiled subgraphs of  $\mathbb{G}(n, p)$ .

**Claim 7.9.** *Asymptotically almost surely  $G \sim \mathbb{G}(n, p)$  does not have a collection of (pairwise) edge-disjoint  $K_4$ -tiled subgraphs,  $H_1, \dots, H_k$ , with  $k \geq 2$ , satisfying  $|V(H_i) \cap V(H_{i+1})| = 1$  for every  $i \in [k-1]$ , such that*

- (i)  $\varphi(H_i) \geq 6$  (equivalently,  $5v(H_i) - 2e(H_i) \leq 2$ ) for  $i \in \{1, k\}$ , and
- (ii)  $\varphi(H_i) \geq 3$  (equivalently,  $5v(H_i) - 2e(H_i) \leq 5$ ) for every  $2 \leq i \leq k-1$ .

**Proof.** Suppose that  $H_1, \dots, H_k$  is such a collection, and let  $H = \bigcup_i H_i$ . Then

$$5v(H) - 2e(H) = \sum_{i=1}^k (5v(H_i) - 2e(H_i)) - 5(k-1) \leq 4 + 5(k-2) - 5k + 5 < 0.$$

The proof can be completed as in the proof of Claim 7.8.  $\square$

Let  $G \sim \mathbb{G}(n, p)$  satisfying all of the above properties (as stated in Claims 7.4 – 7.9) be fixed.

**Claim 7.10.** *Let  $H \in \mathcal{C} := \mathcal{C}(G)$  (that is,  $H$  is a connected union of  $K_4$ -components  $H'$  with  $\varphi(H') \geq 3$ ). Then  $H$  admits a proper edge-colouring with all rainbow copies of  $K_4$  sharing a common colour, say, red.*

**Proof of Claim 7.10 using Lemma 7.3.** Fix  $H \in \mathcal{C}$  and let  $H_1, \dots, H_k \in \mathcal{H}$  be  $K_4$ -components satisfying  $H = \bigcup_{i=1}^k H_i$ . Without loss of generality, we may assume that  $\varphi(H_1) \geq \varphi(H_i)$  for  $2 \leq i \leq k$ . Recall that, by the definition of  $\mathcal{C}$ , we have  $\varphi(H_i) \geq 3$  for every  $i \in [k]$ . Since  $H$  is connected, it follows by Claims 7.6 and 7.8, that  $H$  has a tree-like structure. That is, upon appropriate relabelling (but keeping  $H_1$  unchanged), we may insist on

$$\left| V(H_i) \cap (V(H_1) \cup \dots \cup V(H_{i-1})) \right| = 1$$

holding for every  $2 \leq i \leq k$ ; let  $u_i$  be the unique vertex in this intersection. It follows by Claim 7.9, applied to every *sub-path* of this tree-like structure, and by the assumed maximality of  $\varphi(H_1)$ , that  $\varphi(H_i) \leq 5$  holds for every  $2 \leq i \leq k$ .

It follows by Claim 7.5 that  $\varphi(H_1) \leq 7$ . Hence, one of the alternatives (ii) and (iii) of Lemma 7.3 must hold for  $H_1$ . Either way, it follows that  $H_1$  admits a proper edge-colouring  $\psi_1$  such that every copy of  $K_4$  in  $H_1$  which is rainbow under  $\psi_1$  contains, say, a red edge. Moreover, as  $3 \leq \varphi(H_i) \leq 5$  holds for every  $i \in [2, k]$ , these graphs satisfy alternative (ii) of Lemma 7.3. Consequently, for every  $i \in [2, k]$ , there exists a triangle  $T_i \subseteq H_i$  and a proper edge-colouring  $\psi_i$  of  $H_i$  such that every copy of  $K_4$  in  $H_i$  which is rainbow under  $\psi_i$  contains  $T_i$ . We may assume that the colour sets used by  $\psi_1, \dots, \psi_k$  are pairwise disjoint. For every triangle  $T_i$ , let  $v_i$  and  $w_i$  be distinct vertices in  $V(T_i) \setminus \{u_i\}$ . Then  $\{v_1w_1, \dots, v_kw_k\}$  forms a matching that does not meet  $V(H_1)$ . Recolour the edges of this matching red. The resulting colouring  $\psi$  is a proper edge-colouring of  $H$  such that every copy of  $K_4$  in  $H$  which is rainbow under  $\psi$ , contains a red edge, as required.  $\square$

We are finally ready to derive Proposition 7.1 from Lemma 7.3.

**Proof of Proposition 7.1 using Lemma 7.3.** By Claim 7.10, for every  $H \in \mathcal{C}$ , we can find a proper edge-colouring  $\psi_H$  such that all rainbow copies of  $K_4$  in  $H$  contain, say, a red edge. Since the graphs in  $\mathcal{C}$  are pairwise vertex-disjoint, the union of these colourings is a proper partial edge-colouring of  $G$ . Next, as every  $H \in \mathcal{H}$  which is not a subgraph of a member of  $\mathcal{C}$  satisfies  $\varphi(H) \leq 2$ , every such  $H$  satisfies alternative (i) of Lemma 7.3, i.e., there is a proper edge-colouring  $\varphi_H$  of  $H$  that admits no rainbow copies of  $K_4$ . We assume that this proper edge-colouring uses colours unique to  $H$ . Finally, colour each of the edges of  $G$  that are not contained in any copy of  $K_4$  with a new unique colour. The union of all of these colourings results in a proper edge-colouring of  $G$  with all rainbow copies of  $K_4$  containing, say, a red edge, as required.  $\square$

### 7.3 Proof of Lemma 7.3

Let  $H$  be a  $K_4$ -tiled graph satisfying  $\varphi := \varphi(H) \leq 7$ , and let  $H_0 \cong K_4, H_1, \dots, H_r = H$  be a stretched sequence generating  $H$ . The assumption  $2\gamma + \beta = \varphi \leq 7$  mandates that  $0 \leq \gamma \leq 3$ . We consider each possible value of  $\gamma$  separately.

Having  $\gamma = 0$  means that  $H$  can be obtained from a copy of  $K_4$  through a sequence of standard steps and vertex-steps without missing edges. Consequently,  $H$  is 3-degenerate (with the ordering of the vertices dictated by the steps of the stretched sequence generating  $H$ ) and thus  $K_5$ -free. For higher values of  $\gamma$  this cannot be assumed. We add a fifth case to the case analysis over the possible values of  $\gamma$  in which we consider the case that  $H$  contains a copy of  $K_5$ . Dealing with it separately allows us to assume  $K_5$ -freeness throughout.

**Case 1.**  $\gamma = 0$ .

Having  $\gamma = 0$  implies that  $H$  can be obtained from a copy of  $K_4$  through a sequence of standard steps and vertex-steps without missing edges. Consequently, as noted above,  $H$  is 3-degenerate and thus  $K_5$ -free.

**Observation 7.11.** *All copies of  $K_4$  in  $H$  are the ‘obvious’ ones, namely  $H_0$  and  $\{x_i y_i z_i w_i\}_{i=1}^r$ .*

**Proof.** The proof is by induction on  $i$ . The assertion clearly holds for  $i = 0$ . If the  $i$ th step is a standard step, then every copy of  $K_4$  in  $H_i$  which does not appear in  $H_{i-1}$  contains at least one of  $x_i$  and  $y_i$ . It is easy to see that the only such copy of  $K_4$  is spanned by  $\{x_i, y_i, z_i, w_i\}$ . Similarly, if the  $i$ th step is a vertex-step without missing edges, then the only copy of  $K_4$  in  $H_i$  which is not in  $H_{i-1}$  contains  $x_i$  and is thus spanned by  $\{x_i, y_i, z_i, w_i\}$ .  $\square$

As the graph  $H$  is being built, we partially colour its edges so as to avoid a rainbow copy of  $K_4$ . The construction of this partial colouring can be seen in Figure 5.

