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Relativistic Lamé functions: Completeness vs. polynomial asymptotics

Dedicated to Tom Koornwinder on the occasion of his 60th birthday

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ABSTRACT

In earlier work we introduced and studied two commuting generalized Lamé operators, obtaining in particular joint eigenfunctions for a dense set in the natural parameter space. Here we consider these difference operators and their eigenfunctions in relation to the Hilbert space $L^2((0, \pi/r), w(x)dx)$, with $r > 0$ and the weight function $w(x)$ a ratio of elliptic gamma functions. In particular, we show that the previously known pairwise orthogonal joint eigenfunctions need only be supplemented by finitely many new ones to obtain an orthogonal base. This completeness property is derived by exploiting recent results on the large-degree Hilbert space asymptotics of a class of orthonormal polynomials. The polynomials $p_n(\cos(rx)), n \in \mathbb{N}$, that are relevant in the Lamé setting are orthonormal in $L^2((0, \pi/r), w_P(x)dx)$, with $w_P(x)$ closely related to $w(x)$.

1. INTRODUCTION

In this paper we are primarily concerned with eigenfunctions of second order analytic difference operators with quite special elliptic coefficients. More specifically, the difference operators may be viewed as one-parameter generalizations of the Lamé operator [1]

$$(1.1) \qquad -\frac{d^2}{dx^2} + g(g-1)\wp(x),$$

where $\wp$ is the Weierstrass $\wp$-function and g a coupling constant. (The parameter can be physically interpreted as the speed of light [2].) Although our results pertain to these special difference operators (explicitly given by (1.25)–(1.32) below), they also have a bearing on more general questions that are to date wide open.

Indeed, only a few general results on existence and uniqueness of solutions to analytic difference equations are known, whereas we are not aware of any general Hilbert space theory of analytic difference operators. Therefore, feedback from explicit examples may be of considerable help in the search for a more comprehensive theory of analytic difference operators and their eigenfunctions. In keeping with this angle, we discuss the pertinent issues in a somewhat more general framework, specializing in several steps to the 'relativistic' Lamé case (1.32), and presenting elementary examples along the way to illustrate the domain problems that arise.

We begin by introducing a large class of analytic difference operators (from now on AΔOs), to which the generalized Lamé operators studied in this paper belong. The class involves elliptic functions with real period π/r and imaginary periods ia. More precisely, we fix

$$r, a_+, a_- \in (0, \infty), \tag{1.2}$$

and first consider operators of the form

$$\mathcal{A}_+ \equiv T_{ia_-} + \mathcal{E}_+(x) T_{-ia_-}, \tag{1.3}$$

where T_α denotes the translation

$$(T_\alpha F)(x) \equiv F(x - \alpha), \quad \alpha \in \mathbb{C}^*, \quad F \in \mathcal{M}, \tag{1.4}$$

with $\mathcal{M}$ the space of meromorphic functions, and where $\mathcal{E}_+(x)$ denotes an elliptic function with periods π/r, ia_+. We may and will view $\mathcal{A}_+$ as an operator on $\mathcal{M}$, which leaves $\mathcal{M}$ invariant.

Now suppose that $F \in \mathcal{M}$ solves the eigenvalue equation

$$\mathcal{A}_+ F = E_+ F, \quad E_+ \in \mathbb{C}. \tag{1.5}$$

(Observe that we restrict attention to meromorphic solutions.) Then $\mu(x)F(x)$ yields another solution for any $\mu \in \mathcal{M}$ with period ia_-, so that the solution space is infinite-dimensional (assuming $F \in \mathcal{M}^*$, of course).

On the other hand, whenever two solutions F_1, F_2 exist whose Casorati determinant

$$\mathcal{D}(F_1, F_2; x) \equiv F_1(x + ia_-/2)F_2(x - ia_-/2) - F_1(x - ia_-/2)F_2(x + ia_-/2) \tag{1.6}$$

does not vanish identically, the solution space is two-dimensional over the field $\mathcal{P}_{ia_-}$, where

$$\mathcal{P}_\alpha \equiv \{F \in \mathcal{M} \mid F(x + \alpha) = F(x)\}, \quad \alpha \in \mathbb{C}^*. \tag{1.7}$$

We include a short proof of this well-known result [3], since it involves ingredients we need below.

First, whenever F_1, F_2 are solutions with $F_1/F_2 \notin \mathcal{P}_{ia_-}$, their Casorati determinant satisfies the first order analytic difference equation (henceforth AΔE)

$$(1.8)\qquad \frac{\mathcal{D}(x+ia_-/2)}{\mathcal{D}(x-ia_-/2)}=\frac{1}{\mathcal{E}_+(x)},$$

as is easily checked. Second, assuming F_3 is a third solution, we have

$$(1.9)\qquad \mu_j(x)\equiv \mathcal{D}(F_3,F_j;x+ia_-/2)/\mathcal{D}(F_1,F_2;x+ia_-/2)\in \mathcal{P}_{ia_-},\quad j=1,2.$$

(Indeed, ia_--periodicity follows from (1.8).) Now one readily verifies the identity

$$(1.10)\qquad F_3(x)=\mu_1(x)F_2(x)-\mu_2(x)F_1(x),$$

which completes the proof.

Next, letting $a_+\neq a_-$ from now on, consider again a fixed solution $F(x)\in\mathcal{M}^*$. Since $\mathcal{E}_+(x)$ is ia_+-periodic, the functions $F_\pm(x)\equiv F(x\pm ia_+)$ are solutions, too. Assuming $F_+(x)/F(x)$ is not ia_--periodic, we deduce

$$(1.11)\quad F_-(x)=\mu_1(x)F_+(x)-\mu_2(x)F(x),\quad F_\pm(x)\equiv F(x\pm ia_+),\quad \mu_1,\mu_2\in\mathcal{P}_{ia_-}.$$

Viewing this relation as an additional AΔE satisfied by the given solution $F(x)$ of (1.5), it is an obvious question to ask whether the ia_--periodic 'monodromy coefficients' μ_1 and μ_2 can be *prescribed*. Equivalently, the problem is whether *joint* solutions to (1.5) and (1.11) exist when $\mu_1,\mu_2\in\mathcal{P}_{ia_-}$ are given.

