The homomorphism threshold of $\{C_3, C_5\}$-free graphs

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Abstract

We determine the structure of $\{C_3, C_5\}$-free graphs with n vertices and minimum degree larger than $n/5$: such graphs are homomorphic to the graph obtained from a $(5k-3)$-cycle by adding all chords of length $1 \pmod 5$, for some k. This answers a question of Messuti and Schacht. We deduce that the homomorphism threshold of $\{C_3, C_5\}$-free graphs is $1/5$, thus answering a question of Oberkampf and Schacht.

1 Introduction

We are interested in the structure of graphs of high minimum degree which forbid specific subgraphs. For a fixed graph H, a graph is said to be *H-free* if it does not contain H as a subgraph. Let $\mathrm{Forb}(H)$ denote the class of H-free graphs, and let $\mathrm{Forb}_n(H)$ denote the class of n-vertex graphs in $\mathrm{Forb}(H)$. Furthermore, let $\mathrm{Forb}(H, d)$ denote the class of H-free graphs G with minimum degree at least $d|V(G)|$. Analogous definitions hold if we replace H by some family $\mathcal{H}$ of graphs. Finally, we say that a graph G is homomorphic to a graph H if there exists a map $f : V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. For example, G is homomorphic to K_r if and only if $\chi(G) \leq r$.

A classical result of Andrásfai, Erdős and Sós [4] states that if G is a K_{r+1}-free graph on n vertices with minimum degree $\delta(G) > \frac{3r-4}{3r-1}n$, then G is r-colourable. This result can be viewed as a significant strengthening of the following fact, which is a consequence of Turán's theorem: the minimum degree of a K_{r+1}-free graph on n vertices is at most $(1 - 1/r)n$. Note also here that the chromatic number $\chi(G)$ of G is bounded by a constant independent of n. In general, one may ask whether or not this behaviour persists when the minimum degree condition is weakened. Along these lines, Häggkvist [13] showed that any n-vertex triangle-free graph of minimum degree greater than $3n/8$ is homomorphic to a 5-cycle, and accordingly has chromatic number at most 3. Note that this is indeed an extension of the Andrásfai-Erdős-Sós result when the minimum degree condition is weakened, since a balanced

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blow-up of a 5-cycle exhibits the tightness of that result. Jin [14] took up the investigation and significantly extended the work of Häggkvist: he proved that for all $1 \leq k \leq 9$, any n-vertex triangle-free graph with minimum degree larger than $\frac{k+1}{3k+2}n$ is homomorphic to the graph F_k^2, which is obtained by adding all chords of length $1 \pmod 3$ to a cycle of length $3k-1$. Observe that F_k^2 is triangle-free and 3-colourable for every k. The graphs F_k^2 are a special case of a larger family of graphs, F_k^ℓ, which we shall discuss shortly. We note that Jin's result [14] is best possible, in the sense that such a statement does not hold for $k = 10$. Indeed, by taking a suitably chosen unbalanced blow-up of the Grötzsch graph (also known as the Mycielski graph, see e.g. [7]) one can obtain a triangle-free graph on n vertices and minimum degree $\lfloor 10n/29 \rfloor$ which is not 3-colourable, so in particular it is not homomorphic to F_k^2 for any k. Building on this work, Chen, Jin, and Koh [7] showed, in particular, that any n-vertex 3-colourable triangle-free graph G with $\delta(G) > n/3$ is homomorphic to F_k^2, for some k. Again, the Grötzsch graph shows that the assumption that the graph is 3-colourable is necessary.

In general, one may ask for the *smallest* minimum degree condition we may impose on an H-free graph which guarantees that it has bounded chromatic number. To be precise, this prompts us to define the *chromatic threshold* $\delta_\chi(H)$ of a graph H:

$$\delta_\chi(H) = \inf\{d : \text{there exists } C = C(H, d) \text{ such that if } G \in \text{Forb}(H, d), \text{ then } \chi(G) \leq C\}.$$

In other words, $\delta_\chi(H)$ is the infimum over all $d \in [0, 1]$ such that every H-free graph on n vertices and with minimum degree at least dn has bounded chromatic number (independent of n). This definition was implicit in the works of Andrásfai [2] and Erdős and Simonovits [11], and was first explicitly formulated by Łuczak and Thomassé [17].

For every $\varepsilon > 0$, Hajnal (appearing in [11]) constructed graphs in $\text{Forb}(K_3, 1/3 - \varepsilon)$ with arbitrarily large chromatic number, thereby proving the bound $\delta_\chi(K_3) \geq 1/3$. Thomassen [22] thereafter established the matching upper bound, showing that $\delta_\chi(K_3) = 1/3$. In fact, Brandt and Thomassé [6] strengthened this by showing that triangle-free graphs of minimum degree larger than $n/3$ have chromatic number at most four, answering a question of Erdős and Simonovits [11]. Extensions of these results were obtained by several authors [12, 20], who showed that $\delta_\chi(K_r) = \frac{2r-5}{2r-3}$. Finally, building off of ideas of Łuczak and Thomassé [17] and Lyle [18], Allen, Böttcher, Griffiths, Kohayakawa and Morris [1] determined the value of $\delta_\chi(H)$ for every graph H with $\chi(H) > 2$.

Note that the results of Häggkvist [13], Jin [14], and Chen, Jin, and Koh [7] mentioned earlier not only show that triangle-free graphs of large enough minimum degree have bounded chromatic number, but that they are actually homomorphic to some specific 3-colourable triangle-free graph. One may ask then, with respect to the above discussion, whether we can replace the property of having bounded chromatic number with the property of admitting a homomorphism to a graph of bounded order with additional properties. This question was posed by Thomassen [22] in the specific case of triangle-free graphs, and motivated Oberkampf and Schacht [21] to introduce the *homomorphism threshold* $\delta_{\text{hom}}(H)$ of a graph H:

$$\delta_{\text{hom}}(H) = \inf\{d : \exists\, C = C(H,d) \text{ s.t. } \forall\, G \in \text{Forb}(H,d) \\ \exists\, G' \in \text{Forb}_C(H) \text{ s.t. } G \text{ is homomorphic to } G'\}.$$

In words, $\delta_{\text{hom}}(H)$ is the infimum over all $d \in [0,1]$ such that every H-free graph with n vertices and minimum degree at least dn is homomorphic to an H-free graph of bounded order (independent of n). Note that the definition of $\delta_{\text{hom}}(H)$ extends naturally if we replace H by a family $\mathcal{H}$ of graphs.

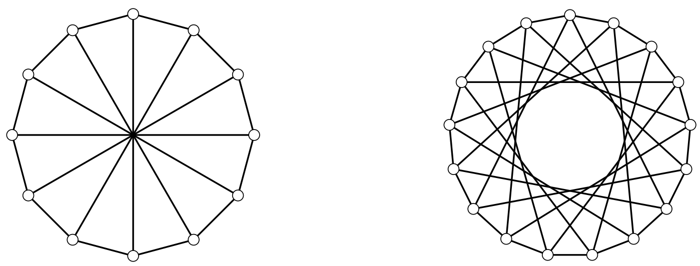
Łuczak [16] proved that $\delta_{\text{hom}}(K_3) \le 1/3$. Note that if G is homomorphic to G', then $\chi(G) \le |V(G')|$. Accordingly, we always have $\delta_{\text{hom}}(H) \ge \delta_\chi(H)$, and so, since $\delta_\chi(K_3) = 1/3$, it follows that $\delta_{\text{hom}}(K_3) = 1/3$. This result was extended by Goddard and Lyle [12] to K_r-free graphs for $r \ge 4$, and, in particular, we know that $\delta_{\text{hom}}(K_r) = \delta_\chi(K_r) = \frac{2r-5}{2r-3}$. Oberkampf and Schacht [21] gave a new proof of this result avoiding the Regularity Lemma (which was used in Łuczak's proof), and asked for the determination of the homomorphism threshold of the odd cycle, $\delta_{\text{hom}}(C_{2\ell-1})$, and $\delta_{\text{hom}}(\{C_3,\dots,C_{2\ell-1}\})$ for $\ell \ge 3$. As our first main result, we determine the value of the second of these two parameters in the case $\ell = 3$.

Theorem 1. *The homomorphism threshold of* $\{C_3, C_5\}$ *is* $1/5$*.*

In other words, Theorem 1 states that, for every $\varepsilon > 0$, if G is a $\{C_3, C_5\}$-free graph on n vertices and minimum degree at least $(1/5+\varepsilon)n$, then G is homomorphic to a $\{C_3, C_5\}$-free graph of order at most C, where C depends on ε but not on n. We also establish an upper bound on $\delta_{\text{hom}}(C_5)$. This is a consequence of Theorem 1, since C_5-free graphs of large minimum degree end up being triangle-free as well. In particular, we have the following.

Corollary 2. *The homomorphism threshold of* C_5 *is at most* $1/5$*.*

In fact, we are able to say much more about the structure of $\{C_3, C_5\}$-free graphs with n vertices and minimum degree larger than $n/5$. First we need to define a family of graphs, sometimes known as *generalised Andrásfai graphs*. For integers $k \ge 1$ and $\ell \ge 2$, denote by F_k^ℓ the graph obtained from a $((2\ell-1)(k-1)+2)$-cycle (an edge, when $k=1$) by adding all chords joining vertices at distances $j(2\ell-1)+1$ for $j = 0, 1, \dots, k-1$. Alternatively, F_k^ℓ can be defined as the complement of the $(\ell-1)$-th power of a cycle of length $(2\ell-1)(k-1)+2$. For $\ell = 2$, these graphs were considered by Erdős [10] and Andrásfai [2, 3]. It is not difficult to check that F_k^ℓ is k-regular, maximal $\{C_3,\dots,C_{2\ell-1}\}$-free, and 3-colourable. For our purposes, ℓ will always be 3 and we shall write F_k instead of F_k^3 for simplicity. In particular, F_1 is an edge, F_2 is a C_7 (a cycle of length 7) and F_3 is the graph obtained by adding all diagonals to a C_{12} (by a *diagonal* in an even cycle $C_{2\ell}$, $\ell \geq 2$, we mean an edge joining vertices at distance ℓ along the cycle). This graph is also known as the Möbius ladder on 12 vertices (see Figure 1a).



(a) the Möbius ladder F_3 **(b)** F_4

Figure 1: examples of graphs F_k

As our second main result, we determine the structure of $\{C_3, C_5\}$-free graphs on n vertices with minimum degree larger than $n/5$, thus answering a question of Messuti and Schacht [19].

