Monochromatic connected matchings in almost complete graphs

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Abstract

A *connected matching* in a graph G is a matching that is contained in a connected component of G. A well-known method due to Łuczak reduces problems about monochromatic paths and cycles in complete graphs to problems about monochromatic matchings in almost complete graphs. We show that these can be further reduced to problems about monochromatic connected matchings in complete graphs.

1 Introduction

The *k-colour Ramsey number* of a graph H, denoted $r_k(H)$, is the minimum N such that every k-edge-colouring of K_N contains a monochromatic H. The study of Ramsey-type problems for paths and cycles was initiated by Gerencsér and Gyárfás [10] in an early paper (1967) in which they determined the 2-colour Ramsey number of a path, showing that $r_2(P_n) = \lfloor \frac{3n-2}{2} \rfloor$. Quite a few results in the area, mostly about two colours, were proved in the following few years (see, e.g., [7, 8, 11, 19]). A while later, in 1999, Łuczak [18] determined, asymptotically, the 3-colour Ramsey number of an odd cycle. Since then, the 3-colour Ramsey number of P_n and C_n has been determined precisely for every large n (see [3, 9, 12, 16]). For $k \geq 4$, the k-colour Ramsey number of a path is still unknown; the best known bounds to date are $(k-1)n + O(1) \leq r_k(P_n) \leq (k-1/2)n + o(n)$ (see Yongqi, Yuansheng, Feng and Bingxi [22] for the lower bound, and Knierim and Su [15] for the upper bound). The same bounds also hold for $r_k(C_n)$ whenever n is even, and are the state of the art in this case too. Remarkably, for long odd cycles the k-colour Ramsey number is known precisely: it is $r_k(C_n) = 2^{k-1}(n-1)+1$ for sufficiently large n (Jenssen and Skokan [14]). Interestingly, this bounds does not holds for all k and n (Day and Johnson [6]).

A *connected matching* is a matching contained in a connected component. A *monochromatic connected matching* is a matching contained in a monochromatic component; similarly, an ℓ-coloured connected matching is a matching contained in a component of colour ℓ. A key method that has been used in the vast majority of results providing upper bounds on Ramsey numbers of paths and

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cycles is the use of *connected matchings* in conjunction with Szemerédi's regularity lemma [20], an idea which originated in the work of Łuczak [18] mentioned above. Łuczak observed that by applying the regularity lemma, problems about finding monochromatic paths and cycles in complete graphs can be reduced to problems about finding monochromatic connected matchings in almost complete graphs. To illustrate this, here is a special case of a lemma by Figaj and Łuczak [9] which formalises this reduction.

Lemma 1 (a special case of Lemma 3 in [9])**.** *Let $\alpha > 0$ and let k be an integer. Suppose that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every sufficiently large n the following holds: every k-colouring of a graph G on at least $(1+\varepsilon)\alpha n$ vertices and with density at least $1-\delta$ contains a monochromatic connected matching of size $\lceil n/2 \rceil$. Then $r_k(C_n) \leq (\alpha + o(1))n$ for every* even *n (and so $r_k(P_n) \leq (\alpha + o(1))n$).*

The connected matchings method, illustrated in Lemma 1, is clearly very handy; connected matchings are more convenient to work with than paths, and there are several structural results at one's disposal when studying matchings, such as Hall and König's theorems (see, e.g., [17]) for bipartite graphs and Tutte's theorem [21] for general graphs. However, the need to switch from complete graphs to almost complete graphs is a drawback. At the very least, it is a nuisance, making proofs more technical and less readable, and in some cases it can be a genuine obstacle.

Our main aim in this paper is to provide a further reduction, replacing the almost complete graphs as in Lemma 1 by complete graphs. To illustrate this, we obtain the following strengthening of Lemma 1.

Corollary 2. *Let $\alpha > 0$ and let k be an integer. Suppose that for every $\varepsilon > 0$ and every sufficiently large n, every k-colouring of $K_{\lceil (1+\varepsilon)\alpha n \rceil}$ contains a monochromatic connected matching of size $\lceil n/2 \rceil$. Then $r_k(C_n) \leq (\alpha + o(1))n$ for every even n.*

We note that while Lemma 1 is stated for the particular case of the k-colour Ramsey number of paths and even cycles, a similar statement can be obtained for various problems of similar nature. For example, the original lemma in [9] can be used to study *asymmetric* Ramsey numbers of paths and even cycles, namely where we require different path or cycle lengths for different colours[1]. Moreover, it is applicable to odd cycles, where an ℓ-coloured C_n, for odd n, corresponds to an ℓ-coloured connected matching of size $n/2$ that is contained in a non-bipartite ℓ-coloured component. Similarly, one can use a variant of Lemma 1 to study *bipartite path-Ramsey numbers*, where the host graph is a complete bipartite graph instead of a complete graph, or, more generally, *multipartite path-Ramsey numbers*[2]. For some examples using this variant, see [2, 4, 5, 13].