**Colouring procedure.**

1. In  $H_0$ , pick any matching of size 2 and colour its edges with the same colour.
2. If the  $i$ th step is standard, colour  $x_i z_i$  and  $y_i w_i$  with the same new colour.
3. If the  $i$ th step is a vertex-step without missing edges, connecting  $x_i$  to the triangle  $T_i := y_i z_i w_i$ , do the following.
  - (i) If there is an edge in  $T_i$ , say  $y_i z_i$ , coloured with a colour  $\chi$  and there is no edge of colour  $\chi$  incident with  $w_i$ , colour the edge  $x_i w_i$  with the colour  $\chi$ . If there is more than one way to do so, choose one arbitrarily.
  - (ii) If the last step was impossible, but there is an edge of  $T_i$ , say  $y_i z_i$ , which is uncoloured, colour it and the edge  $x_i w_i$  with the same new colour.
  - (iii) If steps 3(i) and 3(ii) fail, mark  $T_i$  as *problematic*, and move on to the next step (without colouring any edges).

Figure 5: Partial colouring avoiding rainbow  $K_4$ 's

For a colour  $\chi$  and a triangle  $T$ , we say that  $\chi$  *saturates*  $T$  (at a given moment with respect to a given partial colouring) if  $T$  contains an edge of colour  $\chi$  and the third vertex of  $T$  (not incident with this edge) is also incident to an edge of colour  $\chi$ . The following claim plays a central role in proving Lemma 7.3 in the case  $\gamma = 0$ .

**Claim 7.12.** *If a triangle  $T$  is problematic (see Item 3(iii) in Figure 5), then the sequence generating  $H$  includes at least three vertex-steps in which a new vertex is attached to the triangle  $T$ .*

**Proof.** We start with the following observation.

**Observation 7.13.** *For every colour  $\chi$  and vertex  $u$ , there is no point during the colouring procedure at which the neighbourhood of  $u$  spans two edges of colour  $\chi$ , yet  $u$  is not incident with a  $\chi$ -coloured edge.*

**Proof.** Suppose for a contradiction that at some point there exist a vertex  $u$  and a colour  $\chi$  such that  $u$  is not incident with an edge of colour  $\chi$ , yet there are two  $\chi$ -coloured edges, say,  $ab$  and  $cd$ , such that  $a, b, c$  and  $d$  are in the neighbourhood of  $u$ . Upon its first appearance,  $u$  has degree at most 3, so at least one of the vertices in  $\{a, b, c, d\}$  appears after  $u$ . Without loss of generality, assume that  $d$  is the last vertex to appear amongst  $\{u, a, b, c, d\}$ . Then the edge  $cd$  appears after  $ab$ . As an existing edge can only be coloured with a new colour (i.e. one that did not appear previously), one of the following holds. Either  $cd$  and  $ab$  are coloured  $\chi$  simultaneously, when  $d$  joins the graph; or  $cd$  is coloured by the already existing colour  $\chi$  when  $d$  joins the graph. In the former case  $\{a, b, c, d\}$  forms a copy of  $K_4$  so that  $\{u, a, b, c, d\}$  forms a copy of  $K_5$ , which is impossible as  $H$  is  $K_5$ -free, as noted above. In the latter case,  $d$  joins the graph in a vertex-step connecting  $d$  to a triangle  $T$  containing a  $\chi$ -coloured edge. Since  $d$  is adjacent to  $u$  and  $c$ , it follows that  $T$  contains  $u$  and  $c$ . Note that the  $\chi$ -edge in  $T$  does not include the vertex  $c$  (because  $cd$  is about to receive colour  $\chi$ ), which implies that  $u$  is incident with a  $\chi$ -edge, contrary to our assumption.  $\square$

For a triangle  $T$ , let  $k(T)$  denote the number of vertex-steps that attach a new vertex to  $T$ .

**Observation 7.14.** *The number of colours that saturate a triangle  $T$  at any given moment is at most  $k(T) + 1$ .*

**Proof.** The proof is by induction. Consider the first time that  $T$  appears in the graph. Immediately before this moment, the graph contains at most two vertices of  $T$ , and when  $T$  is added, edges of exactly one colour (new or old) are added. Hence, immediately after  $T$  appears, it is saturated by at most one colour. Similarly, when a step that connects a new vertex to  $T$  is performed, it increases the number of colours that saturate  $T$  by at most 1.

Suppose for a contradiction that there is a step that causes a new colour to saturate  $T$ , but which does not consist of connecting a new vertex to  $T$ . Note that such a step must reuse an old colour. Therefore, it is a vertex-step, and the resulting colouring is performed according to Item 3(i) in the colouring procedure. More precisely, the step consists of connecting a new vertex  $x$  to a triangle  $yzw$ , and without loss of generality, it colours the edge  $xy$  with a colour  $\chi$  that already appears on  $zw$ . By the assumption that after this step the colour  $\chi$  saturates  $T$ , it follows that  $y$  is a vertex in  $T$  and the edge between the other two vertices of  $T$  is coloured  $\chi$ . As this edge is distinct from  $zw$  by the assumption that  $yzw$  is not the triangle  $T$ , we find that before this step the neighbourhood of  $y$  contains two edges of colour  $\chi$ , yet there is no edge of colour  $\chi$  incident with  $y$ . This contradicts Observation 7.13.

To summarise, when it first appears,  $T$  is saturated by at most one colour, and the number of colours that saturate  $T$  can increase (by at most 1) only via vertex-steps that connect a new vertex to  $T$ , as required.  $\square$

The proof of Claim 7.12 follows easily from Observation 7.14. Indeed, a triangle  $T$  is problematic if at some point there is a vertex-step attaching a new vertex to  $T$ , but  $T$  is already saturated by three colours. It thus follows from Observation 7.14 that there are at least three vertex-steps attaching a new vertex to  $T$  (including the one which marks it problematic), as required for Claim 7.12.  $\square$



As mentioned above, the case  $\gamma = 0$  of Lemma 7.3 follows from Claim 7.12. To see this, we consider the possible values of  $\varphi$  as specified by that lemma.

1. If  $\varphi \leq 2$ , then  $\beta \leq 2$ , and thus there are no problematic triangles, i.e. the above colouring procedure can be extended to a proper edge-colouring of  $H$  without rainbow copies of  $K_4$ .
2. If  $\varphi \in \{3, 4, 5\}$ , then there is at most one problematic triangle. This implies the existence of a proper edge-colouring of  $H$  and a triangle  $T$  such that all rainbow copies of  $K_4$  in  $H$  contain  $T$ .
3. Finally, if  $\varphi \in \{6, 7\}$ , then there are at most two problematic triangles. This implies the existence of a proper edge-colouring of  $H$  and two triangles  $T_1, T_2$  such that all rainbow copies of  $K_4$  in  $H$  contain either  $T_1$  or  $T_2$ . It follows that there is a matching  $M$  of size at most 2 (consisting of one edge from each of the triangles  $T_1$  and  $T_2$ ) which meets every rainbow copy of  $K_4$  in  $H$ .

## Case 2. $H$ contains a copy of $K_5$

In the case  $\gamma = 0$ , the graph  $H$  is  $K_5$ -free, which is useful in the proof pertaining to that case. For higher values of  $\gamma$ ,  $K_5$ -freeness is not guaranteed. Hence, prior to pursuing the case analysis for higher values of  $\gamma$  any further, we prove Lemma 7.3 in the case where  $H$  contains a copy of  $K_5$ , so that we can later assume  $K_5$ -freeness.

