Partitioning a graph into monochromatic connected subgraphs

António Girão* Shoham Letzter† Julian Sahasrabudhe‡

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Abstract

We show that every 2-edge-coloured graph on n vertices with minimum degree at least $\frac{2n-5}{3}$ can be partitioned into two monochromatic connected subgraphs, provided n is sufficiently large. This minimum degree condition is tight and the result proves a conjecture of Bal and DeBiasio. We also make progress on another conjecture of Bal and DeBiasio on covering graphs with large minimum degree with monochromatic components of distinct colours.

1 Introduction

It is an old observation of Erdős and Rado that every 2-edge-colouring of the complete graph contains a monochromatic spanning tree. While this fact is easy enough to prove (one line with induction) its discovery opened up a new avenue in graph Ramsey theory; the study of large, sparse structures that appear in every 2-edge-colouring of the complete graph.

A, now classical, example appears in a seminal paper of Erdős, Gyárfás and Pyber [8], that for any r-edge-colouring of K_n (the complete graph on n vertices) the vertices can be covered by $O(r^2 \log r)$ vertex-disjoint, monochromatic cycles. We note that throughout the paper, when we say that the vertices of a graph are covered (or partitioned) by a collection of subgraphs, we mean that the vertices are covered by the *vertex sets* of these subgraphs.

Gyárfás, Ruszinkó, Sárközy and Szemerédi [10] improved the above result by showing that if the edges of the complete graph are r-coloured then the vertices can be partitioned into $O(r \log r)$ monochromatic cycles. In the other direction, Pokrovskiy [15] showed that one needs strictly more than r cycles, disproving a conjecture of Erdős, Gyárfás and Pyber [8]. Conlon and Stein [5] showed similar results for colourings where every vertex is incident with at most r distinct colours. The question of whether one

*Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge, UK; e-mail: A.Girao@dpmms.cam.ac.uk.
†ETH Institute for Theoretical Studies, ETH, 8092 Zurich; e-mail: shoham.letzter@eth-its.ethz.ch.
‡Department of Mathematics, University of Memphis, Memphis, Tennessee; e-mail: julian.sahasra@gmail.com.

can partition an r-coloured graph into $O(r)$ monochromatic cycles remains an enticing open problem in this area.

In a different direction, Erdős, Gyárfás and Pyber [8] conjectured that the vertices of an r-coloured complete graph may be partitioned into at most $r-1$ monochromatic connected subgraphs. This conjecture is known to be tight when $r-1$ is a prime power and n is sufficiently large, due to a well-known construction which requires the existence of an affine plane of an appropriate order. Haxell and Kohayakawa [12] proved a slightly weaker result, showing that one can partition an r-coloured complete graph on n vertices into r monochromatic subgraphs, for sufficiently large n.

Interestingly, this problem is closely related to a well-known conjecture of Ryser on packing and covering edges in r-partite, r-uniform hypergraphs. This link was first noted by Gyárfás [9] in 1997 and leads to the following natural formulation of the conjecture of Ryser, published in [13], where $\alpha(G)$ is the size of the largest independent set in the graph G.

Conjecture 1.1. *The vertex set of an r-coloured graph G can be covered by at most $(r-1)\alpha(G)$ monochromatic connected subgraphs.*

In this form, it is clear that Ryser's conjecture implies the *covering version* of the aforementioned conjecture of Erdős, Gyárfás and Pyber about monochromatic connected subgraphs. Although not much is known about Ryser's conjecture in general, a few special cases are understood. The case $r=2$ is equivalent to König's classical theorem (see [6], for example), while the case $r=3$ was proved by Aharoni [1] in 2001, who built on the earlier advances of Aharoni and Haxell [2]. Beyond this, there are only a few other cases where the conjecture is known to hold: when G is a complete graph and when $r \leq 5$, cumulatively proved by Gyárfás [9] ($r=3$), Duchet [7] and Tuza [17] ($r=4,5$).

Following Schelp [16], who suggested several variants of Ramsey-type problems (e.g. determining the length of the longest monochromatic path in a 2-coloured graphs), we consider variants of the above problems for graphs with large minimum degree. Our first main result proves a conjecture of Bal and DeBiasio [3] on partitioning the vertices of a 2-coloured graph with large minimum degree. Recall that $\delta(G)$ denotes the minimum degree of the graph G.