Corollary 2 above can be generalised similarly. Following [9], given a real number x define $\langle\langle x \rangle\rangle$ to be the largest even number not larger than x and $\langle x \rangle$ to be the largest odd number not larger than x. We now state a generalisation of Corollary 2 where cycle lengths can vary, odd cycles are allowed, and the ground graph is a complete graph.

[1] However, the lengths all need to be of the same order of magnitude.

[2] This generalisation is not mentioned in [9], but it can be proved in a very similar way.

Corollary 3. *Let $k_0 \le k$ be integers and let $\alpha_1, \ldots, \alpha_k > 0$. Suppose that for every $\varepsilon > 0$ and sufficiently large n, every k-colouring of $K_{\lceil (1+\varepsilon)n \rceil}$ yields an ℓ-coloured connected matching M on at least $\alpha_\ell n$ vertices for some $\ell \in [k]$, where M is contained in a non-bipartite ℓ-coloured component if $\ell \in [k_0+1, k]$. Then for every $\varepsilon > 0$ and large n, every k-colouring of $K_{\lceil (1+\varepsilon)n \rceil}$ contains an ℓ-coloured cycle of even length $\langle\langle \alpha_\ell n \rangle\rangle$ if $\ell \in [k_0]$, or odd length $\langle \alpha_\ell n \rangle$ if $\ell \in [k_0+1, k]$.*

Next, we state a generalisation of Corollary 3, where the ground graph can vary and where we obtain a range of cycle lengths in one of the colours. Before stating the theorem, we need a definition. Let F be a graph on vertex set $[s]$, which may have loops, and let $m_1, \ldots, m_s > 0$. Define $F(m_1, \ldots, m_s)$ to be the blow-up of F where vertex i is replaced by a set of $\lceil m_i \rceil$ vertices, for $i \in [s]$. More precisely, the vertex set of $F(m_1, \ldots, m_s)$ is $V_1 \cup \ldots \cup V_s$, where the sets V_i are pairwise disjoint and $|V_i| = \lceil m_i \rceil$, and the edges of $F(m_1, \ldots, m_s)$ are the pairs xy (with $x \ne y$, i.e. loops are not allowed) such that $x \in V_i$ and $y \in V_j$ for some $ij \in E(F)$. In particular, V_i is a clique if F has a loop at i, and is an independent set otherwise.

Theorem 4. *Let s and $k_0 \le k$ be integers, let $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_s > 0$, and let F be a graph on vertex set $[s]$, which may have loops.*

Suppose that for every $\varepsilon > 0$ and large n, every k-colouring of the blow-up $F((\beta_1+\varepsilon)n, \ldots, (\beta_s+\varepsilon)n))$ of F yields an ℓ-coloured connected matching M on at least $\alpha_\ell n$ vertices for some $\ell \in [k]$, where M is contained in a non-bipartite ℓ-coloured component if $\ell \in [k_0+1, k]$.

Then for every $\varepsilon > 0$ there exists T such that for every sufficiently large n, for every k-colouring of $F((\beta_1+\varepsilon)n, \ldots, (\beta_s+\varepsilon)n)$ either for some $\ell \in [k_0]$ there is an ℓ-coloured C_t for every even $t \in [T, \alpha_\ell n]$, or for some $\ell \in [k_0+1, k]$ there is an ℓ-coloured C_t for every integer $t \in [T, \alpha_\ell n]$.

Our proof of Theorem 4 is based on Theorem 6, stated in Section 2, which allows us to reduce problems about monochromatic connected matchings in almost complete graphs (or, more generally, almost blow-ups of F) to complete graphs (or blow-ups of F).

In [4], together with Bucić and Sudakov, we determined, asymptotically, the *bipartite 3-colour Ramsey number* of paths and even cycles. By a variant of Lemma 1, to do so it suffices to determine, asymptotically, the size of the largest monochromatic connected matching guaranteed to exist in a 3-coloured balanced almost complete bipartite graph. As a first step, we determined the size of the largest monochromatic connected matching in a 3-colouring of $K_{n,n}$, for every n. It is often the case that proofs about monochromatic connected matchings in complete graphs (or complete bipartite in this case) can be adapted to work for almost complete graphs. However, due to the inductive nature of our proof, we were not able to find such an adaptation. Instead, we proved (implicitly; see Theorem 12 in [4]) a version of Theorem 6 below, which reduces the 3-colour Ramsey question about connected matchings in almost complete bipartite graphs to complete bipartite graphs. Our proof of Theorem 6 is similar, with the main difference being the use of the Gallai-Edmonds decomposition (see Section 3) which replaces our use of König's theorem in [4].