Suppose then that  $H$  contains a copy of  $K_5$ . Hence, there is a sequence  $K_5 \cong H'_0, H'_1, \dots, H'_r = H$ , such that  $H'_i$  is obtained from  $H'_{i-1}$  via a standard step, a vertex-step, or an edge-step. Strictly speaking, the aforementioned sequence is not a stretched sequence as it starts from a copy of  $K_5$ . Nevertheless, we do away with this technicality and assume that the sequence is *optimal*, which, similarly to the notion of *stretched*, means that the number of edges added between existing vertices is minimised, and that the total number of steps is maximised.

Define  $\alpha', \beta'$  and  $\gamma'$  analogously to the definition of  $\alpha, \beta$  and  $\gamma$ , respectively. Then

$$v(H) = 5 + 2\alpha' + \beta' \quad \text{and} \quad e(H) = 10 + 5\alpha' + 3\beta' + \gamma'.$$

It follows that

$$\varphi := \varphi(H) = 8 - 5v(H) + 2e(H) = 3 + \beta' + 2\gamma' \geq 3.$$

Since, moreover,  $\varphi \leq 7$  holds a.a.s. by Claim 7.5, one should only consider the values of  $\beta'$  and  $\gamma'$  for which  $\beta' + 2\gamma' \leq 4$  holds.

We will make use of the following variant of Claim 7.2 (b); its proof can be found in Appendix B.

**Claim 7.15.** *If the first edge-step in the sequence  $K_5 \cong H'_0, H'_1, \dots, H'_r = H$  is a 1-edge-step, then it is preceded by at least one vertex-step.*

We follow a partial colouring procedure, similar to the one described in Figure 5, with the following modifications.

1. We replace Step 1 in Figure 5 with the proper edge-colouring of  $K_5$  described in Figure 6 (which admits no rainbow copies of  $K_4$ ).

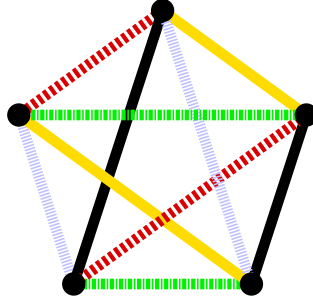


Figure 6: A proper colouring of  $K_5$  with no rainbow  $K_4$ 's

2. For any standard step, we follow Step 2 of the partial colouring procedure given in Figure 5.
3. For any vertex-step, we follow Step 3 of the partial colouring procedure given in Figure 5 with the following additional rule. If this vertex-step is with missing edges, say, it attaches  $x$  to  $y, z$  and  $w$ , and  $yz$  is a non-edge which is added during this vertex-step, then we also allow the colouring of  $xw$  and  $yz$  with the same new colour.
4. We do not colour any edges during an edge-step.

Note that by the optimality assumption on the sequence generating  $H$ , any vertex-step introduces a single new copy of  $K_4$ . Therefore, the above partial colouring guarantees that every  $K_4$  which is coloured upon appearance will not be rainbow.

We will use the following claim, whose proof is similar to that of Observation 7.14 above.

**Claim 7.16.** *The first vertex-step is coloured successfully.*

**Proof.** The assertion of the claim holds by Item 3 above for the first (or any other) vertex-step with missing edges. Hence, we need only consider vertex-steps without missing edges. We monitor the number of colours that saturate each triangle. Triangles contained in  $H'_0$  are initially saturated by two colours; triangles that appear following a standard step are initially saturated by one colour; and triangles that appear following an edge-step are saturated by at most two colours upon appearance (as they contain an edge that was previously missing and which is not coloured upon appearance). Therefore, since standard steps and edge-steps do not increase the number of colours saturating any existing triangle, immediately before the first vertex-step, every triangle is saturated by at most two colours. It follows that the first vertex-step is indeed coloured successfully.  $\square$

To complete the proof of Case 2 of Lemma 7.3, we consider the following five subcases. We will show that if  $\varphi \in \{3, 4, 5\}$ , the partial colouring procedure described above can be extended to a proper edge-colouring with at most one rainbow copy of  $K_4$ ; and if  $\varphi \in \{6, 7\}$ , then it can be extended to a proper edge-colouring such that all rainbow copies of  $K_4$  can be covered by a matching of size at most 3.

1.  $\varphi \in \{3, 4, 5\}$  and  $\gamma' = 0$ . These values of  $\varphi$  and  $\gamma'$  imply that  $\beta' \leq 2$ . Hence, there are at most two non-standard steps, all of which are vertex-steps with no missing edges. By Claim 7.16, the first of these steps is coloured successfully, so we end up with at most one rainbow  $K_4$ .

2.  $\varphi \in \{3, 4, 5\}$  and  $\gamma' = 1$ . These values of  $\varphi$  and  $\gamma'$  imply that  $\beta' = 0$ . Hence, there is one 1-edge-step and no other non-standard steps, contrary to the assertion of Claim 7.15.
3.  $\varphi \in \{6, 7\}$  and  $\gamma' = 0$ . These values of  $\varphi$  and  $\gamma'$  imply that  $\beta' \leq 4$ . Hence, all non-standard steps are vertex-steps without missing edges, and there are at most four such steps. By Claim 7.16, the first such step is coloured successfully, implying that there are at most three rainbow copies of  $K_4$  in  $H$  (it is, in fact, possible to show that there are at most two rainbow copies of  $K_4$ ). It is easy to see that there is a matching  $M$ , of size at most 3, that meets each of these copies.
4.  $\varphi \in \{6, 7\}$  and  $\gamma' = 1$ . These values of  $\varphi$  and  $\gamma'$  imply that  $\beta' \leq 2$ . Hence, either there is one vertex-step with one missing edge and at most one vertex-step without missing edges; or there are at most two vertex-steps without missing edges, and a single 1-edge-step. Since the first vertex-step is coloured successfully by Claim 7.16, all but at most one rainbow copy of  $K_4$  intersect in a given edge (namely, the edge added in the vertex-step with one missing edge in the former case, or the 1-edge-step in the latter case, with the potential exceptional  $K_4$  stemming from the second vertex-step; in the former case, there is in fact at most one rainbow  $K_4$ ). Either way, it readily follows that there is a matching of size at most 2 that meets all rainbow copies of  $K_4$ .
5.  $\varphi \in \{6, 7\}$  and  $\gamma' = 2$ . These values of  $\varphi$  and  $\gamma'$  imply that  $\beta' = 0$ . It follows by Claim 7.15 that the only non-standard step in this case is an edge-step with two missing edges. Therefore, all rainbow copies of  $K_4$  intersect in an edge (in fact, they intersect in at least two adjacent edges, implying that they intersect in a triangle).

**Case 3.**  $\gamma = 1$

Having dealt with Case 2, from now on we assume that  $H$  is  $K_5$ -free. Since  $\gamma = 1$ , there is an edge  $xy$  which is introduced either through a single 1-edge-step or through a vertex-step with a single missing edge. Either way, exactly one copy of  $K_4$  appears during this step (in both cases this follows by the assumption that the sequence generating  $H$  is stretched, where in the former case we also use Claim 7.2); denote its vertex set by  $\{x, y, z, w\}$ . We consider two cases, according to the type of step adding  $xy$ .

**Case 3a.**  $xy$  is introduced via an edge-step

By Claim 7.2, the edge-step is preceded by at least two vertex-steps (without missing edges); in particular, we have  $\varphi = \beta + 2\gamma \geq 4$ .