Theorem 1.2. *There exists an integer n_0 such that every 2-coloured graph G on $n \geq n_0$ vertices and with minimum degree at least $\frac{2n-5}{3}$ can be partitioned into two monochromatic connected subgraphs.*

This result is seen to be sharp by a construction of Bal and DeBiasio [3] which we generalise in Section 5. We also note that Theorem 1.2 generalises the result of Haxell and Kohayakawa [12] to all graphs with sufficiently large minimum degree - in the case of two colours. One can think of this result as saying that $\frac{2n-5}{3}$ is the minimum degree 'threshold' that guarantees a partition of every 2-colouring into two monochromatic connected subgraphs. It is therefore natural to ask what minimum degree condition on a graph G guarantees a partition into t monochromatic connected subgraphs, no matter how the graph is 2-coloured. We conjecture the following.

Conjecture 1.3. *For every t there exists n_0, such that for every 2-colouring of a graph G on $n \geq n_0$ vertices with $\delta(G) \geq \frac{2n-2t-1}{t+1}$ there exists a partition of the vertex set into at most t monochromatic connected subgraphs.*

We support this conjecture by observing an analogous result for *covers* of the vertices by monochromatic components.

Proposition 1.4. *Let t be integer and let G be a 2-coloured graph on n vertices with $\delta(G) \geq \frac{2n-2t-1}{t+1}$. Then the vertices of G can be covered by at most t monochromatic components.*

We also give a construction showing that the inequality in this proposition cannot be improved. It also follows that the conjecture is sharp, if true.

Bal and DeBiasio [3] also considered the problem of covering coloured graphs with monochromatic components of distinct colours. In particular, they conjectured the following.

Conjecture 1.5. *Let G be an r-coloured graph on n vertices with $\delta(G) \geq (1 - 1/2^r)n$. Then the vertices can be covered by monochromatic components of distinct colours.*

Again, Bal and DeBiasio provided examples showing that if true, the bound $(1-2^{-r})n$ is best possible. We prove Conjecture 1.5 for $r = 2, 3$. We state the theorem for $r = 3$ below, as it is the main content of our theorem.

Theorem 1.6. *Let G be a 3-coloured graph on n vertices with $\delta(G) \geq 7n/8$. Then the vertices of G can be covered by monochromatic components of distinct colours.*

We conclude the introduction with a description of the notation that we shall use in this paper. We prove Theorem 1.2 in Section 2. We prove Proposition 1.4 in Section 3 and provide an example to show that it is tight. In Section 4 we prove Theorem 1.6. We conclude the paper in Section 5 with some final remarks and open problems.

1.1 Notation

By an *r-coloured* graph, we mean a graph whose edges are coloured with r colours. When a graph is 2-coloured we call the colours *red* and *blue*; and when it is 3-coloured, we call the colours *red*, *blue* and *yellow*.

For a set of vertices W, we denote by $N_r(W)$ the set of vertices in $V(G) \setminus W$ that are adjacent to a vertex in W by a red edge. If $x \in V(G)$ is a vertex, we define $d_r(x) = |N_r(\{x\})|$ which we refer to as the *red degree* of x. We say that y is a *red neighbour* of x if xy is a red edge. By a *red component* of a graph G, we mean the *vertex set* $C \subseteq V(G)$ of a component of the red graph. We denote the red component that contains x by $C_r(x)$. A *red set* $U \subseteq V(G)$ is a set of vertices for which the red edges induced by U form a connected graph.

All the above definitions and notation, that were defined for red, also works for blue or yellow; e.g. $d_b(x)$ and $d_y(x)$ are the blue and yellow degrees of x, respectively, and a blue set is a set of vertices that is connected in blue.

2 Partitioning into monochromatic connected subgraphs

In this section we prove Theorem 1.2.

Theorem 1.2. *There exists an integer n_0 such that every 2-coloured graph G on $n \geq n_0$ vertices and with minimum degree at least $\frac{2n-5}{3}$ can be partitioned into two monochromatic connected subgraphs.*

We note that the minimum degree condition in this theorem cannot be improved; this can be seen by taking $t = 2$ in Example 3.1, described in Section 5.

Proof of Theorem 1.2. Throughout this proof, we assume that the number of vertices n is sufficiently large. Suppose, for a contradiction, that the vertices of G cannot be partitioned into two monochromatic sets.

Claim 2.1. *There is a blue component of order at most $(n + 2)/6$.*

Proof of Claim 2.1. We may assume that there are at least three red components and at least three blue components (where a single vertex that is not incident to any blue edges counts as a blue component), as otherwise the vertices may be partitioned into two red sets or two blue sets (recall that a *red set* is defined to be a set of vertices that is connected in red, and similarly for blue), contradicting our assumption. Let R be a red component of smallest order, so $|R| \leq n/3$.