Theorem 3. *Let G be a $\{C_3, C_5\}$-free graph on n vertices with $\delta(G) > n/5$. Then G is homomorphic to F_k, for some k.*

We remark that the analogue of this result for graphs of higher odd-girth does not hold in general. We discuss this further in the final section of this paper.

1.1 Organisation and Notation

The remainder of this paper is organised as follows. In Section 2 we shall provide an outline of the technical results needed to prove our main theorem. Many of these state that certain subgraphs cannot appear in maximal $\{C_3, C_5\}$-free graphs of minimum degree larger than $n/5$. In the next three sections (Section 3 to Section 5) we shall prove each of these technical results. In Section 6, we deduce our main theorem, Theorem 3. Finally, Section 7 includes our results concerning homomorphism thresholds, Theorem 1 and Corollary 2.

Our notation is standard. In particular, for a graph G, we use $|V(G)|$ to denote the number of vertices of G, $V(G)$ denotes the vertex set, $E(G)$ the edge set, and $\delta(G)$ denotes the minimum degree. For a vertex v, $N_G(v)$ denotes the neighbourhood of v, and for a subset $X \subseteq V(G)$, $N_G(v, X)$ denotes the neighbourhood of v in X, i.e. $N_G(v, X) = N_G(v) \cap X$. We shall often omit the use of the subscript 'G'. If $X, Y \subseteq V(G)$, then we say an edge e is an $X - Y$ *edge* if one endpoint of e is in X, the other in Y. If $X=\{x\}$, then we simply say e is an $x - Y$ edge. We denote by $(v_1 \dots v_\ell)$ the cycle on vertices $v_1, \dots v_\ell$ taken in this order. Similarly, we denote by $v_0 \dots v_\ell$ the path on vertices $v_0, \dots, v_\ell$ taken in this order. A cycle (path) with ℓ edges is an *ℓ-cycle* (*ℓ-path*).

2 Overview

In this section we provide a tour through the technical results needed to establish our main theorem. Note that in proving Theorem 3 we may assume our graph is maximal $\{C_3, C_5\}$-free. Accordingly, the following results concern maximal $\{C_3, C_5\}$-free graphs. The main tool needed for the proof of Theorem 3 is the following result.

Theorem 4. *Let G be a maximal $\{C_3, C_5\}$-free graph on n vertices with $\delta(G) > n/5$. Then every vertex in G has a neighbour in every 7-cycle in G.*

We remark that Jin [15] proved the analogous theorem for 5-cycles in triangle-free graphs of large enough minimum degree. In order to establish Theorem 4 we shall need a sequence of lemmas which show that certain subgraphs cannot appear in maximal $\{C_3, C_5\}$-free graphs of large minimum degree. The first of these lemmas, which shows that $\{C_3, C_5\}$-free graphs with large minimum degree do not have induced 6-cycles, proves very useful, and we shall use it throughout the paper. Brandt and Ribe-Baumann [5] mention it without proof.

Lemma 5. *Let G be a maximal $\{C_3, C_5\}$-free graph on n vertices with $\delta(G) > n/5$. Then G does not contain an induced 6-cycle.*

We shall also need the fact that a 'partial' Möbius ladder cannot appear as a subgraph in the graphs we consider. More precisely, we need the following lemma.

Lemma 6. *Let G be a maximal $\{C_3, C_5\}$-free graph on n vertices with $\delta(G) > n/5$. If $(x_1 \dots x_{12})$ is a 12-cycle with two consecutive diagonals x_1x_7 and x_2x_8 present. Then either $(x_1 \dots x_{12})$ or $(x_2x_3 \dots x_7x_1x_{12} \dots x_8)$ induces a Möbius ladder.*

We note, and prove, the following useful corollary of Lemma 6.

Corollary 7. *Let G be a maximal $\{C_3, C_5\}$-free graph on n vertices with $\delta(G) > n/5$. If u is a vertex with no neighbours in a 7-cycle C, then u has no neighbour with two neighbours in C.*

Proof. Suppose that $C = (x_1 \dots x_7)$ and u has no neighbours in C, but a neighbour v of u has two neighbours in C. Say, v is adjacent to x_2 and x_7 (see Proposition 10 below).

Since u is not adjacent to x_1 and G is maximal $\{C_3, C_5\}$-free, there must be a path of length 2 or 4 between them; but a path of length 2 is impossible (it will complete the path uvx_2x_1 to a 5-cycle), so there is a 4-path $uy_1y_2y_3x_1$. One may check that none of y_1, y_2, y_3 is equal to one of the vertices of C or to v (see Figure 2). But then $(x_1 \dots x_7vuy_1y_2y_3)$ is a 12-cycle with two consecutive diagonals x_1x_7 and x_2v. It follows from Lemma 6 that all diagonals in the cycle must be present (or we need to consider the 12-cycle $(x_2 \dots x_7x_1y_3y_2y_1uv)$ with diagonals x_1x_2 and x_7v). In particular, u has a neighbour in C, a contradiction. □

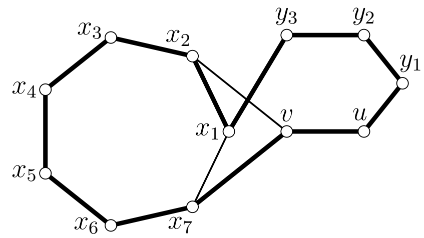


Figure 2

Finally, in order to prove Theorem 4, we establish the following lemma, which is the last of our results regarding forbidden subgraphs in maximal $\{C_3, C_5\}$-free graphs of large minimum degree.

Lemma 8. *Let G be a maximal $\{C_3, C_5\}$-free graph on n vertices with $\delta(G) > n/5$. Then G does not contain, as an induced graph, the graph obtained by two 7-cycles whose intersection is a path of length* 3 *(see Figure 7).*

Before proceeding to the proofs of the above forbidden subgraph lemmas, we shall show how to prove Theorem 4 using Lemmas 5, 6 and 8. The proofs of these lemmas shall be deferred to Sections 3, 4 and 5, respectively. In order to aid in their proofs, we introduce the following definition.

Definition 9. A subgraph H of a graph G is called *well-behaved* (in G) if for every vertex u in G, there is a vertex v in H, such that $N_G(u, H) \subseteq N_H(v)$.

In particular, this implies that $G[H \cup \{u\}]$ is homomorphic to H for every $u \in V(G)$. Many of the subgraphs we consider are actually well-behaved (in their respective host graphs). For example, we note the following useful proposition.

Proposition 10. *Let $\ell \geq 2$ be an integer and let G be a $\{C_3, \dots, C_{2\ell-1}\}$-free graph. Then $C_{2\ell+1}$ is well-behaved in G.*

Proof. Let $\ell \geq 2$ and let $C = (x_1 \dots x_{2\ell+1})$ be a $(2\ell+1)$-cycle in G. Suppose without loss of generality that $w \in V(G) \setminus V(C)$ is joined to x_1. We claim that either $N(w, C) \subseteq N_C(x_2)$ or $N(w, C) \subseteq N_C(x_{2\ell+1})$. Let w' be another neighbour of w in C and suppose to the contrary that $w' \neq x_3, x_{2\ell}$. Let P denote the path $x_1 x_2 x_3 \dots w'$ and P' denote the path $x_1 x_{2\ell+1} x_{2\ell} \dots w'$. Now, note that $l(P) \leq (2\ell+1) - 3 = 2\ell - 2$ and similarly $l(P') \leq 2\ell - 2$ (here $l(P)$ denotes the length of P). Moreover, one of P, P' must have odd length, say P. But then the cycle $(w x_1 P w')$ is odd and has length at most $2\ell - 1$, a contradiction. $\square$

We need the following observation before proving Theorem 4.

Observation 11. *Let G be a maximal $\{C_3, C_5\}$-free graph on n vertices with $\delta(G) > n/5$. Suppose that u has no neighbour in a 7-cycle C. Then u has a common neighbour with at most one of the vertices in C.*

Proof. Suppose that u has no neighbour in the cycle $C = (x_1 \dots x_7)$. Furthermore, suppose that u has a common neighbour v with x_1. By symmetry, it suffices to show that u has no common neighbour with x_2, x_3 or x_4. It easily follows that u and x_2 have no common neighbour (otherwise, a cycle of length 3 or 5 is formed). Suppose that u and x_3 have a common neighbour w. Observe that $w \neq v$ by Corollary 7. Consider the 6-cycle $(vuwx_3x_2x_1)$. Recall that G has no induced 6-cycles; thus one of ux_2, vx_3, wx_1 is an edge in G. But ux_2 is not an edge, by the assumption that u has no neighbour in C, and if one of vx_3 and wx_1 is an edge, a contradiction to Corollary 7 is reached. Finally, if u and x_4 have a common neighbour w (which, as before, is not equal to v), then the set $\{u, v, w, x_1, \dots, x_7\}$ induces a graph that consists of two 7-cycles whose intersection is a path of length 3, contradicting Lemma 8. □

Proof of Theorem 4. Suppose that the theorem is false and choose a vertex u and a 7-cycle C which minimize the distance between u and C such that u has no neighbour in C. Since G must be connected, it easily follows that there is a path of length two between u and C. Therefore, we may assume without loss of generality that u has no neighbour in the 7-cycle $C = (x_1 \dots x_7)$ and v is a common neighbour of u and x_1. Since u is not joined to x_2 and G is maximal $\{C_3, C_5\}$-free, there is a 4-path $uy_1y_2y_3x_2$ between u and x_2 (a 2-path would create a 5-cycle). We note that y_1 cannot be joined to x_1, otherwise a 5-cycle is formed, so in particular $y_1 \neq v$. Thus, by Observation 11, y_1 has no neighbours in C. We note that no two of the four vertices $\{u, x_2, x_3, x_6\}$ have a common neighbour; this follows from Observation 11 and the assumption that G is $\{C_3, C_5\}$-free. It follows from the minimum degree condition that y_1 has a common neighbour with one of u, x_2, x_3, x_6. But y_1 does not have a common neighbour with either u or x_2 (otherwise, a cycle of length 3 or 5 if formed). Thus y_1 has a common neighbour with either x_3 or x_6. Assume that y_1 has a common neighbour with x_3 (with x_6). Then, by Observation 11, y_1 has no common neighbours with any other vertex in C. It follows that no two of the vertices in $\{u, y_1, x_2, x_5, x_6\}$ (in $\{u, y_1, x_3, x_4, x_7\}$) have a common neighbour, a contradiction to the minimum degree condition. □

In the next three sections we shall prove Lemmas 5, 6 and 8. The general strategy is the following. We want to show that some graph F cannot appear in a maximal $\{C_3, C_5\}$-free graph G of large minimum degree. If F is a subgraph of G, and if every vertex has a 'small' number of neighbours in F, then double counting the edges between $V(F)$ and $V(G) \setminus V(F)$ will produce a contradiction with the minimum degree condition. Often the original target graph F does not satisfy this goal, and we shall need to pass to some suitable subgraph of

F which meets our needs. This requires detailed analysis of the possible neighbourhoods of vertices in F (or some subgraph of F).

