Theoretical and Mathematical Physics, Vol. 128, No. 1, pp. 933–945, 2001

SELF-ADJOINT AΔOs WITH VANISHING REFLECTION

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We review our work concerning ordinary linear second-order analytic difference operators (AΔOs) that admit reflectionless eigenfunctions. This operator class is far more extensive than the reflectionless Schrödinger and Jacobi operators corresponding to KdV and Toda lattice solitons. A subclass of reflectionless AΔOs, which generalizes the latter class of differential and discrete difference operators, is shown to correspond to the soliton solutions of a nonlocal Toda-type evolution equation. Further restrictions give rise to AΔOs with isometric eigenfunction transformations, which can be used to associate self-adjoint operators on $L^2(\mathbb{R}, dx)$ with the AΔOs.

1. Introduction

In [1], we used previous findings concerning reflectionless analytic difference operators (AΔOs) of the relativistic Calogero–Moser type [2, 3] as evidence for the conjectured existence of a much larger class of reflectionless AΔOs. In this paper, we sketch an affirmative answer to our existence conjecture together with partial answers to related conjectures concerning self-adjointness issues and associated solitonic evolution equations. Detailed proofs can be found in [3–6].

We begin with some simple observations regarding the "free" AΔO

$$A_0 \equiv e^{-i\partial_x} + e^{i\partial_x} \tag{1.1}$$

viewed as a linear operator on the space $\mathcal{M}$ of meromorphic functions. Evidently, this operator has eigenfunctions $\mathcal{W}_0(x, \pm p)$ with eigenvalues $e^p + e^{-p}$, where

$$\mathcal{W}_0(x,p) \equiv e^{ixp}.$$

Now let $\mu_\pm(x,p)$ be functions that are meromorphic and i-periodic in x for arbitrary $p \in \mathbb{C}$. It is then clear that

$$\mathcal{W}_0^{\mu_+,\mu_-}(x,p) \equiv \mu_+(x,p)\mathcal{W}_0(x,p) + \mu_-(x,p)\mathcal{W}_0(x,-p)$$

is also an A_0-eigenfunction with the eigenvalue $e^p + e^{-p}$. As a consequence, the eigenspaces of the second-order AΔO A_0 are infinite-dimensional, in sharp contrast to the two-dimensionality of the eigenspaces of the second-order differential operator ∂_x^2. Moreover, the multiplier freedom can be used to construct infinite-dimensional eigenspaces of eigenfunctions in $L^2(\mathbb{R}, dx)$ (such as $\mathcal{W}_0(x,p)/\cosh 2\pi(x - x_0)$, $x_0 \in \mathbb{R}$) and to obtain eigenfunctions with an arbitrarily prescribed plane-wave asymptotic behavior as $|\operatorname{Re} x| \to \infty$. For example, taking

$$\mu_-(x,p) \equiv \frac{b(p)e^{-2\pi x}}{e^{2\pi x} + e^{-2\pi x}}, \qquad \mu_+(x,p) \equiv \frac{e^{2\pi x} + a(p)e^{-2\pi x}}{e^{2\pi x} + e^{-2\pi x}}, \tag{1.2}$$

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Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 128, No. 1, pp. 116–132, July, 2001.

we can readily verify that

$$\mathcal{W}_0^{\mu_+,\mu_-}(x,p) \sim \begin{cases} e^{ixp}, & \operatorname{Re} x \to \infty, \\ a(p)e^{ixp} + b(p)e^{-ixp}, & \operatorname{Re} x \to -\infty. \end{cases}$$

Elaborating slightly on the last example, we consider a reflectionless asymptotic function $b(p) = 0$ together with $|a(p)| = 1$, $p \in \mathbb{R}$. We now consider the Hilbert space operator

$$\mathcal{F}_{\mu_+}: \quad L^2(\mathbb{R}, dp) \to L^2(\mathbb{R}, dx), \qquad f(p) \mapsto (2\pi)^{-1/2} \int_{-\infty}^{\infty} dp\, f(p)\mu_+(x,p)e^{ixp}. \tag{1.3}$$

Of course, for $\mu_+(x,p)$ given by (1.2) and $a(p) = 1$, this amounts to the Fourier transformation $\mathcal{F}_0$. Because $\mathcal{F}_0$ is unitary, we can associate a self-adjoint operator $\hat{A}_0$ with A_0 as follows:

$$\hat{A}_0 \equiv \mathcal{F}_0 M \mathcal{F}_0^{-1},$$

where M is the operator of multiplication by $2\cosh p$ on $L^2(\mathbb{R}, dp)$. In the free case considered here, this is the natural way to view A_0 as a self-adjoint operator on $L^2(\mathbb{R}, dx)$. However, there exists an infinite-dimensional space of meromorphic functions $a(p)$ with $|a(p)| = 1$ for real p and associated i-periodic meromorphic multipliers satisfying

$$\mu(x,p) \sim \begin{cases} 1, & \operatorname{Re} x \to \infty, \\ a(p), & \operatorname{Re} x \to -\infty, \end{cases} \tag{1.4}$$

such that the operators $\mathcal{F}_\mu$ given by (1.3) are unitary as well. Taking this assertion for granted, we obtain an infinite-dimensional space of self-adjoint operators

$$\hat{A}_\mu \equiv \mathcal{F}_\mu M \mathcal{F}_\mu^{-1} \tag{1.5}$$

associated with the free AΔO A_0 on the space $\mathcal{M}$. These operators can be compared to the "obvious" free operator $\hat{A}_0$ in the sense of scattering theory, the unsurprising result being that the S-matrix is nontrivial and is (essentially) given by $a(p)$.

The existence of such a space of meromorphic functions can be substantiated via the results in Sec. 5 in [3]. At the end of the present paper, we mention an explicit example of an operator $\hat{A}_\mu$ as just described; more generally, we detail the relation between the present framework and [3]. We only add here that the pertinent multipliers $\mu(x,p)$ are not of the simple form $\mu_+(x,p)$ given by (1.2). Indeed, using the tools developed in [3, 6], it is easy to see that $\mathcal{F}_{\mu_+}$ is not isometric for the latter type of multipliers (unless $a(p) = 1$ and therefore $\mathcal{F}_{\mu_+} = \mathcal{F}_0$).