We note that our method is also handy when proving stability results about monochromatic connected matchings, allowing one to prove such results for complete graphs instead of almost complete graphs. This is likely to be helpful when determining Ramsey numbers of paths and cycles precisely; see Remark 9 below.

In the next section, Section 2, we state Theorem 6 which, as mentioned above, allows us to reduce Ramsey-type problems about connected matchings in almost complete graphs to complete graphs. We then show how to deduce Theorem 4 from Theorem 6 (note that Corollaries 2 and 3 follow directly from Theorem 4). In Section 3 we define the Gallai-Edmonds decomposition of a graph and state the related results that we shall need. We then prove Theorem 6 in Section 4.

2 From almost complete graphs to complete graphs

In this section we state Theorem 4 which allows us to reduce Ramsey-type problems about connected matchings in almost complete graphs to complete graphs, and more generally from almost blow-ups of a fixed graph F to blow-ups of F. We then show how to deduce Theorem 4 from Theorem 6, using a variant of Lemma 1. Before stating Theorem 6, which is quite general but has a long statement, we state the following special case of Theorem 6.

Theorem 5. *Let k be an integer, let $\alpha, \beta > 0$, let $\varepsilon > 0$ be sufficiently small, and set $\delta = \frac{\varepsilon}{2} \cdot \left(\frac{\alpha}{30\beta}\right)^{2k}$. Let $N \leq \beta n$ and $\alpha_1, \ldots, \alpha_k \geq \alpha$. Suppose that G is a k-coloured graph on at least $N + \varepsilon n$ vertices, where every vertex has at most δn non-neighbours, and there is no ℓ-coloured connected matching on at least $\alpha_\ell n$ vertices, for $\ell \in [k]$. Then there exists a k-colouring G' of K_N that contains no ℓ-coloured connected matching on at least $\alpha_\ell n$ vertices, for $\ell \in [k]$. Moreover, there exists such G' that contains an induced subgraph of G on N vertices.*

We now state the main result in this section. Note that Theorem 5 follows from the following theorem, by taking $k_0 = k$, $s = 1$ and F to be a graph on one vertex with a loop.

Theorem 6. *Let s, $k_0 \leq k$ be integers, let $\alpha, \beta > 0$, let $\varepsilon > 0$ be sufficiently small, and set $\delta = \frac{\varepsilon}{2s^2} \cdot 4^{k_0 - k} \cdot \left(\frac{\alpha}{30s\beta}\right)^{2k}$. Let $N_1, \ldots, N_s \leq \beta n$ and let $\alpha_1, \ldots, \alpha_k \geq \alpha$. Let F be a graph on vertex set $[s]$ (possibly with loops), and denote $\mathcal{G} = F(N_1 + \varepsilon n, \ldots, N_s + \varepsilon n)$ and $\mathcal{G}' = F(N_1, \ldots, N_s)$.*

Suppose that G is a k-coloured spanning subgraph of $\mathcal{G}$ such that $d_G(u) \geq d_{\mathcal{G}}(u) - \delta n$ for every vertex u; there is no ℓ-coloured connected matching on at least $\alpha_\ell n$ vertices for $\ell \in [k_0]$; and there is no ℓ-coloured connected matching on at least $\alpha_\ell n$ vertices contained in a non-bipartite ℓ-coloured component for $\ell \in [k_0 + 1, k]$.

Then there exists a k-colouring G' of $\mathcal{G}'$ such that there is no ℓ-coloured connected matching on at least $\alpha_\ell n$ vertices for $\ell \in [k_0]$; and there is no ℓ-coloured connected matching on at least $\alpha_\ell n$ vertices contained in a non-bipartite ℓ-coloured component for $\ell \in [k_0 + 1, k]$. Moreover, there exists such G' which contains a induced subgraph of G with N_i vertices from the set replacing vertex i in F for $i \in [s]$.

Remark 7. *It is sometimes convenient to allow edges to have multiple colours (see, e.g., [1, 12]). We thus allow the edges in the graph G in Theorems 5 and 6 to have multiple colours; this has very little effect on the rest of the paper.*

Before showing how to deduce Theorem 4 from Theorem 6, we state the following lemma, formalising Łuczak's connected matching in a fairly general form. This lemma is a generalisation of a lemma Łuczak [9] (and of Lemma 1). It can be proved very similarly to the lemma in [9]; we do not include a proof here.