3 No induced 6-cycles

Brandt and Ribe-Baumann [5] stated that maximal $\{C_3, C_5\}$-free graphs of high minimum degree forbid induced 6-cycles. However, they did not provide a proof and for completeness we provide one in this section.

Proof of Lemma 5. Suppose G contains an induced 6-cycle $C = (x_1 \dots x_6)$. By the edge-maximality of G there are three 4-paths $P_{14} = x_1y_1y_2y_3x_4$, $P_{25} = x_2z_1z_2z_3x_5$, and $P_{36} = x_3w_1w_2w_3x_6$. It is easily verified that all the vertices in the union of these paths are distinct. Denote by H the graph induced on $V(C) \cup V(P_{14}) \cup V(P_{25}) \cup V(P_{36})$ (see Figure 3). We shall show that G cannot contain H as a subgraph. The proof breaks into two cases:

1. At least two vertices from $\{y_2, z_2, w_2\}$ have a common neighbour.
2. No pairs from $\{y_2, z_2, w_2\}$ have a common neighbour.

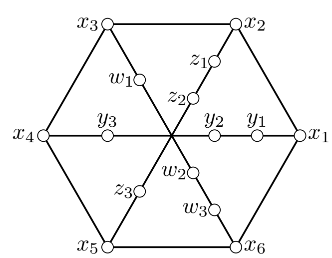


Figure 3: Constructing H from an induced 6-cycle

In each case, we shall find a 10-vertex subgraph of H for which every vertex of G has at most two neighbours in H. We reach a contradiction to the minimum degree condition on G via double counting the edges between this subgraph and the rest of G.

3.1 Case 1

Suppose, without loss of generality, that y_2 and z_2 have a common neighbour v. Denote by H' the 10-vertex graph induced on $V(H) \setminus V(P_{36})$.

Claim 12. *Every vertex of G has at most two neighbours in H'.*

Proof. Observe that x_1 and x_4 cannot have a common neighbour, else a 5-cycle is formed. Thus, a vertex in G can have at most two neighbours in $C \setminus \{x_3, x_6\}$, and if it has two such neighbours, then it is joined either to both x_2 and x_4 or to both x_1 and x_5. It is then routine to check that such a vertex cannot be joined to any other vertex of H': all cases lead to a triangle or pentagon in G.

Let us now consider vertices which have precisely one neighbour in $C \setminus \{x_3, x_6\}$. By symmetry, let u be a vertex joined to x_1. We claim that u can be joined to at most one other vertex in H', which must be from $\{y_2, z_1\}$. Indeed, it is easy to verify that u cannot be joined to any vertices of $H' \setminus \{y_2, z_1\}$, since these cases lead to a triangle or pentagon in G. Suppose u is adjacent to both y_2 and z_1. This, however, produces the 5-cycle $(y_2uz_1z_2v)$, a contradiction.

Finally, we consider vertices which have no neighbour in $C \setminus \{x_3, x_6\}$. First, note that if a vertex u is joined to y_2, then its only other possible neighbour in H' is z_2 (and the same claim holds with the roles of y_2 and z_2 reversed). For example, if u is joined to y_2 and z_3, the 5-cycle $(y_2uz_3z_2v)$ is produced. One may dispense with the other cases similarly. On the other hand, both pairs $\{y_3, z_3\}$ and $\{y_1, z_1\}$ do not have any common neighbours. It follows that any vertex with no neighbour in $C \setminus \{x_3, x_6\}$ has at most two neighbours in H', and this finishes the proof of Claim 12. □

Let us bound the number of edges between $V(H')$ and $V(G) \setminus V(H')$ in two ways. Using the minimum degree condition and Claim 12, we have that every vertex in H' has at most two neighbours in H', and therefore has more than $n/5 - 2$ neighbours outside of H'. Thus, there are more than $10(n/5-2) = 2(n-10)$ such edges. On the other hand, Claim 12 implies that there are at most $2(n-10)$ such edges, a contradiction. This completes the proof of Lemma 5 under Case 1.

3.2 Case 2

Suppose the condition in Case 2 holds; that is, no pairs from $\{y_2, z_2, w_2\}$ have a common neighbour. We begin by examining the size and structure of possible neighbourhoods in H of vertices of G.

Let u be a vertex which is not joined to any vertex in C. If u is joined to a middle vertex, say y_2, then by assumption it cannot be adjacent to z_2 or w_2. Further, u cannot be joined to y_1 or y_3 (else, a triangle is formed), has at most one neighbour in $\{z_3, w_3\}$, and has at most one neighbour in $\{z_1, w_1\}$. Thus, u has at most three neighbours in H. Similarly, one may verify that if u has no neighbour in $\{y_2, z_2, w_2\}$, then u has at most three neighbours in H as well.

Suppose u is a vertex joined to two vertices of C. Say, by symmetry, that u is joined to x_2 and x_4. Then it is easy to check that the only other possible neighbour of u in H is w_1.

Hence u has at most three neighbours in H.

If u is a vertex joined to three neighbours of C, then (up to relabelling) u is adjacent to all vertices in $\{x_1, x_3, x_5\}$, and one may verify that u can have no further neighbour in H. Thus, such vertices have at most three neighbours in H.

Only one case remains: suppose u has precisely one neighbour in C, and, by symmetry, suppose this neighbour is x_1. In this case, u may be joined to all vertices in $\{y_2, z_1, w_3\}$. Accordingly, u has at most four neighbours in H.

Now, if every vertex of G has at most three neighbours in H, then we are done by double counting the edges between $V(H)$ and $V(G) \setminus V(H)$: there are at most $3(n-15)$ such edges, and by the minimum degree condition, more than $15(n/5-3) = 3(n-15)$ such edges, a contradiction. Therefore, we may assume that there is a vertex v of degree 4 in H. By the preceding analysis, we may assume that y_1z_1 and y_1w_3 are edges: if not, replace y_1 by v.

The proof breaks into two cases from here:

(a) z_3 and w_1 have a common neighbour.

(b) z_3 and w_1 do not have a common neighbour.

Assuming (a), let w be a common neighbour of z_3 and w_1, and denote by H'' the 10-vertex graph induced on $V(H) \setminus V(P_{14})$ (see the black vertices in Figure 4).

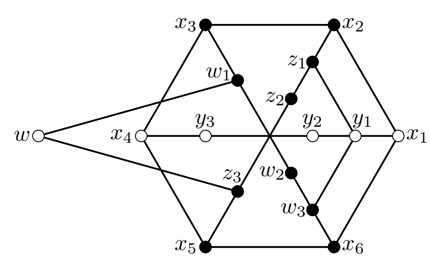


Figure 4: H'', with common neighbour w

Claim 13. *Every vertex of G has at most two neighbours in H''.*

Proof. The proof is similar to that of Claim 12. Any vertex u is adjacent to at most two vertices of $C \cap H''$. If u is joined to x_2 and x_6, then its only other possible neighbour is y_1, but $y_1 \notin H''$. A similar statement holds if u is joined to x_3 and x_5.

Suppose now that u has precisely one neighbour in $C \cap H''$, and suppose this neighbour is x_2. By our preceding analysis of possible neighbourhoods in H, u's only other possible neighbours are y_1, z_2, and w_1. However, $y_1 \notin H''$ and u cannot be joined to both z_2 and w_1: otherwise, the 5-cycle $(z_2z_3ww_1u)$ is formed. Hence, u is joined to at most two vertices of H''.

Similarly, if u is a vertex whose only neighbour in $C \cap H''$ is x_3, then u's only other possible neighbours are y_3, w_2, and z_1. But $y_3 \notin H''$ and u cannot be joined to both w_2 and z_1: otherwise the 5-cycle $(w_2 w_3 y_1 z_1 u)$ is created. The other cases (i.e., u joined to x_5 or x_6) are symmetric.

Finally, suppose u is a vertex with no neighbour in $C \cap H''$. If u is joined to a middle vertex, say, without loss of generality, w_2, then by assumption u cannot be joined to z_2. Hence the only other possible neighbours of u are z_1 and z_3. But u cannot be joined to z_1, since otherwise $(uw_2w_3y_1z_1)$ is a 5-cycle in G. If u is not joined to a middle vertex, then observe that it can be adjacent to at most one vertex from each pair $\{z_3, w_3\}$ and $\{z_1, w_1\}$.

This completes the proof of Claim 13. □

Let us now assume (b), that z_3 and w_1 do not have a common neighbour. Denote by H''' the 10-vertex graph induced on $V(H) \setminus \{x_1, x_2, x_6, y_2, y_3\}$ (see the black vertices in Figure 5).

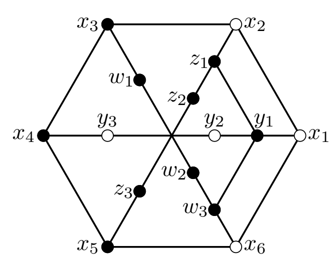


Figure 5: H'''

Claim 14. *Every vertex of G has at most two neighbours in H'''.*

Proof. If a vertex u of G has two neighbours in $C \cap H'''$, then u must be joined to x_3 and x_5. But u's only other potential neighbour is y_3, and $y_3 \notin H'''$.

Suppose u is a vertex with exactly one neighbour in $C \cap H'''$. First, suppose u is joined to x_3. Then u's only other possible neighbours are w_2, y_3, and z_1. But $y_3 \notin H'''$ and u cannot be joined to both z_1 and w_2, as otherwise the 5-cycle $(uw_2w_3y_1z_1)$ is in G. The case when u is joined to x_5 is dealt with symmetrically.

Suppose now that u is joined to x_4. The only other possible neighbours are then y_2, z_3, and w_1. Observe that $y_2 \notin H'''$, and, by assumption, z_3 and w_1 have no common neighbour, so u is joined to at most one of them.