We have begun by giving some facts pertaining to the "free" choice A_0 in order to prepare for an appraisal of the corresponding features for the class of "interacting" AΔOs of the form

$$A \equiv e^{-i\partial_x} + V_a(x)e^{i\partial_x} + V_b(x). \tag{1.6}$$

We assume here that $V_a, V_b \in \mathcal{M}$ and

$$\lim_{|\operatorname{Re} x| \to \infty} V_a(x) = 1, \qquad \lim_{|\operatorname{Re} x| \to \infty} V_b(x) = 0. \tag{1.7}$$

Therefore, A reduces to A_0 for $|\operatorname{Re} x| \to \infty$.

The first natural question is whether the AΔO A admits eigenfunctions with the eigenvalue $e^p + e^{-p}$ and plane-wave asymptotic behavior for $|\operatorname{Re} x| \to \infty$. More specifically, do functions $\mathcal{W}(\,\cdot\,, p) \in \mathcal{M}$ exist that satisfy

$$(A\mathcal{W})(x,p) = (e^p + e^{-p})\mathcal{W}(x,p), \tag{1.8}$$

$$\mathcal{W}(x,p) \sim \begin{cases} e^{ixp}, & \operatorname{Re} x \to \infty, \\ a(p)e^{ixp} + b(p)e^{-ixp}, & \operatorname{Re} x \to -\infty, \end{cases} \tag{1.9}$$

for arbitrary $p \in \mathbb{C}$?

This question can be sharpened into a question that is more pertinent for our present purposes, namely, can *reflectionless* functions $\mathcal{W}(x,p)$ with the latter properties (i.e., $b(p) = 0$ in (1.9)) be found. We immediately stress that if such functions can be found, then the function $a(p)$ is arbitrary. Indeed, it can be changed at will via multiplication of $\mathcal{W}(x,p)$ by suitable i-periodic functions $\mu(x,p)$.

A third question is at issue: if A admits reflectionless eigenfunctions, can they be used to associate a self-adjoint operator $\hat{A}$ on $L^2(\mathbb{R}, dx)$ with A (along the lines already sketched for A_0)? In our opinion, this functional-analytic question is the critical one. Indeed, in the cases where $\mathcal{W}(x,p)$ yields an isometric eigenfunction transformation, the isometry is typically destroyed after multiplication by a nontrivial $\mu(x,p)$.

Before elaborating on these general questions, it may be illuminating to recall that for Schrödinger and Jacobi operators, the second and third questions admit complete answers provided by the inverse scattering transform (IST). In these two cases, there is no ambiguity in the Hilbert space operators to be associated with the given differential and discrete difference operators. Indeed, with appropriate reality restrictions and asymptotic requirements paralleling (1.7), the pertinent Jacobi operators are manifestly bounded and self-adjoint on $l^2(\mathbb{Z})$, whereas any Schrödinger operator $-\partial_x^2 + V(x)$ is obviously self-adjoint on the natural (Sobolev space) domain of ∂_x^2 whenever $V(x)$ is bounded and real-valued. The reflectionless eigenfunctions thus yield eigenfunction transformations giving an explicit realization of the spectral theorem for unambiguously defined self-adjoint operators on $l^2(\mathbb{Z})$ and $L^2(\mathbb{R}, dx)$ respectively.

As is clear from our discussion of free operator (1.1), the situation is vastly different for AΔOs. Returning to our three questions, we first note that the first two questions have not been addressed before. Our results entail the existence of an extensive class of AΔOs for which the second question has an affirmative answer. But we consider it quite unlikely that this yields the most general class of AΔOs admitting reflectionless eigenfunctions.

Similarly, we partially answer the third question by showing that under quite restrictive additional conditions, our reflectionless eigenfunction transformations $\mathcal{F}$ are isometries and can therefore be used to associate a self-adjoint operator $\hat{A}$ on $L^2(\mathbb{R}, dx)$ with A by pulling back the multiplication operator M on $L^2(\mathbb{R}, dp)$. More precisely, this procedure suffices whenever $\mathcal{F}$ maps onto $L^2(\mathbb{R}, dx)$. This is the case for an infinite-dimensional space of potentials (V_a, V_b), again in sharp contrast to the Schrödinger and Jacobi cases, where nontrivial reflectionless potentials *always* have bound states.

In the case where $\mathcal{F}$ is not onto the entire space $L^2(\mathbb{R}, dx)$, we can show that the orthocomplement of $\mathcal{F}(L^2(\mathbb{R}, dp))$ is spanned by finitely many pairwise orthogonal A-eigenfunctions with real eigenvalues, and $\hat{A}$ can therefore be defined to be equal to A on this bound-state subspace. After introducing a suitable step-size scaling in the AΔOs, a subclass of our operators can be shown to converge to the class of reflectionless Schrödinger operators as the step size tends to zero. We mention the latter result (which is detailed in Sec. 3 in [5]) in order to make it clear at the outset that the class of self-adjoint reflectionless AΔOs we construct is quite large.

Even so, it may well be that that a given AΔO A in this class can yield quite different self-adjoint reflectionless operators on $L^2(\mathbb{R}, dx)$. (As explained above, for the special AΔO A_0 given by (1.1), there does exist an infinite-dimensional space of such operators.) More generally, our results should be helpful in studying the direct problem.

In summary, our results can be viewed as exposing the beginnings of a possibly quite rich Hilbert space theory for AΔOs of the above simple form. From the concrete cases we are able to handle, it transpires that a key problem in this area is to single out *isometric* eigenfunction transformations.