Let us assume first that $|R| \leq (n-5)/3$. Since every vertex in R sends at least $(2n-5)/3-(|R|-1) > (n - |R|)/2$ blue edges outside of R, every two vertices in R have a common blue neighbour outside of R. Hence, R is contained in a blue component of order at least $|R| + (2n - 5)/3 - (|R| - 1) \geq (2n-2)/3$. Since there are at least three blue components, there is a blue component of order at most $(n - (2n - 2)/3)/2 = (n + 2)/6$.

We now assume that $(n-4)/3 \leq |R| \leq n/3$. If every two vertices in R have a common blue neighbour, then, again, R is contained in a blue component of order at least $(2n - 2)/3$ and as before there is a blue component of order at most $(n + 2)/6$. Otherwise, there exist two vertices $u, v \in R$ whose blue neighbourhoods do not intersect. But every vertex in R has at least $(n - 5)/3$ blue neighbours outside of R, and therefore every vertex in $R \setminus \{u, v\}$ has a common blue neighbour with either u or v. It follows that there are two blue components (namely, the components $C_b(u)$ and $C_b(w)$) whose union has order at least $|R| + 2(n - 5)/3 > n - 5$, hence there is a blue component of order at most 4. $\square$

Claim 2.2. *There is a red set U of size at most $27 \log n$ such that $|N_r(U)| \geq 2n/3 - 27 \log n$.*

Proof of Claim 2.2. By the previous claim, there is a blue component B of order at most $(n+2)/6$. Note that every vertex in B has at least $(2n-5)/3 - |B|$ red neighbours in $V(G) \setminus B$. Fix a vertex $u \in B$ and let T be the set of red neighbours of u outside B. Every $w \in B$ has at least the following number of red neighbours in T.

$$2 \cdot ((2n-5)/3 - |B|) - (n - |B|) = (n-10)/3 - |B| \geq (n-22)/6.$$

Now let U' be a random subset of T where each vertex $w \in T$ belongs to U', independently, with probability $13 \log n / n$. Let I_w be the event that w (where $w \in B$) does not have a red neighbour in U'. We bound

$$\mathbb{P}\Big(\bigcup_{w \in B} I_w\Big) \leq |B| \cdot \mathbb{P}(I_w) \leq n \cdot \left(1 - \frac{13 \log n}{n}\right)^{\frac{n-22}{6}} \leq n \cdot e^{-2 \log n} < 1/2.$$

Note that since $\mathbb{E}(|U'|) \leq 13 \log n$, we have $\mathbb{P}(|U'| \geq 26 \log n) \leq 1/2$, by Markov's inequality. Therefore, there is a choice of $U' \subseteq T$ such that $|U'| \leq 26 \log n$ and every vertex in B is joined by a red edge to some vertex in U'. We choose $U = U' \cup \{u\}$. Note that

$$\begin{aligned} \big|N_r(U' \cup \{u\})\big| &\geq |T \setminus U'| + |B \setminus \{u\}| \\ &\geq ((2n-5)/3 - |B| - 26 \log n) + (|B| - 1) \\ &= 2n/3 - 27 \log n. \end{aligned}$$

Hence, the set $U = U' \cup \{u\}$ satisfies the requirements of Claim 2.2. $\square$

Let U be a red set as in Claim 2.2 and let $N = N_r(U)$. Now choose a maximal sequence of distinct vertices $x_1, \ldots, x_t \in V \setminus (N \cup U)$ so that x_i has at least $\log n$ red neighbours in the set $N \cup \{x_1, \ldots, x_{i-1}\}$, for every $i \in [t]$. Then put $\overline{N} = N \cup \{x_1, \ldots, x_t\}$ and write $W = V(G) \setminus \big(U \cup \overline{N}\big)$. Note that every vertex in W has at most $\log n$ red neighbours in $\overline{N}$.

Claim 2.3. *There exist two vertices in W that have at most $\log n$ common blue neighbours in $\overline{N}$.*

Proof of Claim 2.3. For a contradiction, suppose that every two vertices in W have at least $\log n$ common neighbours in $\overline{N}$. We shall deduce that the vertices can be partitioned into a red set and a blue set, a contradiction.