One may (as in the proof of Claim 13) dispense with the case when u is a vertex with no neighbour in $C \cap H'''$. Thus, no vertex of G has more than two neighbours in H''' and this completes the proof of Claim 14. □

We may now complete the proof of Lemma 5 in Case 2. Indeed, if (a) holds, then apply Claim 13 together with the usual double counting technique to produce a contradiction. If instead (b) holds, then apply Claim 14 together with double counting. This completes the proof of Lemma 5. □

4 12-cycles with few diagonals

Our aim in this section is to prove Lemma 6. We divide the proof into steps, according to the number of diagonals. Note that the case of having precisely five diagonals is immediate from Lemma 5 that forbids induced 6-cycles. It remains to examine the situation when there are either two, three or four diagonals present.

Proposition 15. *Let G be a maximal $\{C_3, C_5\}$-free graph on n vertices with $\delta(G) > n/5$. Then G has no 12-cycle with exactly four diagonals.*

Proof. Suppose that $(x_1 \dots x_{12})$ is a 12-cycle with exactly four diagonals. Let H be the graph induced by $\{x_1, \dots, x_{12}\}$. In light of Lemma 5, G has no induced 6-cycle, so we may assume that the edges $x_1x_7, x_2x_8, x_3x_9, x_4x_{10}$ are present in the graph and that x_5x_{11}, x_6x_{12} are non-edges. In fact, it is easy to verify that the only edges in H are the edges of the 12-cycle and these four diagonals.

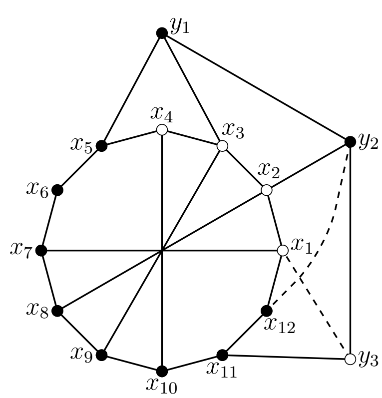


Figure 6: Constructing the graph H' in the case of four diagonals

The pair $\{x_5, x_{11}\}$ is a non-edge in G, and so there is a path of length 2 or 4 between x_5 and x_{11}. In fact, the length must be 4 because, otherwise, a cycle of length 3 or 5 will be created. Let $x_5y_1y_2y_3x_{11}$ be this 4-path. One may verify that $y_2 \notin H$, and possibly $y_3 = x_{12}$ or $y_1 = x_6$, but not both. We shall assume, without loss of generality, that $y_1 \neq x_6$.

Claim 16. *We may assume that y_1x_3 and y_2x_2 are edges of G.*

Proof. No two of the following vertices have a common neighbour: x_3, x_6, x_9, x_{12} (they are at distance one or three from each other). In other words, their neighbourhoods are pairwise disjoint, and so, by the minimum degree condition, every vertex in G has a common neighbour with at least one of these four vertices. Note that y_2 does not have a common neighbour with either x_6 or x_{12} (this will create a C_5). By symmetry, we may assume that y_2 and x_3 have a common neighbour u. If $u = y_1$, Claim 16 follows. Thus, we suppose otherwise. Consider the 6-cycle $(uy_2y_1x_5x_4x_3)$. Since there are no induced 6-cycles, one of the following is an edge: y_1x_3, y_2x_4, ux_5. If y_1x_3 is an edge, the claim follows; y_2x_4 cannot be an edge (because of the 5-cycle $(y_2x_4x_{10}x_{11}y_3)$); if ux_5 is an edge, we replace y_1 by u to prove the first part of the Claim.

To see the second part, by considering the neighbours of x_2, x_5, x_8, x_{11}, we have that y_3 has a common neighbour with x_2 or x_8. If u is a common neighbour of y_3 and x_2, we may assume that $u \neq y_2$ (otherwise, we are done). By considering the 6-cycle $(ux_2x_3y_1y_2y_3)$, either y_2x_2 or uy_1 is an edge. We may assume that uy_1 is an edge. Then, by replacing y_2 by u we obtain the required property. Now suppose that y_3 and x_8 have a common neighbour u. By considering $(ux_8x_9x_{10}x_{11}y_3)$, u is adjacent to x_{10}. This, in turn, implies that u is adjacent to x_3 (see $(ux_8x_2x_3x_4x_{10})$), a contradiction: the 5-cycle $(ux_3y_1y_2y_3)$ is formed. □

Denote by H' the graph induced by $\{x_5, \ldots, x_{12}, y_1, y_2\}$ (see the black vertices in Figure 6). We shall show that every vertex of G has few neighbours in H', yielding a contradiction to the minimum degree condition on G. More precisely, we have the following:

Claim 17. *Every vertex of G has at most two neighbours in H'.*

Proof. We first prove that H is well-behaved. First, note that no vertex u in G can be adjacent to all of $\{x_4, x_6, x_{11}\}$. Indeed, otherwise, $(ux_{11}x_{12}x_1x_7x_6)$ is an induced C_6 (the addition of any chord to this cycle creates a triangle or a pentagon), contradicting Lemma 5. By symmetry, no vertex can be adjacent to all vertices in one of the following sets: $\{x_5, x_7, x_{12}\}$, $\{x_1, x_6, x_{11}\}$, $\{x_5, x_{10}, x_{12}\}$. We conclude that no vertex can be adjacent to both x_6 and x_{11}. Indeed, by considering the 6-cycle $(x_1x_7x_6ux_{11}x_{12})$, since there is no induced C_6 in G, u must be adjacent to x_1, contradicting the above. Similarly, no vertex is adjacent to both x_5 and x_{12}. One may check that any other possible neighbourhood of a vertex of G in H is contained in the neighbourhood of a vertex in H.

Now, as H is well-behaved, no vertex in G has more than two neighbours in $H' \cap H$. Thus, if a vertex u has three neighbours in H', at least one of them is either y_1 or y_2. If u is adjacent to y_1, then the only other neighbours u can have in H' are x_6, x_9, x_{12}, but no two of these vertices may have a common neighbour. Similarly, if u is adjacent to y_2, its other possible neighbours in H' are x_5, x_8, x_{11}, no two of which have a common neighbour. The Claim follows. □

Using Claim 17, we may now finish the proof of Proposition 15 by double counting the number of edges between $V(H')$ and $V(G) \setminus V(H')$, as usual. $\square$

Now we deal with the remaining case, of a 12-cycle with two or three diagonals, and thereby complete the proof of Lemma 6.

Proposition 18. *Let G be a maximal $\{C_3, C_5\}$-free graph on n vertices with $\delta(G) > n/5$. Then G induces no 12-cycle with two consecutive diagonals and at most one additional chord.*

Proof. Suppose that $C = (x_1 \dots x_{12})$ is a 12-cycle with two consecutive diagonals x_1x_7 and x_2x_8, and at most one additional chord. We note that any additional chord (that does not complete a triangle or 5-cycle) is a diagonal in one of the following 12-cycles $(x_1 \dots x_{12})$ or $(x_2 \dots x_7x_1x_{12} \dots x_8)$, both of which have two consecutive diagonals. Hence, and by symmetry, we may assume that the additional chord is either x_6x_{12} or x_5x_{11}. However, if x_5x_{11} is the additional chord, then $(x_1x_7x_6x_5x_{11}x_{12})$ is an induced 6-cycle, contradicting Lemma 5. Thus we assume that, if there is an additional chord, it is x_6x_{12}. Furthermore, if x_6x_{12} is not an edge, we assume that G contains no 12-cycles with two consecutive diagonals and exactly one extra chord. Let H be the graph induced on $\{x_1, \dots, x_{12}\}$ and denote $H' = H \setminus \{x_1, x_7\}$.

Claim 19. *Every vertex in G has at most two neighbours in H'.*

Proof. Suppose that u has three neighbours in H'. It follows by symmetry that u has two neighbours in $\{x_2, \dots, x_6\}$, which we can denote by x_{i-1} and x_{i+1} for some $i \in \{3, 4, 5\}$ (by Proposition 10), and another neighbour x_j for some $j \in \{8, \dots, 12\}$. But then, by replacing x_i by u, we may assume that x_i is joined to x_j. This is a contradiction: either to Proposition 15 (if C had three chords, i.e. if x_6x_{12} is an edge, then now it has four chords); or, if x_6x_{12} is not an edge, to the assumption that there is no 12-cycle with two consecutive diagonals and an additional chord. $\square$

Proposition 18 follows from Claim 19 by double counting the number of edges between $V(H')$ and $V(G) \setminus V(H')$. The proof of Lemma 6 is therefore complete. $\square$

5 Two 7-cycles intersecting in a 3-path

In this section we prove Lemma 8; that is, the graph in Figure 7 cannot appear as an induced subgraph of a maximal $\{C_3, C_5\}$-free graph on n vertices with minimum degree larger than $n/5$.

Proof of Lemma 8. Suppose that H is an induced subgraph of G which is the union of two 7-cycles intersecting in a path of length 3. Denote the two 7-cycles by $(x_1x_2x_3x_4x_5x_6x_7)$ and $(x_1x_2x_3x_4x_8x_9x_{10})$ (see Figure 7). We start by showing that H is a well-behaved subgraph of G (recall Definition 9), a fact that will be useful in the proof.

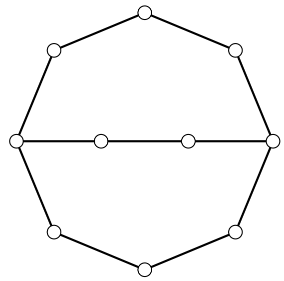


Figure 7: two 7-cycles intersecting in a 3-path

Claim 20. *The graph H is well-behaved.*

Proof. If H is not well-behaved, then, up to relabelling, one of the following two pairs has a common neighbour in G: $\{x_6, x_9\}$ or $\{x_5, x_{10}\}$. If u is a neighbour of x_6 and x_9 then, by Lemma 5, u is also a neighbour of x_1 (consider the 6-cycle $(ux_6x_7x_1x_{10}x_9)$), and of x_4 (consider the 6-cycle $(ux_6x_5x_4x_8x_9)$). But this produces the 5-cycle $(ux_1x_2x_3x_4)$. Now suppose that u is a neighbour of both x_5 and x_{10}. By considering the 6-cycle $(ux_5x_4x_8x_9x_{10})$, u must be adjacent to x_8. Now consider the 7-cycle $(ux_{10}x_1x_2x_3x_4x_8)$. The vertex x_6 has no neighbour in C (x_6 cannot be adjacent to u), but x_5 has two neighbours in C (x_4 and u). This is a contradiction to Corollary 7. □

Arguments as in Claim 20, using Corollary 7 and Lemma 5 will appear frequently in the proof of Lemma 8.