We now specialize this question to a setting that is closer to our specific AΔOs (1.31) (although it is still far more general). Consider a second AΔO of the form

$$(1.12)\qquad \mathcal{A}_-\equiv T_{ia_+}+\mathcal{E}_-(x)T_{-ia_+},$$

with $\mathcal{E}_-(x)$ an elliptic function with periods $\pi/r, ia_-$. Then $\mathcal{A}_-$ commutes with $\mathcal{A}_+$, so we are naturally led to the question whether joint eigenfunctions exist. From the previous more general perspective, the second eigenvalue equation

$$(1.13)\qquad \mathcal{A}_-F=E_-F,\quad E_-\in\mathbb{C},$$

amounts to prescribing $\mu_1(x)=-\mathcal{E}_-(x)$ and $\mu_2(x)=-E_-$ in the monodromy equation (1.11).

To our knowledge, there is no information on these issues in the literature. As will be recalled below, in the relativistic Lamé case there exists a two-dimensional space of joint $\mathcal{A}_\pm$-eigenfunctions for a dense subset of the parameter space [4]. One of the new insights detailed in (Section 3 of) this paper is, however, that at most a one-dimensional subspace can be continuously interpolated to all of the parameter space.

The latter 'no-go' result has a function-theoretic flavor, whereas in most of the paper we address the question whether the commuting Lamé AΔOs $\mathcal{A}_\pm$ can be reinterpreted as commuting self-adjoint operators $\hat{\mathcal{A}}_\pm$ on the Hilbert space

$$(1.14)\qquad \mathcal{H}\equiv L^2((0,\pi/r),dx).$$

To be more precise, the first problem is to find a dense subspace D of $\mathcal{H}$ consisting of functions $F(x)$ that are restrictions to $(0,\pi/r)$ of meromorphic functions, and which is such that the meromorphic functions $(\mathcal{A}_\pm F)(x)$ belong to $\mathcal{H}$.

(Here and below, this means that the restrictions to $(0, \pi/r)$ belong to $\mathcal{H}$.) With such a subspace D given, one therefore obtains Hilbert space operators $\hat{\mathcal{A}}_\pm : D \to \mathcal{H}$. The second problem is whether the operators $\hat{\mathcal{A}}_\pm$ are symmetric. Assuming they are, the third problem is whether D is a core (domain of essential self-adjointness). Assuming D is a core, the fourth problem is to show that the self-adjoint closures of $\hat{\mathcal{A}}_\pm$ commute, in the sense that the associated time evolutions or resolvents commute [5]. Last but not least, the spectral properties of the self-adjoint operators should be elucidated.

Of course, whenever one can exhibit (or prove the existence of) joint eigenfunctions $F_n \in \mathcal{M}$ of the AΔOs $\mathcal{A}_\pm$ with real eigenvalues, which belong to $\mathcal{H}$ and are pairwise orthogonal and complete, then all of the above problems are trivialized by choosing D equal to the linear hull of the eigenfunctions F_n. (To ease the notation, we use the same notation for $F \in \mathcal{M}$ and for its restriction to $(0, \pi/r)$, the distinction always being clear from context.) The simplest situation in which this happens for the above AΔOs is when

$$\mathcal{E}_\delta(x) = c_\delta \in \mathbb{R}, \quad \delta = +, -. \tag{1.15}$$

Indeed, the functions

$$F_n^{(P)}(x) \equiv \exp(2inrx), \quad n \in \mathbb{Z}, \tag{1.16}$$

are then joint eigenfunctions with $\mathcal{A}_\pm$-eigenvalues $\exp(2nra_\mp) + c_\pm \exp(-2nra_\mp)$. This choice of domain amounts to reinterpreting the AΔO $T_{ia}, a \in \mathbb{R}$, as $\exp(a\hat{p})$, where $\hat{p}$ denotes the self-adjoint extension of the symmetric operator $-id/dx$ on $C_0^\infty((0, \pi/r))$ obtained by imposing periodic boundary conditions.

Turning to non-constant elliptic function coefficients, it seems quite unlikely that one can reinterpret the commuting AΔOs $\mathcal{A}_\pm$ as commuting self-adjoint operators on $\mathcal{H}$ unless one imposes at least formal self-adjointness. Interpreting as before the shifts $T_{\pm ia_\delta}$ as $\exp(\mp ia_\delta d/dx)$, this amounts to requiring

$$\mathcal{E}_+^*(x) = \mathcal{E}_+(x - ia_-), \quad \mathcal{E}_-^*(x) = \mathcal{E}_-(x - ia_+). \tag{1.17}$$

(Here and below, F^* denotes the conjugate meromorphic function of $F \in \mathcal{M}$, i.e., $F^*(x) \equiv \overline{F(\overline{x})}$.)