To define the partition, fix $w \in W$ and let $X = N_b(w) \cap \overline{N}$. Let S be a random subset of X, obtained by taking each vertex of X independently with probability $1/2$. We claim that, with positive probability, $(U \cup \overline{N}) \setminus S$ is red *and* $W \cup S$ is blue.

To bound the probability that $W \cup S$ is blue, we consider the probability that every vertex in W is joined by a blue edge to S (an event which would imply that $W \cup S$ is blue). Since every $x \in W$ has at least $\log n$ blue neighbours in X, the probability that a x has no blue neighbours in S is at most

$2^{-\log n} = 1/n$. Thus, the expected number of vertices in W with no edges to S is smaller than $1/2$ (note that $|W| \le n/3$). Hence, $\mathbb{P}(W \cup S \textit{ is blue}) > 1/2$.

We now estimate the probability that $(U \cup \overline{N}) \setminus S$ is red. First note that as $N = N_r(U)$, we have that $U \cup N'$ is red for any subset $N' \subseteq N$. So it remains to show that the vertices of $\{x_1, \ldots, x_t\} \setminus S$ can be joined, via a red path, to $U \cup (N \setminus S)$, with sufficiently high probability. For $i \in [t]$, let E_i be the event that vertex x_i is joined by a red edge to $(N \cup \{x_1, \ldots, x_{i-1}\}) \setminus S$. Note that if the event $E = \bigcap_i^t E_i$ holds, $(U \cup \overline{N}) \setminus S$ is red. Now, to estimate $\mathbb{P}(E_i)$, for $i \in [t]$, note that each vertex x_i has at least $\log n$ forward neighbours, and the probability that one of these vertices is deleted is at most $1/2$. Thus $\mathbb{P}(E_i) \ge 1 - 2^{-\log n} = 1 - 1/n$, therefore $\mathbb{P}\big((U \cup \overline{N}) \setminus S \text{ is red}\big) \ge \mathbb{P}(E) > 1/2$, where the second inequality holds since $t < n/2$.

Thus, with positive probability, $W \cup S$ is blue and $(U \cup \overline{N}) \setminus S$ is red. In particular, the vertices can be partitioned into a blue set and a red one, a contradiction. $\square$

Claim 2.4. *There is a vertex of blue degree at most* $60 \log n$.

Proof of Claim 2.4. By definition of $\overline{N}$ every vertex in W has at least $(2n-5)/3 - (n - |\overline{N}|) - \log n \ge |\overline{N}|/2 - 15 \log n$ blue neighbours in $\overline{N}$ (where the lower bound follows as $|\overline{N}| \ge 2n/3 - 27 \log n$).

By the previous claim, there exist $v, w \in W$ such that $|N_b(v) \cap N_b(w) \cap \overline{N}| < \log n$. Then the at most two blue components containing v and w cover all vertices of W and all but at most $30 \log n$ vertices of $\overline{N}$. Since $|U| \le 27 \log n$, it follows that these two components cover all but at most $60 \log n$ vertices. Recall that there are at least three blue components, hence there is a component of order at most $60 \log n$, and any vertex in that component has blue degree at most $60 \log n$. $\square$

Let u_r be a vertex of blue degree at most $60 \log n$, which exists by the previous claim. By symmetry, there is a vertex u_b of red degree at most $60 \log n$. Then $d_r(u_r), d_b(u_b) \ge 2n/3 - 60 \log n - 2$. Write $A_1 = N_b(u_b) \setminus N_r(u_r)$, $A_2 = N_b(u_b) \cap N_r(u_r)$ and $A_3 = N_r(u_r) \setminus N_b(u_b)$. Then $|A_2| \ge n/3 - 120 \log n - 4$ and $|A_1|, |A_3| \le n/3 + 60 \log n + 2$.

Claim 2.5. *There is a vertex with no blue neighbours in A_1, no red neighbours in A_3, and at most* $2 \log n$ *neighbours in A_2.*

Proof of Claim 2.5. Suppose that the statement does not hold. Let $\{B, R\}$ be a random partition of A_2, obtained by putting vertices in B, independently, with probability $1/2$. Then, with positive probability, every vertex in G has a blue neighbour in $A_1 \cup B \subseteq N_b(u_b)$ or a red neighbour in $A_3 \cup R \subseteq N_r(u_r)$. We thus obtain a partition of the vertices into a red set and a blue set, a contradiction. $\square$

Let x be a vertex with no blue neighbours in A_1, no red neighbours in A_3, and at most $2 \log n$ neighbours in A_2 (its existence is guaranteed by the previous claim). Then $|A_2| \le n/3 + 3 \log n$, so $|A_1|, |A_3| \ge n/3 - 65 \log n$. Furthermore, x has at least $n/3 - 70 \log n$ red neighbours in A_1 and at least $n/3 - 70 \log n$ blue neighbours in A_3. Write $A_1' = A_1 \cap N_r(x)$, $A_2' = A_2 \setminus N(x)$, and $A_3' = A_3 \cap N_b(x)$ (so $|A_1'|, |A_3'| \ge n/3 - 70 \log n$ and $|A_2'| \ge n/3 - 130 \log n$).

