CLASSICAL SOLITON SYSTEMS

(A graduate course by S. Ruijsenaars, Amsterdam '91)

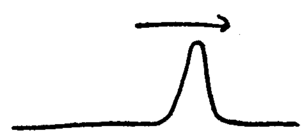
<u>Preamble</u>. Integrable systems are dynamical systems with special features which render them 'soluble'. They occur in the contexts of classical and quantum mechanics and field theory. We shall concentrate on classical systems. More specifically, we consider classical integrable systems giving rise to <u>soliton</u> <u>scattering</u>.

The first partial differential equation for which solitons were discovered is the Korteweg-de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0$$

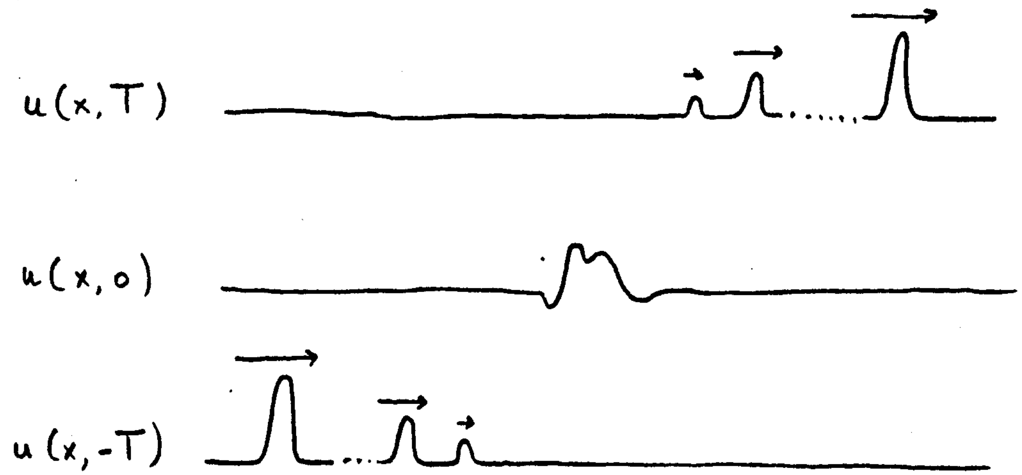
which describes one-dimensional water waves (along a canal, say). The 1-soliton solution was already found in the last century (Russell, Rayleigh). It reads

$$u(x,t) = \frac{2a^2}{\mathrm{ch}^2\left[a(x-x_0)-4a^3t\right]}$$



$v = 4a^2$

The N-soliton solutions were discovered in the late sixties (Kruskal and co-workers):



Thus, soliton scattering is characterized first of all by <u>stability</u>: number and velocities are conserved. The second characteristic of soliton scattering is <u>factorization</u>: the positions of the peaks are shifted as if independent pair collisions were taking place.

Soliton solutions occur for many other nonlinear evolution equations (mainly one-dimensional). They have an ever growing number of applications to physics, chemistry and biology. For instance, the sine-Gordon equation

$$\varphi_{xx} - \varphi_{tt} = \sin\varphi$$

has been used to model the propagation of dislocations in crystals, phase differences across Josephson junctions, torsion waves in strings and pendulas, and waves along lipid membranes. It arose already in the last century in the context of pseudo-spherical surfaces and is also used (in quantized form) as a model for elementary particles.

The soliton PDEs just referred to can be viewed as infinite-dimensional integrable systems. In the first part of these lectures we are concerned with a class of finite-dimensional integrable systems describing N point particles on the line. Their dynamics not only leads to a soliton type scattering, but can also be used to obtain the N-soliton solutions of various PDEs

and soliton lattices, such as the infinite Toda lattice. These so-called Calogero-Moser systems (and relatives thereof) are surveyed in:

[1] M. Olshanetsky, A. Perelomov, Classical integrable finite-dimensional systems related to Lie algebras, Physics Reports 71 (1981) 313-400

[2] S.R., Finite-dimensional soliton systems, in: Integrable and superintegrable systems (B. Kupershmidt, ed.) pp. 165-206, World Scientific, Singapore 1990

The second part of the course concerns infinite-dimensional soliton systems. A selection from the vast literature:

[3] A. Scott, F. Chu, D. McLaughlin, The soliton: a new concept in applied science, Proc. IEEE 61 (1973) 1443-1483

[4] M. Ablowitz, H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia 1981

[5] L. Faddeev, L. Takhtajan, Hamiltonian methods in the theory of solitons, Springer, Berlin 1987

A. Finite-dimensional soliton systems

A1. Classical mechanics / symplectic geometry (first steps)

i) Physical setting

Will consider N point particles on the line. Position of system given by

$$x \in G \subset \mathbb{R}^N \quad (\text{configuration space})$$

and state of system given by point in

$$\Omega \equiv \{ (x,p) \in \mathbb{R}^{2N} \mid x \in G \} \quad (\text{phase space})$$

System dynamics (time evolution) is specified via Hamilton equations:

$$\dot{x}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial x_j}, \qquad j = 1, \ldots, N$$

where $H(x,p) \in C^\infty(\Omega)$ (Hamiltonian / energy function)

Conserved quantities $F \in C^\infty(\Omega)$ satisfy

$$0 = \frac{d}{dt} F(x(t), p(t)) = \partial_x F \cdot \dot{x} + \partial_p F \cdot \dot{p} = \{F, H\}$$

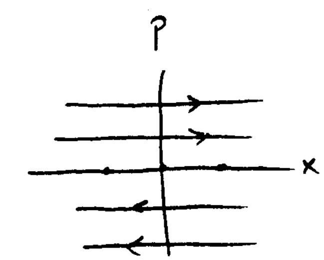
where $\{F_1, F_2\} \equiv \partial_x F_1 \cdot \partial_p F_2 - \partial_p F_1 \cdot \partial_x F_2$ Poisson bracket.

Ex. $H = \sum_{j=1}^N \frac{1}{2} p_j^2 + U(x_1, \ldots, x_N)$ = kinetic + potential energy

For free system $U = 0$; then any $F = f(p)$ conserved and evolution linear in t:

$$(x,p) \mapsto (x_1 + t p_1, \ldots, x_N + t p_N, p)$$

Phase diagram for $N = 1$:



ii) Mathematical setting

Define 2-form $\omega = \sum_{j=1}^N dx_j \wedge dp_j$ on above phase space Ω.

(More generally, can start from differentiable manifold Ω and nondegenerate closed 2-form ω; then $\langle \Omega, \omega \rangle$ is symplectic manifold, ω symplectic form / structure.)

Since ω nondegenerate, can identify 1-form α with vector field $X^{(\alpha)}$ by requiring

$$\omega(X^{(\alpha)}, X) = \alpha(X), \quad X \text{ arbitrary vector field}$$

In particular, if

$$\alpha = dH = \sum_{j=1}^{N} (\partial_{x_j} H \, dx_j + \partial_{p_j} H \, dp_j), \quad H \in C^\infty(\Omega)$$

then will get Hamiltonian vector field

$$X_H \equiv X^{(dH)} = \sum_{j=1}^{N} \left(\partial_{p_j} H \frac{\partial}{\partial x_j} - \partial_{x_j} H \frac{\partial}{\partial p_j} \right)$$

and Hamilton equations read

$$\dot{u} = X_H(u), \quad u \in \Omega$$

or

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \mathcal{S} \begin{pmatrix} \partial_x H \\ \partial_p H \end{pmatrix}, \quad \mathcal{S} \equiv \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix}$$

From ODE lore get (local) Hamiltonian flow

$$\Phi_t : \Omega \to \Omega, \quad u \mapsto u_t$$

also written e^{tX_H} or e^{tH}.

Next, introduce Poisson bracket

$$\{\cdot,\cdot\} : C^\infty(\Omega) \times C^\infty(\Omega) \to C^\infty(\Omega), \quad (F,G) \mapsto \{F,G\}$$

$$\{F,G\} \equiv \omega(X_F, X_G) = -X_F G = X_G F$$
$$= -dG(X_F) = dF(X_G) = \nabla F \cdot \mathcal{S} \, \nabla G^T$$

and verify canonical commutation relations:

$$\{x_i, x_j\} = \{p_i, p_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij}$$

Now let $\langle \hat{\Omega}, \hat{\omega} \rangle$ be symplectic manifold with $\hat{\Omega} \subset \mathbb{R}^{2N}$, $\hat{\omega} \equiv \sum_{j=1}^{N} d\hat{x}_j \wedge d\hat{p}_j$ and Poisson bracket $\{\cdot,\cdot\}^\wedge$, and let

$$\Phi : \Omega \to \hat{\Omega}, \quad (x,p) \equiv u \longmapsto (\hat{x}, \hat{p}) \equiv \hat{u}$$

be diffeomorphism. Then Φ is called canonical/symplectic

$$\Longleftrightarrow \{\hat{x}_i, \hat{x}_j\} = \{\hat{p}_i, \hat{p}_j\} = 0, \quad \{\hat{x}_i, \hat{p}_j\} = \delta_{ij}$$

Equivalent properties are (check):

$$\Phi^* \hat{\omega} = \omega, \quad \Phi^* \{\cdot,\cdot\}^\wedge = \{\cdot,\cdot\},$$

$$(D\Phi)(u) \, \mathcal{S} \, (D\Phi(u))^T = \mathcal{S}, \quad \forall u \in \Omega$$

The key property of canonical transformations is that they leave Hamilton's equations invariant. That is, if Φ canonical, then

$$e^{tH} = \Xi \circ e^{t\hat{H}} \circ \Phi, \text{ where } \Xi \equiv \Phi^{-1}, \ \hat{H} \equiv H \circ \Xi :$$

$$\begin{tikzcd} t & t \arrow[l, "\Xi"'] \\ 0 \arrow[u, "e^{tH}"] \arrow[r, "\Phi"'] & 0 \arrow[u, "e^{t\hat{H}}"'] \end{tikzcd}$$

More concretely, if $\hat{u}(t)$ solves $\dot{\hat{u}} = \mathcal{S}\, \nabla \hat{H}(\hat{u})^T$, then $u(t) \equiv u(\hat{u}(t))$ solves $\dot{u} = \mathcal{S}\, \nabla H(u)^T$. (Check via chain rule.) Thus, to solve Hamilton's equations for given $H \in C^\infty(\Omega)$, should find canonical transformation Φ rendering $\hat{H} \equiv H \circ \Phi^{-1}$ 'as simple as possible'.

iii) <u>Liouville integrability</u>

Let $\mathcal{S} = \langle \Omega, \omega, H \rangle$ be Hamiltonian system with N degrees of freedom (i.e., $\dim \Omega = 2N$). Assume X_H is complete (i.e., e^{tH} is global flow). Then

$\mathcal{S}$ is called (completely) <u>integrable</u> iff there exist $I_1, \ldots, I_N \in C^\infty(\Omega)$ such that:

1) $\{H, I_k\} = 0$, $k = 1, .., N$ ($\Leftrightarrow I_1, .., I_N$ <u>integrals</u>)

2) $\{I_k, I_l\} = 0$, $k, l = 1, .., N$ ($\Leftrightarrow I_1, .., I_N$ <u>in involution</u>)

3) $I_1, .., I_N$ independent (i.e, $dI_1, .., dI_N$ linearly independent on open dense subset of Ω)

<u>Ex.</u> a) $H = \frac{1}{2} \sum_{j=1}^N p_j^2$. (Can take $I_k \equiv \frac{1}{k} \sum_{j=1}^N p_j^k$, $k = 1, .., N$)

b) $H = \frac{1}{2} \sum_{j=1}^N (p_j^2 + x_j^2)$. (Can take $I_k \equiv p_k^2 + x_k^2$, $k = 1, .., N$)

c) Let $\hat{\Omega} \subset \mathbb{R}^{2N}$, $\hat{\omega} = \sum_{j=1}^N d\hat{x}_j \wedge d\hat{p}_j$ and let

$$\Xi : \langle \hat{\Omega}, \hat{\omega} \rangle \to \langle \Omega, \omega \rangle, \ (\hat{x}, \hat{p}) \mapsto (x, p)$$

be canonical diffeo. Now let $\hat{H} = f(\hat{p})$ and define $H = \hat{H} \circ \Xi^{-1}$. (Can take $I_k = \hat{I}_k \circ \Xi^{-1}$, $\hat{I}_k \equiv \sum_{j=1}^N \hat{p}_j^k$. Indeed, since $\hat{H}, \hat{I}_1, \ldots, \hat{I}_N$ commute and Ξ canonical, $H, I_1, \ldots, I_N$ commute as well.)

iv) <u>Liouville-Arnold theorem ('first approximation')</u>

Suppose $H, I_1, \dots, I_N$ yield integrable system on $\langle \Omega, \omega \rangle$. Locally, there exists a canonical transformation $\Phi : (x,p) \mapsto (\hat{x}, \hat{p})$ (<u>action-angle transform</u>) such that $\hat{H} \equiv H \circ \Phi^{-1}$, $\hat{I}_k \equiv I_k \circ \Phi^{-1}$, $k = 1, \dots, N$, depend only on $\hat{p}$.

Here, $\hat{p}_1, \dots, \hat{p}_N$ are called <u>action</u> variables, $\hat{x}_1, \dots, \hat{x}_N$ <u>angle</u> variables ($\hat{x}_j$ varies over S^1 or $\mathbb{R}$). This theorem entails e.g.: $e^{t\hat{H}}(\hat{x}, \hat{p}) = (\hat{x} + t(\partial_{\hat{p}} \hat{H})(\hat{p}), \hat{p})$

Therefore, the solution

$$(x(t), p(t)) = \Xi \circ e^{t\hat{H}} \circ \Phi\,(x(0), p(0)), \qquad \Xi \equiv \Phi^{-1}$$

to Hamilton's equations is explicitly known, once Φ has been explicitly determined. (The L.A. theorem only yields existence, under some technical conditions that will not be detailed here.)

More generally, for F of the form $f(H, I_1, \dots, I_N)$ one gets:

$$\begin{tikzcd} t & t \arrow[l, "\Xi"'] \\ 0 \arrow[u, "F(x,p)\text{-flow (nonlinear in } t)"] \arrow[r, "\Phi"'] & 0 \arrow[u, "\hat{F}(\hat{p})\text{-flow (linear in } t)"'] \end{tikzcd}$$

<u>Ex.</u> Can take $\Phi = \mathrm{id}$ and $\Phi = \Xi^{-1}$ for examples a) and c) in (iii). For ex. b) can define Φ via

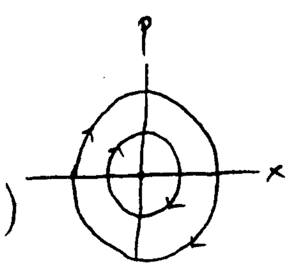
$$\hat{p}_j = \tfrac{1}{2}(p_j^2 + x_j^2) \qquad \Longleftrightarrow \qquad p_j = (2\hat{p}_j)^{1/2} \cos \hat{x}_j$$
$$\hat{x}_j = \operatorname{arctg}(x_j / p_j) \qquad\qquad\qquad x_j = (2\hat{p}_j)^{1/2} \sin \hat{x}_j$$

Here, $\hat{x}_j \in (-\pi, \pi]$ is viewed as polar angle on $\mathbb{R}^2 \setminus 0$; should take $(x_j, p_j) \neq 0$ to avoid singularities.