Lemma 8 (A generalisation of Lemma 3 in [9])**.** *Let s, $k_0 \le k$ be integers, let $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_s > 0$, and let F be a graph on vertex set $[s]$ (possibly with loops).*

Suppose that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every large n, if G is a spanning subgraph of $\mathcal{G} := F((\beta_1 + \varepsilon)n, \dots, (\beta_s + \varepsilon)n)$ with $e(G) \ge (1 - \delta)e(\mathcal{G})$, the following holds: every k-colouring of G has an ℓ-coloured connected matching on at least $\alpha_\ell n$ vertices for some $\ell \in [k]$, which is contained in a non-bipartite ℓ-coloured component if $\ell \in [k_0 + 1, k]$.

Then for every $\varepsilon > 0$ there exists T such that for every sufficiently large n, for every k-colouring of $F((\beta_1 + \varepsilon)n, \dots, (\beta_s + \varepsilon)n)$ either for some $\ell \in [k_0]$ there is an ℓ-coloured C_t for every even $t \in [T, \alpha_\ell n]$, or for some $\ell \in [k_0 + 1, k]$ there is an ℓ-coloured C_t for every integer $t \in [T, \alpha_\ell n]$.

Finally, we are ready to prove Theorem 4; the proof follows easily from Lemma 8 and Theorem 6.

Proof of Theorem 4. Let $\varepsilon > 0$, let $\delta > 0$ be sufficiently small, and let n be large. Denote $\mathcal{G} = F((\beta_1 + \varepsilon)n, \dots, (\beta_s + \varepsilon)n)$ and $\mathcal{G}' = F((\beta_1 + \varepsilon/4)n, \dots, (\beta_s + \varepsilon/4)n)$. By the assumption of Theorem 4, every k-colouring of $\mathcal{G}'$ has an ℓ-coloured connected matching on at least $\alpha_\ell n$ vertices, which is contained in a non-bipartite ℓ-coloured component if $\ell \in [k_0 + 1, k]$.

Let G be a k-colouring of a spanning subgraph of $\mathcal{G}$, with $e(G) \ge (1 - \delta)e(\mathcal{G})$. By Lemma 8, it suffices to show that G contains an ℓ-coloured connected matching on at least $\alpha_\ell n$ vertices, which is contained in an ℓ-coloured non-bipartite component if $\ell \in [k_0 + 1, k]$. Suppose this is not the case.

Let W be the set of vertices w in G with $d_G(w) \le d_{\mathcal{G}}(w) - \sqrt{\delta} n$. As $|E(\mathcal{G}) \setminus E(G)| \le \delta(s\beta n)^2/2$, where $\beta = \max\{\beta_1, \dots, \beta_s\}$, we have $|W| \le s^2\beta^2\sqrt{\delta} n \le (\varepsilon/2)n$ (for sufficiently small δ). Define $G' = G \setminus W$. Assuming δ is sufficiently small, by Theorem 6 (using $\varepsilon' = \varepsilon/4$, $\beta_i' = \beta_i + \varepsilon/4$ for $i \in [s]$, $\delta' = \sqrt{\delta}$), there is a k-colouring of $\mathcal{G}'$ with no ℓ-coloured connected matching on at least $\alpha_\ell n$ vertices, which is contained in a non-bipartite ℓ-coloured component if $\ell \in [k_0 + 1, k]$, a contradiction. □

Our discussion so far showed that in order to asymptotically solve a Ramsey-type problem about paths and cycles, it suffices to consider a similar Ramsey-type question about connected matchings, where the ground graph in both cases is a blow-up of F. Our method turns out to be handy also for solving such problems precisely.

Remark 9. *It is often the case that in order to determine a Ramsey number of paths and cycles precisely, one should prove a* stability *result: either an edge-coloured graph contains a large monochromatic connected matching, or it has a special structure.*

Suppose that we are given an edge-coloured graph H where we wish to find monochromatic paths or cycles of given length. Apply the regularity lemma and let G be the so-called reduced graph *(whose vertices represent the clusters in the regular partition, and edges represent regular pairs and are coloured suitably: e.g. by a majority colour, or by all colours with large enough density in the pair). Luczak's method implies that a monochromatic connected matching in G of size $\alpha|G|$ can be lifted to a monochromatic cycle in H (of the same colour) on at least $(\alpha - \varepsilon)|H|$ vertices (where $\varepsilon > 0$ can be arbitrarily small). Thus, a stability result would tell us that either there is a suitably long monochromatic path or cycle in H, or G has a certain special structure, from which it should follows that H has a special structure. In the latter case, the required monochromatic path or cycle can be found 'by hand' by looking at the structure of H more closely.*