Claim 2.6. *The vertices x and u_b are in distinct blue components; similarly, x and u_r are in distinct red components.*

Proof of Claim 2.6. Suppose that x and u_b are in the same blue component. Then there is a blue path P from $\{x\} \cup A_3'$ to $\{u_b\} \cup A_1' \cup A_2'$. We may assume that the inner vertices of P are outside of $A_1' \cup A_2' \cup A_3' \cup \{x, u_b\}$. Hence, $|P| \leq 300 \log n$.

Now, let $\{B, R\}$ be a random partition of $(A_2' \cup A_3') \setminus V(P)$, obtained by putting each vertex in B, independently, with probability $1/2$. It is easy to see that, with positive probability, every vertex in G has a red neighbour in R or a blue neighbour in B, from which it can be deduced that there is a partition of the vertices into a red set and a blue set, which is a contradiction. Indeed, note that $P \cup \{x, u_b\} \cup B$ is a blue set and $\{u_r\} \cup R$ is a red set. Thus, we have that u_b and x are in distinct blue components; by symmetry, u_r and x are in different red components. $\square$

Note that $|C_b(u_b)|, |C_r(u_r)| \geq 2n/3 - 61 \log n$ and $|C_b(x)|, |C_r(x)| \geq n/3 - 70 \log n$. Recall that there are at least three blue components. Hence, there is a vertex w_r which is not in $C_b(u_b)$ or in $C_b(x)$. It follows that $d_b(w_r)$ is at most $131 \log n$, hence it has red degree at least $2n/3 - 132 \log n$, so $w_r \in C_r(u_r)$. Similarly, there is a vertex w_b which is not in $C_r(u_r)$ or in $C_r(x)$, and therefore it must belong to $C_b(u_b)$. We claim that the set $X = \{w_b, w_r, x\}$ is independent. The pairs $w_r x$ or $w_b x$ are non-edges, for otherwise we obtain a contradiction to either the choice of $w_r \notin C_b(x)$ and $w_b \notin C_r(x)$ or the statement of Claim 2.6. If $w_r w_b$ is a red edge then $w_b \in C_r(w_r)$ which is a contradiction, by definition of w_b, and similarly if $w_r w_b$ is a blue edge. Thus X is independent. So finally, by the minimum degree condition, there must be a vertex y that is adjacent to all three vertices in X. Indeed, if no such y exists, then the number of edges between X and $V(G) \setminus X$ is at most $2(n-3) < 3(2n-5)/3$, a contradiction. Without loss of generality, y sends two red edges into X, implying that two of the vertices in X belong to the same red component, a contradiction. This completes our proof of Theorem 1.2. $\square$

3 Covering by t monochromatic components

In this section we prove Proposition 1.4; this proposition can be thought of as a generalisation of the weaker covering version of Theorem 1.2.

Proposition 1.4. *Let t be integer and let G be a 2-coloured graph on n vertices with $\delta(G) \geq \frac{2n-2t-1}{t+1}$. Then the vertices of G can be covered by at most t monochromatic components.*

Proof of Proposition 1.4. We form an auxiliary graph H, whose vertex set is the vertex set of G, two vertices are joined by a blue edge if and only if they are in the same blue component in G, and red edges are defined similarly. Note that H is a multigraph that contains G as a subgraph. Moreover, the vertex sets of monochromatic components in G and H are the same. Hence, it suffices to show that the vertices of H can be covered by t monochromatic components.