At first sight, the requirement (1.17) may seem very restrictive: it entails that $\mathcal{E}_\delta(x)$ must have a suitable dependence on $a_{-\delta}$, when we view a_+ and a_- as free parameters, constrained only by (1.2). In fact, however, (1.17) can be readily satisfied, for instance as follows. Let $\phi_\delta(x), \delta = +, -$, be arbitrary elliptic functions with periods $\pi/r, ia_\delta$ and no dependence on $a_{-\delta}$. Now set

$$\mathcal{E}_\delta(x) \equiv \phi_\delta^*(x)\phi_\delta(x + ia_{-\delta}), \quad \delta = +, -. \tag{1.18}$$

Then (1.17) is obeyed. Note also that the resulting AΔOs can be rewritten as

$$\mathcal{A}_\delta = T_{ia_{-\delta}} + \phi_\delta^*(x)T_{-ia_{-\delta}}\phi_\delta(x), \quad \delta = +, -. \tag{1.19}$$

From this representation formal self-adjointness on $\mathcal{H}$ can be read off directly. (The generalized Lamé AΔOs can also be written in this form, but the functions V_δ playing the role of ϕ_δ are not elliptic in that case, cf. (1.28)–(1.32) below.)

With (1.18) in effect, the span $D^{(P)}$ of the functions (1.16) is a first candidate for a dense domain on which the AΔO-actions might give rise to symmetric operators on $\mathcal{H}$. Assuming that the factors $\phi_\delta^*(x)$ and $\phi_\delta(x+ia_{-\delta})$ have no poles in $[0, \pi/r]$, we do obtain well-defined operators $\hat{\mathcal{A}}_\delta$ from $D^{(P)}$ to $\mathcal{H}$. But in general these operators are not symmetric on $D^{(P)}$.

To see why this is so, let $a_- < a_+$ and shift contours over ia_+ to test symmetry of the summand $\mathcal{E}_-(x)T_{-ia_+}$ (the first summand T_{ia_+} is of course symmetric on $D^{(P)}$). The vertical parts of the pertinent rectangular contour cancel (by π/r-periodicity), but since $\mathcal{E}_-(x)$ has poles inside the contour, one is left with residue terms that have no reason to vanish. Thus the horizontal parts do not generally cancel, entailing symmetry violation. To remedy this, one might restrict the functions in $D^{(P)}$ by requiring they vanish at the pole locations, but then it is no longer obvious (and probably false) that the resulting subspace is dense in $\mathcal{H}$.

In our special case, the factors have simple poles at $x=0$ and $x=\pi/r$, and the $\mathcal{A}_\delta$-action on $D^{(P)}$ does not even yield a subspace of $\mathcal{H}$. Again, one might be inclined to restrict attention to functions $F \in D^{(P)}$ for which $\mathcal{A}_\delta F$ does belong to $\mathcal{H}$, but we will not explore this avenue. Instead, we work with initial domains that are not subspaces of $D^{(P)}$, but whose definition reflects properties of the joint eigenfunctions from [4].

One of the crucial features of these initial domains is that neither of the two summands of $\mathcal{A}_\pm$ has a symmetric action on it, whereas the sum does yield a symmetric action. As a last example before embarking on the details, we show that the latter state of affairs can already arise for the special case $\mathcal{E}_\pm(x)=1$, so that we are dealing with the 'free' AΔOs

(1.20) $$\mathcal{A}_\delta^{(0)} \equiv T_{ia_{-\delta}} + T_{-ia_{-\delta}}, \qquad \delta = +,-.$$

Specifically, the functions

(1.21) $$F_n^{(D)}(x) \equiv \sin(n+1)rx, \qquad n \in \mathbb{N},$$

and

(1.22) $$F_n^{(N)}(x) \equiv \cos nrx, \qquad n \in \mathbb{N},$$

are pairwise orthogonal and complete in $\mathcal{H}$, and we have

(1.23) $$\mathcal{A}_\delta^{(0)} F_n^{(D)} = 2\cosh((n+1)ra_{-\delta})F_n^{(D)},$$

(1.24) $$\mathcal{A}_\delta^{(0)} F_n^{(N)} = 2\cosh(nra_{-\delta})F_n^{(N)}.$$

Therefore, we are led to two distinct ways to associate self-adjoint operators on $\mathcal{H}$ to the AΔOs (1.20); the four summands involved, however, yield well-defined, but non-symmetric operators on the linear hulls $D^{(D)}$ and $D^{(N)}$ of the functions (1.21) and (1.22). (For example, one has $(F_1^{(N)}, T_{ia_+}F_0^{(N)}) = 0$, but $(T_{ia_+}F_1^{(N)}, F_0^{(N)}) \neq 0$.) This state of affairs may be viewed as a consequence of

$-id/dx$ not being symmetric on $D^{(D)}$ and $D^{(N)}$, whereas $-d^2/dx^2$ is of course essentially self-adjoint on these subspaces. Thus, we are basically reinterpreting the free AΔOs $\mathcal{A}_{\pm}^{(0)}$ as the power series $2\sum_k a_{\mp}^{2k}(-d^2/dx^2)^k/(2k)!$.

Having provided a more general context for the specific problems we address and partly solve in this paper, we now proceed to define the two commuting generalized Lamé operators at issue. We find it convenient to use a relative $s(r, a; x)$ of the Weierstrass σ-function as a building block for elliptic functions with periods π/r and ia. Specifically, using the conventions of [1], we have

$$(1.25) \qquad s(r, a; x) \equiv \sigma(x; \frac{\pi}{2r}, \frac{ia}{2}) \exp(-\eta x^2 r/\pi).$$

Since two distinct periods ia_+, ia_- are involved, we also put

$$(1.26) \qquad s_\delta(x) \equiv s(r, a_\delta; x), \quad \delta = +, -.$$

The functions $s_\delta(x)$ are entire, odd, π/r-antiperiodic functions with simple zeros in the elliptic lattice points $\mathbb{Z}\pi/r + i\mathbb{Z}a_\delta$. They satisfy the A$\Delta$Es

$$(1.27) \qquad \frac{s_\delta(x + ia_\delta/2)}{s_\delta(x - ia_\delta/2)} = -\exp(-2irx),$$

and converge to $\sin(rx)/r$ for $a_\delta \to \infty$ and to $\sinh(\pi x/a_\delta)a_\delta/\pi$ for $r \to 0$.