This paper is organized as follows. In Sec. 2, we sketch our construction of a vast class of AΔOs of form (1.6) that admit reflectionless eigenfunctions, which are explicitly constructed as well. We present complete details in the simplest case ($N = 1$). The key idea is to mimic the IST scheme for reflectionless Schrödinger and Jacobi operators [7–11].

In Sec. 3, we explain the connection of our reflectionless AΔOs to a novel solitonic evolution equation, which can be viewed as an analytic, nonlocal version of the infinite Toda lattice equation. We obtain real-valued N-soliton solutions via a suitable restriction on the spectral data.

With the latter restriction in force, we consider functional-analytic properties in Sec. 4. We develop the special case $N = 1$ in some detail and sketch how our previous work [1–3] fits into the more general framework at hand.

2. Constructing reflectionless AΔOs

We start from "spectral data"

$$(r, \mu) = (r_1, \ldots, r_N, \mu_1(x), \ldots, \mu_N(x)), \quad N \in \mathbb{N}^*, \tag{2.1}$$

restricted as follows. The complex numbers $r_1, \ldots, r_N$ satisfy

$$\operatorname{Im} r_n \in (-\pi, 0) \cup (0, \pi), \quad n = 1, \ldots, N, \tag{2.2}$$

and

$$e^{r_m} \neq e^{\pm r_n}, \quad 1 \leq m < n \leq N. \tag{2.3}$$

The functions $\mu_1(x), \ldots, \mu_N(x)$ ("normalization coefficients") are allowed to be meromorphic functions satisfying

$$\mu_n(x + i) = \mu_n(x), \qquad \lim_{|\operatorname{Re} x| \to \infty} \mu_n(x) = c_n, \quad c_n \in \mathbb{C}^*, \quad n = 1, \ldots, N. \tag{2.4}$$

(The analogue of the Schrödinger and Jacobi operators arises for the special case $\mu_n(x) = c_n$, $n = 1, \ldots, N$, which is included of course.)

The restrictions on r ensure that the Cauchy matrix

$$C(r)_{mn} \equiv \frac{1}{e^{r_m} - e^{-r_n}}, \quad m, n = 1, \ldots, N,$$

is well defined and regular, as in the case of the Schrödinger and Jacobi operators. Next, we define the diagonal matrix

$$D(r, \mu; x) \equiv \operatorname{diag}(d_1(x), \ldots, d_N(x)),$$

where we use the notation

$$d_n(x) = d(r_n, \mu_n; x), \quad n = 1, \ldots, N,$$

with

$$d(\rho, \nu; x) \equiv \begin{cases} \nu(x)e^{-2i\rho x}, & \operatorname{Im}\rho \in (0, \pi), \\ \nu(x)e^{-2i(\rho+i\pi)x}, & \operatorname{Im}\rho \in (-\pi, 0). \end{cases} \tag{2.5}$$

All the remaining quantities are now defined via the solution $R(r, \mu; x)$ of the system

$$(D(r, \mu; x) + C(r))R = \zeta, \quad \zeta \equiv (1, \dots, 1)^{\mathrm{t}}. \tag{2.6}$$

In terms of the auxiliary functions

$$\lambda(r, \mu; x) \equiv 1 + \sum_{n=1}^{N} e^{r_n} R_n(r, \mu; x),$$

$$\Sigma(r, \mu; x) \equiv \sum_{n=1}^{N} R_n(r, \mu; x),$$

the potentials V_a and V_b with the asymptotic behavior given by (1.7) that define A via (1.6) are

$$V_a(r, \mu; x) \equiv \frac{\lambda(r, \mu; x)}{\lambda(r, \mu; x+i)}, \tag{2.7}$$

$$V_b(r, \mu; x) \equiv \Sigma(r, \mu; x-i) - \Sigma(r, \mu; x). \tag{2.8}$$

Moreover, the wave function satisfying (1.8) is given by

$$\mathcal{W}(r, \mu; x, p) \equiv e^{ixp}\left(1 - \sum_{n=1}^{N} \frac{R_n(r, \mu; x)}{e^p - e^{-r_n}}\right). \tag{2.9}$$

It has asymptotic behavior (1.9) with $b(p) = 0$ and

$$a(p) = \prod_{n=1}^{N} \frac{e^p - e^{r_n}}{e^p - e^{-r_n}}.$$

For later use, we also mention the asymptotic behavior

$$\lambda(x) \sim \begin{cases} 1, & \operatorname{Re} x \to \infty, \\ \exp\left(2\sum_{n=1}^{N} r_n\right), & \operatorname{Re} x \to -\infty. \end{cases} \tag{2.10}$$

Of course, neither the asymptotic properties nor the eigenvalue assertion is obvious. They follow from a detailed analysis of the $N \times N$ linear system given by (2.6) (see Sec. 2 in [4]). Here, we only develop the N=1 case. In this special case, the pertinent asymptotic behavior follows immediately, but the proof of the eigenvalue property is already nontrivial. Moreover, it is quite instructive, showing the crux of the argument for the case of arbitrary N and pointing the way to further generalizations.

Accordingly, for $N = 1$, we must consider two cases: $\operatorname{Im} r > 0$ and $\operatorname{Im} r < 0$. In the first case, we obtain

$$C = \frac{1}{2\sinh r}, \qquad d(x) = \mu(x)e^{-2irx}, \quad \operatorname{Im} r \in (0, \pi),$$

$$\mu(x) \in \mathcal{M}, \qquad \mu(x+i) = \mu(x), \qquad \lim_{|\operatorname{Re} x| \to \infty} \mu(x) = c \in \mathbb{C}^*.$$

We therefore have

$$R(x) = \left[\mu(x)e^{-2irx} + \frac{1}{2\sinh r}\right]^{-1} \sim \begin{cases} 0, & \operatorname{Re} x \to \infty, \\ 2\sinh r, & \operatorname{Re} x \to -\infty. \end{cases} \tag{2.11}$$

Hence, it follows from (2.7) and (2.8) that

$$V_a(x) = \frac{1 + e^r R(x)}{1 + e^r R(x+i)} \sim 1, \qquad |\operatorname{Re} x| \to \infty,$$

$$V_b(x) = R(x-i) - R(x) \sim 0, \quad |\operatorname{Re} x| \to \infty.$$

Similarly, from (2.9), we have

$$\mathcal{W}(x,p) = e^{ixp}\left(1 - \frac{R(x)}{e^p - e^{-r}}\right) \sim \begin{cases} e^{ixp}, & \operatorname{Re} x \to \infty, \\ \dfrac{e^p - e^r}{e^p - e^{-r}} e^{ixp}, & \operatorname{Re} x \to -\infty, \end{cases} \tag{2.12}$$

as announced.