If the original graph H is a complete graph, then the reduced graph G is almost complete, and similarly if H is a blow-up of F then G is an almost blow-up of F. Assuming that G does not contain a large enough monochromatic connected matching, by applying Theorem 5 or Theorem 6, we obtain a slightly smaller complete graph (or blow-up of F) G' without a large enough monochromatic connected matching. Moreover, G' is very similar to G: they differ on at most $s\varepsilon n \cdot (N_1 + \ldots + N_s + s\varepsilon n) + \delta n \cdot (N_1 + \ldots + N_s) \leq 2s^2 \varepsilon \beta n^2$ edges. Given a stability result for complete graphs (or blow-ups of F), it follows that G' has a special structure, which implies that G has a similar structure.

To summarise, in order to solve a Ramsey-type problem about paths and cycles precisely, by using Theorem 5 or Theorem 6 it is often enough to prove a stability result about monochromatic connected matchings in complete graphs (blow-ups of F).

3 The Gallai-Edmonds decomposition

We shall use the *Gallai-Edmonds decomposition* (which we shall abbreviate to *GE-decomposition*, following [2]) of a graph G, defined next.

Definition 10. In a graph G, let B be the set of vertices that are covered by every maximum matching in G, let A be the set of vertices in B that have a neighbour outside of B, and set $C := B \setminus A$ and $D := V(G) \setminus B$. The *GE-decomposition* of G is the partition (A, C, D) of $V(G)$.

The following theorem, due to Edmonds and Gallai, lists useful properties of the GE-decomposition. Recall that a graph H is called *factor-critical* if for every vertex u in H the graph $H \setminus \{u\}$ has a perfect matching.

Theorem 11 (Edmonds and Gallai; see Theorem 3.2.1 in [17])**.** *Let (A, C, D) be the GE-decomposition of a graph G, as given in Definition 10. Then*

(i) *Every maximum matching in G covers C and matches A into distinct components of $G[D]$.*

(ii) *Every component of $G[D]$ is factor-critical.*

In the next corollary, we show that by adding to G all edges touching A, or having both ends in either C or some D_i, the size of the maximum matching does not increase.

Corollary 12. *Let (A, C, D) be the GE-decomposition of a graph G, and let $D_1, \ldots, D_r$ be the components in $G[D]$. Let H be the graph on vertex set $V(G)$ whose edges are pairs of vertices uw such that one of the following holds: u or w are in A; $u, w \in C$; or $u, w \in D_i$ for some $i \in [r]$. Then G is a subgraph of H, and the size of a maximum matching is the same in both graphs.*

Proof. It is easy to see that $G \subseteq H$. Indeed, by definition of the GE-decomposition, there are no edges of G between C and D, and by choice of $D_1, \ldots, D_r$, there are no edges of G between distinct D_i's. All other pairs of vertices are edges in H.

Let M be a maximum matching in G. By Theorem 11 (i), without loss of generality, M matches A into the components $D_1, \ldots, D_{|A|}$. As the $G[D_i]$'s are factor-critical, by Theorem 11 (ii), M covers D_i for every $i \in [|A|]$, and leaves exactly one vertex of D_i uncovered, for every $i \in \{|A|+1, \ldots, r\}$. It follows that $|M| = n - (r - |A|)$. A similar argument holds for H: if M' is a maximum matching in H, then it has at most $|A|$ edges from A to D, and thus at least $r - |A|$ sets D_i have at least one vertex which is uncovered by M', as the D_i's are odd. It follows that $|M'| \leq n - (r - |A|)$. As $G \subseteq H$, we have $|M'| \geq |M| = n - (r - |A|)$. We conclude that $|M'| = |M|$, as required. □

4 Proofs

Our main aim in this section is to prove Theorem 6. We start with the proof of Theorem 5, a special case of Theorem 6. The proofs are similar, but the proof of the latter theorem avoids some technicalities, so we hope that including its proof first will be instructive.

Proof of Theorem 5. Fix a k-colouring of G. Note that we may assume that $|G| = \lceil N + \varepsilon n \rceil \leq 2\beta n$, where the inequality holds for sufficiently small ε.