Then get $\hat{H} = \sum_{j=1}^{N} \hat{p}_j$ on action-angle phase space $\hat{\Omega} = (S^1 \times (0, \infty))^N$, and

$$e^{t\hat{H}}(\hat{x}, \hat{p}) = (\hat{x}_1 + t, \dots, \hat{x}_N + t, \hat{p})$$

$$e^{tH}(x,p) = (x_1 \cos t + p_1 \sin t, \dots, p_N \cos t - x_N \sin t)$$



A2. Nonrelativistic Calogero-Moser systems

i) The defining dynamics

The N-particle Hamiltonian of the (nonrelativistic) CMS reads $H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + g^2 \sum_{1\le j<k\le N} V(x_j - x_k)$, with $V(x)$ given by:

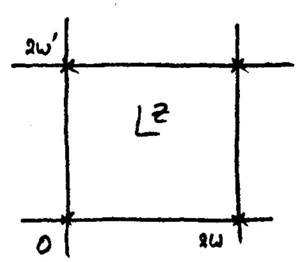
I. $1/x^2$ (rational CMS)

II. $\mu^2/4\,\mathrm{sh}^2\frac{\mu}{2}x$, $\mu>0$ (hyperbolic CMS)

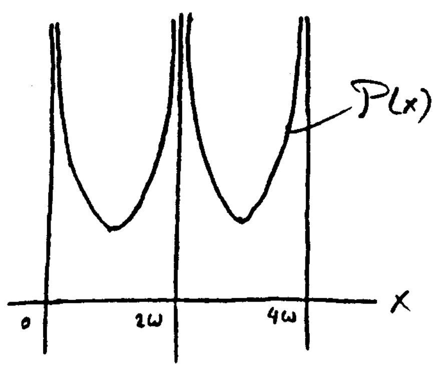
III. $\mu^2/4\sin^2\frac{\mu}{2}x$, $\mu>0$ (trigonometric CMS)

IV. $\wp(x;\omega,\omega')$, $\omega, -i\omega'>0$ (elliptic CMS)

Here, $\wp(z;\omega,\omega')$, $z\in\mathbb{C}$, denotes the Weierstrass $\wp$-function, a prototypical elliptic (i.e., meromorphic, doubly-periodic) function:



x = double poles



N.B. 1) I - III can be viewed as degenerate cases of IV:

$$\wp(z;\omega,\omega') = \begin{cases} 1/z^2 & \omega=\infty \quad \omega'=i\infty \\ \nu^2/3 + \nu^2/\mathrm{sh}^2\nu z & \omega=\infty \quad \omega'=i\pi/2\nu \\ -\nu^2/3 + \nu^2/\sin^2\nu z & \omega=\pi/2\nu \quad \omega'=i\infty \end{cases}$$

2) Taking suitable limits yields the nonrelativistic Toda systems with Hamiltonian

$$H = \frac{1}{2}\sum_j p_j^2 + \lambda_1 e^{\mu(x_2-x_1)} + \lambda_2 e^{\mu(x_3-x_2)} + \dots + \lambda_N e^{\mu(x_1-x_N)}$$

and coupling constants given by

V. $\lambda_1 = \lambda_2 = \dots = \lambda_N$ (periodic Toda systems)

VI. $\lambda_N = 0$ (nonperiodic Toda systems)

3) Connection diagram (detailed in [2]):

$$\begin{tikzcd}
\text{2-period level} & & & \mathrm{IV} \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] \arrow[dd] & & \\
\text{1-period level} & \mathrm{III} \arrow[r, leftrightarrow] \arrow[drr] & \mathrm{II} \arrow[rr] \arrow[dr] & & \mathrm{VI} & \mathrm{V} \arrow[l] \\
\text{0-period level} & & & \mathrm{I} & &
\end{tikzcd}$$

ii) <u>The $N=2$ case</u>

For $N=2$ the canonical transformation

$$y = x_1 + x_2 \qquad k = \tfrac{1}{2}(p_1+p_2)$$
$$x = x_1 - x_2 \qquad p = \tfrac{1}{2}(p_1 - p_2) \qquad \text{(center-of-mass variables)}$$

yields the transformed Hamiltonian (for the CMS)

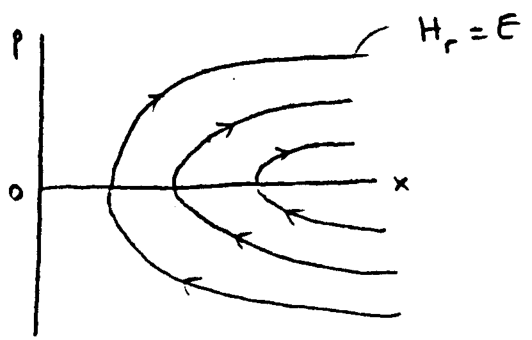
$$k^2 + p^2 + g^2 V(x) \equiv k^2 + H_r(x,p) \quad (H_r \equiv \text{reduced Hamiltonian})$$

Then Hamilton's equations can be integrated:

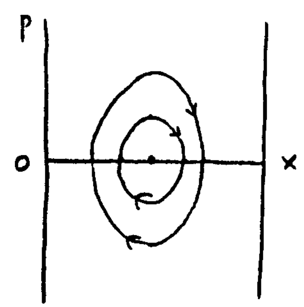
$$\dot{y} = 2k\,,\ \dot{k} = 0 \ \Rightarrow\ k(t) = k(0),\ y(t) = y(0) + 2t\,k(0)$$

$$\dot{x} = 2p,\ \dot{p} = -g^2 V'(x) \Rightarrow \frac{\dot{x}^2}{4} + g^2 V(x) = E \Rightarrow \int_{x(0)}^{x(t)} \frac{ds}{2\sqrt{E - g^2 V(s)}} = t$$

Since H_r is constant along orbits, character of motion can be read off from contour lines:



I, II : <u>scattering</u>



III, IV : <u>oscillation</u>

<u>Explicit solution</u> for case II (check details):

Integral can be done via hyperbolic substitutions:

$$\int^{x} \frac{ds}{\sqrt{E - g^2 V_{II}(s)}} = \frac{2}{\mu}\frac{1}{\sqrt{E}}\,\mathrm{Arch}\left(\sqrt{\frac{4E}{4E+\mu^2 g^2}}\ \mathrm{ch}\,\frac{\mu}{2}x\right) + c^{st}$$

Choosing $t=0$ at turning-point (i.e. $g^2 V_{II}(x(0)) = E, p(0)=0$) yields

$$x(t) = \frac{2}{\mu}\,\mathrm{Arch}\left(\sqrt{1 + \frac{\mu^2 g^2}{4E}}\ \mathrm{ch}\,\mu\sqrt{E}\,t\right)$$

$$\Rightarrow\quad x(t) = \pm 2\sqrt{E}\,t + \frac{1}{\mu}\ln\left(1 + \frac{\mu^2 g^2}{4E}\right) + O\left(e^{\mp 2\mu\sqrt{E}t}\right),\quad t \to \pm\infty$$

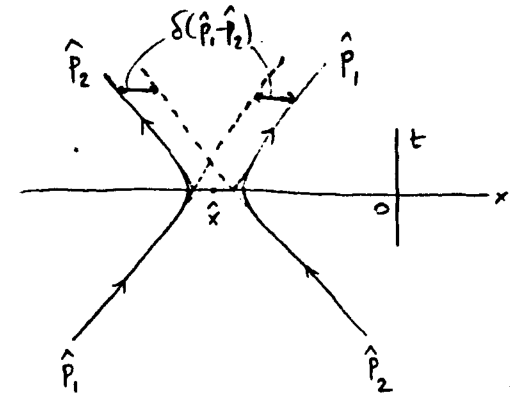
Hence, reduced particle shifted by $\frac{2}{\mu}\ln\left(1+\frac{\mu^2 g^2}{4E}\right)$, as compared to free motion + reflection at origin. Transforming back and setting $\hat{x} \equiv \frac{1}{2}y(0)$, $\hat{p}_{\substack{1\\2}} \equiv k(0) \pm \sqrt{E}$ one gets for $t \to \pm\infty$:

$$p_1(t) \sim \hat{p}_{\substack{1\\2}}\,,\quad p_2(t) \sim \hat{p}_{\substack{2\\1}}$$

$$x_1(t) \sim \hat{x} + t\,\hat{p}_{\substack{1\\2}} + \tfrac{1}{2}\delta(\hat{p}_1 - \hat{p}_2)$$

$$x_2(t) \sim \hat{x} + t\,\hat{p}_{\substack{2\\1}} - \tfrac{1}{2}\delta(\hat{p}_1 - \hat{p}_2)$$

$$\delta(p) \equiv \frac{1}{\mu}\ln\left(1 + \frac{\mu^2 g^2}{p^2}\right)$$



iii) Lax pairs : generalities

Both for finite- and for infinite-dimensional soliton systems the existence of conserved quantities can be obtained as a corollary of the following result.

— Suppose L_t, B_t, $t \in \mathbb{R}$, are operator families on a Hilbert space $\mathcal{H}$ such that $\dot{L} = [B, L]$, $\forall t \in \mathbb{R}$.
(If $\dim \mathcal{H} = \infty$ and L_t, B_t unbounded, extra assumptions needed.)
Then the spectrum of L_t is time-independent.

Proof (Sketch) The linear ODE $\dot{U} = BU$ has a unique solution U_t such that $U_0 = \mathbb{1}$. Defining

$$\tilde{L}_t = U_t L_0 U_t^{-1}$$

it follows that $\tilde{L}_t$ has the same spectrum as L_0. Thus, need only show $\tilde{L}_t = L_t$. Since $\tilde{L}_0 = L_0$, it suffices to prove $\dot{\tilde{L}} = [B, \tilde{L}]$. (Note this is a linear ODE, too.) Using $0 = (UU^{-1})^{\cdot} = B + U(U^{-1})^{\cdot}$ this is readily verified. □

N.B. 1) Iteration of $U_t = \mathbb{1} + \int_0^t B_s U_s \, ds$ yields

$$U_t = \mathbb{1} + \sum_{n=1}^{\infty} \int_0^t ds_1 \cdots \int_0^{s_{n-1}} ds_n \, B_{s_1} \cdots B_{s_n} \equiv T\exp\left(\int_0^t B_s \, ds\right)$$

("Variation of constants formula / Volterra expansion / time-ordered exponential / Dyson expansion")

2) If $\dim \mathcal{H} = N < \infty$, get $L_t, B_t \in M_N(\mathbb{C})$, and result can also be proved without ODE lore: Define

$$I_k \equiv \frac{1}{k} \operatorname{Tr} L^k \qquad \text{(power traces)}$$

and note $\dot{I}_k = \operatorname{Tr}(\dot{L} L^{k-1}) = \operatorname{Tr}(BL^k - LBL^{k-1}) = 0$

$\Rightarrow I_1, \dots, I_N$ integrals. Now set $\det(\mathbb{1}_N + \lambda L) \equiv \sum_{k=0}^{N} \lambda^k S_k$, so that

$$S_k \equiv \sum k^{\text{th}} \text{ order principal minors} \quad \text{(symmetric functions)}$$

Since S_k is polynomial in $I_1, \dots, I_k$ (and vice versa), get $\dot{S}_k = 0$, so L_t has constant spectrum. □

(The relations between the I_k and S_k ('Newton identities') follow e.g. via $1 + \lambda S_1 + \lambda^2 S_2 + \dots = \exp\left(\sum_{k=1}^{\infty} (-)^{k+1} \lambda^k I_k\right)$.)

iv) The Lax pair for case II

By energy conservation the particle ordering is preserved for type II systems. Thus, will take

$$G \equiv \{ x \in \mathbb{R}^N \mid x_N < \dots < x_1 \}$$

$$\Omega \equiv \{ (x,p) \in \mathbb{R}^{2N} \mid x \in G \}$$

Now define the Lax matrix $L : \Omega \to M_N(\mathbb{C})$ by

$$L_{jk} = p_j \delta_{jk} + ig\tfrac{\mu}{2}(1-\delta_{jk}) \frac{1}{\operatorname{sh}\frac{\mu}{2}(x_j - x_k)}, \quad j,k = 1,\dots,N$$

and note

$$H = \tfrac{1}{2} \operatorname{Tr} L^2 .$$

Claim. The spectrum of $L_t \equiv L \circ e^{tH}$ is t-independent.

Proof. In view of (iii), need only exhibit $B : \Omega \to M_N(\mathbb{C})$ such that

$$\frac{d}{dt} L(u_t) = [B(u_t), L(u_t)], \quad u_t \equiv e^{tH} u, \quad \forall u \in \Omega \quad (*)$$

Assertion: (*) is satisfied when

$$B_{jk} \equiv \frac{ig\mu^2}{4}\left[-\delta_{jk} \sum_{l \neq j} \frac{1}{\operatorname{sh}^2\frac{\mu}{2}(x_j - x_l)} + (1-\delta_{jk}) \frac{\operatorname{ch}\frac{\mu}{2}(x_j - x_k)}{\operatorname{sh}^2\frac{\mu}{2}(x_j - x_k)} \right]$$

Indeed,

$$\left.\begin{aligned} \dot{L}_{jj} &= \dot{p}_j = -\partial_{x_j} H \\ [B,L]_{jj} &= \sum_{k \neq j} (B_{jk} L_{kj} - L_{jk} B_{kj}) = \frac{g^2\mu^3}{4} \sum_{k \neq j} \frac{\operatorname{ch}\frac{\mu}{2}(x_j - x_k)}{\operatorname{sh}^3\frac{\mu}{2}(x_j - x_k)} \end{aligned}\right) \text{equal}$$

$$\dot{L}_{jk} = -ig\frac{\mu^2}{4} \frac{\operatorname{ch}\frac{\mu}{2}(x_j - x_k)}{\operatorname{sh}^2\frac{\mu}{2}(x_j - x_k)} (\dot{x}_j - \dot{x}_k)$$

$$[B,L]_{jk} = B_{jk} L_{kk} - L_{jj} B_{jk} + \text{remaining terms}$$

$$\equiv \frac{ig\mu^2}{4} \frac{\operatorname{ch}\frac{\mu}{2}(x_j - x_k)}{\operatorname{sh}^2\frac{\mu}{2}(x_j - x_k)} (p_k - p_j) + R$$

Thus, should check $R = 0$. To this end, combine terms with $l \neq j,k$ and use functional equation

$$(\operatorname{ch}a \operatorname{sh}b - \operatorname{sh}a \operatorname{ch}b)\operatorname{sh}(a+b) + \operatorname{sh}^2 a - \operatorname{sh}^2 b = 0$$

(with $a = \frac{\mu}{2}(x_j - x_l)$, $b = \frac{\mu}{2}(x_l - x_k)$).

$\square$

v) Integrability and soliton scattering for case II

Combining (iii), (iv) it follows that I_k or S_k, $k=1,\dots,N$, are independent integrals. Since the forces are repulsive in case II, one expects

$$x_j(t) \sim x_j^{\pm} + t\, p_j^{\pm}, \quad p_j(t) \sim p_j^{\pm}, \quad t \to \pm\infty \qquad (*)$$

and also

$$p_N^- > \dots > p_1^-, \quad p_N^+ < \dots < p_1^+ \qquad (**)$$

since the ordering $x_N < \dots < x_1$ is preserved. Taking this for granted, one gets

$$L_t \to \mathrm{diag}(p_1^{\pm}, \dots, p_N^{\pm}), \qquad t \to \pm\infty$$

But the spectrum of L_t is conserved ('isospectral flow'), so this entails

$$p_j^+ = p_{N-j+1}^-, \quad j=1,\dots,N \qquad \text{(conservation of momenta)}$$

Moreover, the Jacobi identity for $\{\cdot,\cdot\}$ yields

$$\{C_{k\ell}, H\} = \{\{H, S_\ell\}, S_k\} + \{\{S_k, H\}, S_\ell\} = 0, \quad C_{k\ell} \equiv \{S_k, S_\ell\}$$

(since S_k, S_ℓ are integrals). Therefore, $C_{k\ell}$ is conserved, too.

But each term in $C_{k\ell}$ contains (at least) one factor $1/\mathrm{sh}\,\frac{\mu}{2}(x_i - x_j)$, so $(*)$ implies

$$C_{k\ell}(x,p) = \lim_{t\to\infty} C_{k\ell}(x(t), p(t)) = 0$$

$\therefore$ $S_1, \dots, S_N$ are in involution.

Next, one expects that the scattering map

$$S : (x_1^-, \dots, x_N^-, p_1^-, \dots, p_N^-) \longmapsto (x_1^+, \dots, x_N^+, p_1^+, \dots, p_N^+)$$

is a canonical transformation. Accepting this and setting $x_{N-j+1}^+ = x_j^- + \Delta_j$, it then follows from $p_{N-k+1}^+ = p_k^-$ that $\frac{\partial}{\partial x_k^-} \Delta_j = 0$, $k=1,\dots,N$. Therefore, Δ_j depends only on p^- and can be determined by choosing $x_1^-, \dots, x_N^-$ such that the collisions take place (approximately) pairwise. But then one clearly gets

$$\Delta_j(p^-) = \sum_{k<j} \delta(p_j^- - p_k^-) - \sum_{k>j} \delta(p_j^- - p_k^-) \qquad \text{(factorization)}$$

N.B. These results can be rigorously proved, cf. [2].

A3. Relativistic Calogero-Moser systems

i) Galilei and Poincaré groups in 2D

In two space-time dimensions the space-time symmetry group consists of time and space translations and boosts:

$$(t,x) \mapsto (t+a_0, x+a_1) = e^{a_0 X_t + a_1 X_s}(t,x), \quad X_t = \partial_t, \quad X_s = \partial_x$$

$$(t,x) \mapsto \begin{cases} (t, x+vt) = e^{vX_b}(t,x), \quad X_b = t\partial_x & (\underline{\text{Galilei boost}}) \\ \left(\dfrac{t+vx/c^2}{\sqrt{1-v^2/c^2}}, \dfrac{x+vt}{\sqrt{1-v^2/c^2}}\right) & (\underline{\text{Lorentz boost}}) \end{cases}$$

where c = <u>speed of light</u> ($\Rightarrow$ " $\lim_{c\to\infty}$ Poincaré = Galilei ")

Set $\frac{v}{c} = \operatorname{th}\frac{\alpha}{c}$ to get 1-parameter group w.r.t. α:

$$(t,x) \mapsto \left(t \operatorname{ch}\frac{\alpha}{c} + \frac{x}{c}\operatorname{sh}\frac{\alpha}{c}, x\operatorname{ch}\frac{\alpha}{c} + ct\operatorname{sh}\frac{\alpha}{c}\right) = e^{\alpha X_b}(t,x), \quad X_b = \frac{x}{c^2}\partial_t + t\partial_x$$

Then X_t, X_s, X_b yield basis of Lie algebra:

$$[X_t, X_s] = 0, \quad [X_t, X_b] = X_s, \quad [X_s, X_b] = \begin{cases} 0 \\ \frac{1}{c^2} X_t \end{cases}$$

ii) The defining dynamics

— For <u>any</u> potential $V(x)$ the Hamiltonians

$$H = \frac{1}{2m}\sum_{j=1}^N p_j^2 + \sum_{1\le j<k\le N} V(x_j - x_k)$$

$$P = \sum_{j=1}^N p_j \qquad\qquad B = -m\sum_{j=1}^N x_j$$

represent the Galilei Lie algebra under Poisson brackets:

$$\{H,P\} = 0, \quad \{H,B\} = P, \quad \{P,B\} = mN \simeq 0 \quad (\text{since } X_{mN} = 0)$$

— Now consider the Ansatz

$$H = mc^2 \sum_{j=1}^N \operatorname{ch}(p_j/mc) \prod_{k\neq j} f(x_j - x_k)$$

$$P = mc \sum_{j=1}^N \operatorname{sh}(p_j/mc) \prod_{k\neq j} f(x_j - x_k) \qquad B = -m\sum_{j=1}^N x_j$$

Clearly, $\{H,B\} = P$, $\{P,B\} = \frac{1}{c^2}H$, and for $f(x) = 1$ get $\{H,P\} = 0$ ($\Rightarrow$ system of N free relativistic particles).