We apply the colouring procedure described in Figure 5, with the sole change that until the edge  $xy$  is added, we aim to leave the edge  $zw$  uncoloured. If successful, we may then colour  $xy$  and  $zw$  with the same new colour, thus ensuring that the unique new copy of  $K_4$  created by the edge-step introducing  $xy$  is not rainbow.

Note that this is always possible in standard steps. Indeed, when  $x_i$  and  $y_i$  are added and connected to each other and to  $z_i$  and  $w_i$ , we can choose to either colour  $x_i z_i$  and  $y_i w_i$ , or to colour  $x_i w_i$  and  $y_i z_i$ , and one of these choices would avoid  $zw$ . It is also possible to avoid colouring  $zw$  in

a vertex-step, unless it is a step attaching a new vertex to a triangle  $T$  that contains  $z$  and  $w$  and is saturated by two colours. We further divide this subcase.

1. Suppose, first, that  $zw$  is uncoloured when  $xy$  is introduced into the graph. It follows that only vertex-steps may result in rainbow copies of  $K_4$ . Note that the first two vertex-steps are guaranteed to be coloured successfully (because, similarly to the proof of Observation 7.14, the number of colours saturating a triangle is at most the number of colours introduced by non-standard steps, plus one). Thus, if  $\varphi \in \{4, 5\}$ , the colouring procedure fails at most once, implying the existence of at most one rainbow  $K_4$ . Similarly, if  $\varphi \in \{6, 7\}$ , then the colouring procedure fails at most three times, implying the existence of at most three rainbow copies of  $K_4$ . It is easy to see that there is a matching  $M$ , of size at most 3, that meets each of these copies.
2. Suppose then, that  $zw$  is coloured when  $xy$  is introduced, and let  $T$  be a triangle that contains  $z$  and  $w$  and was extended by vertex-steps at least twice (such a triangle must exist as otherwise we could have avoided colouring  $zw$ ). We modify the colouring so that in these vertex-steps nothing is coloured (if there is more than one such triangle, we avoid colouring in all vertex-steps extending these triangles). This ensures that  $zw$  is uncoloured when  $xy$  is introduced, and thus, as in the previous case, the edge-step adding  $xy$  will not create a rainbow copy of  $K_4$ .

If  $\varphi \in \{4, 5\}$ , there are at most two non-standard steps (namely, the edge-step introducing  $xy$  and possibly one vertex-step) in which we try to colour, so success is guaranteed in each of them. It follows that all rainbow copies of  $K_4$  contain  $T$  (if  $\varphi \in \{4, 5\}$ , there cannot be another triangle that is extended twice).

If  $\varphi \in \{6, 7\}$ , there are at most three vertex-steps in which we try to colour, and we are guaranteed success in at least one of them. Therefore, in total, we fail to colour in at most four vertex-steps, at least two of which extend the same triangle. It follows that there are at most four rainbow copies of  $K_4$ , two of which intersect in a triangle. It is easy to see that there is a matching  $M$ , of size at most 3, that meets each of these copies.

### **Case 3b. $xy$ is introduced via a vertex-step with a missing edge**

In this case  $\gamma = 1$  and  $\beta \geq 1$ , implying that  $\varphi = \beta + 2\gamma \geq 3$ . Suppose that the vertex-step introducing  $xy$  attaches  $z$  to  $x, y, w$ , where  $xw, yw$  are existing edges and  $xy$  is a non-edge. Then the edges  $xy$  and  $zw$  are introduced simultaneously and thus can be coloured using the same new colour. It follows that the vertex-step introducing  $xy$  is guaranteed to be coloured successfully.

If  $\varphi \in \{3, 4, 5\}$ , there are at most three vertex-steps, the first two of which can be coloured successfully. It follows that there is at most one rainbow  $K_4$ . If  $\varphi \in \{6, 7\}$ , there are at most five vertex-steps, and we fail to colour in at most three of them. It follows that there are at most three rainbow copies of  $K_4$ . It is easy to see that there is a matching  $M$ , of size at most 3, that meets each of these copies.

**Case 4.**  $\gamma = 2$

In this case  $\varphi \geq 4$ . We consider two subcases according to the number of steps used to introduce the two missing edges.

1. Assume first that both missing edges are introduced in the same step (either a vertex-step with two missing edges or a 2-edge-step). It then follows by the definition of a stretched sequence, that all the copies of  $K_4$  which appear upon the introduction of these two edges, intersect in some triangle  $T$  (as they all contain the two new edges). By following the colouring procedure described in Figure 5 (not colouring anything in the aforementioned step), the first two vertex-steps (if they exist) can be coloured successfully. Hence, if  $\varphi \in \{4, 5\}$ , there is at most one vertex-step and thus all rainbow copies of  $K_4$  contain the triangle  $T$ . If  $\varphi \in \{6, 7\}$ , there is at most one vertex-step in which we fail to colour, implying that all rainbow copies of  $K_4$  contain one of two given triangles.
2. Assume then that the missing edges are added in two separate steps. Consider first the case  $\varphi \in \{4, 5\}$ , which implies that  $\beta \leq 1$ . If there are two 1-edge-steps, then the first one is preceded by at most one vertex-step without missing edges, contrary to the assertion of Claim 7.2. Therefore, there must be one vertex-step with one missing edge, followed by a 1-edge-step. The vertex-step can be coloured successfully, implying that the only potential rainbow copy of  $K_4$  is the one created by the 1-edge-step (recall that, by Claim 7.2, the 1-edge-step creates only one copy of  $K_4$ ).

Next, consider the case  $\varphi \in \{6, 7\}$ , which implies that  $\beta \leq 3$ . Suppose first that at least one of the missing edges is added in a vertex-step. Since the first two vertex-steps can be coloured successfully (recall that we do not colour anything during edge-steps), we fail to colour in at most one vertex-step. It follows that all rainbow copies of  $K_4$  contain either a given edge or a given triangle. It is easy to see that there is a matching  $M$ , of size at most 2, that meets each of these copies.

We may thus assume that there are two 1-edge-steps. Moreover, it follows by Claim 7.2 that the first 1-edge-step is preceded by two vertex-steps without missing edges and it creates exactly one new copy of  $K_4$ . Since the first two vertex-steps can be coloured successfully, we obtain a proper edge-colouring in which all rainbow copies of  $K_4$  contain either a given  $K_4$  (stemming from the first edge-step), a given triangle (stemming from the third vertex-step, if it exists), or a given edge (stemming from the second edge-step). It is easy to see that there is a matching  $M$ , of size at most 3, that meets each of these copies.

**Case 5.**  $\gamma = 3$

In this case  $\varphi \geq 6$ , implying that  $\varphi \in \{6, 7\}$  and  $\beta \leq 1$ . We consider three subcases according to the number of steps used to introduce the three missing edges.

1. Suppose, first, that all of the three missing edges are introduced in the same step. It follows by the definition of a stretched sequence that all copies of  $K_4$  which appear upon the introduction of these edges, intersect in a triangle. There can be at most one additional non-standard step

(which is a vertex-step), giving rise to at most one additional rainbow copy of  $K_4$ . It is easy to see that there is a matching  $M$ , of size at most 2, that meets each of the rainbow copies of  $K_4$ .