For each colour ℓ, we partition the vertices of G into so-called *virtual components* $V_{\ell,1}, \ldots, V_{\ell,t_\ell}$, such that each $V_{\ell,i}$ is either an ℓ-coloured component of G, or it is a disjoint union of ℓ-coloured components of G with $|V_{\ell,i}| < \alpha n$. We may assume that at most one virtual component $V_{\ell,i}$'s has order smaller than $(\alpha/2)n$, because if there are two such virtual components, we can replace them by their union. Under this assumption, the number t_ℓ of virtual components of colour ℓ satisfies

$$t_\ell \leq \frac{|G|}{(\alpha/2)n} + 1 \leq \frac{4\beta}{\alpha} + 1 \leq \frac{5\beta}{\alpha}. \tag{1}$$

Form a k-coloured graph H as follows. For each ℓ-coloured virtual component U in H, let (A, C, D) be its GE-decomposition and let $D_1, \ldots, D_r$ be the ℓ-coloured components in D. We add to H, in colour ℓ, all pairs of vertices uw with one of u and w in A, or with $u, w \in C$, or with $u, w \in D_i$ for some $i \in [r]$. (There may be some edges in H with multiple colours; either keep one colour per edge arbitrarily, or keep all colours).

We mention the following properties of H.

(a) $G \subseteq H$. In particular, every vertex in H has at most δn non-neighbours.

(b) There are no ℓ-coloured edges in H between distinct virtual components $V_{\ell,i}$.

(c) H has no ℓ-coloured connected matching on at least $\alpha_\ell n$ vertices, for $\ell \in [k]$.

Property (a) follows from the choice of H and the assumption on G. Property (b) follows from the definition of H: each ℓ-coloured edge that is added to G when forming H has both ends in the same virtual component $V_{\ell,i}$. We claim that (c) follows from Corollary 12. Indeed, by (b), the size of a maximum matching in $V_{\ell,i}$ is the same in G and in H, for every $i \in [t_\ell]$. As $V_{\ell,i}$ is either an ℓ-coloured component in G, or it has order smaller than αn, (c) follows.

We assign a type $(\sigma_1, \ldots, \sigma_k, \phi_1, \ldots, \phi_k)$ to each vertex u, so that $u \in V_{\ell,\sigma_\ell}$ and ϕ_ℓ points to the set in the GE-decomposition of V_{ℓ,σ_ℓ} that contains u;[3] namely, denoting the corresponding partition by (A, C, D), if $u \in A$ then $\phi_\ell = A$, if $u \in C$ then $\phi_\ell = C$, and if $u \in D$ then $\phi_\ell = D$. Recall that the number t_ℓ of virtual components of colour ℓ is at most $5\beta/\alpha$; see (1). It follows that the number of types of vertices is at most $(15\beta/\alpha)^k$.

Let $\overline{H}$ be the complement of H, namely the graph on vertex set $V(H)$ whose edges are the non-edges of H.

Claim 13. *$\overline{H}$ does not have a matching of size larger than $(30\beta/\alpha)^{2k} \cdot \delta n$.*

Proof. Suppose that M is a matching in $\overline{H}$ of size larger than $(30\beta/\alpha)^{2k} \cdot \delta n$. As there are at most $(15\beta/\alpha)^{2k}$ pairs of types, there exist two types $t_1 = (\sigma_1, \ldots, \sigma_k, \phi_1, \ldots, \phi_k)$ and $t_2 = (\tau_1, \ldots, \tau_k, \psi_1, \ldots, \psi_k)$ and a submatching M_0 of M with $|M_0| > 4^k \delta n$, such that one end of each edge in M_0 has type t_1, and the other end of each edge has type t_2. Let X_0 be the set of vertices in $V(M_0)$ of type t_1 and let Y_0 be the set of vertices in $V(M_0)$ of type t_2.

We will find submatchings $M_0 \supseteq M_1 \supseteq \ldots \supseteq M_k$ such that $|M_\ell| > 4^{k-\ell}\delta n$ and there are no ℓ-coloured edges in H between X_ℓ and Y_ℓ, where $X_\ell = X_0 \cap V(M_\ell)$ and $Y_\ell = Y_0 \cap V(M_\ell)$ for $\ell \in [k]$. This would lead to a contradiction to (a), as it would follow that there are no edges in H between X_k and Y_k, yet $|X_k|, |Y_k| > \delta n$.

Suppose that we have defined $M_0 \supseteq \ldots \supseteq M_{\ell-1}$ for some $\ell \in [k]$. If $\sigma_\ell \neq \tau_\ell$ then there are no ℓ-coloured edges between $X_{\ell-1}$ and $Y_{\ell-1}$, as the two sets belong to distinct virtual components of colour ℓ; we may thus take $M_\ell = M_{\ell-1}$.