Assuming $f(x) = f(-x)$, get equivalence (check)

$$\{H,P\} = 0 \iff \sum_{j=1}^N \partial_j \prod_{k\neq j} f^2(x_j - x_k) = 0 \qquad (FE)_N$$

$\therefore$ Get system of N interacting relativistic particles if and only if f^2 satisfies $(FE)_N$.

— $(FE)_2$ is clearly satisfied for any (even) f.

— $(FE)_3$ can be rewritten as the functional equation

$$\begin{vmatrix} f^2(u) & f(u)f'(u) & 1 \\ f^2(v) & f(v)f'(v) & 1 \\ f^2(u+v) & -f(u+v)f'(u+v) & 1 \end{vmatrix} = 0 \qquad (FE)$$

— <u>Facts.</u> (FE) holds true iff $f^2(x) = a + b\wp(x)$, $a, b \in \mathbb{C}$.

$(FE)_N$, $N>3$, holds when $f^2(x) = a + b\wp(x)$.

$\therefore$ Get relativistic generalizations $I_{rel} - IV_{rel}$.

— Now replace $\prod_{k \neq j} f(x_j - x_k)$ by $f_T(x_{j+1}-x_j)f_T(x_j - x_{j-1})$ and check:

$$\{H, P\} = 0 \iff \sum_{j=1}^{N} \partial_j\, f_T^2(x_{j+1}-x_j)\, f_T^2(x_j - x_{j-1}) = 0$$

These functional equations are satisfied (check) when

$$f_T^2(x) = 1 + \alpha e^{\mu x}, \quad \alpha, \mu > 0, \quad x_0 \equiv \begin{cases} x_N \\ \infty \end{cases}, \quad x_{N+1} \equiv \begin{cases} x_1 & V_{rel} \\ -\infty & VI_{rel} \end{cases}$$

— <u>Fact.</u> The 'connection diagram' in A2 (i) applies again.

iii) <u>Integrability</u>

<u>N.B.</u> From now on: $m \equiv 1$, $\beta \equiv 1/c$.

Can write $H = (S_1 + S_{-1})/2\beta^2$, $P = (S_1 - S_{-1})/2\beta$, where

$$S_{\pm 1} \equiv \sum_{\substack{I \subset \{1,\dots,N\} \\ |I| = 1}} \exp\Big(\pm\beta \sum_{i \in I} p_i\Big) \cdot \begin{cases} \prod_{\substack{i \in I \\ j \notin I}} f(x_i - x_j) & I - IV \\ \prod_{\substack{i \in I \\ i+1 \notin I}} f_T(x_{i+1} - x_i) \prod_{\substack{i \in I \\ i-1 \notin I}} f_T(x_i - x_{i-1}) & V - VI \end{cases}$$

Substitute $|I| = 1 \to |I| = N$ to get

$$S_{\pm N} = \exp(\pm\beta[p_1 + \dots + p_N]), \quad \{S_{\pm 1}, S_{\pm N}\} = 0$$

Now substitute $|I| = 1 \to |I| = 2, \dots, N-1$ and (try to) check:

$$\{S_k, S_l\} = 0, \ \forall k, l \in \{\pm 1, \dots, \pm N\}, \ \forall N > 1 \iff$$

$$\sum_{\substack{I \subset \{1,\dots,N\} \\ |I| = k}} \Big(\sum_{i \in I} \partial_i\Big) \cdot \begin{cases} \prod_{\substack{i \in I \\ j \notin I}} f^2(x_i - x_j) = 0 & (FE)_{CM} \\ \prod_{\substack{i \in I \\ i+1 \notin I}} f_T^2(x_{i+1} - x_i) \prod_{\substack{i \in I \\ i-1 \notin I}} f_T^2(x_i - x_{i-1}) = 0 & (FE)_T \end{cases} \quad \forall k \in \{1,\dots,N\},\ \forall N > 1$$

<u>Fact.</u> $(FE)_{CM}$, $(FE)_T$ valid for $f^2(x) = a + b\wp(x)$, $f_T^2(x) = 1 + \alpha e^{\mu x}$

$\therefore$ The dynamics $H(I_{rel}) - H(VI_{rel})$ yield <u>integrable systems</u>.

iv) The Lax matrix for case II_{rel}

— Start from Cauchy's identity (try to check)

$$\det\left(\frac{1}{w_j - z_k}\right)_{j,k=1}^{N} = \prod_j \frac{1}{w_j - z_j} \prod_{j<k} \frac{(w_j - w_k)(z_j - z_k)}{(w_j - z_k)(z_j - w_k)}$$

— Substitute $w_j \to e^{\mu(x_j + \frac{i}{2}\beta g)}$, $z_k \to e^{\mu(x_k - \frac{i}{2}\beta g)}$, $\beta, \mu, g > 0$,

to get (check)

$$\det C = e^{-\mu \sum_j x_j} \prod_{j<k} 1/f^2(x_j - x_k) \qquad (|C|)$$

where

$$f(x) \equiv \sqrt{1 + \frac{\sin^2\tau}{\operatorname{sh}^2\frac{\mu}{2}x}}\,, \qquad \tau \equiv \tfrac{1}{2}\beta\mu g$$

$$C(\mu,\tau;x)_{jk} \equiv e^{-\frac{\mu}{2}(x_j + x_k)} \frac{\operatorname{sh}(i\tau)}{\operatorname{sh}(\frac{\mu}{2}(x_j - x_k) + i\tau)} \qquad (C)$$

— Now set

$$e(\beta,\mu,\tau;x,p)_j \equiv e^{\frac{\mu}{2}x_j + \frac{\beta}{2}p_j} \prod_{\ell \neq j} f(x_j - x_\ell)^{1/2}$$

$$L(\beta,\mu,\tau;x,p)_{jk} \equiv (e_j C_{jk} e_k)(\beta,\mu,\tau;x,p)$$

Then principal minor $\{i_1,\ldots,i_k\} \equiv I$ given by $L(I) = \left(\prod_{i\in I} e_i^2\right) C(I)$.

Using (C) and (|C|) this can be rewritten

$$L(I) = \Big(\prod_{i\in I} e^{\beta p_i}\Big)\Big(\prod_{i\in I}\prod_{\ell\neq i} f(x_i - x_\ell)\Big) \prod_{\substack{i<\ell \\ i,\ell\in I}} 1/f^2(x_i - x_\ell)$$

$$= e^{\beta\sum_{i\in I} p_i} \prod_{\substack{i\in I \\ \ell\notin I}} f(x_i - x_\ell)\,.$$

$\therefore$ The commuting functions $S_1,\ldots,S_N$ from (iii) are equal to the symmetric functions of L.

— Next, note: $L_{rel} = \mathbb{1}_N + \beta L_{nr} + O(\beta^2)$, $\beta \to 0$

cf. A2 (iv). Hence, one gets

$$S_{k,nr} = \lim_{\beta\to 0} \beta^{-k} \sum_{\ell=0}^{k} (-)^{k+\ell} \binom{N-\ell}{N-k} S_{\ell,rel}$$

(Write out $\det\left(\frac{L_{rel} - \mathbb{1}_N}{\beta} - \lambda \mathbb{1}_N\right)$ to check this.)

$\therefore$ Relativistic integrability $\xRightarrow{c\to\infty}$ nonrelativistic integrability

N.B. Since the spectrum of $L(e^{tH}(x,p))$ is constant, the 'expected' asymptotics $(*)$, $(**)$ in A2 (v) entails once more soliton scattering; again, this can be substantiated, cf. [2] and below.

A 4. Action-angle transforms

i) Preview

Problem: find action-angle transform $\Phi : (x,p) \mapsto (\hat{x},\hat{p})$

in explicit form. Some results:

— Problem solved for systems of type I–III and VI

— For I–III key tool is commutation relation of $L(x,p)$

and $A(x) \equiv \mathrm{diag}(a(x_1),\dots,a(x_N))$, $a(u) \equiv \begin{cases} u & \mathrm{I} \\ \exp(\mu u) & \mathrm{II} \\ \exp(i\mu u) & \mathrm{III} \end{cases}$

— Since symmetric functions $S^d_k(x)$ of $A(x)$ commute,

$\hat{S}^d_1(\hat{x},\hat{p}),\dots,\hat{S}^d_N(\hat{x},\hat{p})$, $\quad \hat{S}^d_k \equiv S^d_k \cdot \Phi^{-1}$

commute, too, yielding dual integrable systems $\hat{\mathrm{I}}$–$\hat{\mathrm{III}}$.

— For I–II one can take $\Omega = \{(x,p) \in \mathbb{R}^{2N} \mid x_N < \dots < x_1\}$,

$\hat{\Omega} = \{(\hat{x},\hat{p}) \in \mathbb{R}^{2N} \mid \hat{p}_N < \dots < \hat{p}_1\} \simeq P\Omega$, $\quad P(x,y) \equiv (y,x)$

With this identification I_{nr} and II_{rel} are self-dual, whereas $\hat{\mathrm{II}}_{nr} \simeq \mathrm{I}_{rel}$, $\hat{\mathrm{I}}_{rel} \simeq \mathrm{II}_{nr}$.

— Using Φ, can get detailed information on various Hamiltonian flows.

ii) The construction of $\Phi(\mathrm{II}_{rel})$

— Fix arbitrary $(x,p) \in \Omega$ and set $A(x) \equiv \mathrm{diag}(e^{\mu x_1},\dots,e^{\mu x_N})$.

Then key commutation relation reads (check)

$$\tfrac{1}{2}\coth(i\tau)[A,L] = e \otimes e - \tfrac{1}{2}(AL + LA) \qquad \text{(CR)}$$

where e.g. $L = L(\beta,\mu,\tau;x,p)$ (cf. A3 (iv)).

— Since $L = L^*$, there exists a unitary U such that

$\hat{L} \equiv U L U^* = \mathrm{diag}(\lambda_1,\dots,\lambda_N)$, $\quad \lambda_i \in \mathbb{R}$

Transforming (CR) with U and setting

$\hat{A} \equiv U A U^*$, $\quad \hat{e} \equiv Ue$.

one obtains (check)

$$\tfrac{1}{2}\hat{A}_{jk}\left[\coth(i\tau)(\lambda_k - \lambda_j) + \lambda_k + \lambda_j\right] = \hat{e}_j \bar{\hat{e}}_k \qquad (jk)$$

— Since $A > 0$, one has $\hat{A} > 0$, so $\hat{A}_{jj} > 0$. Also, using A3 (iv): $\prod_j \lambda_j = |L| = \exp\beta(p_1 + \dots + p_N) \neq 0$

Hence (jj) entails $\hat{e}_j \neq 0$, $\lambda_j > 0$. Then may define $\hat{p} \in \mathbb{R}^N$ by setting $\lambda_j \equiv e^{\beta \hat{p}_j}$, $j = 1,\dots,N$.

— Now rewrite (jk) as (check)

$$\hat{A}_{jk} = \hat{e}_j \bar{\hat{e}}_k \exp\left(-\tfrac{\beta}{2}(\hat{p}_j+\hat{p}_k)\right) \frac{\operatorname{sh}(i\tau)}{\operatorname{sh}\left(\frac{\beta}{2}(\hat{p}_k-\hat{p}_j)+i\tau\right)}$$

and use Cauchy's identity (1C1) in A3 (iv) to get

$$|\hat{A}| = \prod_j |\hat{e}_j|^2 e^{-\beta\hat{p}_j} \cdot \prod_{j<k} \left[\frac{\operatorname{sh}^2 \frac{\beta}{2}(\hat{p}_j-\hat{p}_k)}{\operatorname{sh}^2 \frac{\beta}{2}(\hat{p}_j-\hat{p}_k)+\sin^2\tau} \right]$$

Since $|\hat{A}| = |A| = \exp\mu(x_1+\dots+x_N) \neq 0$, must have $\hat{p}_j \neq \hat{p}_k$.

$\therefore$ $\sigma(L)$ non-degenerate and gauge ambiguity in U given by permutation and diagonal phase matrix.

— To render U unique, need <u>gauge fixing</u>: Require

$$\hat{L} = \operatorname{diag}(e^{\beta\hat{p}_1},\dots,e^{\beta\hat{p}_N}), \qquad \hat{p}_N < \dots < \hat{p}_1$$

$$\hat{e}_j > 0, \qquad j=1,\dots,N.$$

(Can be done, since $\hat{e} = Ue$ and $\hat{e}_j \neq 0$.) Then define $\hat{x} \in \mathbb{R}^N$ by parametrizing $\hat{e}_j$ as

$$\hat{e}_j = \exp\left(\tfrac{\beta}{2}\hat{p}_j + \tfrac{\mu}{2}\hat{x}_j\right) \prod_{l\neq j} \left[1 + \frac{\sin^2\tau}{\operatorname{sh}^2 \frac{\beta}{2}(\hat{p}_j-\hat{p}_l)} \right]^{1/4}$$

<u>Upshot</u>: The map Φ defined by

$$\Phi : \Omega \to \hat{\Omega}, \quad (x,p) \mapsto (\hat{x},\hat{p})$$

satisfies the self-duality relations (cf. A3 (iv))

$$\hat{A}(\hat{x},\hat{p}) = L(\mu,\beta,\tau;\hat{p},\hat{x})^{tr.} \qquad (\hat{A})$$

$$\hat{e}(\hat{x},\hat{p}) = e(\mu,\beta,\tau;\hat{p},\hat{x}) \qquad (\hat{e})$$

Now construct a map $\mathcal{E} : \hat{\Omega} \to \Omega$ such that

$$\mathcal{E} \circ \Phi = id_{\Omega}, \qquad \Phi \circ \mathcal{E} = id_{\hat{\Omega}}$$

by 'running Φ backwards'. This entails $\mathcal{E} = \Phi^{-1}$ and

$$\mathcal{E}(\beta,\mu,\tau;\hat{x},\hat{p}) = P \circ \Phi(\mu,\beta,\tau;\hat{p},\hat{x}), \qquad P(x,y) \equiv (y,x) \qquad (\mathcal{E})$$

<u>Fact.</u> Φ is a canonical diffeomorphism.

(Can be proved by exploiting scattering theory, cf. below.)

<u>N.B.</u> 1) Taking $\beta=\mu$ and $P\hat{\Omega} \simeq \Omega$, $(\mathcal{E})$ says that Φ is an <u>involution</u> (i.e. $\Phi^2 = id_{\Omega}$).

2) The above reasoning can be specialized to the II_{nr}, I_{rel} and I_{nr} cases, yielding $\hat{II}_{nr} \simeq I_{rel}$, $\hat{I}_{rel} \simeq II_{nr}$, $\hat{I}_{nr} \simeq I_{nr}$. (Try to check, taking $A = \operatorname{diag}(x_1,\dots,x_N)$ for I_{rel} and I_{nr}.)

iii) <u>Explicit solutions to II_{rel} Hamilton equations</u>

Since $L>0$, $\ln L$ can be defined via the functional calculus, i.e., since

$$U L(x,p) U^* = \mathrm{diag}\,(e^{\beta\hat p_1},\dots,e^{\beta \hat p_N}),$$

may set

$$(\ln L)(x,p) \equiv U^* \,\mathrm{diag}\,(\beta\hat p_1,\dots,\beta\hat p_N)\,U$$

Similarly, for $h\in C^\infty(\mathbb{R})$ may define

$$h(\beta^{-1}\ln L(x,p)) = U^*\,\mathrm{diag}\,(h(\hat p_1),\dots h(\hat p_N))\,U$$

Now consider Hamiltonians of the form

$$H_h \equiv \mathrm{Tr}\, h(\beta^{-1}\ln L), \qquad h\in C^\infty(\mathbb{R})$$

<u>Ex</u>. $h(y)=e^{\pm\beta y} \Rightarrow H_h = S_{\pm 1}$

$$h(y)=\beta^{-2}\,\mathrm{ch}\,\beta y \Rightarrow H_h = H,\ h(y)=\beta^{-1}\mathrm{sh}\,\beta y \Rightarrow H_h = P$$

<u>Claim</u>. Let $t\in\mathbb{R}$ and $Q\equiv(x,p)\in\Omega$. Then the matrix

$$A_h(t,Q)\equiv A(Q)\exp(t\mu h'(\beta^{-1}\ln L(Q)))$$

has simple and positive spectrum. The integral curve $e^{tH_h}(Q)$

is complete and its configuration part is given by

$$x_j(t)=\mu^{-1}\ln\gamma_j(t), \qquad 0<\gamma_N(t)<\dots<\gamma_1(t) \qquad (\mathrm{Conf.})$$

where $\gamma_1,\dots,\gamma_N$ are the ordered eigenvalues of A_h.