2. Next, suppose that the missing edges are introduced in two steps; in particular, one of these steps introduces two missing edges. As in the previous case, it again follows by the definition of a stretched sequence that all copies of  $K_4$  which appear upon the introduction of these edges intersect in a triangle. By following our usual colouring procedure (not colouring any edges which are introduced in edge-steps), we can ensure that the only vertex-step, if it exists (recall that  $\beta \leq 1$ ), is coloured successfully. We thus obtain a proper edge-colouring such that all rainbow copies of  $K_4$  contain a given triangle (stemming from the step with two missing edges, if it is an edge-step) or a given edge (stemming from the step with one missing edge, if it is an edge-step). It is easy to see that there is a matching  $M$ , of size at most 2, that meets each of the rainbow copies of  $K_4$ .
3. Finally, suppose that the missing edges are added in three separate steps. Since  $\beta \leq 1$ , it follows by Claim 7.2 that there is one vertex-step with a single missing edge followed by two 1-edge-steps, and the first edge-step creates at most one new copy of  $K_4$ . Since the vertex-step can be coloured successfully, we obtain a proper edge-colouring where all rainbow copies of  $K_4$  contain a given copy of  $K_4$  (stemming from the first 1-edge-step) or a given edge (stemming from the second 1-edge-step). It is easy to see that there is a matching  $M$ , of size at most 2, that meets each of the rainbow copies of  $K_4$ .

## References

- [1] E. Aigner-Horev, O. Danon, D. Hefetz, and S. Letzter, *Large rainbow cliques in randomly perturbed dense graphs*, preprint.
- [2] E. Aigner-Horev and Y. Person, *Monochromatic Schur triples in randomly perturbed dense sets of integers*, *SIAM Journal on Discrete Mathematics* **33** (2019), no. 4, 2175–2180.
- [3] J. Balogh, A. Treglown, and A. Z. Wagner, *Tilings in randomly perturbed dense graphs*, *Combinatorics, Probability and Computing* **28** (2019), no. 2, 159–176.
- [4] G.F. Barros, B.P. Cavalari, G.O. Mota, and O. Parczyk, *Anti-Ramsey Threshold of Cycles for Sparse Graphs*, *Electronic Notes in Theoretical Computer Science* **346** (2019), 89–98.
- [5] W. Bedenknecht, J. Han, Y. Kohayakawa, and G. O. Mota, *Powers of tight Hamilton cycles in randomly perturbed hypergraphs*, *Random Structures & Algorithms* **55** (2019), no. 4, 795–807.
- [6] T. Bohman, A. Frieze, M. Krivelevich, and R. Martin, *Adding random edges to dense graphs*, *Random Structures & Algorithms* **24** (2004), no. 2, 105–117.
- [7] T. Bohman, A. Frieze, and R. Martin, *How many random edges make a dense graph Hamiltonian?*, *Random Structures & Algorithms* **22** (2003), no. 1, 33–42.
- [8] T. Bohman, A. Frieze, O. Pikhurko, and C. Smyth, *Anti-Ramsey properties of random graphs*, *Journal of Combinatorial Theory. Series B* **100** (2010), no. 3, 299–312. MR 2595673

- [9] J. Böttcher, J. Han, Y. Kohayakawa, R. Montgomery, O. Parczyk, and Y. Person, *Universality for bounded degree spanning trees in randomly perturbed graphs*, Random Structures & Algorithms **55** (2019), no. 4, 854–864, online.
- [10] J. Böttcher, R. H. Montgomery, O. Parczyk, and Y. Person, *Embedding spanning bounded degree graphs in randomly perturbed graphs*, Mathematika (2019), (to appear).
- [11] D. Conlon, W. T. Gowers, W. Samotij, and M. Schacht, *On the KLR conjecture in random graphs*, Israel Journal of Mathematics **203** (2014), no. 1, 535–580.
- [12] S. Das, P. Morris, and A. Treglown, *Vertex Ramsey properties of randomly perturbed graphs*, arXiv:1910.00136v1, 2019.
- [13] S. Das and A. Treglown, *Ramsey properties of randomly perturbed graphs: cliques and cycles*, arXiv:1901.01684, 2019.
- [14] A. Dudek, C. Reiher, A. Ruciński, and M. Schacht, *Powers of Hamiltonian cycles in randomly augmented graphs*, arXiv preprint arXiv:1805.10676, 2018.
- [15] S. Fujita, C. Magnant, and K. Ozeki, *Rainbow generalizations of Ramsey theory: a survey*, Graphs and Combinatorics **26** (2010), no. 1, 1–30. MR 2606615
- [16] J. Han and Y. Zhao, *Embedding Hamilton  $\ell$ -cycles in randomly perturbed hypergraphs*, arXiv:1802.04586, 2018.
- [17] S. Janson, *New versions of Suen’s correlation inequality*, Random Structures & Algorithms **13** (1998), no. 3-4, 467–483.
- [18] S. Janson, T. Łuczak, and A. Ruciński, *Random graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000. MR 1782847
- [19] Y. Kohayakawa, P. B. Konstantinidis, and G. O. Mota, *On an anti-Ramsey threshold for random graphs*, European Journal of Combinatorics **40** (2014), 26–41. MR 3191486
- [20] ———, *On an anti-Ramsey threshold for sparse graphs with one triangle*, Journal of Graph Theory **87** (2018), no. 2, 176–187. MR 3742177
- [21] Y. Kohayakawa, G. O. Mota, O. Parczyk, and J. Schnitzer, *The anti-Ramsey threshold of complete graphs*, arXiv:1902.00306v1, 2018.
- [22] J. Komlós and M. Simonovits, *Szemerédi’s regularity lemma and its applications in graph theory*, Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993), Bolyai Soc. Math. Stud., vol. 2, János Bolyai Math. Soc., Budapest, 1996, pp. 295–352. MR 1395865
- [23] B. Kreuter, *Threshold functions for asymmetric Ramsey properties with respect to vertex colorings*, Random Structures & Algorithms **9** (1996), no. 3, 335–348. MR 1606853
- [24] M. Krivelevich, M. Kwan, and B. Sudakov, *Cycles and matchings in randomly perturbed digraphs and hypergraphs*, Combinatorics, Probability and Computing **25** (2016), no. 6, 909–927.

- [25] ———, *Bounded-degree spanning trees in randomly perturbed graphs*, SIAM Journal on Discrete Mathematics **31** (2017), no. 1, 155–171.
- [26] M. Krivelevich, B. Sudakov, and P. Tetali, *On smoothed analysis in dense graphs and formulas*, Random Structures & Algorithms **29** (2006), no. 2, 180–193.
- [27] T. Łuczak, A. Ruciński, and B. Voigt, *Ramsey properties of random graphs*, Journal of Combinatorial Theory, Series B **56** (1992), no. 1, 55–68.
- [28] A. McDowell and R. Mycroft, *Hamilton  $\ell$ -cycles in randomly perturbed hypergraphs*, Electronic Journal of Combinatorics **25** (2018), no. 4, Paper 4.36, 30.
- [29] G. O. Mota, *Advances in anti-Ramsey theory for random graphs*, Encontro de Teoria da Computacao, 2017.
- [30] R. Nenadov, Y. Person, N. Škorić, and A. Steger, *An algorithmic framework for obtaining lower bounds for random Ramsey problems*, Journal of Combinatorial Theory. Series B **124** (2017), 1–38. MR 3623165
- [31] R. Nenadov and A. Steger, *A short proof of the random Ramsey theorem*, Combinatorics, Probability and Computing **25** (2016), no. 1, 130–144.
- [32] E. Powierski, *Ramsey properties of randomly perturbed dense graphs*, arXiv:1902.02197, 2019.
- [33] V. Rödl and A. Ruciński, *Lower bounds on probability thresholds for Ramsey properties*, Combinatorics, Paul Erdős is eighty, 1, Bolyai Society Mathematical Studies, vol. 1, János Bolyai Mathematical Society, 1993, pp. 317–346.
- [34] ———, *Random graphs with monochromatic triangles in every edge coloring*, Random Structures & Algorithms **5** (1994), no. 2, 253–270.
- [35] ———, *Threshold functions for Ramsey properties*, Journal of the American Mathematical Society **8** (1995), no. 4, 917–942.
- [36] V. Rödl and Z. Tuza, *Rainbow subgraphs in properly edge-colored graphs*, Random Structures & Algorithms **3** (1992), no. 2, 175–182. MR 1151360
- [37] B. Sudakov and J. Vondrák, *How many random edges make a dense hypergraph non-2-colorable?*, Random Structures & Algorithms **32** (2008), no. 3, 290–306.
- [38] E. Szemerédi, *Regular partitions of graphs*, Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), Colloq. Internat. CNRS, vol. 260, CNRS, Paris, 1978, pp. 399–401.