Since x_6 and x_8 are nonadjacent, there is a 4-path with ends x_6 and x_8 (a 2-path would create a C_5). Up to relabelling, three cases arise:

1. There is a 3-path $x_6y_1y_2x_9$ between x_6 and x_9. The vertices y_1 and y_2 are not in H.
2. There is a 3-path $x_7y_1y_2x_8$ between x_7 and x_8. The vertices y_1 and y_2 are not in H.
3. There is a 4-path $x_6y_1y_2y_3x_8$ between x_6 and x_8. The vertices y_1, y_2, y_3 are not in H.

In the rest of the proof, we show that each of the three cases is impossible, thus completing the proof of Lemma 8. Case 2 will be the most difficult to resolve.

5.1 **Case 1:** a 3-path between x_6 and x_9

Denote by H' the graph induced by $\{x_1, \ldots, x_{10}, y_1, y_2\}$.

Claim 21. *H' is well-behaved.*

Proof. Suppose that H' is not well-behaved. Up to relabelling, it follows that y_1 and x_3 have a common neighbour u (recall that H is well-behaved). By considering the 6-cycle

$(ux_3x_4x_5x_6y_1)$ and in light of Lemma 5, it follows that u is adjacent to x_5. Consider the 7-cycle $C = (x_1x_2x_3ux_5x_6x_7)$. Observe that x_4 has two neighbours in C (x_3 and x_5), but x_8 has no neighbour in C (x_8 cannot be adjacent to u). This contradicts Corollary 7. □

Now consider the graph $H'' = H' \setminus \{x_5, x_{10}\}$. It follows from Claim 21 that every vertex in G has at most two neighbours in H''. The usual argument, of double counting the edges between $V(H'')$ and $V(G)\setminus V(H'')$, leads to a contradiction to the minimum degree condition, thus completing the proof of Lemma 8 in Case 1.

5.2 **Case 2:** a 3-path between x_7 and x_8

Denote by H' the graph induced by $\{x_1, \dots, x_{10}, y_1, y_2\}$ (see Figure 8).

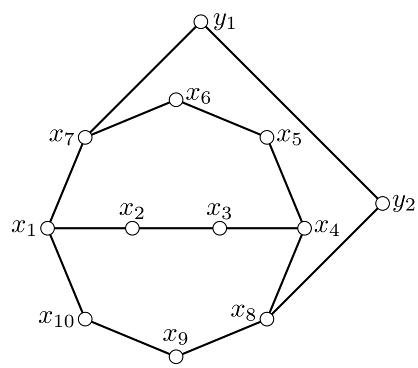


Figure 8: Case 2: a path of length 3 between x_7 and x_8

Claim 22. *The graph H' is well-behaved.*

Proof. If H' is not well-behaved, then up to relabelling, y_1 and x_3 have a common neighbour u (recall that H is well-behaved by Claim 20). Consider the 6-cycle $(uy_1x_7x_1x_2x_3)$. Since there is no induced 6-cycle (Lemma 5), either y_1 is adjacent to x_2, or u is adjacent to x_1. The former case leads to a contradiction similarly to Claim 21: then x_1 has two neighbours in the 7-cycle $(x_2x_3x_4x_5x_6x_7y_1)$ whereas its neighbour x_{10} has no neighbour there, contradicting Corollary 7. So, suppose the latter case holds, i.e. u is adjacent to x_1. But then u has two neighbours in the 7-cycle $(x_1x_2x_3x_4x_8x_9x_{10})$ whereas y_1 has none, a contradiction. □

As before, in light of the missing edge x_6x_{10}, one of the following three cases holds.

(a) There is a 3-path $x_6z_1z_2x_9$ between x_6 and x_9.

(b) There is a 3-path $x_5z_1z_2x_{10}$ between x_5 and x_{10}.

(c) There is a 4-path $x_6z_1z_2z_3x_{10}$ between x_6 and x_{10}.

However, (a) does not hold, as we have seen in the previous subsection. So it remains to consider (b) and (c).

Case 2b: 3-paths between x_7 and x_8 and between x_5 and x_{10}

Denote by F the graph induced by $\{x_1, \ldots, x_{10}, y_1, y_2, z_1, z_2\}$ (see Figure 9). It is easy to check that the vertices y_1, y_2, z_1, z_2 are distinct. Define $F' = F \backslash \{x_1, x_4, x_7, x_8\}$ (see Figure 9).

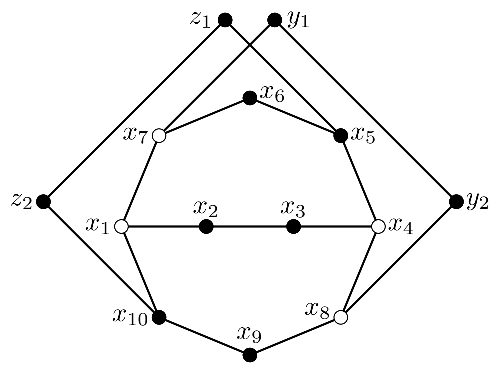


Figure 9: Case 2b: the graphs F and F' (marked in black)

Claim 23. *Every vertex of G has at most two neighbours in F'.*

Proof. Suppose that there is a vertex u with at least three neighbours in F'. We note that u is adjacent to one of y_1 and y_2 and also to one of z_1 and z_2. Indeed, otherwise, it is easy to check that u has at most two neighbours in F' using the fact that H is well-behaved (and thus also the graph induced by $\{x_1, ..., x_{10}, z_1, z_2\}$; see Claim 20). By symmetry, we may assume that u is adjacent to y_1. Suppose that u is also adjacent to z_1. By considering the 6-cycle $(uz_1x_5x_6x_7y_1)$, it follows that u is adjacent to x_6. But, then, x_7 has two neighbours in the 7-cycle $(uy_1y_2x_8x_4x_5x_6)$ while x_1 has none. This is a contradiction to Corollary 7.

It remains to consider the case where u is adjacent to both y_1 and z_2. It follows that u is adjacent to x_1 (consider $(uy_1x_7x_1x_{10}z_2)$). This is a contradiction to Corollary 7: x_{10} has two neighbours in $(uz_2z_1x_5x_6x_7x_1)$ whereas x_9 has none. □

Claim 23 leads to a contradiction by double counting the edges between $V(F')$ and $V(G) \backslash V(F')$. This completes the proof of Lemma 8 in this case.

Case 2c: a 3-path between x_7 and x_8 and a 4-path between x_6 and x_{10}

Denote by F the graph induced by $\{x_1, \ldots, x_{10}, y_1, y_2, z_1, z_2, z_3\}$ (see Figure 10).

Claim 24. *The only edges spanned by F are those spanned by H and the edges of the two paths $x_6z_1z_2z_3x_{10}$ and $x_7y_1y_2x_8$.*

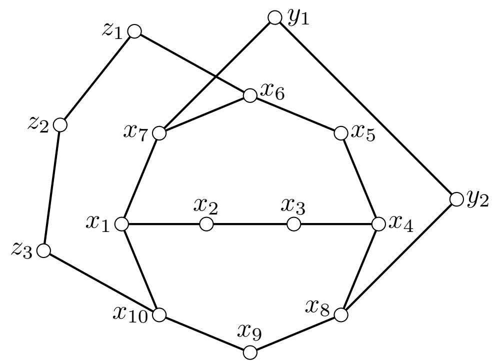


Figure 10: Case 2c: the graph F

Proof. First we note that y_1 and y_2 do not have additional neighbours in $\{x_1, \ldots, x_{10}\}$. Indeed, by symmetry we assume that y_1 has an additional neighbour in H. The only possible such neighbour is x_2. We reach a contradiction to Corollary 7 (consider the 7-cycle $(y_1x_2x_3x_4x_5x_6x_7)$ and the vertices x_1 and x_{10}).

We now show that z_1, z_2 and z_3 do not send additional edges into H. Using the fact that H' is well-behaved, the only possible additional neighbour of z_1 is x_4. But then, by replacing x_5 by z_1, we may assume that there is a 3-path from x_5 to x_{10}. This leads to a contradiction, as we have seen in Case 2b. Similarly, the possible additional neighbours of z_3 in H are x_8 and x_2. If z_3 is adjacent to x_8 then, by replacing x_9 by z_3, we may assume that there is a 3-path between x_6 and x_9, contradicting Case 1. If z_3 is adjacent to x_2 we reach a contradiction to Corollary 7 (x_1 has two neighbours in the 7-cycle $(z_3x_2x_3x_4x_8x_9x_{10})$ while x_7 has none). The possible neighbours of z_2 in H are x_3, x_5 and x_9. But z_2 is not adjacent to x_5 or x_9, because, otherwise, there is a 3-path between x_5 and x_{10} or between x_6 and x_9, contradicting previous cases. Furthermore, z_2 is not adjacent to x_3 because, otherwise, $(z_1z_2x_3x_4x_5x_6)$ is an induced 6-cycle, contradicting Lemma 5.

Finally, we show that there are no edges between $\{z_1, z_2, z_3\}$ and $\{y_1, y_2\}$. The only such edges that do not create a triangle or pentagon are z_1y_1 and z_2y_2. If z_1 is adjacent to y_1 we reach a contradiction to Corollary 7 (see $(z_1y_1y_2x_8x_4x_5x_6)$ and the vertices x_1, x_7), and if z_2y_2 is an edge, a contradiction to Lemma 5 is reached (consider the induced 6-cycle $(z_1z_2y_2y_1x_7x_6)$). This completes the proof of Claim 24. □

The following claim states that no vertex has more than three neighbours in F. Since $|V(F)| = 15$, this is a contradiction to the minimum degree condition on G by the usual double counting argument, hence the proof of Lemma 8 in this case follows.

Claim 25. *No vertex has more than three neighbours in F.*

Proof. Since H' is well-behaved (see Claim 22; recall that H' is the graph induced by the set $\{x_1, \ldots, x_{10}, y_1, y_2\}$) and has maximum degree 3, if there is a vertex u with four neighbours in

F, it must be adjacent to at least one of z_1, z_2, z_3. We note that u cannot be adjacent to both z_1 and z_3 because then, by replacing z_2 by u, we may assume that z_2 has an additional edge in F, a contradiction to Claim 24. It follows that u has one neighbour among z_1, z_2, z_3 and at least three neighbours in H'. Since H' is well-behaved, u is adjacent to all three neighbours of a vertex v in H' of degree three (in H'). But then, by replacing v by u, we may assume that v has an additional edge in F, a contradiction to Claim 24. □

5.3 **Case 3:** a 4-path between x_6 and x_8

Denote by H' the graph induced by $\{x_1, \dots, x_{10}, y_1, y_2, y_3\}$, and let $H'' = H' \setminus \{x_5, x_7, y_3\}$ (see Figure 11).