<u>Proof</u>. Setting $\hat Q\equiv \Phi(Q)=(\hat x,\hat p)$, $\hat H_h\equiv H_h\circ\Phi^{-1}$, one has

$$\hat H_h(\hat Q) = \mathrm{Tr}\, U^*\,\mathrm{diag}\,(h(\hat p_1),\dots,h(\hat p_N))\,U = \sum_{j=1}^N h(\hat p_j)$$

Clearly,

$$e^{t\hat H_h}(\hat Q) = (\hat x_1+th'(\hat p_1),\dots,\hat x_N+th'_N(\hat p_N),\hat p)$$

so that $e^{t\hat H_h}$ is global. Since one has

$$e^{tH_h} = \mathcal{E}\circ e^{t\hat H_h}\circ\Phi$$

the flow e^{tH_h} is global, too. Denoting similarity by $\sim$, one now obtains (check using (ii)):

$$A_h(t,Q)\sim \hat A(\hat Q)\underbrace{e^{t\mu h'(\beta^{-1}\ln\hat L(\hat Q))}}_{=\hat A(\hat x,\hat p)\,\mathrm{diag}\,(e^{t\mu h'(\hat p_1)},\dots)}$$

$$\sim \hat A(\hat x_1+th'(\hat p_1),\dots,\hat x_N+th'(\hat p_N),\hat p) = \hat A(e^{t\hat H_h}(\hat Q))$$

$$\sim A(e^{tH_h}(Q)).$$

entailing the claim. $\square$

iv) Soliton scattering revisited

In view of (iii) the $t\to\pm\infty$ asymptotics of $e^{tH_h}(x,p)_{conf.}$ (with conf. = projection on configuration space) amounts to the spectral asymptotics for $t\to\pm\infty$ of

$$\hat{A}(\hat{x},\hat{p})\,\mathrm{diag}(e^{td_1},\dots,e^{td_N}),\quad d_j\equiv\mu h'(\hat{p}_j)$$

Assume from now on $h''(y)>0$, so that $h'(y)$ is strictly increasing. (Holds true for S, and H, e.g.) Since $\hat{A}$ is positive (cf. (ii)), one needs the spectral asymptotics of matrices of the form

$$E(t)=Me^{tD},\quad M>0,\ D\equiv\mathrm{diag}(d_1,\dots,d_N),\ d_N<\dots<d_1 .$$

For $t\to\infty$ ($t\to-\infty$) this can be expressed in terms of upper (lower) corner principal minors of M. Specifically, introduce

$$m_1^+\equiv M(1),\ m_2^+\equiv M(1,2)/M(1),\ m_3^+\equiv M(1,2,3)/M(1,2),\dots$$

$$m_1^-\equiv M(N),\ m_2^-\equiv M(N-1,N)/M(N),\dots$$

(so $m_1^+=M_{11}$, $m_2^+=(M_{11}M_{22}-M_{12}M_{21})/M_{11}$, etc.)

<u>Fact</u>. The (ordered) eigenvalues $\gamma_1(t),\dots,\gamma_N(t)$ of $E(t)$ obey

$$e^{-td_j}\gamma_{\substack{j\\N-j+1}}(t)-m^\pm_{\substack{j\\N-j+1}}=O(e^{\mp tR}),\qquad t\to\pm\infty$$

where $R\equiv\min\{d_1-d_2,\dots,d_{N-1}-d_N\}$

<u>Consequences</u>. To apply this to case at hand, need $m_k^\pm(\hat{A})$. To this end use $(\hat{A})$ in (ii) and Cauchy's identity (cf. A3 (iv)) to obtain (check)

$$m^\pm_{\substack{j\\N-j+1}}(\hat{A}(\hat{x},\hat{p}))=\exp\left[\mu\hat{x}_j\mp\tfrac{\mu}{2}\Delta_j(\hat{p})\right]$$

where $\Delta_j(p)\equiv\Big(\sum_{k<j}-\sum_{k>j}\Big)\delta(p_j-p_k)$

$$\delta(p)\equiv\frac{1}{\mu}\ln\left(1+\frac{\sin^2\tau}{\sh^2\frac{\beta}{2}p}\right)$$

From (conf.) in (iii) one now gets (check)

$$x_{\substack{j\\N-j+1}}(t)=\hat{x}_j\mp\tfrac{1}{2}\Delta_j(\hat{p})+th'(\hat{p}_j)+O(e^{\mp tR}),\quad t\to\pm\infty$$

and using $\sigma(L_t)=\sigma(L_0)$: $p_{\substack{j\\N-j+1}}(t)=\hat{p}_j+O(e^{\mp tR}),\quad t\to\pm\infty$

<u>N.B.</u> For technical details see S.R., Action-angle maps and scattering theory ...., Comm. Math. Phys. <u>115</u> (88) 127-165.

B. Intermezzo: scattering and inverse scattering

B1. Scattering theory: generalities

N.B. An excellent and comprehensive reference is:
M. Reed, B. Simon, Methods of modern mathematical physics. III. Scattering theory. Academic Press, New York '79

i) Setup

Consider a dynamical system $\mathcal{S} = \langle \Sigma, T_t \rangle$, i.e., Σ is a set of system states, and $T_t : \Sigma \to \Sigma, \varphi \mapsto \varphi_t$ a 1-parameter group of transformations. The notion of 'scattering' is relevant for $\mathcal{S}$ whenever the long-time asymptotics of T_t (on subsets of Σ) 'looks like' a 'free' dynamics. ('Free' = rectilinear motion, linear dynamics, no coupling, ..., depending on concrete context.) Thus, this notion involves a comparison of the 'interacting' system $\mathcal{S}$ to 'free' systems $\mathcal{S}^\pm = \langle \Sigma^\pm, T_t^\pm \rangle$.

More precisely, the scattering states $\Sigma_s \subset \Sigma$ are those $\varphi \in \Sigma$ for which φ_t resembles states $\varphi_t^\pm \in \Sigma^\pm$ for $t \to \pm\infty$, in the sense that (in a suitable topology)

$$\varphi_t \simeq J^\pm \varphi_t^\pm, \quad t \to \pm\infty$$

Here, one has

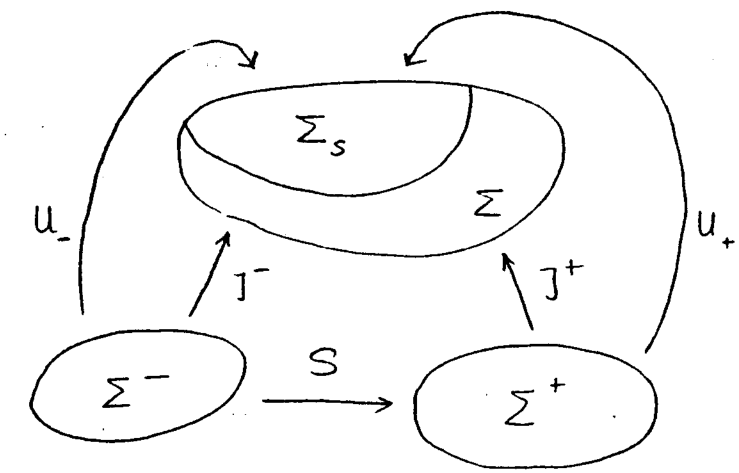
$$\varphi_t \equiv T_t \varphi, \quad \varphi_t^\pm \equiv T_t^\pm \varphi^\pm$$

and $J^\pm : \Sigma^\pm \to \Sigma$ are comparison maps. The wave maps $U_\pm : \Sigma^\pm \to \Sigma_s$ are then given by

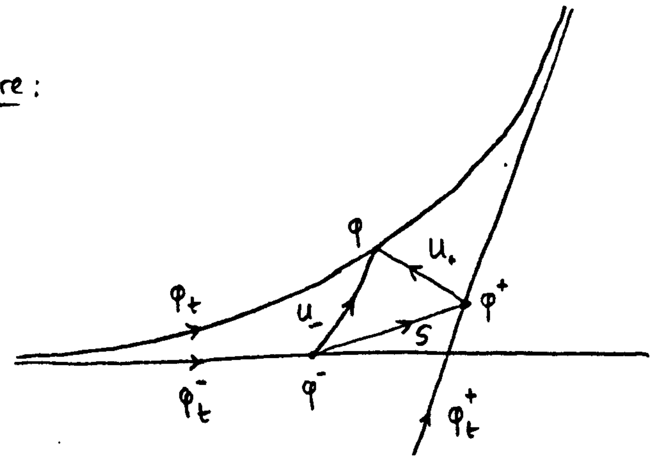
$$U_\pm \equiv \lim_{t \to \pm\infty} T_{-t} J^\pm T_t^\pm \qquad (U_\pm)$$

and the S-map: $S : \Sigma^- \to \Sigma^+$ by $S \equiv U_+^{-1} U_-$.

Set picture:



Point picture:



N.B. A crucial consequence of $(U_\pm)$ are the so-called intertwining relations:

$$T_t U_\delta = U_\delta T_t^\delta \qquad \delta = +,- \qquad \forall t \in \mathbb{R} \quad \text{(ITR)}$$

(ITR) picture:

$$\begin{tikzcd} t & t \arrow[l, "U_\delta"'] \\ 0 \arrow[u, "T_t"] \arrow[r, "U_\delta^{-1}"'] & 0 \arrow[u, "T_t^\delta"'] \\ \Sigma_s & \Sigma^\delta \end{tikzcd}$$

Proof of (ITR): One has, e.g.,

$$T_t U_+ = \lim_{s\to\infty} T_{t-s} J^+ T_s^+ = \lim_{s'\to\infty} T_{-s'} J^+ T_{s'}^+ T_t^+$$
$$= U_+ T_t^+ .$$

$\square$

ii) Scattering in classical and quantum mechanics

1) Classical mechanics. Here, $\Sigma = \Omega$, $\Sigma_s = \Omega_s$, $\Sigma^\delta = \Omega^\delta$ are 2N-dimensional phase spaces with symplectic forms ω, ω^δ; $T_t = e^{tH}$, $T_t^\delta = e^{tH^\delta}$ (with $H, H^\delta \in C^\infty(\Omega), C^\infty(\Omega^\delta)$, resp.) are canonical maps; if J^δ canonical and convergence in $(U_\pm)$ 'sufficiently strong', then U_δ canonical, so that S canonical; if U_δ canonical, then (ITR) equivalent to

$$H(x,p) = H^\delta \circ U_\delta^{-1}(x,p) \qquad \delta = +,- \qquad \forall (x,p) \in \Omega_s$$

2) Quantum mechanics. Here, $\Sigma = \mathcal{H}$, $\Sigma_s = \mathcal{H}_s$, $\Sigma^\delta = \mathcal{H}^\delta$ are Hilbert spaces; $T_t = e^{-itH}$, $T_t^\delta = e^{-itH^\delta}$ (with $H = H^*$, $H^\delta = H^{\delta *}$) are unitaries; if J^δ isometries and $(U_\pm)$ converges in strong topology, then U_δ and S are isometries; if U_δ isometric, then (ITR) $\Leftrightarrow$

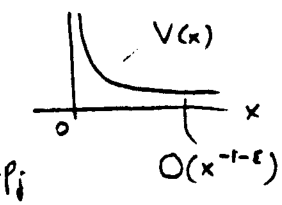
$$H v = U_\delta H^\delta U_\delta^{-1} v \qquad \delta = +,- \qquad \forall v \in D(H) \cap \mathcal{H}_s$$

B2. Scattering and inverse scattering vs. classical solitons

i) Classical scattering for repulsive systems on the line

Suppose $\mathcal{S}$ is a nonrelativistic N-particle system with repulsive pair interactions:

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + g^2 \sum_{1\le j<k\le N} V(x_j - x_k)$$



$$\Omega \equiv \{(x,p)\in\mathbb{R}^{2N} \mid x_N < \dots < x_1\}, \quad \omega \equiv \sum_{j=1}^{N} dx_j \wedge dp_j$$

<u>Fact.</u> Every state is scattering state:

$$x_j(t) \sim x_j^{\pm} + t p_j^{\pm}, \quad p_j(t) \sim p_j^{\pm}. \qquad t\to\pm\infty$$

$$p_N^- > \dots > p_1^-, \quad p_N^+ < \dots < p_1^+$$

Introduce $\Omega^{\pm} \equiv \{(x^{\pm},p^{\pm})\in\mathbb{R}^{2N} \mid p_N^{\pm} \lessgtr \dots \lessgtr p_1^{\pm}\}$, $\omega^{\delta} \equiv \sum_j dx_j^{\delta}\wedge dp_j^{\delta}$

$$T_t^{\delta} \equiv e^{tH^{\delta}}, \qquad H^{\delta}(x^{\delta},p^{\delta}) \equiv \frac{1}{2}\sum_{j=1}^{N}(p_j^{\delta})^2$$

$$J^{\delta}(x^{\delta},p^{\delta}) \equiv \begin{cases} (x^{\delta},p^{\delta}) & \text{if } (x^{\delta},p^{\delta})\in\Omega \\ \text{arbitrary point in } \Omega & \text{if } (x^{\delta},p^{\delta})\notin\Omega \end{cases}$$

(Note $e^{tH^{\delta}}(x^{\delta},p^{\delta}) = (x^{\delta}+tp^{\delta}, p^{\delta}) \in \Omega$ for $t\to\delta\infty$.)

<u>Expectations.</u> The limits in $(U_{\pm})$ exist uniformly on compacts; $U_{\delta}: \Omega^{\delta}\to\Omega$ and $S:\Omega^-\to\Omega^+$ are symplectic diffeos.

ii) Classical scattering vs. integrability

Taking canonicity of the wave maps for granted, we claim that the system $\mathcal{S}$ from (i) is integrable.

<u>Proof.</u> Define $I_k^{\delta}(x^{\delta},p^{\delta}) \equiv \frac{1}{k}\sum_{j=1}^{N}(p_j^{\delta})^k$ and set $\check{I}_k^{\delta}(x,p) \equiv I_k^{\delta}\circ U_{\delta}^{-1}(x,p)$. Then one has $I_2^{\pm} = H^{\pm}$; so that $\check{I}_2^{\pm} = H$ due to (ITR). Since $I_1^{\delta},\dots,I_N^{\delta}$ commute and U_{δ}^{-1} canonical, $\check{I}_1^{\delta},\dots,\check{I}_N^{\delta}$ commute, too.

$\square$

— As action-angle map one can take e.g. U_+^{-1}. Then one has $\hat{\Omega} = \Omega^+$, $(\hat{x},\hat{p}) = (x^+,p^+)$, $\hat{I}_k = I_k^+$.

— This class of integrable systems is far too big; for general V one cannot determine $U_{\pm}$ and S explicitly. Therefore, need more restrictive definition:

<u>Def.</u> $\mathcal{S}$ is <u>soliton system</u> iff

$$p_{N-j+1}^+ = p_j^-, \qquad j=1,\dots,N \qquad (*)$$

N.B. 1) For $N=2$ $(*)$ is satisfied for any repulsive V, and one gets (cf. also A2 (ii))

$$x_1^+ = x_2^- + \delta(p_2^- - p_1^-), \quad x_2^+ = x_1^- - \delta(p_1^- - p_2^-)$$

with $\delta(p)$ even. (Use conservation of p_1+p_2 to check this.)

2) If $\mathcal{S}$ is soliton system, then the heuristic argument in A2 (v) yields $S(x^-,p^-) = (x_N^- + \Delta_N(p^-), \ldots, x_1^- + \Delta_1(p^-), p_N^-, \ldots, p_1^-)$, with $\Delta_j(p^-) = (\sum_{k<j} - \sum_{k>j})\,\delta(p_j^- - p_k^-)$.