In the second case $\operatorname{Im} r \in (-\pi, 0)$, the only difference from the first case is changes in the formulas for $d(x)$ and $R(x)$:

$$R(x) = \left[\mu(x)e^{-2irx+2\pi x} + \frac{1}{2\sinh r}\right]^{-1}, \quad \operatorname{Im} r \in (-\pi, 0). \tag{2.13}$$

We now prove the A-eigenfunction property, handling both cases at once. We must show that $\mathcal{W}(x,p)$ satisfies the second-order analytic difference equation

$$F(x-i) + V_a(x)F(x+i) + [V_b(x) - e^p - e^{-p}]F(x) = 0. \tag{2.14}$$

Clearly, it suffices to prove that the auxiliary wave function

$$\mathcal{A}(x,p) \equiv (e^p - e^{-r})\mathcal{W}(x,p) = e^{ixp}(e^p - e^{-r} - R(x)) \tag{2.15}$$

satisfies (2.14). To do this, we substitute $F(x) = \mathcal{A}(x,p)$ in the left-hand side of (2.14) and obtain a function $\mathcal{D}(x,p)$ of the form

$$\mathcal{D}(x,p) = e^{ixp}\big(e^p c_1(x) + c_0(x) + e^{-p} c_{-1}(x)\big).$$

For this function to vanish, it is obviously necessary and sufficient that the coefficients c_1, c_0, and c_{-1} vanish. We now readily verify the equivalences

$$c_1(x) = 0 \quad \Leftrightarrow \quad V_b(x) = R(x-i) - R(x),$$

$$c_{-1}(x) = 0 \quad \Leftrightarrow \quad V_a(x) = \frac{1 + e^r R(x)}{1 + e^r R(x+i)}.$$

Because we have *defined* V_a and V_b such that the equalities in the right-hand sides hold, we can infer that c_1 and c_{-1} vanish.

The nontrivial claim is that $c_0(x)$ also vanishes. Even in this quite simple case, verifying this directly already involves a substantial calculation. But we can avoid this direct verification by appealing to a uniqueness argument that generalizes to the case of arbitrary N. Specifically, we first note that because $R(x)$ satisfies the system given by (2.6) for N=1, it follows from (2.15) and (2.5) that

$$\begin{aligned}\mathcal{A}(x,r) &= e^{irx}(e^r - e^{-r})(1 - CR(x)) = \\ &= e^{irx}(e^r - e^{-r})d(x)R(x) = \alpha(x)\mathcal{A}(x,-r), \\ \alpha(x+i) &= \alpha(x).\end{aligned} \tag{2.16}$$

The key point here is that a function

$$G(x,p) = e^{ixp}(e^p + c(x)) \tag{2.17}$$

satisfying the relation

$$G(x,r) = \alpha(x)G(x,-r) \tag{2.18}$$

for a given i-periodic function $\alpha(x)$ is *unique*. Indeed, substituting (2.17) in (2.18) shows that $c(x)$ is uniquely determined. To exploit this, we recall that we have already shown that both $\mathcal{A}(x,p)$ and $\mathcal{A}(x,p) - \mathcal{D}(x,p)$ have form (2.17). Also, $\mathcal{A}$ satisfies (2.18) in view of (2.16). Now, $\mathcal{D}$ arises from the substitution $F(x) \to \mathcal{A}(x,p)$ in the left-hand side of (2.14), and because $\alpha(x)$ is i-periodic, $\mathcal{D}$ also satisfies (2.18). But then both $\mathcal{A}$ and $\mathcal{A} - \mathcal{D}$ satisfy (2.18), and therefore $\mathcal{A} = \mathcal{A} - \mathcal{D}$ by uniqueness. Hence, $\mathcal{D}(x,p) = 0$, which completes the proof.

We stress that the assumption $\mu(x) \to c \in \mathbb{C}^*$ as $|\operatorname{Re} x| \to \infty$ is not used in this proof (but the i-periodicity of $\mu(x)$ is used). As a consequence, we can enlarge the above class of reflectionless AΔOs for $N = 1$ by relaxing the requirements on $\mu(x)$. For example, we can readily verify that with the choice $\mu(x) \equiv c_1 + c_2 \tanh \pi(x - x_0)$, we obtain the same conclusions. Choosing $\mu(x) \equiv c_1 + c_2 \cosh 4\pi(x - x_0)$ instead, we obtain $R(x) \to 0$ as $|\operatorname{Re} x| \to \infty$ and therefore (1.7)–(1.9) with $a(p) = 1$ and $b(p) = 0$.

The latter example shows that an *interacting* AΔO A given by (1.6) (i.e., with potentials $(V_a, V_b) \neq (1, 0)$ satisfying (1.7)) may admit eigenfunctions $\mathcal{W}(x,p)$ with *trivial* scattering. We have already pointed out that the *free* AΔO A_0 given by (1.1) admits eigenfunctions that yield *nontrivial* scattering and a *unitary* eigenfunction transformation. Any general theory starting from (1.6) and (1.7) must take such phenomena, which have no counterparts for Schrödinger and Jacobi operators, into account.

Our aim in [4–6] was to isolate a *special* class of reflectionless AΔOs, whose properties can be determined in great detail. Restrictions (2.1)–(2.4) on the spectral data determine this special class, which is already quite large and which can be studied systematically from an algebraic standpoint (see [4]). For later purposes, we note two general aspects of the objects in this class.