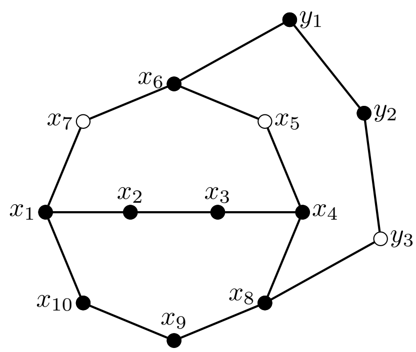


Figure 11: Case 3: the graphs H' and H'' (marked in black)

Claim 26. *The only edges in H' are those spanned by H or by the path $x_6y_1y_2y_3x_8$.*

Proof. Suppose that there are additional edges. These must be between $\{y_1, y_2, y_3\}$ and $V(H)$. The only possible neighbour (that is not already accounted for) of y_1 in H' is x_1. But then, by replacing x_7 by y_1, we reach a contradiction to Case 2.

The only possible additional neighbours of y_3 in H are x_3 and x_{10}. If y_3 is adjacent to x_3, then x_4 has two neighbours in $(x_1x_2x_3y_3x_8x_9x_{10})$ whereas x_5 has none, a contradiction to Corollary 7. If y_3 is adjacent to x_{10} then, by replacing x_9 with y_3, we reduce to Case 1.

The only possible additional neighbours of y_2 in H are x_2, x_7, x_9. If y_2 is adjacent to x_9 or x_7, we reduce to Case 1 or 2, respectively. Finally, if y_2 is adjacent to x_2 then $(x_6x_7x_1x_2y_2y_1)$ is an induced 6-cycle, a contradiction to Lemma 5. □

Claim 27. *No vertex in G has more than two neighbours in H''.*

Proof. Suppose that there is a vertex u in G with three neighbours in H''. Since H is well-behaved (see Claim 20), u must be a neighbour of either y_1 or y_2.

Suppose first that u is a neighbour of y_1. The other possible neighbours of u in H'' are x_2, x_3, x_9, x_{10}. Out of these four vertices, the only two that may have a common neighbour are x_2 and x_{10}. By considering the 6-cycle $(ux_2x_1x_7x_6y_1)$, it follows that u is adjacent also to x_7, i.e. u is adjacent to x_2, x_7, x_{10}, y_1. By replacing x_1 by u, we may assume that y_1 is adjacent to x_1, a contradiction to Claim 26.

We may now assume that u is adjacent to y_2. The other possible neighbours of u in H'' are $x_1, x_2, x_3, x_6, x_8, x_{10}$. If u is adjacent to x_6 or x_8, then by replacing y_1 or y_3 by u we see that u cannot have any additional neighbours in H'': otherwise we reach a contradiction to Claim 26. It follows that u is not adjacent to x_1, because otherwise, $(ux_1x_7x_6y_1y_2)$ is an induced 6-cycle. Similarly, u is not adjacent to x_{10} (see $(x_{10}x_9x_8y_3y_2u)$). This completes the proof of Claim 27, since the only remaining possible neighbours of u are x_2 and x_3, and these do not have a common neighbour. □

By Claim 27, we reach a contradiction using the usual double counting argument. This completes the proof of Lemma 8. □

6 The proof of Theorem 3

In this section we shall finish the proof of Theorem 3 by combining Theorem 4 along with some facts we have obtained regarding forbidden substructures in maximal $\{C_3, C_5\}$-free graphs of large minimum degree. Recall that F_k is the graph obtained from a $(5k-3)$-cycle (an edge, when $k = 1$) by adding all chords joining vertices at distances along the cycle of the form $5j + 1$ for $j = 1, \dots, k-2$. First, we prove the following proposition, which records several useful properties of the graphs F_k that we shall need in the sequel.

Proposition 28. *The following properties of F_k hold.*

(a) *Every two distinct vertices in F_k ($k \geq 2$) are contained in a 7-cycle.*

(b) *Let x and y be distinct vertices in F_k. Then there is a path of length* 1*,* 3 *or* 5 *between x and y.*

(c) *Let F be a copy of F_k in a maximal $\{C_3, C_5\}$-free graph G with $\delta(G) > n/5$. Then every vertex in G has either $k-1$ or k neighbours in F.*

(d) *Let F be a copy of F_k in a maximal $\{C_3, C_5\}$-free graph G with $\delta(G) > n/5$. Denote the vertices of F by $x_1, \dots, x_{5k-3}$ and its edges by the pairs x_ix_j for which $|i-j| \equiv 1 \pmod 5$.*

   Then for every vertex u in G there is a vertex x_i in F such that the neighbours of u in F are the neighbours of x_i in F, except at most one of x_{i-1} and x_{i+1}. In particular, F is well-behaved as a subgraph of G.

Proof. To see (a), denote the vertices and edges of F_k as above (see (d)). Note that (a) holds for $k = 2$. Now suppose that $k \geq 3$ and let x_i and x_j be two distinct vertices in F_k. Suppose that $i < j$. If $j \leq i + 6$, the two vertices are in the 7-cycle $(x_i \dots x_{i+6})$. Otherwise, x_i and x_j are two vertices in the graph induced by $F_k \setminus \{x_{i+1}, \dots, x_{i+6}\}$ which is a copy of F_{k-1}. Then, by induction, x_i and x_j are in a copy of a 7-cycle in F_k. Next, observe that (b) follows immediately from (a).

We prove (c) by induction on k. For $F_1 = K_2$ the result is clear, and for $F_2 = C_7$ the result follows from Theorem 4. So suppose that $k \geq 3$ and the result holds for smaller values of k. Let F be a copy of F_k in G as in the statement of (c), denote its vertices and edges as before, and let u be a vertex of G. Assume first that u has $k + 1$ neighbours in F. If u has at most one neighbour in some consecutive interval $x_i, \dots, x_{i+4}$ of five vertices, then u has at least k neighbours in the copy of F_{k-1} induced on $F \setminus \{x_i, \dots, x_{i+4}\}$, a contradiction to the induction hypothesis. Therefore, u has at least two neighbours in every consecutive interval of five vertices. Suppose, without loss of generality, that u is adjacent to x_1. Then u has at least $1 + 2(k - 2) \geq k$ neighbours (recall that $k \geq 3$) in the copy of F_{k-1} induced on $F \setminus \{x_{5k-7}, \dots, x_{5k-3}\}$, a contradiction. If u has at most $k - 2$ neighbours in F, one of which is, say, x_1, then u has at most $k - 3$ neighbours in the copy of F_{k-1} induced on $F \setminus \{x_1, \dots, x_5\}$, contradicting the induction hypothesis. It follows that u has either $k - 1$ or k neighbours in F, as required.

Finally, let us prove (d). Let F and G be as in the statement of (d), and suppose that u has $k - 1$ neighbours in F. Then there must exist five consecutive vertices $x_\ell, \dots, x_{\ell+4}$ which are not neighbours of u. Let F' be the copy of F_{k-1} given by $F \setminus \{x_\ell, \dots, x_{\ell+4}\}$. Then by induction there is a vertex x of F' such that u is joined to all neighbours of x in F'. We claim that $x = x_{\ell-1}$ or $x = x_{\ell+5}$. Indeed, note that x must be adjacent to precisely one of $x_{\ell-1}, x_{\ell+5}$ (it cannot be adjacent to both); otherwise, u has no neighbour in the 7-cycle $(x_{\ell-1}x_\ell \dots x_{\ell+5})$, contradicting Theorem 4. Suppose, without loss of generality, that x is joined to $x_{\ell-1}$. Since u must have a neighbour in the 7-cycle $(x_\ell x_{\ell+1} \dots x_{\ell+6})$, u is also adjacent to $x_{\ell+6}$. It follows that $x = x_{\ell+5}$ and u has $k - 1$ neighbours in F, which are precisely the neighbours of $x_{\ell+5}$, except for $x_{\ell+4}$. Now, suppose that u has precisely k neighbours in F. Then we may find two neighbours of u that are at distance at most four. We claim that this implies there must be two neighbours at distance two apart. Indeed, they cannot be at distance three (this would produce a 5-cycle). So suppose these neighbours are at distance four and suppose they are x_i and x_{i+4}. Then $(ux_{i+4}x_{i+5}x_{i+6}x_i)$ is a 5-cycle in G, a contradiction. Accordingly, we may assume without loss of generality that u is adjacent to both x_2 and x_{5k-3}. Consider the copy of F_{k-1} given by $F \setminus \{x_3, \dots, x_7\}$ and apply induction. Clearly, we must have u joined to x_7 (u's only possible neighbour in $\{x_3, \dots, x_7\}$) and the neighbourhood of u in F is precisely the neighbourhood of x_1 in F. This completes the proof of (d). □

We actually prove the followinbg theorem, which clearly implies Theorem 3. It is the odd-girth 7 analogue of a result of Chen, Jin, and Koh [7] concerning triangle-free graphs of large minimum degree.

Theorem 29. *Let G be a maximal $\{C_3, C_5\}$-free graph on n vertices with $\delta(G) > n/5$. For every integer $k \geq 2$, if G contains no copy of F_k, then G is homomorphic to F_{k-1}.*

Proof. We shall use induction on k. For $k = 2$, it is easy to show that if G contains no 7-cycle, then it must be bipartite. So fix $k \geq 3$ and suppose the result holds for smaller values of k. Let G be as in the statement of the theorem and suppose it contains no copy of F_k. If G contains no copy of F_{k-1}, then by induction G is homomorphic to F_{k-2}. But F_{k-1} contains F_{k-2}, so we are done. Hence we may assume that G contains a copy of F_{k-1}. Let H be a vertex-maximal blow-up of F_{k-1} in G with vertex classes $X_1, \ldots, X_{5k-8}$, where the edges of H are $X_i - X_j$ edges for which $|i - j| \equiv 1 \pmod 5$. Our aim is to show that G is a blow-up of F_{k-1}, or, in other words, that H spans all vertices in G. Note that by (c) of Proposition 28, every vertex in $V(G) \setminus V(H)$ has at most $k - 1$ neighbours in F_{k-1}.

Suppose $u \in V(G) \setminus V(H)$ is adjacent to vertices in precisely $k - 1$ of the classes of H. Without loss of generality, by (d), we may assume that these classes are those in the neighbourhood of vertices in X_1, i.e., $X_2, X_7, \ldots, X_{5k-8}$, and let $J = \{2, 7, \ldots, 5k - 8\}$ be the set of indices j such that u has a neighbour in X_j. We claim that u must be adjacent to every vertex in each of these classes, contradicting the assumption that H is a vertex-maximal blow-up in G. Suppose this is not the case. By (c), u has a non-neighbour in at most one of the sets X_j with $j \in J$ (indeed, otherwise we find a copy of F_{k-1} in which u has at most $k - 3$ neighbours). Furthermore, by (d), we may assume that this set is X_2. Let $y \in X_2$ be a neighbour of u and let $z \in X_2$ be a non-neighbour of u.