We claim that $\alpha(H) \le t$. Indeed, suppose that A is an independent set of size $t+1$. By definition of H, every two vertices of A belong to distinct red and blue components. In particular, every vertex outside of A sends at most two edges into A, so $e(G[A, V \setminus A]) \le 2(n-t-1)$ (here $V = V(G)$). On the other hand, by the minimum degree condition, $e(G[A, V \setminus A]) \ge \delta(G) \cdot (t+1) \ge 2(n-t-1)+1$, a contradiction. So, indeed, $\alpha(H) \le t$. Hence, by Ryser's conjecture for $r = 2$ (which follows from König's theorem, see [6]), we find that the vertices of H can be covered by at most $\alpha(G) \le t$ monochromatic components, as required. $\square$

We remark that a similar argument can be used to show that if Ryser's conjecture holds for r, and $t \ge r$, then the following holds. If G is a r-coloured graph on n vertices with minimum degree larger than $\frac{r(n-t-1)}{t+1}$ then its vertices can be covered by at most $(r-1)t$ monochromatic components. However, this bound is not tight[1].

We note that the restriction on the minimum degree in Proposition 1.4 cannot be improved. The special case of this example, where $t = 2$, appears in [3] and shows that the minimum degree condition in Theorem 1.2 is best possible.

Example 3.1. *Let U be a set of size $n \ge t+1$, and let $\{X, A_1, \ldots, A_{t+1}\}$ be a partition of U, where $|X| = t+1$ and the sizes of $A_1, A_2, \ldots, A_{t+1}$ are as equal as possible; write $X = \{x_1, \ldots, x_{t+1}\}$. We define a 2-coloured graph G on vertex set U as follows.*

- *the sets A_i are cliques, and we colour them arbitrarily;*
- *we add all possible edges between A_i and A_{i+1}, where $i \in [t]$, and colour them red if i is odd, and blue otherwise;*
- *we add all edges between x_i and $A_i \cup A_{i+1}$, for $i \in [t+1]$ (addition is taken modulo $t+1$). We colour these edges red if i is in $[t]$ and i is odd; and blue if i is in $[t]$ and i is even. Finally, we colour the edges from x_{t+1} to A_1 blue, and colour the edges from x_{t+1} to A_{t+1} red if t is even and blue if t is odd.*

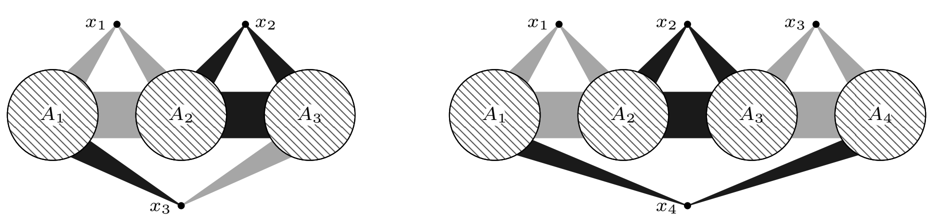

Figure 1: an illustration of Example 3.1 for $t = 2$ and $t = 3$ (here grey represents red and black represents blue).

[1] It can be shown that if G is a 3-coloured graph on n vertices with minimum degree larger than $(3/4 - 1/(12 \cdot 13))n$ then its vertices can be covered by at most six monochromatic components, which is smaller than the bound $3(n-4)/4$ suggested by this arguments; for the sake of brevity we do not present the proof of this claim here.

An easy calculation shows that G has minimum degree[2] $\lceil (2n-2t-1)/(t+1) \rceil - 1$, *and that no two vertices in X belong to the same monochromatic component; in particular, the vertices of G cannot be covered by at most t monochromatic components.*

4 Covering with monochromatic components of distinct colours

In this section we verify Conjecture 1.5 for $r \in \{2,3\}$. Most of the difficulty is in the proof for $r = 3$, but we include a short proof for $r = 2$ for completeness. Actually, the $r = 2$ case (for n large) already follows from a difficult result of Letzter [14], who showed that when $\delta(G) \geq 3n/4$, the vertices can be partitioned into two monochromatic *cycles* of different colours, for every 2-colouring of G. Before turning to the proofs, we mention the following construction of Bal and DeBiasio [3], which shows that the minimum degree condition in Conjecture 1.5 cannot be improved.

Example 4.1. *Let $n \geq 2^r$; we shall define a graph on vertex set $[n]$ as follows. Partition $[n]$, as equally as possible, into 2^r sets which are indexed by the sequences $s \in \{0,1\}^r$. We write*

$$[n] = \bigcup_{s \in \{0,1\}^r} A(s)$$

and define the following, where $\mathbb{1} = (1, \ldots, 1)$.

$$E = [n]^{(2)} \setminus \bigcup_{s \in \{0,1\}^r} \{xy : x \in A(s), y \in A(\mathbb{1} - s)\}.$$

In other words, we include all edges in the graph except for the edges between parts of the partition corresponding to antipodal elements of $\{0,1\}^r$. Now, colour all edges xy, where $x \in A(s), y \in A(s')$, by the first coordinate on which s, s' agree; e.g. the edge between $(0,1,0,0)$ and $(1,0,0,1)$ is coloured 3.