## A Emergence of small graphs in $\mathbb{G}(n, p)$

In this section we prove several claims that we used in previous sections regarding the appearance of fixed graphs in certain subgraphs of  $G(n, p)$ . Throughout this section, we make repeated appeals



to a result of Janson [17] (see also [18, Theorems 2.18]) regarding random variables of the form  $X = \sum_{A \in \mathcal{S}} I_A$ . Here,  $\mathcal{S}$  is a family of non-empty subsets of some ground set  $\Omega$  and  $I_A$  is the indicator random variable for the event  $A \subseteq \Omega_p$ , where  $\Omega_p$  is the so-called *binomial random set* arising from including each element of  $\Omega$  independently with probability  $p$ . For such random variables, set  $\lambda := \mathbb{E}[X]$ , and define

$$\Delta := \frac{1}{2} \sum_{\substack{A, B \in \mathcal{S}: \\ A \neq B \text{ and } A \cap B \neq \emptyset}} \mathbb{E}[I_A I_B].$$

The following result is commonly referred to as the *probability of nonexistence* (see [18]).

**Theorem A.1.** [18, Theorem 2.18] *For  $X$ ,  $\lambda$ , and  $\Delta$  as above it holds that  $\mathbb{P}[X = 0] \leq \exp\left(-\frac{\lambda^2}{\lambda + 2\Delta}\right)$ .*

Of specific interest to us is the random variable  $X_H := X_H(n, p)$  which for a prescribed graph  $H$  accounts for the number of (labelled) occurrences of  $H$  in  $\mathbb{G}(n, p)$ . More specifically, for a prescribed  $H$ , let  $\mathcal{H} := \mathcal{H}_n$  denote the family of (labelled) copies of  $H$  in  $K_n$ . For every  $\tilde{H} \in \mathcal{H}$ , let  $Z_{\tilde{H}}$  denote the indicator random variable for the event  $\tilde{H} \subseteq \mathbb{G}(n, p)$ . Then,  $X_H := \sum_{\tilde{H} \in \mathcal{H}} Z_{\tilde{H}}$  counts the number of copies of  $H$  in  $\mathbb{G}(n, p)$ . Note that

$$\mathbb{E}(X_H) = \sum_{\tilde{H} \in \mathcal{H}} p^{e(\tilde{H})} = \binom{n}{v(H)} \frac{(v(H))!}{|Aut(H)|} \cdot p^{e(H)} = \Theta\left(n^{v(H)} p^{e(H)}\right),$$

where  $Aut(H)$  is the automorphism group of  $H$ . Writing  $H_i \sim H_j$  whenever  $(H_i, H_j) \in \mathcal{H} \times \mathcal{H}$  are distinct and not edge-disjoint, we define

$$\begin{aligned} \Delta(H) &:= \sum_{\substack{(H_i, H_j) \in \mathcal{H} \times \mathcal{H} \\ H_i \sim H_j}} \mathbb{E}[Z_{H_i} Z_{H_j}] = \sum_{\substack{(H_i, H_j) \in \mathcal{H} \times \mathcal{H} \\ H_i \sim H_j}} p^{e(H_i) + e(H_j) - e(H_i \cap H_j)} \\ &= \sum_{J \subsetneq H: e(J) \geq 1} \sum_{\substack{(H_i, H_j) \in \mathcal{H} \times \mathcal{H} \\ H_i \cap H_j \cong J}} p^{2e(H) - e(J)} = O_H \left( n^{2v(H)} p^{2e(H)} \cdot \sum_{J \subsetneq H: e(J) \geq 1} n^{-v(J)} p^{-e(J)} \right). \quad (9) \end{aligned}$$

Given a set  $\mathcal{C} \subseteq \binom{[n]}{v(H)}$ , we write  $X_H(\mathcal{C})$  to denote the number of copies of  $H$  in  $\mathbb{G}(n, p)$  supported on the members of  $\mathcal{C}$ , that is,

$$X_H(\mathcal{C}) = \{\tilde{H} \in \mathcal{H} : V(\tilde{H}) \in \mathcal{C} \text{ and } \tilde{H} \subseteq \mathbb{G}(n, p)\}.$$

Put

$$\Delta(H, \mathcal{C}) := \sum_{\substack{(H_i, H_j) \in \mathcal{H}(\mathcal{C}) \times \mathcal{H}(\mathcal{C}) \\ H_i \sim H_j}} \mathbb{E}[Z_{H_i} Z_{H_j}], \quad (10)$$

where  $\mathcal{H}(\mathcal{C})$  serves as the analogue of  $\mathcal{H}$  for the copies of  $H$  supported on  $\mathcal{C}$ . In particular,  $\Delta(H, \mathcal{C}) \leq \Delta(X_H)$ . For  $Y \subseteq [n]$ , we abbreviate  $X_H\left(\binom{Y}{v(H)}\right)$  to  $X_H(Y)$  and  $\Delta\left(H, \binom{Y}{v(H)}\right)$  to  $\Delta(H, Y)$ .

**Corollary A.2.** *Let  $H$  be a graph, let  $\eta > 0$  be fixed, and let  $p = p(n)$ . Suppose that  $n^{v(J)} p^{e(J)} = \omega(1)$  for every induced subgraph  $J \subseteq H$  that contains at least one edge. Let  $\mathcal{C} \subseteq \binom{[n]}{v(H)}$  be a fixed family of size at least  $\eta \binom{n}{v(H)}$ . Then a.a.s.  $G \sim \mathbb{G}(n, p)$  satisfies  $X_H(\mathcal{C}) \geq 1$ .*

**Proof.** Write  $\Delta := \Delta(H, \mathcal{C})$  and  $\lambda := \mathbb{E}[X_H(\mathcal{C})]$ . Then  $\lambda = |\mathcal{C}| \cdot p^{e(H)} = \Theta(n^{v(H)}p^{e(H)}) = \omega(1)$ . Moreover

$$\Delta \leq \Delta(H) = O_H \left( \left( n^{v(H)}p^{e(H)} \right)^2 \sum_{J \subsetneq H: e(J) \geq 1} n^{-v(J)}p^{-e(J)} \right) = o(\lambda^2)$$

holds by (9) (Note that, by assumption,  $n^{v(J)}p^{e(J)} = \omega(1)$  for every induced subgraph  $J$  of  $H$  with at least one edge, but this implies that  $n^{v(J)}p^{e(J)} = \omega(1)$  holds for every subgraph  $J$  of  $H$  with at least one edge). It then follows by Theorem A.1 that  $\mathbb{P}[X = 0] \leq \exp\left(-\frac{\lambda^2}{\lambda + 2\Delta}\right) = o(1)$ .  $\square$