Owing to the missing edge uz, and by the edge-maximality of G, there must exist a 4-path $uw_1w_2w_3z$ in G between u and z (a 2-path is impossible). Consider the $(5k-3)$-cycle $C = (uw_1w_2w_3zx_1x_{5k-8} \ldots x_3y)$, where $x_i \in X_i$ (see Figure 12).

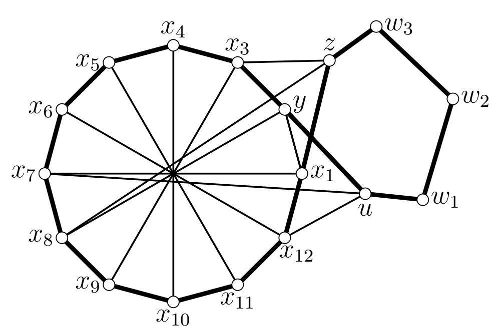


Figure 12: the $(5k-3)$-cycle C obtained from u and H, $k = 4$

Our aim is to show that $V(C)$ induces a copy of F_k, contrary to our assumption on G. Relabel the cycle C in order as $(z_0 z_1 \dots z_{5k-4})$, so that $z_0 = u, z_i = w_i$ for $i = 1, 2, 3$, $z_4 = z, z_5 = x_1$, $z_i = x_{5k-2-i}$ for $6 \leq i \leq 5k-5$, and $z_{5k-4} = y$. We must check that all chords of lengths $1 + 5t$ for $t = 0, \dots k-1$ are present in the graph induced on $V(C)$. Note that all possible chords of these lengths that are not incident with a vertex in $S = \{u, w_1, w_2, w_3\} = \{z_0, z_1, z_2, z_3\}$ are present, since all vertices in $V(C) \setminus S$ are in an appropriate copy of F_{k-1}. So we must check that all possible chords incident with a vertex in S are present. This is summarized in the following claim, where we temporarily revert to the original labelling of C:

Claim 30. *The following hold:*

- $N(u, C) = \{w_1, y\} \cup \{x_{5\ell+2} : 1 \leq \ell \leq k-2\}$.
- $N(w_1, C) = \{u, w_2\} \cup \{x_{5\ell+1} : 1 \leq \ell \leq k-2\}$.
- $N(w_2, C) = \{w_1, w_3\} \cup \{x_{5\ell} : 1 \leq \ell \leq k-2\}$.
- $N(w_3, C) = \{z, w_2\} \cup \{x_{5\ell-1} : 1 \leq \ell \leq k-2\}$.

Proof. Observe that the first item is immediate from our choice of u. Fix some ℓ with $1 \leq \ell \leq k-2$. Note that every vertex in X_2 is joined to $x_{5(\ell-1)+3} = x_{5\ell-2}$. In particular, y and z are joined to $x_{5\ell-2}$. Consider the 12-cycle $C' = (uw_1w_2w_3zx_{5\ell-2} \dots x_{5\ell+2}x_1y)$, with two consecutive diagonals $yx_{5\ell-2}$ and x_1z. Observe that C' gives rise to another 12-cycle $C'' = (uw_1w_2w_3zx_1x_{5\ell+2}x_{5\ell+1} \dots x_{5\ell-2}y)$ with two consecutive diagonals yx_1 and $ux_{5\ell+2}$. By Lemma 6, either C' or C'' has all of its diagonals present. However, it cannot be C', since u cannot be adjacent to $x_{5\ell-1}$. Therefore, C'' has all diagonals present: $w_1x_{5\ell+1}$, $w_2x_{5\ell}$, and $w_3x_{5\ell-1}$ are edges in G. This completes the proof of Claim 30. □

It remains to check that Claim 30 produces chords of the right lengths. We do this for chords incident with w_1; the other cases follow identically. Indeed, $w_1 = z_1$ so we must check that z_1 is joined to $z_{1+(1+5t)}$ for $t = 0, 1, \dots, k-1$. This is obviously true for $t = 0$ and $t = k-1$, so let $1 \leq t \leq k-2$. Then the above is equivalent to w_1 being joined to $x_{5k-2-(1+(1+5t))} = x_{5(k-t-1)+1}$, where $1 \leq k-t-1 \leq k-2$, which clearly follows by Claim 30. Accordingly, there is a copy of F_k in G contrary to our assumption, so u must be adjacent to every vertex in X_j for all $j \in J$. But then we may place u in X_1 and produce a blow-up of F_{k-1} of larger order, which is a contradiction to the choice of H. It follows that every vertex in $V(G) \setminus V(H)$ is adjacent to vertices in at most $k-2$ of the sets X_i. In fact, by (d) of Proposition 28, it follows that every vertex in $V(G) \setminus V(H)$ is adjacent to precisely $k-2$ of the X_i's.

Before proceeding, let us introduce a bit of notation and terminology. Let $\widetilde{H}$ be the graph with vertex set $\{X_1, \ldots, X_{5k-8}\}$, where an edge X_iX_j is present whenever the pair $\{X_i, X_j\}$ induces a complete bipartite graph in G. As H is a blow-up of F_{k-1}, $\widetilde{H}$ is isomorphic to F_{k-1}. We say that a vertex v is *joined* to a subset $X \subseteq V(G)$ if v is adjacent to every vertex of X.

If a vertex v is joined to vertices in the neighbourhood of X_i, then by (d) of Proposition 28 we have that v misses vertices in at most the two sets X_{i-1}, X_{i+1}; by symmetry, we may assume that each such vertex v misses X_{i-1}. Thus the following sets Y_i, where $i = 1, \ldots, 5k-8$, defined below, form a partition of $V(G) \setminus V(H)$ (see Figure 13). Note that each of these sets is independent (as G is triangle-free):

$$Y_i = \{u \in V(G) \setminus V(H) : u \text{ is joined to } X_{i+1}, X_{i+6}, \ldots, X_{i+5k-14} \text{ (indices modulo } 5k-8)\}$$

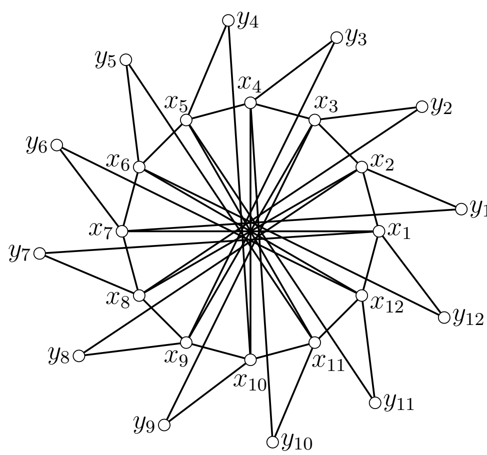


Figure 13: the sets X_i and Y_i

Claim 31. *Let $k \geq 3$ and $1 \leq i, j \leq 5k-8$. If j is such that $X_j \notin N_{\widetilde{H}}(X_i)$, then there are no edges between Y_i and Y_j.*

Proof. Without loss of generality, set $i = 1$. Suppose j is such that $X_j \notin N_{\widetilde{H}}(X_1)$. We may assume that $j \neq 1$, as each Y_i is independent. Then $j = 5l + r$, where $l \in \{0, \ldots, k-3\}$ and $r \in \{3, 4, 5, 6\}$. Towards a contradiction, suppose there is an edge y_1y_j between Y_1 and Y_j. We consider four cases, according to the value of r. Suppose first that $r = 3$. Then we find the following 5-cycle $(y_1x_2x_{5l+3}x_{5l+4}y_j)$. If $r = 4$, we find the induced 6-cycle $(y_1x_2x_{5l+3}x_{5l+4}x_{5l+5}y_j)$. If $r = 5$, there is, again, an induced 6-cycle $(y_1x_2x_1x_{5l+7}x_{5l+6}y_j)$. Finally, if $r = 6$, there is a 5-cycle $(y_1x_2x_{5l+8}x_{5l+7}y_j)$.

For each of the possible values of r, we reached a contradiction by showing that G contains either a 5-cycle or an induced 6-cycle. Claim 31 follows. □

Let $Z_i = X_i \cup Y_i$. Note that the sets Z_i are independent and they partition $V(G)$. It follows from Claim 31 that there are no $Z_i - Z_j$ edges if $X_i X_j \notin E(\widetilde{H})$. By maximality of G, all $Z_i - Z_j$ edges are present if $X_i X_j \in E(\widetilde{H})$, implying that G is a blow-up of F_{k-1}. In particular, G is homomorphic to F_{k-1}, as required to complete the proof of Theorem 29. $\square$

We close this section by showing the following consequence of Theorem 29.

Corollary 32. *Let G be a $\{C_3, C_5\}$-free graph on n vertices with $\delta(G) > \frac{k}{5k-3}n$. Then G is homomorphic to F_{k-1}.*

Proof. Note that we may assume that G is maximal $\{C_3, C_5\}$-free. By Theorem 29, if G is not homomorphic to F_{k-1}, it contains a copy F of F_k. The number of edges between $V(F)$ and $V(G) \setminus V(F)$ is at most $k(n-(5k-3))$, since every vertex in G has at most k neighbours in F, by Proposition 28. It follows that there is a vertex u in F with at most $\frac{kn}{5k-3} - k$ neighbours outside of F. Since u has k neighbours in F, it follows that u has degree at most $\frac{kn}{5k-3}$, a contradiction to the minimum degree condition. $\square$

7 Homomorphism thresholds

Recall that, given a family of graphs $\mathcal{H}$, the homomorphism threshold $\delta_{\text{hom}}(\mathcal{H})$ of $\mathcal{H}$ is the infimum of d such that every $\mathcal{H}$-free graph with n vertices and minimum degree at least dn is homomorphic to a bounded $\mathcal{H}$-free graph. In this section, we prove Theorem 1, thereby determining the value of $\delta_{\text{hom}}(\{C_3, C_5\})$. We also prove that $\delta_{\text{hom}}(C_5) \leq 1/5$ by showing that C_5-free graphs of large enough minimum degree are also triangle-free.