Next, we define the factor functions

$$(1.28) \qquad V_\delta(b; x) \equiv \exp(-rb)s_\delta(x - ib)/s_\delta(x), \quad \delta = +, -, \quad b \in \mathbb{R},$$

and coefficient functions $C_\delta(b; x)$ via (1.18). More precisely, we have

$$(1.29) \qquad C_\delta(b; x) \equiv \exp(-2rb)\frac{s_\delta(x + ib)s_\delta(x - ib + ia_{-\delta})}{s_\delta(x)s_\delta(x + ia_{-\delta})}, \quad \delta = +, -.$$

Clearly, the functions $C_\delta(b; x)$ are indeed elliptic, whereas $V_\delta(b; x)$ is π/r-periodic, but not ia_δ-periodic. We also point out the invariance property

$$(1.30) \qquad C_\delta(a_+ + a_- - b; x) = C_\delta(b; x).$$

The generalized Lamé AΔOs now read

$$(1.31) \qquad \mathcal{A}_\delta(b) \equiv T_{ia_{-\delta}} + C_\delta(b; x)T_{-ia_{-\delta}}, \quad \delta = +, -,$$

or, equivalently,

$$(1.32) \qquad \mathcal{A}_\delta(b) \equiv T_{ia_{-\delta}} + V_\delta^*(b; x)T_{-ia_{-\delta}}V_\delta(b; x), \quad \delta = +, -.$$

Due to (1.30), they satisfy

$$(1.33) \qquad \mathcal{A}_\delta(a_+ + a_- - b) = \mathcal{A}_\delta(b), \quad \delta = +, -.$$

From now on, we often suppress the b-dependence.

There exist several distinct avatars of the Lamé AΔOs that each have their pros and cons. To suit our present purposes, we mostly work with operators $A_\pm$ arising from $\mathcal{A}_\pm$ via a similarity transformation with the function

$$(1.34)\qquad c(b;x) \equiv \frac{G_{\mathrm{ell}}(x - ib + i(a_+ + a_-)/2)}{G_{\mathrm{ell}}(x + i(a_+ + a_-)/2)}.$$

Here, G_{ell} denotes the elliptic gamma function

$$(1.35)\qquad G_{\mathrm{ell}}(x) \equiv \prod_{m,n=0}^{\infty} \frac{1 - \exp[-(2m+1)ra_+ - (2n+1)ra_- - 2irx]}{1 - \exp[-(2m+1)ra_+ - (2n+1)ra_- + 2irx]},$$

introduced and studied in [6]. The scattering function

$$(1.36)\qquad u(b;x) \equiv -e^{-2irx} c(b;x)/c(b;-x),$$

and weight function

$$(1.37)\qquad w(b;x) \equiv 1/c(b;x)c(b;-x),$$

from [6] also play important roles below. Note that each of these functions is invariant under interchange of a_- and a_+. Moreover, $u(b;x)$ is invariant under taking $b \to a_+ + a_- - b$, but the c- and w-functions are not.

As a consequence, the commuting AΔOs

$$(1.38)\qquad A_\delta(b) \equiv c(b;x)\mathcal{A}_\delta(b)c(b;x)^{-1}, \quad \delta = +,-,$$

are not invariant under $b \to a_+ + a_- - b$. But in contrast to $\mathcal{A}_\delta$, they commute with the parity operator

$$(1.39)\qquad (PF)(x) \equiv F(-x), \quad F \in \mathcal{M},$$

as will now be made clear. The point is that $V_\delta(x)$ (1.28) can be written as

$$(1.40)\qquad V_\delta(x) = c(x)/c(x - ia_{-\delta}), \quad \delta = +,-,$$

a representation that readily follows from the AΔEs satisfied by the elliptic gamma function [6]. Thus we obtain from (1.32) and (1.38)

$$(1.41)\qquad A_\pm = V_\pm(x)T_{ia_\mp} + V_\pm(-x)T_{-ia_\mp},$$

whence the vanishing of the commutators,

$$(1.42)\qquad [P, A_\pm] = 0,$$

is plain.

This parity property is crucial in Section 2, where we show that when the natural elliptic parameter space

$$(1.43)\qquad \mathcal{E} \equiv \{(r, a_+, a_-, b) \in \mathbb{R}^4 \mid r, a_+, a_- > 0\},$$

is restricted to

$$(1.44)\qquad \mathcal{C} \equiv \{(r, a_+, a_-, b) \in \mathcal{E} \mid b \in (0, a_+ + a_-)\},$$

then the AΔOs $A_\pm$ admit a reinterpretation as symmetric operators on a dense subspace D_w of

$$(1.45)\qquad \mathcal{H}_w \equiv L^2((0, \pi/r), w(x)dx).$$

(This also involves detailed information on the weight function, obtained in Subsection VB of [6].)

We would like to stress that on the one hand this symmetry result is independent of our findings concerning joint eigenfunctions in [4]. On the other hand, the definition of D_w is *inspired* by the latter. Indeed, we allow the meromorphic functions in D_w to have simple poles at those locations in the strip

$$(1.46) \qquad \mathcal{S} \equiv \{x \in \mathbb{C} \mid |\mathrm{Im}\, x| \leq \max(a_+, a_-)\},$$

for which the pertinent eigenfunctions can have simple poles as well.