The first is that the presence and location of eventual poles of $\mathcal{W}(x,p)$ in x is governed by the presence and location of zeros of the multipliers $\mu_1(x), \ldots, \mu_N(x)$ and of the τ-function

$$\tau(x) \equiv |\mathbf{1}_N + CD(x)^{-1}|. \tag{2.19}$$

Indeed, solving (2.6) by Cramer's rule, we can see this from (2.9).

The second consists of the conditions encoding *formal* self-adjointness of the AΔO A given by (1.6) on $L^2(\mathbb{R}, dx)$, i.e., without regard to the domains of definition, singularities, etc. Clearly, we need $V_b(x)$ to be real-valued for real x, and we also need that $V_a(x)\exp(i\partial_x)$ be equal to

$$[V_a(x)e^{i\partial_x}]^* = e^{i\partial_x}\overline{V_a(x)} = \overline{V_a(x-i)}e^{i\partial_x}, \quad x \in \mathbb{R}.$$

With the notation

$$f^*(x) \equiv \overline{f(\overline{x})}, \quad x \in \mathbb{C}, \quad f \in \mathcal{M},$$

formal self-adjointness therefore amounts to

$$V_b^*(x) = V_b(x), \qquad V_a^*(x) = V_a(x-i).$$

In turn, this can be shown to be equivalent to

$$\mathrm{Re}(r_n) = 0, \qquad\qquad n = 1, \dots, N, \tag{2.20}$$

$$\mathrm{Re}(e^{-r_n}\mu_n(x)) = 0, \quad n = 1, \dots, N, \quad x \in \mathbb{R}, \tag{2.21}$$

(see Appendix D in [4]). (For $N = 1$, these assertions can be easily verified directly from the explicit formulas above.)

3. A related nonlocal Toda-type soliton equation

To relate the above objects to a solitonic evolution equation, we must introduce a suitable time dependence in the normalization coefficients $\mu(x)$. More precisely, this is the procedure that yields the soliton solutions of the KdV and Toda lattice equations in the case of reflectionless Schrödinger and Jacobi operators. Taking a hint from the situation with the latter discrete difference operators, we define the time evolution by

$$\mu_n(x) \to \mu_n(x)e^{2it\sinh r_n}, \quad n = 1, \dots, N.$$

With this substitution, the quantities R, V_a, V_b, $\mathcal{W}$, and τ all depend on t. However, we usually suppress this time dependence for simplicity of notation. Once again exploiting properties following solely from (now time-dependent) system (2.6), we can now prove the equations

$$\dot{V}_a(x) = iV_a(x)[V_b(x+i) - V_b(x)], \qquad \dot{V}_b(x) = i[V_a(x) - V_a(x-i)],$$

$$\dot{\mathcal{W}}(x,p) = (B\mathcal{W})(x,p) + ie^p\mathcal{W}(x,p), \qquad B \equiv -i(e^{-i\partial_x} + V_b(x)),$$

and

$$\ddot{\tau}(x)\tau(x) - \dot{\tau}(x)^2 = \tau(x)^2 - \tau(x+i)\tau(x-i). \tag{3.1}$$

(See Sec. 2 in [5]; these equations can be easily verified directly for $N = 1$.)

Clearly, (3.1) is a Hirota-type bilinear equation. Rewriting it as

$$\partial_t^2 \log\tau(x) = 1 - \frac{\tau(x+i)\tau(x-i)}{\tau(x)^2}$$

and introducing

$$\Psi(x) \equiv i\log\frac{\tau(x-i)}{\tau(x)},$$

we now readily find that Ψ satisfies the nonlocal Toda-type equation

$$\ddot{\Psi}(x) = ie^{i[\Psi(x+i)-\Psi(x)]} - ie^{i[\Psi(x)-\Psi(x-i)]}.$$

(Such nonlocal evolution equations have been encountered before; see, e.g., Santini's review [12] and the references therein.)

The requirement that the function $\Psi(x,t)$ be real-valued for real x and t can be shown to be satisfied when the AΔO $A(r,\mu(x,t))$ is formally self-adjoint, i.e., when (2.20) and (2.21) hold. (The point is that this ensures that $\tau(x-i/2)$ is real-valued for real x.)

We now consider the $N{=}1$ solution with $\operatorname{Im} r \in (0,\pi)$

$$\Psi(x,t) = i \log \frac{1 + (2\sinh r)^{-1}(\mu(x))^{-1} e^{2ir(x-i)-2it\sinh r}}{1 + (2\sinh r)^{-1}(\mu(x))^{-1} e^{2irx-2it\sinh r}}.$$

Unless $\mu(x)$ is constant, this function is not of the traveling-wave form $f(x-vt)$. However, setting

$$r = i\alpha, \quad \alpha \in (0,\pi), \qquad \mu(x) = \frac{e^{i\alpha}}{2\sinh(i\alpha)} e^{-2\alpha x_0}, \quad x_0 \in \mathbb{R}, \tag{3.2}$$

we obtain a right-moving kink-type soliton

$$\Psi(x,t) = i \log \frac{1 + e^{i\alpha} e^{-2\alpha[(x-x_0)-v(\alpha)t]}}{1 + e^{-i\alpha} e^{-2\alpha[(x-x_0)-v(\alpha)t]}}, \qquad v(\alpha) \equiv \frac{\sin\alpha}{\alpha}. \tag{3.3}$$

Choosing

$$r = i\alpha - i\pi, \quad \alpha \in (0,\pi), \qquad \mu(x) = -\frac{e^{i\alpha}}{2\sinh(i\alpha)} e^{-2\alpha x_0}, \quad x_0 \in \mathbb{R}, \tag{3.4}$$

we similarly obtain (3.3) with $t \to -t$ in the right-hand side, i.e., a left-moving soliton.