Proof of Theorem 1. Denote $\delta = \delta_{\text{hom}}(\{C_3, C_5\})$. First, we show that $\delta \geq 1/5$. We note that F_k is not homomorphic to a $\{C_3, C_5\}$-free graph H with fewer than $|V(F_k)|$ vertices. Indeed, suppose otherwise. Then two vertices x and y in F_k are mapped to the same vertex u in H. By (b) there is a path P of length 1, 3 or 5 between x and y. Clearly, P cannot have length 1 (because the set of vertices mapped to the same vertex is independent). It follows that P has length 3 or 5. This implies that the path P is mapped to a cycle of length 3 or 5, a contradiction. It follows that, for each $k \geq 1$, F_k is a $\{C_3, C_5\}$-free graph with minimum degree at least $|V(F_k)|/5$, which is not homomorphic to a $\{C_3, C_5\}$-free graph on fewer than $|V(F_k)|$ vertices. Hence, indeed, $\delta \geq 1/5$.

It remains to show that $\delta \leq 1/5$. Let $\varepsilon > 0$ be fixed. Suppose that G is a $\{C_3, C_5\}$-free on n vertices and minimum degree at least $(1/5 + \varepsilon)n$. Let k be such that $\frac{k}{5k-3} < 1/5 + \varepsilon$. Then, by Corollary 32, G is homomorphic to F_{k-1}. This shows that $\delta \leq 1/5 + \varepsilon$. Since ε was arbitrary, we conclude that $\delta \leq 1/5$. $\square$

It would be interesting to determine the homomorphism threshold of C_5. The following lemma enables us to easily obtain an upper bound.

Lemma 33. *Let G be a C_5-free graph on n vertices and let $\gamma > 0$ with $\delta(G) \geq n/6 + \gamma n$. Then G is triangle-free provided n is sufficiently large.*

We remark that with a little extra work we are able to prove the same result under the weaker condition that $\delta(G) > n/6 + 1$, which is in fact tight whenever 12 divides n; we omit the details for brevity. Before proving Lemma 33, we use it to prove Corollary 2, which provides an upper bound on the homomorphism threshold $\delta_{\text{hom}}(C_5)$. We currently do not have any nontrivial lower bound on $\delta_{\text{hom}}(C_5)$.

Proof of Corollary 2. Suppose that G is a C_5-free graph on n vertices and minimum degree at least $(1/5 + \varepsilon)n$ for some fixed $\varepsilon > 0$. Then, by Lemma 33, G is also triangle-free. It follows from Theorem 1 that G is homomorphic to a C_5-free (and C_3-free) graph H of order at most $C = C(\varepsilon)$. Hence, indeed, $\delta_{\text{hom}}(C_5) \leq 1/5$. □

We now turn to the proof of Lemma 33.

Proof of Lemma 33. We start by showing that every vertex in G is incident with at most 13 *triangular edges* (i.e. edges on triangles). To see this, suppose that u is incident with at least 14 triangular edges. In other words, the neighbourhood $N(u)$ of u contains edges that span at least 14 vertices. The following claim implies that there is a set X of seven neighbours of u such that every vertex in X has a neighbour in $N(u) \setminus X$.

Claim 34. *Let H be a graph with n vertices and no isolated vertices. Then there is a set X of size at least $n/2$ such that every vertex in X has a neighbour outside of X.*

Proof. We note that it suffices to prove the claim under the assumption that H is connected. Indeed, for each component H_i of H, we may pick a set X_i as in the claim, and let X be the union of the X_i's. So now we assume that H is connected. Because of the assumption that there are no isolated vertices, we may assume that $|V(H)| \geq 2$.

Let u be a vertex for which $H \setminus \{u\}$ is connected. Let v be a neighbour of u. Consider the graph $H' = H \setminus \{u, v\}$. Let $H_1, \ldots, H_t$ be the connected components of H'. We pick a set X_i for each $i \in [t]$ as follows: if H_i consists of a single vertex x_i, then x_i must be adjacent to v, and we take $X_i = \{x_i\}$; otherwise, if H_i has at least two vertices, then by induction there is a set X_i of size at least $|V(H_i)|/2$ such that every vertex in X_i has a neighbour outside of X_i (but in H_i). Let $X = \bigcup_{i=1}^{t} X_i \cup \{u\}$. It is easy to check that X satisfies the requirements of Claim 34. □

Let Y be a set of at most seven neighbours of u, which is disjoint from X and satisfies that every vertex in X has a neighbour in Y. Due to the minimum degree condition, if n is sufficiently large, then we may find two distinct vertices x_1 and x_2 in X that have a common neighbour z outside of $X \cup Y \cup \{u\}$. Let $y \in Y$ be a neighbour of x_1. Then we find the 5-cycle $(x_1 y u x_2 z)$, a contradiction. Thus, indeed, every vertex is incident with at most 13 triangular edges.

With this in mind, the proof of Lemma 33 is nearly complete. Indeed, suppose that $T = x_1x_2x_3$ is a triangle in G, and remove an edge from each triangle (except T) to form a new graph G'. The minimum degree only drops by at most 13, so if n is sufficiently large, we obtain $\delta(G') > n/6 + 1$. Consider the neighbourhoods N_1, N_2, N_3 of x_1, x_2, x_3 outside of T. These sets are pairwise disjoint and independent in G'. Pick $y_i \in N_i$ and consider $N(y_i) \setminus \{x_i\}$ for $i = 1, 2, 3$. These sets are pairwise disjoint (from the assumption that G contains no 5-cycle), and independent. Moreover, they are disjoint from $N_1 \cup N_2 \cup N_3$. Letting M be the union of these six sets and T, we have

$$|M| > 3\left(\frac{n}{6} - 1\right) + 3 \cdot \frac{n}{6} + 3 = n,$$

a contradiction. Thus G is triangle-free provided n is sufficiently large, completing the proof. □

8 Final remarks

We are able to determine precisely the structure of $\{C_3, C_5\}$-free graphs with high minimum degree, and thereby deduce the value of the homomorphism threshold $\delta_{\text{hom}}(\{C_3, C_5\})$. It would be very interesting to extend this result to $\{C_3, \ldots, C_{2\ell-1}\}$-free graphs. Recall that, for integers $k \geq 2, \ell \geq 3$, F_k^ℓ is the graph obtained from a $((2\ell-1)(k-1)+2)$-cycle by adding all chords joining vertices at distances $j(2\ell-1)+1$ for $j = 0, 1, \ldots, k-1$. In light of our Theorem 3 it is natural to ask whether or not a $\{C_3, C_5 \ldots, C_{2\ell-1}\}$-free graph on n vertices with minimum degree larger than $\frac{n}{2\ell-1}$ is homomorphic to F_k^ℓ for some k. Rather surprisingly it turns out that this is false when $\ell \geq 4$ is even, as shown by the following construction due to Oliver Ebsen [9]. Suppose that $\ell \geq 4$ is even. Starting with a complete graph on four vertices, subdivide two independent edges by an additional $2\ell - 6$ vertices and subdivide the remaining four edges by an additional two vertices each. Denote the resulting graph by T_ℓ. It is easy to check that this graph is maximal $\{C_3, C_5 \ldots, C_{2\ell-1}\}$-free. To obtain large minimum degree assign weight 2 to each vertex of the original K_4 and to $\ell - 4$ vertices of the 'long' subdivided edges, and assign weight 1 to the remaining vertices. This may be done in such a way that each vertex has weight 3 in its neighbourhood (as ℓ is even). To obtain an unweighted graph of order n simply blow up each vertex with an independent set of size proportional to its weight. Then the resulting graph T_ℓ^* is maximal $\{C_3, C_5 \dots, C_{2\ell-1}\}$-free and $\delta(T_\ell^*) = \frac{3n}{6\ell-4} > \frac{n}{2\ell-1}$. However, it is not hard to show that T_ℓ is not homomorphic to F_k^ℓ, for any k (and therefore no blow-up of T_ℓ is homomorphic to any F_k^ℓ). We do not know whether Theorem 3 extends naturally to $\{C_3, C_5, \dots, C_{2\ell-1}\}$-free graphs when $\ell \geq 5$ is odd, and it would be interesting to pursue this line of research further.

Recall that the homomorphism threshold of a family of graphs $\mathcal{H}$ is the infimum of d satisfying that every $\mathcal{H}$-free graph with n vertices and minimum degree at least dn is homomorphic to an $\mathcal{H}$-free graph of bounded order (depending on d but not on n). Despite the above remarks concerning the extension of Theorem 3 to general odd-girth graphs, we still make the following conjecture concerning the homomorphism threshold of $\{C_3, C_5, \dots, C_{2\ell-1}\}$-free graphs for $\ell \geq 4$.

Conjecture 35. *Let $\ell \geq 4$ be an integer. Then $\delta_{\text{hom}}(\{C_3, C_5 \dots, C_{2\ell-1}\}) = \frac{1}{2\ell-1}$.*

We have also obtained an upper bound on $\delta_{\text{hom}}(C_5)$, namely, that it is at most $1/5$. We ask if it is true that $1/5$ is the correct value.

Question 36. *Is it true that $\delta_{\text{hom}}(C_5) = 1/5$?*

In fact, any nonzero lower bound on $\delta_{\text{hom}}(C_5)$ would be interesting. In order to obtain such a lower bound, one would have to find, in particular, a family of graphs that have large minimum degree, are C_5-free and are not 4-colourable (indeed, otherwise, the graphs are homomorphic to K_4, which is clearly C_5-free). Although it is well known that such graphs exist, it seems hard to find explicit examples, especially with the added condition that they are not homomorphic to C_5-free graphs of bounded order.

Note added in the proof. After the paper was submitted Ebsen and Schacht [8] proved that the homomorphism threshold of $\{C_3, \dots, C_{2\ell-1}\}$ is $\frac{1}{2\ell-1}$, thereby proving Conjecture 35. We note that Question 36 is still left unanswered. Furthermore, in light of this new result, we pose the following question which suggests a strengthening of Conjecture 35.

Question 37. *Let $\ell \geq 4$ be integer and let $\varepsilon > 0$, and suppose G is a $\{C_3, \dots, C_{2\ell-1}\}$-free graph on n vertices with minimum degree at least $(\frac{1}{2\ell-1} + \varepsilon)n$. Is it true that G is 3-colourable?*

Acknowledgments

This research was initially conducted at the University of Cambridge. The second author is grateful to the Department of Mathematical Sciences at the University of Memphis, and especially to Béla Bollobás, for making that possible, and for the hospitality of the Combinatorics Group at Cambridge. We should also like to thank Julian Sahasrabudhe for suggesting the problem, and Béla Bollobás, Tomasz Łuczak, and Mathias Schacht for very helpful comments. Finally, we would like to thank the anonymous referees for their careful reading and helpful suggestions.

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