Without previous explicit information on joint eigenfunctions of the AΔOs, however, we do not know how to proceed beyond the symmetry results in Section 2. Even though it is easy to see that the associated symmetric operators on D_w admit self-adjoint extensions, the properties of the latter seem quite inaccessible without having such information available. In any event, only for parameters in the dense subset

$$(1.47) \qquad \mathcal{C}_{\mathrm{irr}} \equiv \mathcal{C} \cap \mathcal{D}_{\mathrm{irr}},$$

where

$$(1.48) \qquad \mathcal{D}_{\mathrm{irr}} \equiv \{(r, a_+, a_-, (N_+ + 1)a_+ - N_- a_-) \in \mathcal{E} \mid a_+/a_- \notin \mathbb{Q}, N_+, N_- \in \mathbb{N}\},$$

we are going to obtain detailed answers to the Hilbert space questions.

The key point is that for parameters in (1.48) we do know explicit joint eigenfunctions $\Psi(\pm x, y)$ of the AΔOs A_δ from [4]. (We focus on $\mathcal{D}_{\mathrm{irr}}$ for simplicity; the larger set $\mathcal{D}$ given by (3.33)–(3.35) in [4] can be treated by making some rather obvious changes.) In the first part of Section 3 we summarize some algebraic and function-theoretic aspects of these functions. The second part concerns the question whether the functions $\Psi(\pm x, y)$ can be continuously interpolated to all $b \in \mathbb{R}$ (for fixed a_+, a_- with a_+/a_- irrational). Here we report new results on this interpolation problem, which are however not used for the Hilbert space analysis undertaken in Section 4.

More in detail, we obtain representations for Casorati determinants that can be exploited to study the interpolation question. For the hyperbolic specialization we invoke results from [7] to answer it in the negative. (We refer to our lecture notes [8] and [9] for discursive accounts covering the joint eigenfunctions and the associated interpolation problem.)

To be sure, in the hyperbolic case the even combination $\Psi(x, y) + \Psi(-x, y)$ does admit an analytic interpolation (cf. [9]), and in the elliptic case it seems plausible that the sequence of even joint eigenfunctions relevant for the Hilbert space arena also admits an analytic interpolation, at least for parameters in $\mathcal{C}$. At any rate, this is strongly suggested when the Hilbert space results in Section 4 (which pertain to $\mathcal{C}_{\mathrm{irr}}$) are combined with the ones in Section 2 (which hold on all of $\mathcal{C}$).

To sketch the results of Section 4, we should first recall that the Hilbert space results in [4] are incomplete, even for parameters in $\mathcal{C}_{\mathrm{irr}}$. The findings reported in Section 4 complete our previous results for $\mathcal{C}_{\text{irr}}$, inasmuch as we solve all of the problems mentioned in the general setting below (1.14).

The key new ingredient compared to [4] is a comparison of the pertinent even eigenfunctions $\chi_n(x) \equiv \Psi(x, nr) + \Psi(-x, nr)$ for large n to the orthonormal base of polynomials $p_n(\cos(rx))$ for the Hilbert space

$$\mathcal{H}_P \equiv L^2((0, \pi/r), w_P(x)dx). \tag{1.49}$$

Here the weight function $w_P(x)$ is constructed such that the dominant large-n asymptotics of the functions $w(x)^{1/2}\chi_n(x)$ is proportional to that of $w_P(x)^{1/2}p_n(\cos(rx))$.

The $n \to \infty$ asymptotics of the latter functions in $\mathcal{H}$ (1.14) follows from our recent paper [10]. Its relevance for the comparison argument just mentioned hinges on a completeness result that can be found in a monograph by Higgins [11]. To be quite precise, the *reasoning* in the proof of Theorem A on p. 72 of [11] can be adapted to our situation. In this connection we point out that our starting point differs significantly from *loc.cit.* This is because [4] only yields pairwise orthogonal functions $\chi_n(x)$ for $n > K/r$, and we know very little about the minimal choice of K.

In two important special cases, however, we do know that K can be chosen negative. These are the cases $(N_+, N_-) = (0, 0), (1, 0)$. The first case gives rise to the 'free' AΔOs (1.20). More precisely, we have

$$\mathcal{A}_\delta(a_+) = e^{-ra_{-\delta}}e^{-irx}\mathcal{A}_\delta^{(0)}e^{irx}, \quad \delta = +, -, \tag{1.50}$$

$$\begin{aligned} &c(a_+; x) = \mathcal{N}e^{irx}/s_-(x), \\ &\chi_n(a_+; x) = \mathcal{N}(e^{i(n+1)rx} - e^{-i(n+1)rx})/s_-(x), \quad n \in \mathbb{N}, \end{aligned} \tag{1.51}$$

with $\mathcal{N}$ a constant. Hence orthogonality and completeness are immediate, cf. (1.21).

The second case is not elementary. This special case is studied in considerable detail in [12], and the results in Section 4 entail that the relativistic $b = 2a_+$ Lamé functions $\chi_n(x), n \in \mathbb{N}$, of [12] are *complete* in $\mathcal{H}_w$ for $b \in (0, a_+ + a_-)$, a conjecture left open in [12].

Returning to the general case $(N_+, N_-) \in \mathbb{N}^2$, our adaptation of the completeness argument only proves that the pairwise orthogonal joint eigenfunctions χ_n with $n \geq M \geq 0, n, M \in \mathbb{N}$, have an orthogonal complement of dimension M. But once this finite-dimensionality is known, we can show by additional arguments that it consists of functions in the symmetry domain D_w defined in Section 2. It is then straightforward to establish the existence of an orthonormal base of joint eigenfunctions with real eigenvalues. (This proves conjectures we already made in Section IV of [4].) Thus all of the above-mentioned problems are solved for parameters in $\mathcal{C}_{\text{irr}}$.