Turning to the case of arbitrary N, we choose spectral data of form (3.2) for $N_+ \in \{0,1,\dots,N\}$ numbers among $r_1,\dots,r_N$ and of form (3.4) for the $N_- \equiv N - N_+$ remaining ones (this choice guarantees that (2.20) and (2.21) are satisfied, and $\Psi(x,t)$ is therefore real-valued for real x and t). Then $\Psi(x,t)$ can be viewed as an N-soliton solution: its large-time asymptotic behavior involves the N_+ right-moving and N_- left-moving 1-kink solutions detailed above.

It is not obvious that the latter assertion is valid, but it can be proved rather easily in a weak form (see Proposition 6.1 in [5]). A more detailed analysis (including a study of soliton space–time trajectories) hinges on a somewhat intricate reparametrization of the $2N$ real numbers $\alpha_1,\dots,\alpha_N, x_{0,1},\dots,x_{0,N}$ in terms of which $\Psi(x,t)$ is defined. This reparametrization is also crucial for studying self-adjointness issues. We now describe it in general terms (the details can be found in Sec. 5 in [5]). For this, we recall that the well-known Calogero–Moser N-particle systems admit a generalization to a relativistic setting [13]. Just as in the nonrelativistic case, there is a version describing N_+ particles and N_- antiparticles, in the sense that a particle and an antiparticle have an attractive interaction, whereas two particles or two antiparticles repel each other. We studied this version in considerable detail [14], together with the particle-like solutions of the KdV, sine-Gordon, and modified KdV equations. Indeed, the latter soliton-type solutions can also be parametrized by the relativistic N-particle systems, a soliton–particle correspondence that can be exploited to great advantage (see Chap. 7 in [14]).

The reparametrization needed for the above nonlocal Toda solitons entails that τ-function (2.19) is related to the Lax matrix L of the particle systems via

$$\tau\left(x - \frac{i}{2}, t\right) = |\mathbf{1}_N + L(x,t)|, \quad x,t \in \mathbb{R}. \tag{3.5}$$

Here, $L(x,t)$ denotes the Lax matrix evaluated in an (x,t)-dependent point of the pertinent $2N$-dimensional phase space. Unless N_+ or N_- vanishes, this is a novel type of relation. Indeed, in previous cases the soliton

τ-functions were related to the *dual* Lax matrices of the particle systems (see also [15]). Unfortunately, in the present case, the analytic consequences of the soliton–particle correspondence (as expressed in (3.5)) seem less powerful than before. In particular, even when N_+ or N_- vanishes, we are unable to obtain uniform bounds on the large-time asymptotic behavior. Moreover, clear-cut soliton space–time trajectories exist only for sufficiently large times (unless N_+ or N_- vanishes).

4. Isometry and self-adjointness issues

Retaining the above choice of spectral data that yields N-soliton solutions, we now study Hilbert space aspects (see [6] for proofs of the assertions made below). The key property we need for establishing orthogonality and completeness is the absence of poles of $\mathcal{W}(x,p)$ in the strip $\operatorname{Im} x \in [-1,0]$. We first explain how relation (3.5) with $t=0$ can be used to study this pole issue. For this, we observe that the absence of poles in the critical strip is guaranteed by the absence of zeros of $\tau(x)$ in the same strip. Indeed, this follows from the paragraph containing (2.19) (we recall that $\mu_1(x),\ldots,\mu_N(x)$ are constants in this section). By virtue of (3.5), no such zeros occur whenever the spectrum of $L(x)$ does not contain the number -1 for $\operatorname{Im} x \in [-1/2, 1/2]$.

At this point, the spectral analysis of the Lax matrix in [14] can be invoked. Specifically, it can be used to deduce the absence of poles provided that

$$r_n \in i(-\pi,-\pi/2) \cup i(0,\pi/2), \quad n=1,\ldots,N. \tag{4.1}$$

This condition can be relaxed to

$$r_n \in i(0,\pi), \quad n=1,\ldots,N,$$

for $N=N_+$ and to

$$r_n \in i(-\pi,0), \quad n=1,\ldots,N,$$

for $N=N_-$. Equivalently, in the latter two special cases, we can allow arbitrary phase-space points as spectral data. It seems likely that (4.1) can also be considerably relaxed, but when $N_+N_- > 0$, there do exist phase-space points yielding τ-zeros in the critical strip.

Assuming from now on that the spectral data are such that $\tau(x)$ has no zeros in the strip, we now detail some salient Hilbert space features. First, the A-eigenfunctions $\mathcal{W}(x,r_n)$ with $r_n \in i(0,\pi)$ are in $L^2(\mathbb{R},dx)$ and are pairwise orthogonal. Second, the eigenfunction transformation

$$\mathcal{F}: \quad L^2(\mathbb{R},dp) \to L^2(\mathbb{R},dx), \qquad f(p) \mapsto (2\pi)^{-1/2} \int_{-\infty}^{\infty} dp \mathcal{W}(x,p) f(p),$$

is isometric. Third, the orthocomplement of the range $\mathcal{F}(L^2(\mathbb{R},dp))$ is spanned by the bound states $\mathcal{W}(x,r_n)$ with $r_n \in i(0,\pi)$ (in particular, for $N_+=0$, there are no bound states, and $\mathcal{F}$ is *unitary*).

These properties allow associating a self-adjoint operator $\hat{A}$ on $L^2(\mathbb{R},dx)$ with the AΔO A on $\mathcal{M}$ as follows. We define $\hat{A}$ as multiplication by $2\cosh r_n$ on the N_+ bound states $\mathcal{W}(x,r_n)$ with $r_n \in i(0,\pi)$. On the orthocomplement $\mathcal{F}(L^2(\mathbb{R},dp))$ of the bound-state subspace, we define $\hat{A}$ as the pullback of the self-adjoint multiplication operator M on $L^2(\mathbb{R},dp)$ with the domain $\mathcal{D}(M)$:

$$\hat{A}\mathcal{F}f \equiv \mathcal{F}Mf \quad \forall f \in \mathcal{D}(M).$$

For $f(p) \in C_0^\infty(\mathbb{R})$, it is then easy to verify that $(\mathcal{F}f)(x)$ belongs to $\mathcal{M}$ and that the action of $\hat{A}$ on $\mathcal{F}f$ coincides with the action of A, just as the action of $\hat{A}$ on the bound states.