Now suppose that $\sigma_\ell = \tau_\ell$, let U be the ℓ-coloured virtual component that contains $X_{\ell-1}$ and $Y_{\ell-1}$, and let (A, C, D) be the GE-decomposition of U. We note that $\phi_\ell, \psi_\ell \neq A$, because every vertex in A is joined to all other vertices in U, contrary to the assumption that there are non-edges of H between $X_{\ell-1}$ and $Y_{\ell-1}$. Similarly, we do not have $\phi_\ell = \psi_\ell = C$, because C induces a clique in H.

[3] To be precise, we take the GE-decomposition of the ℓ-coloured graph induced by V_{ℓ,σ_ℓ} in G, though it is easy to check, following the proof of Corollary 12, that the ℓ-coloured graph induced by V_{ℓ,σ_ℓ} in H has the same GE-decomposition.

We may thus assume that $\phi_\ell = D$. If $\psi_\ell = C$, then there are no ℓ-coloured edges between $X_{\ell-1}$ and $Y_{\ell-1}$, so we may take $M_\ell = M_{\ell-1}$. It remains to consider the case $\phi_\ell = \psi_\ell = D$.

Let $D_1, \ldots, D_r$ be the ℓ-coloured components in D; recall that each of them induces a clique in H and there are no ℓ-coloured edges between them. Let (I, J) be a partition of $[r]$, chosen uniformly at random, and let $M_\ell(I, J)$ be the submatching of $M_{\ell-1}$ consisting of edges $xy \in M_{\ell-1}$ with $x \in X_{\ell-1} \cap (\bigcup_{i\in I} D_i)$ and $y \in Y_{\ell-1} \cap (\bigcup_{j\in J} D_j)$. Let xy be some edge in $M_{\ell-1}$ with $x \in X_{\ell-1}$ and $y \in Y_{\ell-1}$, and let i, j be such that $x \in D_i$ and $y \in D_j$. Note that $i \neq j$, because xy is a non-edge in H, and D_i is a clique in H. Thus, the probability that $xy \in M_\ell(I, J)$ is $1/4$, implying that the expected size of $M_\ell(I, J)$ is $|M_{\ell-1}|/4$. Take $M_\ell = M_\ell(I, J)$ for some partition (I, J) of $[r]$ such that $|M_\ell| \geq |M_{\ell-1}|/4$. As the are no ℓ-coloured edges between distinct D_i's, there are no ℓ-coloured edges between X_ℓ and Y_ℓ, as required. $\square$

Let M be a maximum matching in $\overline{H}$. By Claim 13, $|M| \leq (30\beta/\alpha)^{2k} \cdot \delta n = (\varepsilon/2) \cdot n$. Consider the graph $H' = H \setminus V(M)$. This is a k-coloured complete graph, as the existence of a non-edge in H' would imply the existences of a larger matching in $\overline{H}$ than M, contrary to the choice of M. By the upper bound on the size of M, we have $|H'| \geq |H| - 2|M| \geq |H| - \varepsilon n \geq N$. Theorem 5 follows by taking G' to be any induced subgraph of H' on N vertices. $\square$

We now prove Theorem 4. The proof is similar to the above proof, so we allow ourselves to skip some details.

Proof of Theorem 4. Fix a k-colouring of G. By taking ε to be sufficiently small, we have $|G| = \sum_{i\in[s]} \lceil N_i + \varepsilon n \rceil \leq 2s\beta n$.

As in the proof of Theorem 5, for each colour ℓ we partition the vertices of G into 'virtual components' $V_{\ell,1}, \ldots, V_{\ell,t_\ell}$, such that $V_{\ell,i}$ is either an ℓ-coloured component of G, or a disjoint union of components satisfying $|V_{\ell,i}| < \alpha n$, and

$$t_\ell \leq \frac{|G|}{(\alpha/2)n} + 1 \leq \frac{5s\beta}{\alpha}. \tag{2}$$

We form a k-coloured graph H similarly to the proof of Theorem 5. For each ℓ-coloured virtual component U in G, denote by (A, C, D) its GE-decomposition and let $D_1, \ldots, D_r$ be the ℓ-coloured components in D. If U is bipartite and $\ell \in [k_0 + 1, k]$ fix a bipartition (X, Y) of U. Now add to H, in colour ℓ, all pairs of vertices uw such that uw is an edge in $\mathcal{G}'$ and one of the following holds: one of u and w is in A; $u, w \in C$; or $u, w \in D_i$ for some $i \in [r]$. If (X, Y) was defined, we only add such pairs uw which have one end in X and one in Y.