Since $\mathcal{C}_{\text{irr}}$ is dense in $\mathcal{C}$, it is eminently plausible that these results interpolate continuously to all of $\mathcal{C}$, as already suggested above. But it is better to have a proof than to have no doubt. In this connection, we would like to mention recent results by Komori [13] (see also his earlier paper [14]). He studies multivariable generalizations of one of the above AΔOs $A_\pm$, proving essential self-adjointness on suitable domains, and also the existence of an orthonormal base of eigenfunctions for suitable parameters. Specializing his results to the above one-variable case, it is unfortunately not clear whether they have a bearing on the conjectured interpolation of our results to all of C. The problem is that Komori focuses on only one of the AΔOs, using perturbation theory to compare it to the 'free' case $b = a_+$. In view of possible differences in domains (whose ambiguities are largely unexplored to date), the precise relation to his findings is elusive. In fact, since Komori's Hilbert space eigenfunctions arise after taking closures, it is not even clear whether they are restrictions of meromorphic functions to $(0, \pi/r)$.

2. SYMMETRY DOMAINS FOR PARAMETERS IN $\mathcal{C}$

Consider the $\mathcal{M}$-subspace

$$\mathcal{P}^{(e)}_{2\pi/r} \equiv \{F \in \mathcal{P}_{2\pi/r} \mid F(x) = F(-x)\} \tag{2.1}$$

of $2\pi/r$-periodic even functions, cf. (1.7). Setting

$$F^{(\pm)}(x) \equiv F(x) \pm F(x + \pi/r), \quad F \in \mathcal{P}^{(e)}_{2\pi/r}, \tag{2.2}$$

one easily checks $F^{(\pm)} \in \mathcal{P}^{(e)}_{2\pi/r}$. Moreover, $F^{(+)}$ and $F^{(-)}$ satisfy

$$F^{(\pm)}(x + \pi/r) = \pm F^{(\pm)}(x), \tag{2.3}$$

and

$$F^{(\pm)}(\pi/r - x) = \pm F^{(\pm)}(x). \tag{2.4}$$

It follows that $\mathcal{P}^{(e)}_{2\pi/r}$ is the direct sum of its subspaces $\mathcal{P}^{(\pm)}$ of functions satisfying (2.3), or equivalently (2.4). Now the AΔOs A_δ (1.41) not only commute with parity, but also leave the $\mathcal{M}$-subspaces of π/r-periodic and π/r-antiperiodic functions invariant. Thus they leave the decomposition

$$\mathcal{P}^{(e)}_{2\pi/r} = \mathcal{P}^{(+)} \oplus \mathcal{P}^{(-)} \tag{2.5}$$

invariant.

The weight function $w(b; x)$ (1.37) is clearly even and π/r-periodic, so we have $w \in \mathcal{P}^{(+)}$ and

$$w(\pi/r - x) = w(x). \tag{2.6}$$

Accordingly, the Hilbert space (1.45) is a direct sum

$$\mathcal{H}_w = \mathcal{H}_w^{(+)} \oplus \mathcal{H}_w^{(-)} \tag{2.7}$$

of orthogonal subspaces

$$\mathcal{H}_w^{(\pm)} \equiv \{f \in \mathcal{H}_w \mid f(\pi/r - x) = \pm f(x)\}. \tag{2.8}$$

It is convenient to write $w(x)$ as

$$(2.9) \qquad w(b;x) = Cs_+(x)s_-(x)w_r(b;x).$$

Here, C is a positive constant depending on r, a_+, a_-, and w_r is the 'reduced' weight function

$$(2.10) \qquad w_r(b;x) \equiv G_{\rm ell}(x + ib - i(a_+ + a_-)/2)G_{\rm ell}(-x + ib - i(a_+ + a_-)/2),$$

cf. [6] (5.41). The s_δ-functions yield zeros at

$$(2.11) \qquad x = ima_+, \quad x = ina_-, \quad m, n \in \mathbb{Z},$$

and the $G_{\rm ell}$-factors yield poles at

$$(2.12) \qquad x = \pm i(b + ka_+ + la_-), \quad k, l \in \mathbb{N},$$

and zeros at

$$(2.13) \qquad x = \pm i(b - ma_+ - na_-), \quad m, n \in \mathbb{N}^*,$$

cf. (1.35).

For the remainder of this section we assume that the parameters belong to $\mathcal{C}$. This entails in particular that the w-poles (2.12) are at a distance $b > 0$ from the real axis. Therefore, the vector spaces $\mathrm{Pol}^{(p)}, p = +, -$, of polynomials in $\cos rx$ that are even/odd for $p = +/-$ are dense subspaces of $\mathcal{H}_w^{(p)}$. (Indeed, if $f \in \mathcal{H}_w^{(p)}$ is orthogonal to $\mathrm{Pol}^{(p)}$, then all Fourier-Neumann coefficients $(f, F_n^{(N)})_w, n \in \mathbb{N}$, vanish, cf. (1.22). Hence $f(x)w(x) = 0$, so $f(x) = 0$.)

We now define subspaces $D_w^{(p)}$ of $\mathcal{H}_w^{(p)}$, as follows. Functions in $D_w^{(p)}$ are restrictions to $(0, \pi/r)$ of functions in $\mathcal{P}^{(p)}$, whose poles in the strip $\mathcal{S}$ (1.46) are at most simple and occur at

$$(2.14) \qquad x = \pm i(ma_+ + na_- - b) + j\pi/r, \quad m, n \in \mathbb{N}^*, \quad j \in \mathbb{Z}.$$

Thus, setting

$$(2.15) \qquad a_s \equiv \min(a_+, a_-), \quad a_l \equiv \max(a_+, a_-),$$

the pertinent pole locations can be rewritten

$$(2.16) \qquad x = \pm i(a_l + a_s - b + ka_s) + j\pi/r, \quad k \in \mathbb{N}, \quad (k+1)a_s \leq b, \quad j \in \mathbb{Z}.$$

Since we have $\mathrm{Pol}^{(p)} \subset D_w^{(p)}$, the subspaces $D_w^{(p)}$ are dense in $\mathcal{H}_w^{(p)}$.