3) The technical difficulty in proving canonicity of $U_{\mathcal{S}}$ is to justify the interchange of the $t \to \pm\infty$ limit in $(U_\pm)$ and the partials implied in $\{\cdot,\cdot\}$. For systems of type I–II this can be handled via analyticity arguments. From A4 one obtains

$$x_j^+ = \hat{x}_j - \tfrac{1}{2}\Delta_j(\hat{p}), \; p_j^+ = \hat{p}_j \; ; \; x_{N-j+1}^- = \hat{x}_j + \tfrac{1}{2}\Delta_j(\hat{p}), \; p_{N-j+1}^- = \hat{p}_j$$

Hence, $\Omega^+ = \hat{\Omega}$, $\Omega^- = R\,\Omega^+$ (with $R(x,y) \equiv (x_N, \ldots, x_1, y_N, \ldots, y_1)$)

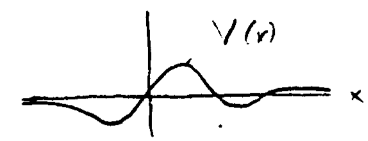
Also, setting $T : \hat{\Omega} \to \hat{\Omega}, (\hat{x}, \hat{p}) \mapsto (\hat{x}_1 - \tfrac{1}{2}\Delta_1(\hat{p}), \ldots, \hat{x}_N - \tfrac{1}{2}\Delta_N(\hat{p}), \hat{p})$ one gets $U_- = \mathcal{E}TR$, $U_+ = \mathcal{E}T^{-1}$, where $\mathcal{E} \equiv \Phi^{-1}$; hence,

$S = U_+^{-1} U_- = T^2 R$. (For I_{nr} and I_{rel} one has $\delta(p) = 0$, so that $S = R$: 'billiard ball scattering'.)

iii) 1D quantum scattering: direct transform

Consider the quantum dynamical system $\langle \mathcal{H}, H \rangle$, where $\mathcal{H} = L^2(\mathbb{R}, dx)$ and H is the Schrödinger operator

$$H = -\frac{d^2}{dx^2} + V(x), \quad V \in S(\mathbb{R}):$$



Since $V(x)$ rapidly decreases for $x \to \pm\infty$, there exist solutions $f_j(\lambda, x)$, $j = 1,2$ (the so-called Jost solutions) to the eigenvalue problem

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\Psi = \lambda^2 \Psi, \qquad \lambda \in \mathbb{C} \qquad \text{(EV)}$$

obeying

$$f_1 \sim \begin{cases} e^{i\lambda x} & x \to \infty \\ a(\lambda)e^{i\lambda x} - b(-\lambda)e^{-i\lambda x} & x \to -\infty \end{cases}$$

$$f_2 \sim \begin{cases} a(\lambda)e^{-i\lambda x} + b(\lambda)e^{i\lambda x} & x \to \infty \\ e^{-i\lambda x} & x \to -\infty \end{cases}$$

The scattering states of $\langle \mathcal{H}, H \rangle$ can then be obtained via wave packets made up from $\{f_j(p,x)\}_{p \in \mathbb{R}^*}$. cf. below. It is customary to set

$$t(p) \equiv \frac{1}{a(p)}, \quad r(p) \equiv \frac{b(p)}{a(p)} \qquad \text{'transmission/reflection coefficients'}$$

Any $V \in S(\mathbb{R})$ yields either no or finitely many eigenvalues $\lambda_1^2 < \dots < \lambda_N^2 < 0$. Setting $\lambda \equiv i\kappa$, these correspond to zeros $0 < \kappa_N < \dots < \kappa_1$ of $a(i\kappa)$. (Note $f_j(i\kappa, x) \in L^2(\mathbb{R}) \Leftrightarrow a(i\kappa) = 0$.) Thus, the bound state subspace reads

$$\mathcal{H}_{bs} = \mathrm{Span}\{f_1(i\kappa_1, x), \dots, f_1(i\kappa_N, x)\}$$

Defining <u>normalization coefficients</u> by

$$\nu_j \equiv 1 \Big/ \int_{\mathbb{R}} f_1(i\kappa_j, x)^2 \, dx, \qquad j = 1, \dots, N$$

it can be shown that the <u>direct transform</u>

$$\mathcal{D}: V(x) \mapsto \{ r(p), p \in \mathbb{R}^*, \ \nu_j, \kappa_j, j = 1, \dots, N \}$$

($\{\dots\} \equiv$ <u>scattering data</u>) is injective on $S(\mathbb{R})$.

<u>Ex.</u> $V(x) = -\dfrac{2a^2}{\mathrm{ch}^2 ax}$, $a > 0$. Then one gets (check)

$$f_{\substack{1\\2}}(\lambda, x) = \pm e^{\pm i\lambda x} (\mathrm{th}\, ax \mp i\lambda/a) / (1 - i\lambda/a)$$

$$a(\lambda) = \frac{1 + i\lambda/a}{-1 + i\lambda/a}, \quad b(\lambda) = 0, \quad t(p) = \frac{p + ia}{p - ia}, \quad r(p) = 0$$

$$f_1(ia, x) = 1/2\,\mathrm{ch}\, ax, \quad \nu = 1 \Big/ \int \frac{dx}{4\,\mathrm{ch}^2 ax} = 2a$$

<u>N.B.</u> The relation to the time-dependent scattering theory of B1 is as follows. One has

$$\mathcal{H}_s = \mathcal{H}_{bs}^{\perp}, \quad \mathcal{H}^{\pm} = L^2(\mathbb{R}, dp) \equiv \hat{\mathcal{H}}$$

$$H^{\pm} = \text{multiplication by } p^2$$

$$J^{\pm} = \text{Fourier transformation } \mathcal{F}$$

The wave operators $U_{\pm}$ are then isometries from $\hat{\mathcal{H}}$ onto $\mathcal{H}_s$, explicitly given by

$$(U_{\pm}\varphi)(x) = \int \Psi^{\pm}(x, p)\, \varphi(p)\, dp$$

Their kernels Ψ^-/Ψ^+ are the so-called <u>incoming / outgoing</u> solutions of (EV):

$$\Psi^-(x, p) \equiv \begin{cases} \frac{1}{\sqrt{2\pi}} f_1(x, p)/a(p), & p > 0 \\ \frac{1}{\sqrt{2\pi}} f_2(x, -p)/a(-p), & p < 0 \end{cases} \qquad \Psi^+(x, p) \equiv \overline{\Psi^-(x, -p)}$$

Then one can show: $(S\varphi)(p) = T(p)\varphi(p) + R(-p)\varphi(-p)$

where

$$T(p) = \frac{1}{a(|p|)}, \qquad R(p) = \begin{cases} \dfrac{-b(-p)}{a(p)}, & p > 0 \\ \dfrac{b(-p)}{a(-p)}, & p < 0 \end{cases}$$

iv) 1D quantum scattering: inverse transform

The inverse $\mathcal{J}$ of the direct transform $\mathcal{D}$ is called the Inverse Scattering Transform (IST):

$$\mathcal{J} : \{ r(p),\ p \in \mathbb{R}^*,\ \nu_j, \kappa_j,\ j=1,\dots,N \} \longmapsto V(x)$$

For given scattering data the potential can be determined by solving the Gelfand-Levitan-Marchenko integral equation

$$K(x,y) + B(x+y) + \int_x^\infty K(x,z) B(z+y)\, dz = 0, \quad y > x \qquad (GLM)$$

$$B(x) \equiv \sum_{j=1}^N \nu_j e^{-\kappa_j x} + \frac{1}{2\pi} \int e^{ipx} r(p)\, dp$$

Then one has $V(x) = -2 \frac{d}{dx} K(x,x)$, where $K(x,y)$ is the (unique) solution to (GLM), cf. [3,4].

Ex. 1) The '1-soliton potentials', i.e., $r(p) = 0$, $\kappa_1 \equiv a$, $\nu_1 \equiv \nu$.

Then the solution to (GLM) reads (check)

$$K(x,y) = -\nu e^{-ay} / (e^{ax} + \tfrac{\nu}{2a} e^{-ax})$$

so that $V(x) = -2a^2 / \mathrm{ch}^2 (ax - \frac{1}{2} \ln \frac{\nu}{2a})$

2) The 'N-soliton potentials', i.e., $B(x) = \sum_{j=1}^N \nu_j e^{-\kappa_j x}$.

Setting $K(x,y) \equiv \sum_{j=1}^N c_j(x) e^{-\kappa_j y}$, (GLM) amounts to the system (check)

$$c_j(x) + \nu_j \sum_{\ell=1}^N \frac{e^{-(\kappa_j+\kappa_\ell)x}}{\kappa_j + \kappa_\ell} c_\ell(x) = -\nu_j e^{-\kappa_j x}, \qquad j = 1,\dots,N$$

Solving this by Cramer's rule, and setting

$$\tau(x) \equiv \det \begin{pmatrix} 1 + \nu_1 \frac{e^{-2\kappa_1 x}}{2\kappa_1} & \cdots & \nu_1 \frac{e^{-(\kappa_1+\kappa_N)x}}{\kappa_1+\kappa_N} \\ \vdots & & \vdots \\ \nu_N \frac{e^{-(\kappa_N+\kappa_1)x}}{\kappa_N+\kappa_1} & \cdots & 1 + \nu_N \frac{e^{-2\kappa_N x}}{2\kappa_N} \end{pmatrix}$$

one infers (check)

$$K(x,x) = \frac{d}{dx} \ln \tau(x) \Rightarrow V(x) = -2 \frac{d^2}{dx^2} \ln \tau(x)$$

N.B. For later use define the coordinate change

$$\mathcal{C} : D \equiv \{ (\nu,\kappa) \in (0,\infty)^{2N} \mid \kappa_N < \cdots < \kappa_1 \} \to \hat{\Omega},\ (\nu,\kappa) \mapsto (\hat{x}, \hat{p})$$

$$\kappa_j \equiv \tfrac{1}{2} e^{\hat{p}_j}, \quad \nu_j \equiv e^{\hat{p}_j + \hat{x}_j} \prod_{\ell \neq j} \left| \coth \tfrac{1}{2} (\hat{p}_j - \hat{p}_\ell) \right|$$

Then (check) $\tau(x) = d(\hat{x}_1 - x e^{\hat{p}_1}, \dots, \hat{x}_N - x e^{\hat{p}_N}, \hat{p})$, where

$$d(\hat{x}, \hat{p}) \equiv \det (\mathbb{1}_N + \hat{A}(\hat{x}, \hat{p})) \qquad (d)$$

and $\hat{A}$ is given by ($\hat{A}$) in A4 (ii) with $\beta = \mu = 1$, $\tau = \pi/2$.

C. Infinite-dimensional soliton systems

C1. The solution to the KdV Cauchy problem

i) The KdV Lax pair

Consider the Korteweg-de Vries equation

$$u_t + 6\alpha u u_x + u_{xxx} = 0, \qquad \alpha \in \mathbb{R}^* \qquad (\mathrm{KdV})_\alpha$$

and the pair of operators on $\mathcal{H} \equiv L^2(\mathbb{R}, dx)$ (with $\partial \equiv \frac{d}{dx}$)

$$L(u) \equiv -\partial^2 - \alpha u, \quad B(u) \equiv -4\partial^3 - 3\alpha(u_x + 2u\partial)$$

Claim. $u(x,t)$ solves $(\mathrm{KdV})_\alpha \Leftrightarrow \frac{d}{dt} L(u) = [B,L](u), \ \forall t \in \mathbb{R}$

Proof. One has $[B,L] = 4\alpha[\partial^3, u] + 3\alpha[u_x, \partial^2] + 6\alpha[u\partial, \partial^2 + \alpha u]$

$$= 4\alpha(u_{xxx} + \underline{3u_{xx}\partial} + 3u_x\partial^2) - 3\alpha(u_{xxx} + \underline{2u_{xx}\partial}) - 6\alpha(\underline{u_{xx}\partial} + 2u_x\partial^2) + 6\alpha^2 u u_x$$

$$= \alpha(u_{xxx} + 6\alpha u u_x), \text{ so the claim follows.}$$

$\square$

N.B. In the present case the Lax pair result in A2 (iii) involves technicalities, since L and B are unbounded. However, modulo these problems one infers constancy of $\sigma(L(u(x,t)))$ when u solves $(\mathrm{KdV})_\alpha$ (isospectral flow).

ii) The solution strategy

N.B. 1) One has: u solves $(\mathrm{KdV})_\alpha \Leftrightarrow \tilde{u} \equiv \alpha u$ solves $(\mathrm{KdV})_1$.

2) From now on: $\alpha = 1$.

Assume $u(x,t)$ is solution to $(\mathrm{KdV})_1$ satisfying $u(\cdot,t) \in S(\mathbb{R})$, $\forall t \in \mathbb{R}$; and assume $\Psi_\lambda(x,t)$ solves the eigenvalue problem

$$(-\partial^2 - u(x,0))\Psi = \lambda^2 \Psi, \qquad \lambda \in \mathbb{C} \qquad \cdots (\mathrm{EV})_0$$

cf. B2 (iii). Ignoring domain problems, set

$$U(t) \equiv T\exp\left(\int_0^t B_s\, ds\right), \quad B_s \equiv B(u(\cdot, s))$$

and recall (cf. A2 (iii))

$$\dot{U}(t) = B_t U(t), \qquad L_t U(t) = U(t) L_0$$

From this one deduces that

$$\Psi_\lambda(x,t) \equiv U(t)\Psi_\lambda(x,0)$$

solves the PDE system

$$(-\partial_x^2 - u(x,t))\Psi = \lambda^2 \Psi \qquad (\mathrm{EV})_t$$

$$\partial_t \Psi = (-4\partial_x^3 - 3u_x(x,t) - 6u(x,t)\partial_x)\Psi \qquad (\dot{\Psi})_t$$

Since $u(x,t) \to 0$ rapidly for $|x| \to \infty$, $(EV)_t$ implies

$$\psi_\lambda \sim \begin{cases} \alpha_+ e^{-i\lambda x} + \beta_+ e^{i\lambda x} & x \to \infty \\ \alpha_- e^{-i\lambda x} + \beta_- e^{i\lambda x} & x \to -\infty \end{cases}$$

and then $(\dot\psi)_t$ entails

$$\alpha_\delta(t) = e^{-4i\lambda^3 t}\alpha_\delta(0). \qquad \beta_\delta(t) = e^{4i\lambda^3 t}\beta_\delta(0). \qquad \delta = +,-$$

Choosing $\alpha_-(0) = 1$, $\beta_-(0) = 0$ one infers

$$e^{4i\lambda^3 t}\psi_\lambda(x,t) = f_{2,t}(\lambda,x)$$

where $f_{2,t}$ = Jost solution to $(EV)_t$, cf. B2(iii). Thus:

$$t(p,t) = t(p,0), \qquad r(p,t) = e^{8ip^3 t} r(p,0)$$

Next, choose $\alpha_+(0) = 0$, $\beta_+(0) = 1$. Then one gets

$$e^{-4i\lambda^3 t}\psi_\lambda(x,t) = f_{1,t}(\lambda,x)$$

so that

$$e^{-4\kappa_j^3 t}\psi_{i\kappa_j}(x,t) = f_{1,t}(i\kappa_j,x). \qquad j = 1,\dots,N$$

From this one finally deduces

$$\nu_j(t) = e^{8\kappa_j^3 t}\nu_j(0). \qquad j = 1,\dots,N$$

(Indeed, since iB_t is hermitean, $U(t)$ is unitary:
$\frac{d}{dt} U^* U = U^* \dot U + \dot U^* U = U^*(B + B^*)U = 0$. Therefore,
$\|\psi_{i\kappa_j}(\cdot,t)\| = \|U(t)\psi_{i\kappa_j}(\cdot,0)\| = \|\psi_{i\kappa_j}(\cdot,0)\|$, from which the last relation follows.)

<u>Upshot</u>. If $u(x,0)$ has scattering data $\{r(p), \nu_j, \kappa_j\}$, then $u(x,t)$ has data $\{e^{8ip^3 t} r(p), e^{8\kappa_j^3 t}\nu_j, \kappa_j\}$.