As above, we have

(a') $G \subseteq H \subseteq \mathcal{G}$. In particular, every vertex in H is incident to at most δn edges in $\mathcal{G}$ that are not in H.

(b') There are no ℓ-coloured edges between distinct virtual components $V_{\ell,i}$.

(c') H has no ℓ-coloured connected matching on at least $\alpha_\ell n$ vertices, for $\ell \in [k_0]$; and has no ℓ-coloured connected matching on at least $\alpha_\ell n$ vertices which is contained in a non-bipartite ℓ-coloured component, for $\ell \in [k_0+1, k]$.

For (c'), we use the observation that, for $\ell \in [k_0+1, k]$, any non-bipartite ℓ-coloured component on at least αn vertices in H is a non-bipartite ℓ-coloured component in G (with the same matching number).

We assign to each vertex u a type $(\sigma_1, \ldots, \sigma_k, \phi_1, \ldots, \phi_k, \zeta_{k_0+1}, \ldots, \zeta_k, \mu)$, where σ_ℓ and ϕ_ℓ are defined as in the proof above; if, for $\ell \in [k_0+1, k]$ the ℓ-coloured virtual component U containing u is bipartite then ζ_ℓ denote the part in the bipartition of U that u belongs to (if U is non-bipartite, ζ_ℓ can be ignored); and μ denotes the vertex in F that u corresponds to, so $\mu \in [s]$, and u belongs to the clique or independent set in $\mathcal{G}$ that replaced the vertex μ in F. The number of types of vertices is at most $s \cdot 2^{k-k_0} \cdot (15s\beta/\alpha)^k$.

Let $\overline{H}$ be the spanning subgraph of $\mathcal{G}'$ with edge set $E(\mathcal{G}') \setminus E(G)$.

Claim 14. *$\overline{H}$ does not have a matching of size larger than $s^2 \cdot 4^{k-k_0} \cdot (30s\beta/\alpha)^{2k} \cdot \delta n$.*

Proof. Suppose that M is a matching in $\overline{H}$ with $|M| > s^2 \cdot 4^{k-k_0} \cdot (30s\beta/\alpha)^{2k} \cdot \delta n$. By the upper bound on the number of types, there are two types $t_1 = (\sigma_1, \ldots, \sigma_k, \phi_1, \ldots, \phi_k, \zeta_{k_0+1}, \ldots, \zeta_k, \mu)$ and $t_2 = (\tau_1, \ldots, \tau_k, \psi_1, \ldots, \psi_k, \eta_{k_0+1}, \ldots, \eta_k, \nu)$ and a submatching $M_0 \subseteq M$ with $|M_0| > 4^k \delta n$, such that each edge in M_0 has one end of type t_1 and the other of type t_2. Let X_0 and Y_0 be the vertices of $V(M_0)$ of types t_1 and t_2, respectively. Note that $\mu\nu$ is an edge in F, by definition of $\overline{H}$. It follows that X_0 is fully joined to Y_0 in $\mathcal{G}'$. Following a similar argument as in the proof of Claim 13, we will obtain submatchings $M_0 \supseteq M_1 \supseteq \ldots \supseteq M_k$ such that $|M_\ell| > 4^{k-\ell}\delta n$ and there are no ℓ-coloured edges between $X_\ell = V_0 \cap V(M_\ell)$ and $Y_\ell = V_0 \cap V(M_\ell)$, contradicting (a').

Indeed, the argument from the proof of Claim 13 can be repeated as is, except that when $\ell \in [k_0+1, k]$, $\sigma_\ell = \tau_\ell$ and the ℓ-coloured virtual component U that corresponds to σ_ℓ is bipartite, we distinguish between two cases: if $\zeta_\ell = \eta_\ell$, there are no ℓ-coloured edges between X_0 and Y_0 (as both sets are in the same part of the bipartition of U), so we can take $M_\ell = M_{\ell-1}$; and if $\zeta_\ell \neq \eta_\ell$, the argument can be repeated without changes, as all edges with one end in X_0 and one in Y_0 respect the bipartition of U. □

Take a maximum matching M in $\overline{H}$, and let $H' = H \setminus V(M)$. By Claim 14, $|M| \le (\varepsilon/2) \cdot n$. As $|V(M)| \le \varepsilon n$, the graph H' contains a copy of $F(N_1, \ldots, N_s)$; let G' be any such copy (with the k-colouring inherited from H'). Then G' satisfies the requirements of Theorem 6. □

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