Defining now

$$(2.17) \qquad D_w \equiv D_w^{(+)} \oplus D_w^{(-)},$$

it is not hard to see that the AΔOs A_δ give rise to well-defined operators

$$(2.18) \qquad \hat{A}_\delta : D_w \to \mathcal{H}_w, \quad \delta = +, -,$$

the action of $\hat{A}_\delta$ being defined via the A_δ-action on the meromorphic extension of $F(x) \in D_w$. Indeed, letting $F \in D_w^{(p)}$, consider the meromorphic function

$$(2.19) \qquad (A_\delta F)(x) = e^{-rb}\left(\frac{s_\delta(x - ib)}{s_\delta(x)}F(x - ia_{-\delta}) + \frac{s_\delta(x + ib)}{s_\delta(x)}F(x + ia_{-\delta})\right),$$

taking e.g. $a_l = a_+$. Clearly, we have $A_\delta F \in \mathcal{P}^{(p)}$. Poles of the functions $F(x \pm$

$ia_-)$ on the real axis can only occur when b equals a_l; in that case they must be located at $x = j\pi/r, j \in \mathbb{Z}$, and they are at most simple, cf. (2.16). Now for $b = a_l$, the functions $s_+(x \pm ib)/s_+(x)$ are entire. Since A_+F is even, no simple poles at $x = 0$ can occur. Since A_+F is also π/r-periodic or π/r-antiperiodic, no real poles occur at all, and so $A_+F \in \mathcal{H}_w^{(p)}$. For $b \neq a_l$ the functions $s_+(x \pm ib)/s_+(x)$ yield simple poles for $x = j\pi/r$, but since $A_+F \in \mathcal{P}^{(p)}$, no real poles occur for A_+F. Thus we infer again $A_+F \in \mathcal{H}_w^{(p)}$.

Likewise, poles of $F(x \pm ia_+)$ for real x can only arise for $b = (k+1)a_s$; then they are located at $x = j\pi/r$ and are at most simple. But for these b-values the functions $s_-(x \pm ib)/s_-(x)$ are entire, so we conclude as before $A_-F \in \mathcal{H}_w^{(p)}$.

Denoting the restrictions of $\hat{A}_\delta$ to $D_w^{(p)}$ by $\hat{A}_\delta^{(p)}$, we have

$$(2.20) \qquad \hat{A}_\delta^{(p)} : D_w^{(p)} \to \mathcal{H}_w^{(p)}, \quad \delta = +,-, \quad p = +,-,$$

as just demonstrated. The main result of this section is that the operators $\hat{A}_\delta^{(p)}$ and (hence) $\hat{A}_\delta$ are symmetric on their definition domains $D_w^{(p)}$ and D_w, resp.

Theorem 2.1. *Let $b \in (0, a_+ + a_-)$ and $F, G \in D_w^{(p)}, p \in \{+,-\}$. Then we have*

$$(2.21) \qquad (\hat{A}_\delta F, G)_w = (F, \hat{A}_\delta G)_w, \quad \delta = +,-,$$

where $(\cdot,\cdot)_w$ denotes the inner product on $\mathcal{H}_w$ (1.45).

Proof. To ease the notation we detail the case $\delta = -$. (To handle $\delta = +$ one need only interchange all subscripts $+$ and $-$.) Our task is to prove equality of

$$(2.22) \quad I_L \equiv \int_0^{\pi/r} \left(\frac{s_-(x+ib)}{s_-(x)} F^*(x+ia_+) + \frac{s_-(x-ib)}{s_-(x)} F^*(x-ia_+) \right) G(x) w(x) dx,$$

and

$$(2.23) \quad I_R \equiv \int_0^{\pi/r} F^*(x) \left(\frac{s_-(x-ib)}{s_-(x)} G(x-ia_+) + \frac{s_-(x+ib)}{s_-(x)} G(x+ia_+) \right) w(x) dx.$$

In order to do so we introduce

$$(2.24) \qquad I(x) \equiv w(x-e) \frac{s_-(x+ib-e)}{s_-(x-e)} F^*(x+e) G(x-e), \quad e \equiv \frac{ia_+}{2}.$$

From [6] (5.43) we have

$$(2.25) \qquad \frac{w(x+e)}{w(x-e)} = \frac{s_-(x+ib-e)\, s_-(x+e)}{s_-(x-ib+e)\, s_-(x-e)},$$

so $I(x)$ can also be written as

$$(2.26) \qquad I(x) = w(x+e) \frac{s_-(x-ib+e)}{s_-(x+e)} F^*(x+e) G(x-e).$$

Since $w(x), F^*(x), G(x)$ are even and $s_-(x)$ is odd, we obtain

$$I_L - I_R = \int_0^{\pi/r} [I(x+e) + I(-x+e) - I(x-e) - I(-x-e)]dx$$
$$(2.27) \qquad = \int_{-\pi/r}^{\pi/r} [I(x+e) - I(x-e)]dx, \quad e = ia_+/2.$$

Now $I(x)$ is π/r-periodic, so to prove $I_L = I_R$ it suffices to show $I(x)$ has no poles for $|\text{Im}\, x| \le a_+/2$.

To this end we set

$$(2.28) \qquad J(x) \equiv I(x-e) = w(x)\frac{s_-(x-ib)}{s_-(x)}F^*(x)G(x-ia_+),$$

and prove $J(x)$ is pole-free for $\text{Im}\, x \in [0, a_+]$. To begin with, we observe that the factor $1/s_-(x)$ is matched by the factor $s_-(x)$ in $w(x)$, cf. (2.9). Since $J(x)$ is π/r-periodic, it remains to show that the poles of $w_r(x)$ given by (2.12) and the (eventual) poles of $F^*(x)$ and $G(x-ia_+)$ are matched by zeros when x varies over $i[0, a_+]$.