<u>Consequence</u>. The solution $u(x,t)$ to (KdV), with initial value $u(x,0)$ can be determined as follows:

$$\begin{tikzcd} -u(x,t) & \{e^{8ip^3t} r(p), e^{8\kappa_j^3 t}\nu_j, \kappa_j\} \arrow[l, "\mathcal{J}"'] \\ -u(x,0) \arrow[u] \arrow[r, "\mathcal{D}"'] & \{r(p), \nu_j, \kappa_j\} \arrow[u] \end{tikzcd}$$

That is, the <u>nonlinear</u> Cauchy problem can be reduced to two <u>linear</u> problems:

— Solving $(EV)_0$ (whence scattering data at time t)

— Solving $(GLM)_t$ (i.e., the GLM equation at time t, cf. B2(iv))

iii) <u>The KdV solitons and their scattering</u>

Taking $x \to y$ to avoid ambiguities (cf. below), the KdV N-soliton solutions $u(y,t)$ correspond to potentials $-u(y,0)$ yielding scattering data $r(p)=0$, $(\nu,\kappa)\in D$. Then $(GLM)_t$ can be solved, yielding (cf. B2 (iv)):

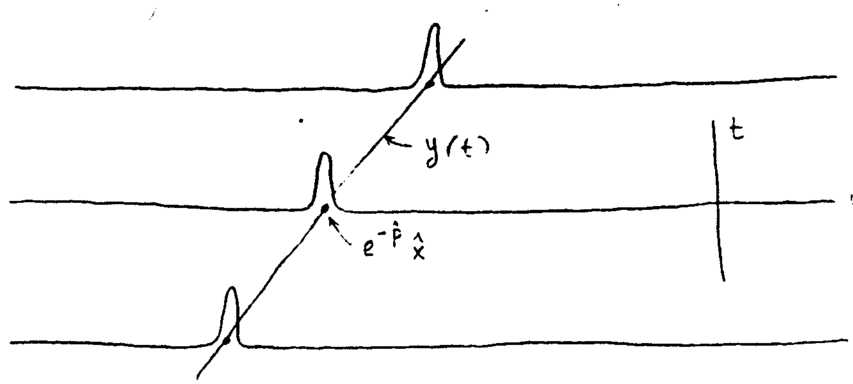
$$u(y,t) = 2\,\partial_y^2 \ln \tau(y,t)$$

$$\tau(y,t) = d(\hat{x}_1 - ye^{\hat{p}_1} + te^{3\hat{p}_1}, \ldots, \hat{x}_N - ye^{\hat{p}_N} + te^{3\hat{p}_N}, \hat{p})$$

Ex. $N=1$. Then $\tau(y,t) = 1+\exp(\hat{x} - ye^{\hat{p}} + te^{3\hat{p}})$,

$$u(y,t) = \frac{e^{2\hat{p}}}{2\,\mathrm{ch}^2[\frac{1}{2}(\hat{x} - ye^{\hat{p}} + te^{3\hat{p}})]}$$

Hence, maximum at $y(t) = e^{-\hat{p}}(\hat{x} + te^{3\hat{p}})$:



Next, use A4 (ii) to deduce

$$\tau(y,t) = |\mathbb{1}_N + \mathrm{diag}(e^{x_1(y,t)}, \ldots, e^{x_N(y,t)})|$$

$$u(y,t) = 2\sum_{j=1}^{N} \partial_y^2 \ln(1+e^{x_j(y,t)})$$

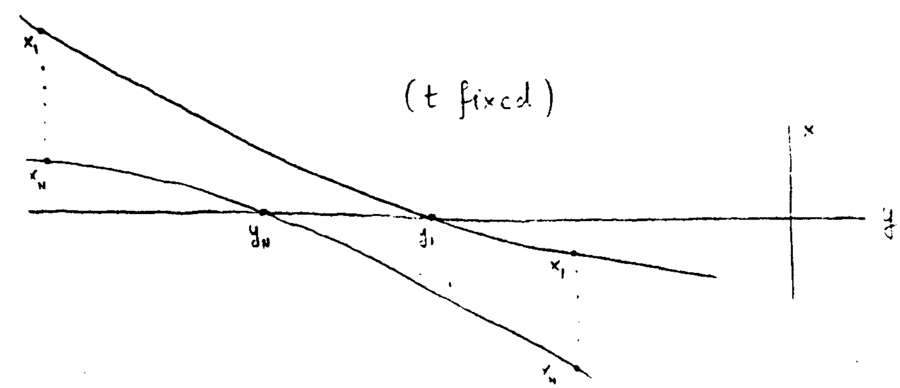
where
$$x(y,t) \equiv e^{-yH_{h_1} + tH_{h_0}}(x,p)_{conf.},$$

$$(x,p) \equiv \mathcal{E}(\hat{x},\hat{p}), \quad h_1(\hat{p}) \equiv e^{\hat{p}}, \quad h_0(\hat{p}) \equiv \tfrac{1}{3}e^{3\hat{p}}$$

(For $N=1$ take $\mathcal{E} = id.$) Now one has

$$\partial_y x_j(y,t) = -e^{p_j(y,t)} \prod_{\ell\neq j} |\mathrm{cth}\tfrac{1}{2}(x_j(y,t) - x_\ell(y,t))| \in [a,b] \subset (-\infty,0)$$

where a,b depend only on the initial point (x,p). (Recall $H = \sum_j \mathrm{ch}\,p_j \prod_{\ell\neq j} |\mathrm{cth}\tfrac{1}{2}(x_j - x_\ell)|$ is conserved.) Thus, unique <u>soliton space-time trajectories</u> $y_1(t), \ldots, y_N(t)$ are determined by requiring $x_j(y_j(t),t) = 0$, $j = 1,\ldots,N$:

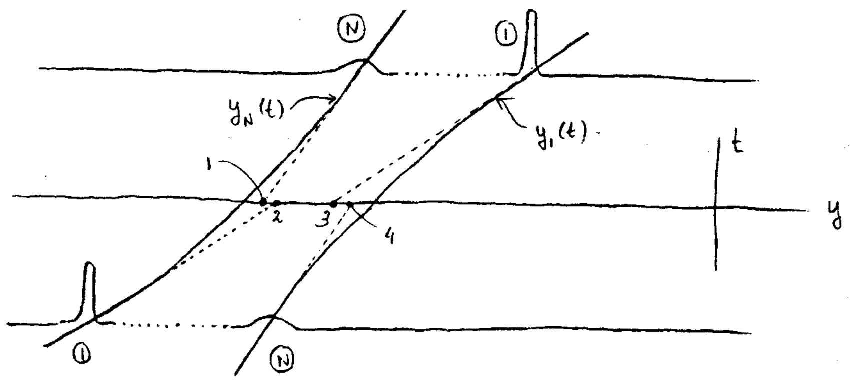


The following facts justify this terminology (cf. A4 (iv)):

$$\lim_{t\to\pm\infty}\left[\, y_{\substack{j\\N-j+1}}(t) - e^{-\hat p_j}\left(\hat x_j \mp \tfrac12 \Delta_j(\hat p) + t\, e^{3\hat p_j}\right)\right] = 0\,,\quad j=1,\dots,N$$

$$\Delta_j(\hat p) \equiv \Big(\sum_{k<j} - \sum_{k>j}\Big)\,\delta(\hat p_j - \hat p_k)\,,\qquad \delta(p) \equiv \ln\left(\mathrm{cth}^2 \tfrac12 p\right)$$

$$\sup_{y\in\mathbb{R}}\left|\, u(y,t) - \sum_{j=1}^{N} e^{2\hat p_j}\Big/ 2\,\mathrm{ch}^2\left[\tfrac12\left(\hat x_j \mp \tfrac12 \Delta_j(\hat p) - y\, e^{\hat p_j} + t\, e^{3\hat p_j}\right)\right]\right| \xrightarrow{t\to\pm\infty} 0$$



$1 = e^{-\hat p_N}(\hat x_N - \tfrac12 \Delta_N)$, $2 = e^{-\hat p_1}(\hat x_1 + \tfrac12 \Delta_1)$, $3 = e^{-\hat p_1}(\hat x_1 - \tfrac12 \Delta_1)$, $4 = e^{-\hat p_N}(\hat x_N + \tfrac12 \Delta_N)$

$\therefore$ Soliton 1 shifted by $3-2 = -e^{-\hat p_1}\Delta_1 = e^{-\hat p_1}\sum_{k>1}\delta(\hat p_1 - \hat p_k)$

Soliton N shifted by $1-4 = -e^{-\hat p_N}\Delta_N = -e^{-\hat p_N}\sum_{k<N}\delta(\hat p_N - \hat p_k)$

General formula: soliton j shifted by

$$e^{-\hat p_j}\Big(\sum_{k>j} - \sum_{k<j}\Big)\,\delta(\hat p_j - \hat p_k) \qquad (\underline{\text{factorization}})$$

iv) <u>Radiation and mixed solutions</u>

The KdV radiation solutions $u(x,t)$ correspond to potentials $-u(x,0)$ without bound states (such as $V(x) = 1/\mathrm{ch}\,x$). For asymptotic times these solutions behave (approximately) according to the linearized KdV equation $w_t + w_{xxx} = 0$, whose solutions read (check)

$$w(x,t) = \frac{1}{\sqrt{2\pi}}\int e^{-ipx - ip^3 t}\,\hat w(p)\,dp\,,\qquad \hat w(p) = \frac{1}{\sqrt{2\pi}}\int e^{ipx}\, w(x,0)\,dx$$

These linear waves move from right to left (the phase is stationary when $x = -3p^2 t$) and are $O(|t|^{-1/2})$ as $t\to\pm\infty$. Using this linear dynamics as comparison dynamics, the scattering of radiation solutions can be determined. In fact, even the scattering of mixed solutions (both $r(p)\neq 0$ and $N>0$) is explicitly known, cf.:

V. Buslaev, L. Faddeev, L. Takhtajan, Scattering theory for the KdV equation and its Hamiltonian interpretation, Physica <u>18D</u> (86) 255–266.

C 2. The KdV hierarchy

i) Local integrals of the KdV flow

Consider functionals

$$F : S(\mathbb{R}) \to \mathbb{R},\ u \mapsto \int P(u, u^{(1)}, \dots, u^{(k)})\,dx, \quad P = \text{polynomial}$$

For such F the Gateaux / Fréchet / variational / functional derivative $\delta F/\delta u$ exists; here,

$$\int \varphi(x) \frac{\delta F}{\delta u(x)}\,dx \equiv \lim_{\varepsilon \to 0} \frac{F(u+\varepsilon\varphi) - F(u)}{\varepsilon}, \quad \forall \varphi \in S(\mathbb{R})$$

Ex. $I_{-1} \equiv -\frac{1}{2}\int u\,dx$, $I_1 \equiv -\frac{1}{2}\int u^2 dx$, $I_3 \equiv \frac{1}{2}\int (u_x^2 - 2u^3)\,dx$

Then one gets (check): $\frac{\delta I_{-1}}{\delta u} = -\frac{1}{2}$, $\frac{\delta I_1}{\delta u} = -u$, $\frac{\delta I_3}{\delta u} = -u_{xx} - 3u^2$

Next, consider the KdV flow : $\mathbb{R} \times S(\mathbb{R}) \to S(\mathbb{R})$, $(t, u(x)) \mapsto u(x,t)$

It follows from C.1 (ii) that there exist non-local integrals

$$|r(p,t)| = |r(p,0)|, \quad p \in \mathbb{R}^*, \quad \kappa_j(t) = \kappa_j(0), \quad j = 1, \dots, N$$

Local integrals are functionals F of the above form such that $F(u(\cdot,t))$ is conserved. Thus, one has

$$\dot F = \int \dot u \frac{\delta F}{\delta u}\,dx = \int (-u_{xxx} - 6uu_x)\frac{\delta F}{\delta u}\,dx = 0$$

Ex. $\dot I_{-1} = \frac{1}{2}\int \partial_x (u_{xx} + 3u^2)\,dx = 0$, $\dot I_1 = \int \partial_x (uu_{xx} - \frac{1}{2}u_x^2 + 2u^3)\,dx = 0$

$$\dot I_3 = \int (u_{xxx} + 6uu_x)(u_{xx} + 3u^2)\,dx = \int \partial_x\left(\tfrac{1}{2}u_{xx}^2 + 3u^2 u_{xx} + \tfrac{9}{2}u^4\right)dx = 0$$

Fact. There exist further local integrals $I_5, I_7, \dots$ such that

$$\frac{d}{dx}\frac{\delta I_{2n+1}}{\delta u} = \mathcal{R}\,\frac{\delta I_{2n-1}}{\delta u}, \quad n \in \mathbb{N} \qquad (\mathcal{R})$$

where $\mathcal{R} \equiv \left(\frac{d}{dx}\right)^3 + 4u\frac{d}{dx} + 2u_x$ (recursion operator)

Ex. $\frac{d}{dx}\frac{\delta I_5}{\delta u} = \mathcal{R}(-u_{xx} - 3u^2) = -u^{(5)} - 20u^{(1)}u^{(2)} - 10uu^{(3)} - 30u^2u^{(1)}$

so that $I_5 = -\frac{1}{2}\int (u_{xx}^2 - 10uu_x^2 + 5u^4)\,dx$

(Check; try to check $\dot I_5 = 0$.)

N.B. 1) Define degree$(\frac{d}{dx}) \equiv 1$, degree$(u^{(n)}) \equiv n+2$. Clearly,

$$\deg \mathcal{R} = 3, \quad \deg\left(\frac{\delta I_{2n-1}}{\delta u}\right) = 2n, \quad n \in \mathbb{N}$$

2) From $(\mathcal{R})$ one infers (check)

$$\frac{\delta I_{2n+1}}{\delta u} = -u^{(2n)} + \text{nonlinear terms}, \quad n \in \mathbb{N}$$

$$I_{2n+1}(u) = \frac{(-)^{n+1}}{2}\int \left(u^{(n)2} + \text{cubic and higher terms}\right)dx \qquad (\text{lin})_n$$

ii) <u>The hierarchy of higher KdV flows</u>

<u>Fact</u>. There exist unique differential operators

$$B_n(u) = -4^n \partial^{2n+1} + \text{lower order in } \partial, \quad n \in \mathbb{N} \qquad (B_n)$$

of degree $2n+1$, such that

$$X_n(u) \equiv [L(u), B_n(u)], \quad L(u) \equiv -\partial^2 - u$$

is a polynomial in $u, u^{(1)}, \dots, u^{(2n+1)}$. Moreover, one has

$$B_n^* = -B_n, \quad X_n = \partial \frac{\delta I_{2n+1}}{\delta u} = \mathcal{R} \frac{\delta I_{2n-1}}{\delta u}, \quad n \in \mathbb{N} \qquad (\text{Eq.})_n$$

<u>Ex</u>. $B_0 = -\partial$, $B_1 = -4\partial^3 - 6u\partial - 3u^{(1)}$ (cf. C1 (i))

$$B_2 = -16\partial^5 - 40u\partial^3 - 60u^{(1)}\partial^2 - (50u^{(2)} + 30u^2)\partial - 15u^{(3)} - 30uu^{(1)}$$

(Check; also check $(\text{Eq.})_n$.)

<u>Consequence</u>. Consider the PDEs

$$\frac{\partial}{\partial t_n} u = X_n(u) \quad \left(\Leftrightarrow \frac{\partial}{\partial t_n} L = [B_n, L]\right) \qquad (n\text{-KdV})$$

(Note (1-KdV) = (KdV), and (0-KdV) has solution $u(x-t_0)$.)

Arguing just as in C1 (ii), it follows that the flows

$$e^{t_n X_n} : S(\mathbb{R}) \to S(\mathbb{R}), \quad u(x,0) \mapsto u(x,t_n), \quad n \in \mathbb{N}$$

can be viewed as the flows $\mathcal{I} \circ e^{t_n \tilde{X}_n} \circ \mathcal{D}$, where

$$e^{t_n \tilde{X}_n} : \{r(p), \nu_j, \kappa_j\} \mapsto \{e^{-(2ip)^{2n+1} t_n} r(p), e^{(2\kappa_j)^{2n+1} t_n} \nu_j, \kappa_j\}$$

Since the latter obviously commute, one deduces

$$e^{t_k X_k} \circ e^{t_n X_n} = e^{t_n X_n} \circ e^{t_k X_k}, \quad \forall k, n \in \mathbb{N}, \ \forall t_k, t_n \in \mathbb{R}$$

<u>N.B.</u> In the context of B2 (iii), (iv) one can prove

$$|t(p)|^2 + |r(p)|^2 = 1 \quad (S \text{ is unitary}) \qquad (\text{un})$$

$$\ln a(p) \sim -2i \sum_{n=-1}^{\infty} \frac{(-)^n}{(2p)^{2n+3}} I_{2n+1}(-V), \quad V \in S(\mathbb{R}), \ p \to \infty \qquad (\text{as})$$

$$\ln a(p) = \sum_{j=1}^{N} \ln\left(\frac{p - i\kappa_j}{p + i\kappa_j}\right) - \frac{1}{2\pi i p} \int_{-\infty}^{\infty} \frac{\ln |a(q)|^2}{1 - q^2/p^2} dq, \ \operatorname{Im} p > 0 \qquad (C)$$

where (C) follows from Cauchy's theorem. Thus (check)

$$I_{2n+1}(u) = \frac{-1}{2n+3} \sum_{j=1}^{N} (2\kappa_j)^{2n+3} + \frac{(-)^n}{\pi} \int_0^\infty (2p)^{2n+2} \ln(1 - |r(p)|^2) dp \qquad (I_{2n+1})$$

These so-called <u>trace identities</u> entail that any I_{2n+1} is an integral of all (k-KdV) flows; they express the local integrals in terms of the non-local ones.