It is instructive to consider the case $N = 1$ in detail. In particular, the different character of the choices $N = N_+$ and $N = N_-$ can be easily illustrated for $N = 1$. Taking $N = N_+$ (see (3.2)), we can set $x_0 = 0$ because x_0 is simply a translation parameter. We then calculate the wave function

$$\mathcal{W}_+(x,p) = e^{ixp}\frac{e^{i\alpha}\sinh(\alpha x + p/2) - \sinh(\alpha x - p/2)}{2\cosh(\alpha x + i\alpha/2)\sinh(p/2 + i\alpha/2)}, \quad N = N_+ = 1, \tag{4.2}$$

from (2.9) and (2.11). Similarly, the $N = N_-$ choice (3.4) together with (2.9) and (2.13) yields

$$\mathcal{W}_-(x,p) = e^{ixp}\frac{e^{i\alpha}\cosh(\alpha x + p/2) + \cosh(\alpha x - p/2)}{2\cosh(\alpha x + i\alpha/2)\cosh(p/2 + i\alpha/2)}, \quad N = N_- = 1. \tag{4.3}$$

From these explicit formulas, we obtain

$$\mathcal{W}_\pm(x,p) \sim e^{ixp}, \quad \operatorname{Re} x \to \infty,$$

and

$$\mathcal{W}_+(x,p) \sim e^{i\alpha}\frac{\sinh(p/2 - i\alpha/2)}{\sinh(p/2 + i\alpha/2)}e^{ixp}, \qquad \mathcal{W}_-(x,p) \sim e^{i\alpha}\frac{\cosh(p/2 - i\alpha/2)}{\cosh(p/2 + i\alpha/2)}e^{ixp}, \quad \operatorname{Re} x \to -\infty,$$

in accordance with (2.12). To get square integrability at $x = \infty$, we therefore need $\operatorname{Im} p > 0$. This restriction on p can only be compatible with square integrability at $x = -\infty$ if $p = i\alpha + 2ki\pi$, $k \in \mathbb{N}$, for $\mathcal{W}_+$ and $p = i\alpha + (2k+1)i\pi$, $k \in \mathbb{N}$, for $\mathcal{W}_-$. We now calculate

$$\mathcal{W}_+(x, i\alpha + 2ki\pi) = \frac{e^{-2k\pi x}}{e^{\alpha x} + e^{-i\alpha - \alpha x}}, \qquad \alpha \in (0,\pi), \quad k \in \mathbb{Z}, \tag{4.4}$$

$$\mathcal{W}_-(x, i\alpha + (2k+1)i\pi) = \frac{e^{-(2k+1)\pi x}}{e^{\alpha x} + e^{-i\alpha - \alpha x}}, \quad \alpha \in (0,\pi), \quad k \in \mathbb{Z}. \tag{4.5}$$

Therefore, we must choose $k = 0$ for (4.4) to yield a function in $L^2(\mathbb{R}, dx)$, whereas (4.5) does not yield a square-integrable function for any $k \in \mathbb{Z}$.

We can use (4.3) to illustrate another issue. Specifically, with

$$\mathcal{W}_-\left(\frac{\pi x}{\alpha}, \frac{\alpha p}{\pi}\right) = \mu(x,p)e^{ixp},$$

the function $\mu(x,p)$ is clearly i-periodic and satisfies

$$\mu(x,p) \sim \begin{cases} 1, & \operatorname{Re} x \to \infty, \\ \dfrac{e^{\alpha p/\pi} + e^{i\alpha}}{e^{\alpha p/\pi} + e^{-i\alpha}}, & \operatorname{Re} x \to -\infty; \end{cases}$$

therefore, it satisfies (1.4). Because $(2\pi)^{-1/2}\mathcal{W}_-(x,p)$ is the kernel of a unitary operator, this also holds for the scaled kernel $(2\pi)^{-1/2}\mu(x,p)e^{ixp}$. As a result, we obtain an example of an interacting self-adjoint Hilbert space operator $\hat{A}_\mu$ of form (1.5) associated with the free AΔO A_0 given by (1.1).

We next sketch how the conventions and results in [3] are related to the above. First, the parameter triple $(\hbar, \nu, \beta)$ used in [3] should be specialized to $(1, \alpha, 1/2)$. We note that the functions $\mathcal{W}_\pm(x,p)$ above can also be viewed as joint eigenfunctions of interacting AΔOs $A_\pm$ and the free AΔO

$$A_{\mathrm{f}} \equiv e^{i\pi\alpha^{-1}\partial_x} + e^{-i\pi\alpha^{-1}\partial_x}$$

with the eigenvalues $2\cosh p$ and $2\cosh(\pi p/\alpha)$. After a unitary similarity transformation, the eigenfunctions studied in [3] amount to wave functions $\mathcal{W}(x,p)$ of the form delineated in Sec. 2, one special feature being that all of them are joint eigenfunctions of A and A_{f} with the latter eigenvalues.