**Corollary A.3.** *Let  $H$  be a graph, let  $\eta > 0$  be fixed, and let  $p = p(n)$ . Suppose that  $n^{v(J)}p^{e(J)} = \omega(n)$  for every induced subgraph  $J \subseteq H$  that contains at least one edge. Then a.a.s.  $X_H(Y) \geq 1$  holds for every subset  $Y \subseteq [n]$  of size  $|Y| \geq \eta n$ .*

**Proof.** Given  $Y \subseteq [n]$  of size  $|Y| \geq \eta n$ , let  $\lambda_Y := \mathbb{E}[X_H(Y)]$  and  $\Delta_Y := \Delta(H, Y)$ . Then  $\lambda_Y = \Theta(n^{v(H)}p^{e(H)}) = \omega(n)$  and, by (9),  $\Delta_Y = o(\lambda_Y^2/n)$ . It then follows by Theorem A.1 that

$$\mathbb{P}[X = 0] \leq \exp\left(-\frac{\lambda^2}{\lambda + 2\Delta}\right) = o(2^{-n}).$$

The result follows by a union bound over all the choices of  $Y \subseteq [n]$  of size  $|Y| \geq \eta n$ .  $\square$

In the remainder of this section, we use Corollaries A.2 and A.3 to prove Claims A.4 to A.10.

**Claim A.4.** *Let  $\eta > 0$  be fixed, let  $p = p(n) = \omega(n^{-1})$ . Then a.a.s.  $X_{K_3}(\mathcal{T}) \geq 1$ , where  $\mathcal{T} \subseteq \binom{[n]}{3}$  is a prescribed fixed set of size  $|\mathcal{T}| \geq \eta n^3$ .*

**Proof.** By Corollary A.2, it suffices to show that  $n^{v(J)}p^{e(J)} = \omega(1)$  for every induced subgraph  $J$  of  $K_3$  containing at least one edge, that is, for  $J \cong K_3$  and  $J \cong K_2$ . Recalling that  $p = \omega(n^{-1})$ , we observe that if  $J \cong K_3$ , then  $n^{v(J)}p^{e(J)} = n^3p^3 = \omega(1)$ , and if  $J \cong K_2$ , then  $n^{v(J)}p^{e(J)} = n^2p = \omega(1)$ ; the claim readily follows.  $\square$

**Claim A.5.** *Let  $\eta > 0$  be fixed, and let  $p = p(n) = \omega(n^{-1})$ . Then a.a.s.  $X_{K_{1,4}}(Y) \geq 1$  for every  $Y \subseteq [n]$  of size  $|Y| \geq \eta n$ .*

**Proof.** Let  $J$  be an induced subgraph of  $K_{1,4}$  with at least one edge, that is,  $J \cong K_{1,r}$  for some  $r \in [4]$ . Then,  $n^{v(J)}p^{e(J)} = n^{r+1}p^r = \omega(n)$  and thus the claim follows by Corollary A.3.  $\square$

Let  $R_7$  denote the graph obtained from  $K_{1,2}$  by attaching two triangles to each of its edges, that is,  $V(R_7) = \{u_1, u_2, u_3, w_1, w_2, w_3, w_4\}$  and

$$E(R_7) = \{u_1u_2, u_2u_3, u_1w_1, u_1w_2, u_2w_1, u_2w_2, u_2w_3, u_2w_4, u_3w_3, u_3w_4\}.$$

See Figure 1a for an illustration.

**Claim A.6.** *Let  $\eta > 0$  and  $k \in \mathbb{N}$  be fixed, let  $R$  be the vertex-disjoint union of  $k$  copies of  $R_7$ , and let  $p = p(n) = \omega(n^{-2/3})$ . Let  $\mathcal{Z} \subseteq \binom{[n]}{7k}$  be a fixed set of size  $|\mathcal{Z}| \geq \eta n^{7k}$ . Then a.a.s.  $X_R(\mathcal{Z}) \geq 1$ .*

**Proof.** A routine examination reveals that every subgraph of  $R_7$  has average degree strictly less than 3. Consequently, every induced subgraph  $J \subseteq R$  with  $e(J) \geq 1$  maintains this property; in particular,  $2e(J) < 3v(J)$ . Thus, for any such  $J$ , it holds that

$$n^{v(J)}p^{e(J)} = \omega(n^{v(J)-(2/3)e(J)}) = \omega(1).$$

Therefore, the claim follows by Corollary A.2.  $\square$

**Remark A.7.** The condition imposed on  $p$  in Claim A.6 can be mitigated to  $p = \omega(n^{-7/10})$ .

Let

$$T_k = (\{x, v_1, \dots, v_{2k}\}, \{xv_i : 1 \leq i \leq 2k\} \cup \{v_{2i-1}v_{2i} : 1 \leq i \leq k\})$$

denote the graph obtained by *gluing*, so to speak,  $k$  edge-disjoint triangles along a single (central) vertex. See Figure 1a for an illustration.

**Claim A.8.** *Let  $\eta > 0$  and  $k \in \mathbb{N}$  be fixed, and let  $p = p(n) = \omega(n^{-2/3})$ . Then a.a.s.  $X_{T_k}(Y) \geq 1$  for every  $Y \subseteq [n]$  of size  $|Y| \geq \eta n$ .*

**Proof.** We claim that  $v(J) - (2/3)e(J) \geq 1$  holds for every induced subgraph of  $T_k$ . If  $\delta(J) \geq 2$ , then  $J \cong T_\ell$  for some  $\ell \in [k]$ , in which case  $v(J) = 2\ell + 1$  and  $e(J) = 3\ell$  entailing the required inequality. If  $\delta(J) < 2$ , repeatedly remove vertices of degree at most 1 until the remaining induced subgraph  $J'$  consists of a single vertex or satisfies  $\delta(J') \geq 2$ . Then  $v(J') - (2/3)e(J') \geq 1$  holds for  $J'$ . The subgraph  $J$  can be obtained from  $J'$  by repeatedly adding vertices of degree at most 1, and thus  $v(J) - (2/3)e(J) \geq 1$  holds as required.

It thus follows that

$$n^{v(J)}p^{e(J)} = \omega(n^{v(J)-(2/3)e(J)}) = \omega(n)$$

holds whenever  $J$  is an induced subgraph of  $T_k$  with  $e(J) \geq 1$ . Therefore the claim follows by Corollary A.3.  $\square$

Let  $\widehat{K}_{3,4}$  be the graph obtained from the complete bipartite graph  $K_{3,4}$  by placing a triangle on its part of size 3.

**Claim A.9.** *Let  $\eta > 0$  and  $k \in \mathbb{N}$  be fixed, let  $K$  be the vertex-disjoint union of  $k$  copies of  $\widehat{K}_{3,4}$ , and let  $p = p(n) = \omega(n^{-7/15})$ . Let  $\mathcal{Z} \subseteq \binom{[n]}{7k}$  be a fixed set of size  $|\mathcal{Z}| \geq \eta n^{7k}$ . Then a.a.s.  $X_{\widehat{K}_{3,4}}(\mathcal{Z}) \geq 1$ .*

**Proof.** We claim that  $15v(J) \geq 7e(J)$  holds whenever  $J$  is an induced subgraph of  $K$  satisfying  $e(J) \geq 1$ . It suffices to prove this assertion for the induced subgraphs of  $\widehat{K}_{3,4}$ . For the latter, suppose for a contradiction that  $J'$  is an induced subgraph of  $\widehat{K}_{3,4}$  for which  $15v(J') < 7e(J')$  holds. Then, the average degree of  $J'$  is strictly larger than 4. This, in turn, implies that such a  $J'$  satisfies  $v(J') \geq 6$ . There are three non-isomorphic induced subgraphs of  $\widehat{K}_{3,4}$  with the above traits and it is easy to verify that all of them satisfy the aforementioned inequality, contrary to our assumption.