C3. The Hamiltonian formulation

i) Hamiltonian formalism: generalities

To rephrase the results of C2 in terms of infinite-dimensional integrable systems, one first needs a notion of infinite-dimensional Hamiltonian system. Thus, the geometric structures of the finite-dimensional context should be generalized. To this end, recall first some notions for the simplest type of state space, $\Omega \equiv \{u \,|\, u \in \mathbb{R}^{2N}\}$:

— state functions: real-valued $F \in C^\infty(\Omega)$

— tangent space in u: 2 equivalent definitions are

1) $T_u\Omega \equiv \{$tangent vectors φ at u to curves through u$\}$

2) $T_u\Omega \equiv \{$directional derivatives X_u in u of state functions$\}$

Relation: $X_u F = \lim_{\varepsilon \to 0} \frac{F(u+\varepsilon\varphi) - F(u)}{\varepsilon} = \sum_{j=1}^{2N} \varphi_j \, (\partial_{u_j} F)|_u)$

Thus, $T_u\Omega = \{X_u \,|\, X_u = \sum_{j=1}^{2N} \varphi_j \frac{\partial}{\partial u_j}\big|_u\} \simeq \{\varphi \,|\, \varphi \in \mathbb{R}^{2N}\}$

— cotangent space in u:

$$T_u^*\Omega = \{\alpha_u \,|\, \alpha_u = \sum_{j=1}^{2N} L_j \, du_j|_u\} \simeq \{L \,|\, L \in \mathbb{R}^{2N*}\}$$

($\mathbb{R}^{2N*}$ = dual of $\mathbb{R}^{2N}$ = $\{$linear functionals on $\mathbb{R}^{2N}\} \simeq \mathbb{R}^{2N}$;

$\{du_j|_u\}$ = dual base of $\{\frac{\partial}{\partial u_k}\big|_u\}$, i.e. $du_j(\frac{\partial}{\partial u_k}) = \delta_{jk}$)

— tangent bundle: $T\Omega = \bigcup_{u\in\Omega} T_u\Omega \simeq \{(u,\varphi) \,|\, u, \varphi \in \mathbb{R}^{2N}\}$

— cotangent bundle: $T^*\Omega = \bigcup_{u\in\Omega} T_u^*\Omega \simeq \{(u,L) \,|\, u \in \mathbb{R}^{2N}, L \in \mathbb{R}^{2N*}\}$

— vector fields: $\{$sections of $T\Omega\} \equiv \Gamma(\Omega, T\Omega)$, i.e.,

$X: \Omega \to T\Omega, \; u \mapsto (u, \varphi(u))$

— 1-forms: $\{$sections of $T^*\Omega\} \equiv \Gamma(\Omega, T^*\Omega)$, i.e.,

$\alpha: \Omega \to T^*\Omega, \; u \mapsto (u, L(u))$

— gradient of F: 1-form dF such that

$dF(X) = XF \qquad \forall X \in \Gamma(\Omega, T\Omega)$

(so that $dF = \sum_{j=1}^{2N} (\partial_{u_j} F) \, du_j$)

— k-forms: $\{$ sections of $T^*\Omega \wedge \cdots \wedge T^*\Omega$ (k×) $\}$

Now introduce generalizations for the infinite-dimensional state space $\Omega \equiv \{u \mid u(x) \in S(\mathbb{R})\}$:

— state functions: functionals F as defined in C2 (i) (a quite restrictive, but analytically convenient choice)

— tangent space in u: as above, yielding

$$T_u\Omega = \{X_u \mid X_u = \int dx\, \varphi(x) \frac{\delta}{\delta u(x)}\Big|_u\} \simeq \{\varphi \mid \varphi \in S(\mathbb{R})\}$$

(i.e., $X_u F = \int dx\, \varphi(x) \frac{\delta F}{\delta u(x)}$)

— cotangent space in u:

$$T_u^*\Omega = \{\alpha_u \mid \alpha_u = \int dx\, L(x)\, \delta u(x)|_u\} \simeq \{L \mid L \in S(\mathbb{R})^*\}$$

($S(\mathbb{R})^*$ = topological dual of $S(\mathbb{R})$ = {tempered distributions}; $\{\delta u(x)|_u\}$ = dual 'base' of $\{\frac{\delta}{\delta u(y)}\Big|_u\}$, i.e., $\delta u(x)\left(\frac{\delta}{\delta u(y)}\right) = \delta(x-y)$)

— tangent bundle: $T\Omega \simeq \{(u,\varphi) \mid u, \varphi \in S(\mathbb{R})\}$

— cotangent bundle: $T^*\Omega \simeq \{(u,L) \mid u \in S(\mathbb{R}), L \in S(\mathbb{R})^*\}$

— vector fields, 1-forms, k-forms: as above

— gradient of F: as above (so that $dF = \int dx \frac{\delta F}{\delta u(x)} \delta u(x)$)

Next, reconsider the symplectic / Hamiltonian / Poisson structures of the finite-dimensional context (cf. A1 (ii)), embodied in:

1) a symplectic form ω on Ω;

2) an identification map

$$\iota : \Gamma(\Omega, T^*\Omega) \to \Gamma(\Omega, T\Omega), \quad \alpha \mapsto X^{(\alpha)}$$

3) a Poisson bracket, i.e., a bilinear antisymmetric map

$$\{\cdot,\cdot\} : C^\infty(\Omega) \times C^\infty(\Omega) \to C^\infty(\Omega); \ (F,G) \mapsto \{F,G\}$$

that satisfies the Jacobi identity.

Starting from 1) one defines 2) and 3) via

$$\omega(X^{(\alpha)}, X) \equiv \alpha(X), \ \forall X \in \Gamma(\Omega, T\Omega), \quad \{F,G\} \equiv \omega(X^{(dF)}, X^{(dG)})$$

However, starting from 2) one can still define Hamiltonian vector fields $F \mapsto X^{(dF)}$ and Hamilton's equations $\dot{u} = X^{(dF)}(u)$; starting from 3) one can still define Hamilton's equations by

$$\dot{u}_j = \{u_j, F\}, \quad j = 1, \ldots, 2N$$

Choices for the structures (1), (2), (3) that are relevant to soliton PDEs often lead to technicalities that have not been rigorously dealt with to date; such difficulties will be largely ignored below. To fix the thoughts, we first consider

$$\omega_u(\varphi_1,\varphi_2) \equiv \int \varphi_1(x)\left(\frac{d}{dx}\right)^k \varphi_2(x) \equiv \langle \varphi_1, \partial^k \varphi_2\rangle, \quad \varphi_1,\varphi_2 \in S(\mathbb{R})$$

For $k \in \{1,3,5,\dots\}$ ω_u is a well-defined bilinear antisymmetric nondegenerate form on $T_u\Omega$, giving rise to a symplectic form ω on Ω (ω is closed since ω_u does not depend on u). However, the map $\iota_u : T_u^*\Omega \to T_u\Omega$, $L \mapsto \varphi^{(L)}$ does not even satisfy $\iota_u(S(\mathbb{R})) \subset S(\mathbb{R})$. Indeed, for $L \in S(\mathbb{R})$ the corresponding $\varphi^{(L)}$ should obey

$$\omega_u(\varphi^{(L)},\varphi) = L(\varphi) \Rightarrow \langle \varphi^{(L)}, \partial^k\varphi\rangle = \langle L,\varphi\rangle \Rightarrow L = -\partial^k\varphi^{(L)}$$

But when $\int L(x)dx \neq 0$ (e.g.), then $\varphi^{(L)} \notin S(\mathbb{R})$.

Next, take $\ell \equiv -k \in \{1,3,5,\dots\}$. Then ω_u is well defined on the space $D^{(\ell)} \subsetneq S(\mathbb{R})$ for which the integral

$$\int \tilde\varphi_1(-p)\, p^{-\ell}\, \tilde\varphi_2(p)\, dp \qquad (\tilde\varphi_j \equiv \text{Fourier transform of } \varphi_j)$$

converges absolutely. In particular, any φ of the form $\partial^\ell\psi$, $\psi \in S(\mathbb{R})$, belongs to $D^{(\ell)}$. Now one has (check)

$$\varphi^{(L)} = -\partial^\ell L, \qquad \ell \in \{1,3,5,\dots\} \qquad (\iota^{(\ell)})$$

so that $\iota_u^{(\ell)}$ sends $D \equiv S(\mathbb{R}) \cup \{\text{constants}\}$ to $D^{(\ell)}$. Since the above gradients dF_u belong to D, the choices

$$\omega_u^{(\ell)}(\varphi_1,\varphi_2) \equiv \langle \varphi_1, \partial^{-\ell}\varphi_2\rangle, \quad \ell \in \{1,3,5,\dots\} \qquad (\omega^{(\ell)})$$

are suitable for Hamiltonian formulations of linear flows, as will be detailed in (ii); the case $\ell = 1$ is a natural choice for KdV, cf. (iii). The Poisson brackets corresponding to $\omega^{(\ell)}$ read (check)

$$\{F_1,F_2\}^{(\ell)}(u) = -\int \frac{\delta F_1}{\delta u(x)}\left(\frac{d}{dx}\right)^\ell \frac{\delta F_2}{\delta u(x)}\, dx \qquad (P^{(\ell)})$$

<u>N.B.</u> $\iota_u^{(\ell)}$ annihilates constants and $\{F,\cdot\}^{(\ell)}$ vanishes for $F = \int u\,dx$, so $\iota^{(\ell)}$ is not 1-1 and $\{\cdot,\cdot\}^{(\ell)}$ is degenerate.

ii) The Fourier transform as action-angle map

N.B. From now on the above Schwartz space will be denoted $S_{\mathbb{R}}(\mathbb{R})$, whereas $S_{\mathbb{C}}(\mathbb{R})$ will be written $S(\mathbb{R})$.

Consider the linear PDEs and vector fields

$$\frac{\partial}{\partial t_n} u = X_n(u), \qquad X_n(u) \equiv -u^{(2n+1)}, \qquad n \in \mathbb{N}$$

on $\Omega \equiv \{u \mid u \in S_{\mathbb{R}}(\mathbb{R})\}$. Using Fourier transformation

$$\mathcal{F} : \Omega \to \tilde{\Omega}, \quad u(x) \mapsto \tilde{u}(p)$$

$$\tilde{u}(p) \equiv \int e^{-2ipx} u(x)dx \qquad \text{('KdV' normalization')}$$

$$\tilde{\Omega} \equiv \{\tilde{u} \mid \tilde{u} \in S(\mathbb{R}), \bar{\tilde{u}}(p) = \tilde{u}(-p)\}$$

one infers that the solution flows $e^{t_n X_n}$ are complete and commute. Indeed, one has (check)

$$e^{t_n X_n} = \mathcal{F}^{-1} \circ e^{t_n \tilde{X}_n} \circ \mathcal{F}$$

where $e^{t_n \tilde{X}_n}$ are the complete and commuting flows

$$e^{t_n \tilde{X}_n} : \tilde{u}(p) \longmapsto e^{-(2ip)^{2n+1} t_n} \tilde{u}(p), \qquad n \in \mathbb{N}$$

(Note that for $n = \frac{1}{2}$ one obtains the time-reversed heat flow: complete for $t \to -\infty$, but not for $t \to \infty$.)

To obtain a Hamiltonian formulation, consider

$$H_n^{(\ell)}(u) \equiv \frac{(-)^{(2n-\ell+1)/2}}{2} \int u^{((2n-\ell+1)/2)}(x)^2 dx, \quad \ell \in \{1,3,5,\dots\}$$

Provided $2n \geq \ell - 1$, this is a well-defined functional satisfying (check)

$$\frac{\delta H_n^{(\ell)}}{\delta u} = u^{(2n-\ell+1)}, \qquad \iota_u^{(\ell)} \frac{\delta H_n^{(\ell)}}{\delta u} = X_n(u).$$

Thus one may write the commuting flows $e^{t_n X_n}$ as $e^{t_n H_n^{(\ell)}}$; moreover, one gets

$$\{H_m^{(\ell)}, H_n^{(\ell)}\}^{(\ell)}(u) = -\int u^{(2m-\ell+1)} \left(\frac{d}{dx}\right)^{\ell} u^{(2n-\ell+1)} dx = 0$$

as expected. (Note $\int u \partial^j u = 0$ when j is odd.)

$\therefore$ The hierarchy of commuting flows can be viewed as arising from ∞-dimensional integrable systems $S^{(\ell)} \equiv \langle \Omega, \omega^{(\ell)}, H_{\frac{\ell-1}{2}}^{(\ell)}, H_{\frac{\ell+1}{2}}^{(\ell)}, \dots \rangle$.

Next, in order to construct an action-angle map for $\mathcal{S}^{(\ell)}$, first transform to $\tilde{\Omega}$. This yields (check)

$$\tilde{H}_n^{(\ell)}(\tilde{u}) \equiv H_n^{(\ell)} \circ \mathcal{F}^{-1}(\tilde{u}) = \frac{(-)^{(2n-\ell+1)/2}}{2\pi} \int (2p)^{2n-\ell+1} |\tilde{u}(p)|^2 dp$$

$$\tilde{\omega}_{\tilde{u}}^{(\ell)}(\psi_1,\psi_2) \equiv (\mathcal{F}_* \omega^{(\ell)})_{\tilde{u}}(\psi_1,\psi_2) = \omega_u^{(\ell)}(\mathcal{F}_*^{-1}\psi_1, \mathcal{F}_*^{-1}\psi_2)$$

$$= \frac{1}{\pi} \int (2ip)^{-\ell} \psi_1(-p)\psi_2(p)\, dp, \quad \forall \psi_1,\psi_2 \in \tilde{\mathcal{D}}^{(\ell)}$$

Now define a coordinate change

$$\mathcal{C}^{(\ell)} : \tilde{\Omega} \to \hat{\Omega}^{(\ell)}, \quad \tilde{u}(p) \longmapsto (\hat{X}(p), \hat{P}^{(\ell)}(p))$$

$$\hat{X}(p) \equiv \arg \tilde{u}(p), \quad \hat{P}^{(\ell)}(p) \equiv \frac{(-)^{(\ell+1)/2}}{\pi} (2p)^{-\ell} |\tilde{u}(p)|^2, \quad p>0$$

and introduce

$$\Phi^{(\ell)} \equiv \mathcal{C}^{(\ell)} \cdot \mathcal{F}, \quad \mathcal{E}^{(\ell)} \equiv \Phi^{(\ell)-1}$$

$$e^{t_n \hat{X}_n^{(\ell)}} \equiv \Phi^{(\ell)} \circ e^{t_n X_n} \circ \mathcal{E}^{(\ell)}$$

$$\hat{H}_n^{(\ell)} \equiv H_n^{(\ell)} \circ \mathcal{E}^{(\ell)}$$

$$\hat{\omega}^{(\ell)} \equiv \Phi_*^{(\ell)} \omega^{(\ell)}$$

Then one obtains (check)

$$e^{t_n \hat{X}_n^{(\ell)}} : (\hat{X}(p), \hat{P}^{(\ell)}(p)) \mapsto (\hat{X}(p) + (-)^{n+1}(2p)^{2n+1} t_n, \hat{P}^{(\ell)}(p))$$

$$\hat{H}_n^{(\ell)}(\hat{X}, \hat{P}^{(\ell)}) = (-)^{n+1} \int_0^\infty (2p)^{2n+1} \hat{P}^{(\ell)}(p)\, dp$$

$$\hat{\omega}^{(\ell)}_{(\hat{X},\hat{P}^{(\ell)})}((\alpha_1,\beta_1),(\alpha_2,\beta_2)) = \int_0^\infty [\alpha_1(p)\beta_2(p) - \beta_1(p)\alpha_2(p)]\, dp$$

Consequently, on $\hat{\Omega}^{(\ell)}$ the commuting flows are linear in t_n and $\hat{\omega}^{(\ell)}$ can be rewritten as

$$\hat{\omega}^{(\ell)} = \int_0^\infty dp\, \delta\hat{X}(p) \wedge \delta\hat{P}^{(\ell)}(p)$$

$\therefore$ $\hat{X}(p)$ and $\hat{P}^{(\ell)}(p)$, $p>0$, may be viewed as angle and action coordinates, resp.; cf. A1 (iv).

<u>N.B.</u> It should be emphasized that the above vector fields X_n and their flows do not depend on the choice of symplectic structure, whereas the action-angle variables do depend on this choice. In contrast, in the finite-dimensional context a symplectic form is given to begin with (cf. A1 (iii)), so that the action-angle variables are essentially unique.

iii) Hamiltonian formalism: KdV

The 'obvious' choice of symplectic structure for the KdV situation is the form (Gardner structure)

$$\omega^G_u(\varphi_1,\varphi_2) \equiv \langle \varphi_1, \partial^{-1}\varphi_2 \rangle \qquad (\omega^G)$$

Indeed, then one has $\omega^G = \omega^{(1)}$, cf. (i), so that

$$\iota^G_u \frac{\delta F}{\delta u} = -\partial \frac{\delta F}{\delta u} \qquad (\iota^G)$$

$$\{F_1, F_2\}^G(u) = -\int \frac{\delta F_1}{\delta u}\, \partial\, \frac{\delta F_2}{\delta u}\, dx \qquad (P^G)$$

(Formally, (P^G) can be written $\{u(x),u(y)\}^G = -\delta'(x-y)$.)

Therefore, setting

$$H^{G}_n \equiv -I_{2n+1}, \qquad n \in \mathbb{N}$$

it follows from C2 (ii) that

$$\iota^G_u \frac{\delta H^G_n}{\delta u} = X_n(u), \qquad \{H^G_n, H^G_k\}^G = 0$$

Thus, the KdV hierarchy gives rise to an ∞-dimensional integrable system $\mathcal{S}^G = \langle \Omega, \omega^G, H^G_0, H^G_1, \dots \rangle$.

However, a second choice can be made (Magri structure):

$$\omega^M_u(\varphi_1,\varphi_2) \equiv \langle \varphi_1, \mathcal{R}(u)^{-1}\varphi_2 \rangle \qquad (\omega^M)$$

This entails (check)

$$\iota^M_u \frac{\delta F}{\delta u} = -\mathcal{R}(u) \frac{\delta F}{\delta u} \qquad (\iota^M)$$

$$\{F_1, F_2\}^M(u) = -\int \frac{\delta F_1}{\delta u}\, \mathcal{R}(u)\, \frac{\delta F_2}{\delta u}\, dx \qquad (P^M)$$

Setting now

$$H^M_n \equiv -I_{2n-1}, \qquad n \in \mathbb{N}$$

one obtains $\iota^M_u \frac{\delta H^M_n}{\delta u} = X_n(u)$, $\{H^M_n, H^M_k\}^M = 0$.

Hence, the KdV hierarchy can also be viewed as an ∞-dimensional integrable system $\mathcal{S}^M = \langle \Omega, \omega^M, H^M_0, H^M_1, \dots \rangle$.