It is illuminating to detail the $N=1$ case before describing the situation with arbitrary N. For real x and p and for $N=1$, the functions $F_a(x,p)$ and $F_e(x,p)$ in Secs. 4 and 5 in [3] are given by

$$F_a(x,p) = e^{-i\alpha/2}\left(\frac{\cosh(\alpha x + i\alpha/2)}{\cosh(\alpha x - i\alpha/2)}\right)^{1/2}\left(\frac{\sinh(p/2 + i\alpha/2)}{\sinh(p/2 - i\alpha/2)}\right)^{1/2}\mathcal{W}_+(x,p),$$

$$F_e(x,p) = e^{-i\alpha/2}\left(\frac{\cosh(\alpha x + i\alpha/2)}{\cosh(\alpha x - i\alpha/2)}\right)^{1/2}\left(\frac{\cosh(p/2 + i\alpha/2)}{\cosh(p/2 - i\alpha/2)}\right)^{1/2}\mathcal{W}_-(x,p).$$

(The branch of the square root is fixed by requiring that the pertinent functions have the limit $e^{i\alpha/2}$ for $x,p\to\infty$.) The function $F_r(x,p)$ in Sec. 3 in [3] is similarly related to the wave function $\mathcal{W}_r(x,p)$ that corresponds to the spectral data

$$r = i\alpha, \qquad \mu(x) = -\frac{e^{i\alpha}}{2\sinh(i\alpha)}, \quad \alpha\in(0,\pi). \tag{4.6}$$

Indeed, we can readily verify that (cf. (4.2))

$$\mathcal{W}_r(x,p) = e^{\pi p/(2\alpha)}\mathcal{W}_+\left(x + \frac{i\pi}{2\alpha}, p\right) =$$

$$= e^{ixp}\frac{e^{i\alpha}\cosh(\alpha x + p/2) - \cosh(\alpha x - p/2)}{2\sinh(\alpha x + i\alpha/2)\sinh(p/2 + i\alpha/2)},$$

and then F_r is given by

$$F_r(x,p) = e^{-i\alpha/2}\left(\frac{\sinh(\alpha x + i\alpha/2)}{\sinh(\alpha x - i\alpha/2)}\right)^{1/2}\left(\frac{\sinh(p/2 + i\alpha/2)}{\sinh(p/2 - i\alpha/2)}\right)^{1/2}\mathcal{W}_r(x,p).$$

The sign change of μ in (4.6) as compared with the spectral data for $\mathcal{W}_+(x,p)$ has a drastic consequence: the eigenfunction transformation corresponding to $\mathcal{W}_r(x,p)$ is no longer isometric. On the other hand, the eigenfunction transformation with the kernel $(2\pi)^{-1/2}F_r(x,p)$ is an isometry from the odd subspace of $L^2(\mathbb{R},dp)$ onto the odd subspace of $L^2(\mathbb{R},dx)$ (see Sec. 3 in [3]).

More generally, parity issues play a crucial role in [3]. The above unitary transformations of the three operators A_+, A_-, and A_r associated with $\mathcal{W}_+$, $\mathcal{W}_-$, and $\mathcal{W}_r$ yield the parity-invariant AΔOs $H_a^2 - 2$, $H_e^2 + 2$, and $H_r^2 - 2$ with $N=1$ in [3]. In this connection, we note that our AΔOs A given by (1.6) are manifestly not parity invariant (unless $V_a(x)=1$ and $V_b(x)$ is even).

To relate to [3] for arbitrary N, a unitary similarity transformation is needed that generalizes the one for $N=1$ detailed above. To specify this transformation, we recall formula (2.7) defining $V_a(x)$. It entails that A can be rewritten as

$$A = e^{-i\partial_x} + \lambda(x)e^{i\partial_x}\lambda(x)^{-1} + V_b(x).$$

It is not obvious, but true, that formal self-adjointness requirements (2.20) and (2.21) entail $\lambda^*(x) = \lambda(x)^{-1}$. Therefore, $\lambda(x)$ is a phase factor for real x. The unitary similarity is now given by

$$A \to \tilde{A} \equiv \overline{\lambda(x)}^{1/2}A\lambda(x)^{1/2} =$$

$$= \left(\frac{\lambda(x-i)}{\lambda(x)}\right)^{1/2}e^{-i\partial_x} + \left(\frac{\lambda(x)}{\lambda(x+i)}\right)^{1/2}e^{i\partial_x} + V_b(x), \tag{4.7}$$

$$\mathcal{W}(x,p) \to \widetilde{\mathcal{W}}(x,p) \equiv \overline{\lambda(x)}^{1/2}\mathcal{W}(x,p). \tag{4.8}$$

The square roots are chosen such that we obtain the limit 1 for $x \to \infty$ (see (2.10)).

The parity-invariant operators $H_a^2 - 2$, $H_e^2 + 2$, and $H_r^2 - 2$ in [3] have form (4.7) for arbitrary N. But they form only a tiny subset of the operators $\tilde{A}$ at issue. Indeed, they correspond to spectral data $r_1, \ldots, r_N, \mu_1, \ldots, \mu_N$ that are already determined once α is fixed. Another point worth noting is that (4.8), (1.9), and (2.10) entail

$$\widetilde{\mathcal{W}}(x,p) \sim \begin{cases} e^{ixp}, & \operatorname{Re} x \to \infty, \\ \exp\left(-\sum_{n=1}^{N} r_n\right) a(p) e^{ixp}, & \operatorname{Re} x \to -\infty. \end{cases}$$

The transmission thus gains an extra phase.

The last few paragraphs are also relevant for a comparison with the scenario envisaged in [1]. With the results in [3] as a starting point, it seems natural to seek a class of reflectionless AΔOs of the form

$$\tilde{A} = V_+(x)^{1/2} e^{-i\partial_x} + V_-(x)^{1/2} e^{i\partial_x} + V_0(x), \tag{4.9}$$

with further properties specified in [1]. The results surveyed here can be adapted to the picture sketched in [1], but our present picture is somewhat different. Indeed, the altered starting point AΔO A given by (1.6) has *meromorphic* coefficients; there are no square root branch points as in $\tilde{A}$. To obtain associated self-adjoint operators on $L^2(\mathbb{R}, dx)$ that are parity invariant under suitable extra conditions, a unitary similarity transformation involving square roots of meromorphic functions must be allowed, as explained above. The parity properties arising in the quite specialized setting in [3] would have been invisible when working with AΔOs of form (1.6). At any rate, the class of AΔOs $\tilde{A}$ given by (4.9) (and specified in [1]) would also merit further consideration (we note that it involves three a priori independent potentials, in contrast to the two potentials in (1.6)).

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