It follows that

$$n^{v(J)}p^{e(J)} = \omega(n^{v(J)-(7/15)e(J)}) = \omega(1)$$

holds for all induced subgraphs of  $K$ . Therefore, the claim follows by Corollary A.2.  $\square$

Let  $K_{1,25}^\Delta$  denote the graph obtained from  $K_{1,25}$  by attaching 49 triangles to each of its edges, where the vertex not in  $K_{1,25}$  is unique for every triangle.

**Claim A.10.** Write  $H = K_{1,25}^\Delta$ . Let  $\eta > 0$  be fixed and let  $p = p(n) = \omega(n^{-7/15})$ . Then a.a.s.  $X_H(Y) \geq 1$  holds for every  $Y \subseteq [n]$  of size  $|Y| \geq \eta n$ .

**Proof.** We claim that  $v(J) - (7/15)e(J) \geq 1$  holds whenever  $J$  is an induced subgraph of  $H$  with at least one edge. Suppose for a contradiction that the assertion is false and let  $J'$  be a minimal induced subgraph of  $H$  with at least one vertex for which  $v(J') - (7/15)e(J') < 1$  holds; note that in fact  $v(J') > 1$ . Since  $H$  is 2-degenerate,  $J'$  admits a vertex  $u$  of degree at most 2. The graph  $J'' := J' \setminus \{u\}$  satisfies

$$v(J'') - (7/15)e(J'') \leq v(J') - 1 - (7/15)(e(J') - 2) = v(J') - (7/15)e(J') - 1/15 < 1$$

contrary to the minimality of  $J'$ .

It thus follows that

$$n^{v(J)} p^{e(J)} = \omega(n^{v(J) - (7/15)e(J)}) = \omega(n)$$

holds whenever  $J$  is an induced subgraph of  $H$  with  $e(J) \geq 1$ . Therefore, the claim follows by Corollary A.3.  $\square$

## B Proof of Claims 7.2 and 7.15

**Proof of Claim 7.2.** Starting with Part (a), note that since the edge  $xy$  is added in an edge-step, at least one of  $x$  and  $y$  does not belong to  $H_0$ . Up to relabelling, there are the following five options regarding the last step before all of  $x, y, z, w$  appear in the graph:  $x$  appears last (amongst  $\{x, y, z, w\}$ ) in a vertex-step;  $x$  appears last in a standard step (together with some vertex  $x' \notin \{y, z, w\}$ );  $z$  appears last in a vertex-step;  $z$  and  $w$  appear last in a standard step; or  $x$  and  $z$  appear last in a standard step. The latter three options all imply that  $x$  and  $y$  are adjacent by the time the last of  $x, y, z, w$  appears, contradicting the assumption that  $xy$  is a non-edge at this point. (For instance, if  $z$  appears last in a vertex-step, then it must be attached to a triangle consisting of the vertices  $x, y, w$ .) Hence, we may assume that  $x$  appears last amongst  $\{x, y, z, w\}$ .

Suppose for a contradiction that the addition of  $xy$  completes two distinct copies of  $K_4$  given by  $\{x, y, z, w\}$  and  $\{x, y, z', w'\}$ ; without loss of generality we may assume that  $w \neq w'$ . A similar argument to the one used above to establish that we may assume that  $x$  appears last amongst  $\{x, y, z, w\}$ , can be used again so that we may further assume that  $x$  appears last amongst  $\{x, y, z, z', w, w'\}$ . It follows that  $x$  has at most three neighbours amongst  $\{w, w', z, z'\}$ , implying that  $z = z'$  and that  $x$  is added in a vertex-step connecting it to  $z, w$  and  $w'$ . Therefore,  $\{x, y, z, w, w'\}$  forms a copy of  $K_5$  (after  $xy$  is added to the graph), contradicting the assumption that  $H$  is  $K_5$ -free.

For the proof of Part (b), we may again assume, as above, that  $x$  appears after  $y, z$ , and  $w$ . The vertex  $x$  appears either in a standard step or in a vertex-step. If the former occurs, then in this step  $x$  and some vertex  $x' \notin \{y, z, w\}$  are added to the graph and are connected to each other and to  $z$  and  $w$ . We may then replace the latter step and the edge-step in which  $xy$  is added to the graph with two consecutive vertex-steps: one attaching  $x$  to  $\{y, z, w\}$  and the second attaching  $x'$  to  $\{x, z, w\}$ .

This results in a smaller value of  $\gamma$ , contradicting the minimality (as stated in **(T1)**) of the stretched sequence generating  $H$ .

Suppose then that  $x$  appears last in a vertex-step. If this vertex-step is with missing edges, the claim follows (as a vertex-step with missing edges preceded the edge-step in which  $xy$  was added). We may thus assume that this vertex-step is without missing edges. Therefore, in this step  $x$  is attached to a triangle, spanned by  $\{x', z, w\}$  for some  $x' \neq y$ .

Assume first that  $y$  appears after  $x', z$  and  $w$  have all appeared. Then, the order of the steps can be altered so that the step adding  $x$  is performed immediately after the appearance of  $x'zw$ . This means that  $y$  is then the last vertex to appear amongst  $\{x, y, z, w\}$ . If  $y$  appears in a standard step, we again obtain a contradiction to the minimality of  $\gamma$  of the stretched generating sequence, as seen in the previous paragraph. Otherwise,  $y$  appears in a vertex-step, implying that there are at least two vertex-steps before the first edge-step. This concludes the proof in this case, as the two vertex-steps (which add  $x$  and  $y$ ) precede the first edge-step also in the original sequence.

Assume then that at least one of  $\{x', z, w\}$  does not appear before  $y$ . If all of  $\{x', y, z, w\}$  appear together, they belong to  $H_0$  and thus form a  $K_4$ ; together with  $x$  they thus eventually form a  $K_5$ , contrary to the assumption that the graph is  $K_5$ -free. Similarly, if  $z$  or  $w$  appear last amongst  $\{x', z, w\}$  (possibly together with another of these three vertices), then  $y$  and  $x'$  must be adjacent, again implying that  $\{x', x, y, z, w\}$  forms a  $K_5$  in  $H$ . It follows that  $x'$  appears after  $y, z, w$ . If  $x'$  appears in a vertex-step, then there are at least two vertex-steps before the first edge-step, as required. If  $x'$  appears in a standard step, then it is added together with a vertex  $x''$  and they are both connected to one another and to  $z$  and  $w$ . We can then modify the sequence as follows: immediately after  $\{y, z, w\}$  all appear, attach  $x$  to  $\{y, z, w\}$ , then attach  $x'$  to  $\{x, z, w\}$ , and then attach  $x''$  to  $\{x', z, w\}$ ; this decreases  $\gamma$ , contradicting the minimality of the stretched generating sequence of  $H$ .  $\square$

**Proof of Claim 7.15.** The proof is similar to the proof of Claim 7.2, provided above; therefore we provide only a sketch. Denote the edge that is added in this first edge-step by  $xy$ , and suppose that it completes a  $K_4$  whose vertex set is  $\{x, y, z, w\}$ . As in the proof of Claim 7.2, without loss of generality, we may assume that  $x$  appears after  $y, z, w$ . Moreover, if it appears in a standard step, we obtain a contradiction to the optimality of the sequence  $H'_0, \dots, H'_r$  (here we retain the notation of the original setting in which Claim 7.15 is stated). This implies that  $x$  appears in a vertex-step, as required.  $\square$