N.B. Viewing the (1-KdV) and (0-KdV) Hamiltonians as 'energy' and 'momentum' (time and space translation generators), the trace identities from C2 (ii) entail ($s \equiv$ solitons, $r \equiv$ radiation):

$$E^G_s,\ -E^G_r,\ M^G_s,\ M^G_r,\ E^M_s,\ E^M_r,\ M^M_s,\ -M^M_r > 0$$

Therefore, the Magri choice appears more physical, cf. C1 (iv).

Next, the action-angle map Φ^G will be detailed:

Define a coordinate change

$$\mathcal{C}^G : \tilde{\Omega} \to \hat{\Omega}^G , \quad \{ r(p), v_j, \kappa_j \} \mapsto \{ \hat{X}(p), \hat{P}^G(p), \hat{x}_j, \hat{p}_j^G \}$$

$$\hat{X}(p) \equiv \arg r(p), \quad \hat{P}^G(p) \equiv \frac{2p}{\pi} \ln (1 - |r(p)|^2), \qquad p>0$$

$$\hat{x}_j \equiv \ln \left(\frac{v_j}{2\kappa_j} \prod_{l \neq j} \left| \frac{\kappa_j - \kappa_l}{\kappa_j + \kappa_l} \right| \right), \quad \hat{p}_j^G \equiv 2\kappa_j^2, \qquad j=1,\dots,N$$

and introduce

$$\Phi^G \equiv \mathcal{C}^G \circ \mathcal{D}, \quad \mathcal{E}^G \equiv \Phi^{G\,-1}$$

$$e^{t_n \hat{X}_n^G} \equiv \Phi^G \circ e^{t_n X_n} \circ \mathcal{E}^G, \quad \hat{H}_n^G \equiv H_n^G \circ \mathcal{E}^G, \quad \hat{\omega}^G \equiv \Phi_*^G \omega^G$$

Then one obtains (check using C2 (ii))

$$e^{t_n \hat{X}_n^G} : (\hat{X}(p), \hat{P}^G(p), \hat{x}_j, \hat{p}_j^G) \mapsto (\hat{X}(p) + (-)^{n+1}(2p)^{2n+1} t_n, \hat{P}^G(p), \hat{x}_j + (2\hat{p}_j^G)^{(2n+1)/2} t_n, \hat{p}_j^G)$$

$$\hat{H}_n^G(\hat{X}, \hat{P}^G, \hat{x}, \hat{p}^G) = (-)^{n+1} \int_0^\infty (2p)^{2n+1} \hat{P}^G(p)\, dp + \frac{1}{2n+3} \sum_{j=1}^N (2\hat{p}_j^G)^{(2n+3)/2}$$

Just as in (ii), the crux is that these variables may be viewed as action-angle variables: One has (don't try to check)

$$\Phi_*^G \omega^G \equiv \hat{\omega}^G = \int_0^\infty dp\, \delta \hat{X}(p) \wedge \delta \hat{P}^G(p) + \sum_{j=1}^N d\hat{x}_j \wedge d\hat{p}_j^G \quad \text{(up to } \delta\text{-functions)}$$

Now change $\mathcal{C}^G, \hat{\Omega}^G, \dots$ to $\mathcal{C}^M, \hat{\Omega}^M, \dots$ by taking

$$\hat{P}^M(p) \equiv -\frac{1}{2p\pi} \ln(1 - |r(p)|^2), \quad p>0, \quad \hat{p}_j \equiv \ln(2\kappa_j), \quad j=1,\dots,N$$

Then one gets (check)

$$e^{t_n \hat{X}_n^M} : (\hat{X}(p), \hat{P}^M(p), \hat{x}_j, \hat{p}_j) \mapsto (\hat{X}(p) + (-)^{n+1}(2p)^{2n+1} t_n, \hat{P}^M(p), \hat{x}_j + e^{(2n+1)\hat{p}_j} t_n, \hat{p}_j)$$

$$\hat{H}_n^M(\hat{X}, \hat{P}^M, \hat{x}, \hat{p}) = (-)^{n+1} \int_0^\infty (2p)^{2n+1} \hat{P}^M(p)\, dp + \frac{1}{2n+1} \sum_{j=1}^N e^{(2n+1)\hat{p}_j}$$

It is plausible (but appears not to be known) that

$$\Phi_*^M \omega^M \equiv \hat{\omega}^M = \int_0^\infty dp\, \delta \hat{X}(p) \wedge \delta \hat{P}^M(p) + \sum_{j=1}^N d\hat{x}_j \wedge d\hat{p}_j \qquad (?)$$

<u>N.B.</u> 1) When $r=0$ the above $\hat{x}_j, \hat{p}_j$ equal those of B2 (iv) (check).

2) The soliton S-map (cf. C1 (iii)) is not G-canonical, whereas it is M-canonical (assuming (?)); the natural comparison maps $J^\pm$ (cf. B1) are not canonical in either case. For more on this, cf. [2], pp. 195-197 and refs. given there.

3) Introducing α as in C1 (i), one has

$$r(p) = \frac{-\alpha}{2ip} \tilde{u}(p) + O(\alpha^2), \quad \alpha \to 0 \quad \text{(\underline{Born approximation})}$$

Moreover, $N \to 0$ for $\alpha \to 0$; hence (check using (ii) and C2 (i))

$$\mathcal{S}_\alpha^G \to \mathcal{S}^{(1)}, \quad \Phi_\alpha^G \to \Phi^{(1)}, \qquad \alpha \to 0$$

'IST = nonlinear Fourier transformation'

C 4. Soliton PDEs and lattices : a sample

i) Generalities

The soliton PDEs and lattices known to date have properties paralleling those of KdV discussed above. In particular, there exist an associated Lax pair and linear integral equation playing the role of $-\partial^2+V(x)$ and the GLM equation. For example, a number of PDEs (including KdV) can be handled via a Lax operator of Dirac form,

$$L=\begin{pmatrix}-i\partial & 0\\ 0 & i\partial\end{pmatrix}+V(x)\,,\quad V(x)\in M_2(\mathbb{C})$$

(This yields the so-called Zakharov/Shabat or AKNS system.) Just as for KdV, there exist hierarchies of commuting nonlinear vector fields, which correspond to an infinity of conserved quantities admitting several Hamiltonian interpretations. Additional correspondences include:

bound states and their energies ~ solitons and their velocities
reflection coefficient $\neq 0$ ~ radiation present
scattering data, direct transform ~ action-angle variables / map
soliton scattering ~ Calogero-Moser scattering

There are quite a few other common features, including:

— a zero curvature and r-matrix formulation;

— a reformulation in terms of Hirota bilinear equations yielding so-called τ-function solutions;

— connections to Kac-Moody and Virasoro algebras via fermion and boson Fock space objects (the corresponding groups yield Bogoliubov transformations, which act on τ-functions);

— relations to algebraic geometry (Grasmannians, theta and Baker-Achiezer functions, moduli spaces, ...);

— Bäcklund transformations, Wahlquist-Estabrook prolongations, ...

N.B. There also exist integrable nonlinear PDEs and lattices without solitons, but with several of the above features (e.g. repulsive nonlinear Schrödinger, massless Thirring, Federbush, ...).

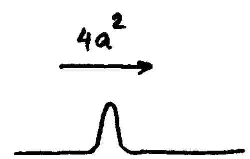
ii) Soliton PDEs in 2D

1. Modified KdV

$$v_t + 24 v^2 v_x + v_{xxx} = 0 \qquad (\text{mKdV})$$

1-soliton solution (check):

$$v = \frac{a}{\mathrm{ch}\,[2a(x-x_0) - 8a^3 t]}, \quad a>0$$



Next, put

$$g \equiv \frac{1}{2i}\left(|\mathbb{1}_N + i\hat{A}| - |\mathbb{1}_N - i\hat{A}| \right)$$

$$f \equiv \frac{1}{2}\left(|\mathbb{1}_N + i\hat{A}| + |\mathbb{1}_N - i\hat{A}| \right)$$

$$\hat{A} \equiv \hat{A}(\hat{x}_1 - x\sigma_1 + t\rho_1, \dots, \hat{x}_N - x\sigma_N + t\rho_N, \hat{p}), \quad \sigma_j \equiv e^{\hat{p}_j}, \quad \rho_j \equiv e^{3\hat{p}_j}$$

where rhs given by $(\hat{A})$ in A4 (ii) with $\beta = \mu = 1$, $\tau = \pi/2$.

Then N-soliton solutions read

$$-v = \partial_x \,\mathrm{arctg}(g/f) = \partial_x \,\mathrm{Tr}\,\mathrm{Arctg}\,\hat{A}$$

Proceeding as in C1 (iii) one obtains

$$-v = \sum_{j=1}^{N} \partial_x \,\mathrm{Arctg}\,(e^{x_j(x,t)}),$$

from which space-time trajectories and scattering follow.

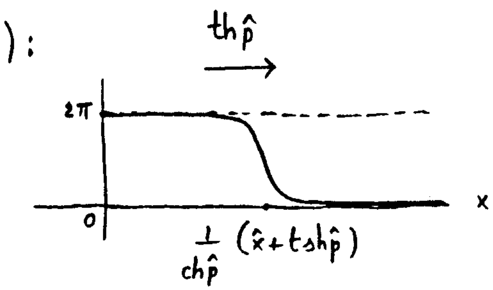
2. Sine-Gordon

$$\varphi_{xx} - \varphi_{tt} = \sin\varphi \qquad (\text{sG})$$

1-soliton solution (check):

$$\varphi = 4\,\mathrm{Arctg}\left(e^{\hat{x} - x\,\mathrm{ch}\hat{p} + t\,\mathrm{sh}\hat{p}}\right)$$

$\hat{x}, \hat{p} \in \mathbb{R}$



Defining $g, f, \hat{A}$ as for mKdV, but now with $\sigma_j \equiv \mathrm{ch}\hat{p}_j$, $\rho_j \equiv \mathrm{sh}\hat{p}_j$, $j = 1, \dots, N$, the N-soliton solutions read

$$\varphi = 4\,\mathrm{arctg}(g/f) = 4\,\mathrm{Tr}\,\mathrm{Arctg}\,\hat{A} = 4 \sum_{j=1}^{N} \mathrm{Arctg}\,(e^{x_j(x,t)})$$

Proceeding again as in C1 (iii), this yields space-time trajectories and S-map. These are Poincaré invariant, in agreement with the Poincaré invariance of (sG).

Here, the Calogero-Moser space and time translation generators H_{h_1} and H_{h_0} are P and H from A3 resp. (with $\beta = \mu = 1$, $\tau = \pi/2$).

For the following 2 equations only the 1-soliton solutions are detailed (try to check). Note that an internal degree of freedom is present. Such 'breathers' occur also for the Calogero-Moser systems and for (mKdV) and (sG), cf. [2], [5].

3. Nonlinear Schrödinger (attractive case)

$$i\Psi_t = -\Psi_{xx} + 2g|\Psi|^2\Psi, \qquad g<0 \qquad \text{(NLS)}$$

$$\Psi = \frac{a}{\sqrt{-2g}}\,\frac{e^{i(a^2t+\frac{1}{2}vx-\frac{1}{4}v^2t+c)}}{\operatorname{ch} a(x-x_0-vt)}, \quad v,x_0\in\mathbb{R},\ a>0,\ c\in[0,2\pi)$$

4. Isotropic Heisenberg magnet

$$\dot S = S\times S_{xx}, \qquad S\in S^2\subset\mathbb{R}^3 \qquad \text{(XXX)}$$

$$S_1+iS_2 = \frac{2u\,e^{i(\frac{1}{2}vx+\frac{1}{4}(u^2-v^2)t+c)}}{(u^2+v^2)\operatorname{ch}^2 y}\left(-u\operatorname{sh}y+iv\operatorname{ch}y\right)$$

$$S_3 = 1-\frac{2u^2}{(u^2+v^2)\operatorname{ch}^2 y}, \quad y\equiv\frac{u}{2}(x-x_0-vt),\quad v,x_0\in\mathbb{R},\ u>0,\ c\in[0,2\pi)$$

N.B. 1) These 2 equations are 'gauge equivalent', cf. [5], pp. 315-319.

2) Adding $S\times\operatorname{diag}(J_1,J_2,J_3)S$, $J_1<J_2<J_3$, to the rhs of (XXX), one gets (XYZ)/Landau-Lifshitz, a soliton PDE related to a special case of the IV_{rel} systems, cf. [2], p. 199.

iii) The nonrelativistic Toda lattice

This is the ∞-dimensional version of the V_{nr} and VI_{nr} systems. Thus, its equation of motion reads (cf. A2(i))

$$\ddot x_n = e^{x_{n+1}-x_n} - e^{x_n-x_{n-1}}, \qquad n\in\mathbb{Z} \qquad \text{(Toda)}$$

The 1-soliton solutions are given by

$$e^{x_{n+1}-x_n} = 1+\frac{\rho^2}{4\operatorname{ch}^2\frac{1}{2}(\hat x-n\sigma+t\rho)}, \quad \sigma\equiv\ln(\operatorname{cth}^2\tfrac{\hat p}{2}),\ \rho\equiv\frac{2}{\operatorname{sh}\hat p},\ \hat p\in\mathbb{R}^+$$

and the N-soliton solutions read $e^{x_{n+1}-x_n} = 1+\partial_t^2\ln\tau(n,t)$, with

$$\tau\equiv\left|\mathbb{1}_N+\hat A(\hat x_1-n\sigma_1+t\rho_1,\dots,\hat x_N-n\sigma_N+t\rho_N,\hat p)\right|$$

N.B. 1) It is customary to require $\lim_{n\to-\infty} x_n = 0$. Viewing $x_n=0$ as the equilibrium of the n^{th} lattice point, one may then regard $c\equiv\lim_{n\to\infty}x_n$ as the total elongation of the lattice.

2) The space-time generators H_{h_1}, H_{h_0} are here defined on the dense subset $\mathcal{E}(\hat\Omega\setminus\{\prod_{j=1}^N \hat p_j=0\})$ of Ω.

3) As continuum limits one can obtain both KdV and Boussinesq (a 2D soliton PDE).

4) The relativistic generalization (cf. A3(ii)) is a soliton lattice, too.

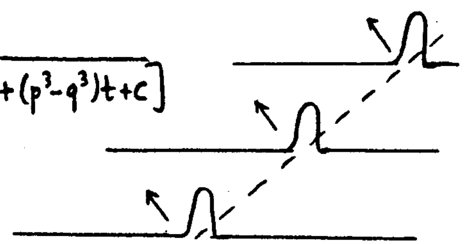
iv) The Kadomtsev-Petviashvili equation

This is the 3D generalization of KdV given by

$$u_{yy} = \partial_x \left[\tfrac{4}{3} u_t - 2 u u_x - \tfrac{1}{3} u_{xxx} \right] \qquad \text{(KP)}$$

1-soliton solution: (t fixed)

$$u(x,y,t) = \frac{(p-q)^2}{2\,\mathrm{ch}^2 \frac{1}{2}\left[(p-q)x + (p^2-q^2)y + (p^3-q^3)t + c\right]}$$



Hirota bilinear form of (KP):

$$\left(\partial_{y_1}^4 - 4\partial_{y_1}\partial_{y_3} + 3\partial_{y_2}^2\right)\tau(x+y)\tau(x-y)\Big|_{y=0} = 0$$

where

$$x_1 \equiv x, \quad x_2 \equiv y, \quad x_3 \equiv t, \qquad u = 2\partial_1^2 \ln\tau$$

The KP hierarchy can be coded via additional 'times' $x_4, x_5, \dots$ and Hirota polynomials. The N-soliton solutions to the hierarchy read

$$\tau = \left| \mathbb{1}_N + DC \right|, \quad C_{jk} \equiv \frac{p_j - q_j}{p_j - q_k}, \quad j,k = 1,\dots,N$$

$$D \equiv \mathrm{diag}(\lambda_1^x, \dots, \lambda_N^x), \quad \lambda_j^x \equiv \lambda_j \exp\Big(\sum_{\ell=1}^{\infty} x_\ell (p_j^\ell - q_j^\ell) \Big)$$

(τ is positive for suitable $p, q, \lambda \in C^N$.)

Reductions of the KP hierarchy lead to various soliton PDEs and lattices. For instance, when τ does not depend on $x_{k(n+1)}$, $k=1,2,\dots$, one obtains τ-functions for:

$n=1$: KdV, $n=2$: Boussinesq, $n=3$: Hirota/Satsuma, ...

In particular, one gets the soliton τ-functions of the latter equations by taking $q_j = p_j \exp(2\pi i/(n+1))$, $j=1,\dots,N$. These solitons correspond to the II_{rel} ($\tau = \pi/(n+1)$) systems of A3(iv) in the same way as detailed for $n=1$ in C1 (iii).

N.B. 1) The above KP solitons have unphysical features: They are not localized and collide for all time. Yet, they model various wave phenomena quite well.

2) Taking $y \to iy$ one gets $-u_{yy}$ at the lhs of (KP). Then there is no choice of p, q, λ such that the above soliton functions $u(x, iy, t)$ are real and non-singular, but now there are real and regular solitons that are $O((x^2+y^2)^{-1})$ at ∞. However, the latter behave as free particles (no scattering).

CONTENTS CLASSICAL SOLITON SYSTEMS