We now show that G cannot be covered by components of distinct colours. Suppose that it can, and note that the i-coloured components are of the form $\bigcup_{s \in S_i} A(s)$ where S_i is a set of elements that agree on their i-th coordinate; denote this coordinate by a_i. It follows that the vertices of $A((1-a_1, \ldots, 1-a_r))$ are not covered by any of these components, a contradiction.

We now prove Conjecture 1.5 for $r = 2$.

Lemma 4.2. *Let G be a 2-coloured graph with $\delta(G) \geq 3n/4$. Then the vertices of G can be covered by a red component and a blue component.*

Proof. We first show that there is a monochromatic component of order greater than $n/2$. If G is red connected we are done. Hence, there exists a red component R with $|R| \leq n/2$. Then, any two vertices

[2] In fact, we need to be a bit more careful here. Write $n = a(t+1) + r$, where a and r are integers and $0 \leq r \leq t$. We consider two cases: $r < \lceil (t+1)/2 \rceil$ and $r \geq \lceil (t+1)/2 \rceil$. In the former case, it is easy to see that $\delta(G) = \lceil (2n-2t-1)/(t+1) \rceil - 1$. In the latter case, note that exactly r of the sets A_i have size a, and the rest have size $a-1$. Then, again, one can check that $\delta(G) = \lceil (2n-2t-1)/(t+1) \rceil - 1$ if $|A_i| = a$ for every odd $i \in [t+1]$ (which is possible as $r \geq (t+1)/2$).

$u, w \in R$ have a common blue neighbour, as $|N_b(u) \cap N_b(w) \cap \overline{R}| \geq 2 \cdot (3n/4 - (|R|-1)) - (n - |R|) > 0$. So $R \subseteq C_b(u)$ and $C_b(u)$ is a blue component of order at least $3n/4$, as required.

Without loss of generality, there is a red component R of order larger than $n/2$. Note that there is a vertex x which is not in R (otherwise we are done), and $|N_b(x) \cap R| = |N(x) \cap R| > n/4$, as x does not send red edges to R. In particular, $|C_b(x) \cap R| > n/4$. It follows that every vertex sends at least one edge to $C_b(x) \cap R$ and thus the components R and $C_b(x)$ cover the whole graph. □

Before we turn to prove Conjecture 1.5 for $r = 3$, we mention the following proposition, which is due to Bal, DeBiasio and McKenny [4].

Proposition 4.3 (Bal, DeBiasio and McKenny [4]). *Let G be a 3-coloured graph on n vertices with minimum degree at least $7n/8$. Then G contains a monochromatic component of order at least $n/2$.*

This proposition strengthens a result of Gyárfás and Sárközy [11] who proved the same statement for graphs with minimum degree at least $9n/10$, and also conjectured that the lower bound on the minimum degree could be lowered to $7n/8$. We note that there are examples of 3-coloured complete graphs on n vertices with no monochromatic component of order larger than $n/2$, hence the lower bound on the order of the monochromatic component is best possible. We present the proof of Proposition 4.3 here for the sake of completeness and because we are not aware of a published version of this proof.

Proof of Proposition 4.3. We assume that all monochromatic components have order smaller than $n/2$. Denote by B the monochromatic component of largest order; without loss of generality it is blue, and by assumption we have $|B| < n/2$. Let R be a monochromatic component, distinct from B, that intersects B and maximises $|R \setminus B|$; without loss of generality R is red. Denote $A_1 = B \setminus R$, $A_2 = R \setminus B$, $U = B \cap R$, and $W = V(G) \setminus (R \cup B)$. Note that $|A_1| \geq |A_2|$, all of A_1, A_2, U, W are non-empty, and all edges between A_1 and A_2 and between U and W are yellow.

We claim that $|A_1 \cup A_2| < n/2$. Indeed, suppose otherwise. Then $|A_1| \geq n/4$ because $|A_1| \geq |A_2|$. It follows that every two vertices in A_2 have a common (yellow) neighbour in A_1 (since every vertex in A_2 has at most $n/8 - 1$ non-neighbours in A_1). Moreover, we have that $|A_2| \geq (3n/4 - (|B|-1))/2 \geq 3n/16$; this is by choice of R and because every vertex in B sends at least this number of red edges, or at least this number of yellow edges, outside of B. It follows that every vertex in A_2 has a yellow neighbour in A_1 (again, because it can have at most $n/8 - 1$ non-neighbours). We conclude that $A_1 \cup A_2$ is contained in a yellow component of order at least $n/2$, a contradiction.

We may now assume that $|U \cup W| > n/2$. We note that $|W| > |U|$. Indeed, otherwise, $|B| = |A_1| + |U| \geq |A_2| + |W| = n - |B|$, implying that $|B| \geq n/2$, a contradiction. In particular, $|W| > n/4$. As before, it follows that every two vertices in U have a (yellow) common neighbour in W. If $|U| \geq n/8$, then every vertex in W has a (yellow) neighbour in U, which implies that $U \cup W$ is contained in a yellow component of order at least $n/2$, a contradiction. So, we have $|U| < n/8$ and thus $|W| > 3n/8$. It follows from the minimum degree condition that every vertex in U sends at least $n/4$ yellow edges to W. In particular, there is a yellow component Y that intersects B and satisfies $|Y \setminus B| \geq n/4$. By choice of R, it follows that $|A_2| = |R \setminus B| \geq n/4$. But since $|A_1| \geq |A_2|$, we find that $|A_1 \cup A_2| \geq n/2$, contradicting the above. We have thus reached a contradiction to the assumption that $|B| < n/2$, which completes the proof. $\square$

We now prove Theorem 1.6.

Theorem 1.6. *Let G be a 3-coloured graph on n vertices with $\delta(G) \geq 7n/8$. Then the vertices of G can be covered by monochromatic components of distinct colours.*

Proof. By Proposition 4.3 there is a monochromatic component of order at least $n/2$; without loss of generality it is red, and we denote it by R. Suppose that $R \neq V(G)$ (otherwise, we are done) and let $u \notin R$. Consider the blue and yellow components containing u by B and Y. By the minimum degree condition, u sends at least $|R| - n/8$ edges to R, none of which are red. So $|(B \cup Y) \cap R| \geq |R| - n/8$. Suppose that R, B and Y do not cover the whole graph (again, otherwise we are done). Let $w \notin R \cup B \cup Y$, and denote the blue and yellow components containing w by B' and Y'. By the same argument as before, $|(B' \cap Y') \cap R| \geq |R| - n/8$, which implies the following.

$$|(B \cup Y) \cap (B' \cup Y') \cap R| \geq |R| - n/4 \geq n/4.$$

Since $B \cap B' = \emptyset$ and $Y \cap Y' = \emptyset$, either $|B \cap Y' \cap R| \geq n/8$ or $|B' \cap Y \cap R| \geq n/8$. Without loss of generality, the former holds; denote $U = B \cap Y' \cap R$. Since $|U| \geq n/8$ and by the minimum degree condition, every vertex not in U has a neighbour in U, which implies that every vertex in the graph belongs to one of the components B, Y' or R, thus completing the proof of Theorem 1.6. $\square$

5 Concluding remarks

To conclude, we remind the reader of a few conjectures of Bal and DeBiasio [3] before recalling our own conjecture.

First, we recall Conjecture 1.5 due to Bal and DeBiasio, which we have settled for $r \leq 3$.

Conjecture 1.5. *Let G be an r-coloured graph on n vertices with $\delta(G) \geq (1 - 1/2^r)n$. Then the vertices can be covered by monochromatic components of distinct colours.*

Second, we highlight a conjecture from the same paper of Bal and Debiasio which concerns the minimum degree required to ensure that an r-coloured graph can be covered by at most r monochromatic components (here colours need not to be distinct).

Conjecture 5.1. *Let G be an r-coloured graph on n vertices with $\delta(G) \geq \frac{r(n-r-1)+1}{r+1}$. Then the vertices of G can be covered by at most r monochromatic components.*

Finally, we recall our conjecture for the best minimum degree condition that guarantees a partition of every 2-coloured graph into t-monochromatic components.

Conjecture 1.3. *For every t there exists n_0, such that for every 2-colouring of a graph G on $n \geq n_0$ vertices with $\delta(G) \geq \frac{2n-2t-1}{t+1}$ there exists a partition of the vertex set into at most t monochromatic connected subgraphs.*

If true, the minimum degree condition in this conjecture is tight by our Example 3.1.

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