We continue to prove this, assuming first $a_l = a_+$. Then the only w_r-poles (2.12) in $i[0, a_+]$ are of the form $i(b + la_-), l \in \mathbb{N}$, and they are simple (since we need $la_- < a_+$). These poles are matched by the zeros

$$(2.29) \qquad ib + ila_-, \quad l \in \mathbb{N},$$

of the factor $s_-(x-ib)$. The pertinent poles of $F^*(x)$ are of the form $i(a_+ + a_- - b + ka_-), k \in \mathbb{N}$, (cf. (2.16)), so they are matched by w_r-zeros occurring in (2.13).

Finally, consider the poles of $G(x - ia_+)$ on the imaginary axis. They must be located at

$$(2.30) \qquad x = ia_+ + i(a_+ + a_- - b + ka_-), \quad k \in \mathbb{N},$$

and

$$(2.31) \qquad x = ia_+ - i(a_+ + a_- - b + ka_-), \quad k \in \mathbb{N}.$$

Clearly, the locations (2.30) are above ia_+, so these poles are innocuous. The poles (2.31) belong to $i[0, a_+]$ for $b \ge (k+1)a_-$, but they are matched by zeros

$$(2.32) \qquad ib - ima_-, \quad m \in \mathbb{N}^*,$$

of the factor $s_-(x - ib)$ that are distinct from the zeros (2.29) already invoked. Thus $J(x)$ has no poles for $\text{Im}\, x \in [0, a_+]$, as asserted.

It remains to study the case $a_l = a_-$. Then the only relevant poles of $w(x), F^*(x)$ and $G(x - ia_+)$ occur at $ib, i(a_+ + a_- - b)$ and $ib - ia_-$, so as in the previous case they are matched by zeros of $s_-(x-ib), w_r(x)$ and $s_-(x-ib)$, resp. Therefore, $J(x)$ is again pole-free for $\text{Im}\, x \in [0, a_+]$. □

It is obvious that the symmetric operators $\hat{A}_\delta^{(p)}$ and $\hat{A}_\delta$ commute with complex conjugation. Therefore they admit self-adjoint extensions [5]. In fact, however, we believe the operators are essentially self-adjoint. Far stronger yet, we state the following conjecture.

Conjecture 2.2. *Assume $b \in (0, a_+ + a_-)$. Then there exists an orthogonal base of joint $\hat{A}_\pm$-eigenvectors $\chi_n \in D_w, n \in \mathbb{N}$, with $\chi_n \in D_w^{(+)}$ for n even and $\chi_n \in D_w^{(-)}$ for n odd, and with positive eigenvalues satisfying*

$$E_{n,\pm} \sim \exp(nra_\mp), \quad n \to \infty. \tag{2.33}$$

In Section 4 we show that this conjecture holds true for the dense subset $C_{\rm irr}$ (1.47).

We proceed with some observations that are valid for all of C. From the above proof we have

$$(F, \hat{A}_-^{(p)} G)_w = 2e^{-rb} \int_0^{\pi/r} I(x) dx, \quad F, G \in D_w^{(p)}, \tag{2.34}$$

where $I(x)$ is given by (2.24). Choosing $G = F$ and recalling the alternative representation (2.26), we deduce that $I(x)$ is real-valued on $(0, \pi/r)$. Now the function

$$K(x) \equiv w(x-e) \frac{s_-(x+ib-e)}{s_-(x-e)} = w(x+e) \frac{s_-(x-ib+e)}{s_-(x+e)} \tag{2.35}$$

has no zeros for $x \in (0, \pi/r)$, so it is either positive or negative. One readily verifies $K(\pi/2r) > 0$, so in fact we have

$$K(x) > 0, \quad x \in (0, \pi/r). \tag{2.36}$$

Since (2.34) and (2.24) entail

$$(F, \hat{A}_-^{(p)} F)_w = 2e^{-rb} \int_0^{\pi/r} K(x) |F(x - ia_+/2)|^2 dx, \quad F \in D_w^{(p)}, \tag{2.37}$$

we deduce

$$(F, \hat{A}_-^{(p)} F)_w \geq 0, \quad F \in D_w^{(p)}. \tag{2.38}$$

Similarly, we obtain

$$(F, \hat{A}_+^{(p)} F)_w \geq 0, \quad F \in D_w^{(p)}. \tag{2.39}$$

From these positivity properties it follows once again that the symmetric operators $\hat{A}_\delta^{(p)}$ admit self-adjoint extensions. Moreover, whenever the operators are essentially self-adjoint, their self-adjoint closures are positive operators on $\mathcal{H}_w^{(p)}$.

To conclude this section, we briefly examine the above in terms of the AΔOs $\mathcal{A}_\pm$ (1.31) and the Hilbert space $\mathcal{H}$ (1.14), as this yields useful information for the more general contexts considered in the introduction. Clearly, $\mathcal{H}$ can be identified with $\mathcal{H}_w$ (1.45) via the unitary similarity transformation

$$I : \mathcal{H}_w \to \mathcal{H}, \quad f(x) \mapsto f(x)/c(x), \tag{2.40}$$

cf. (1.37). Then we obtain symmetric operators

$$(2.41) \qquad \hat{\mathcal{A}}_\delta \equiv I\hat{A}_\delta I^{-1}, \quad \delta = +, -,$$

on the domain

$$(2.42) \qquad D \equiv ID_w,$$

whose action on D coincides with the $\mathcal{A}_\delta$-action.

We now discuss some salient features of the meromorphic functions belonging to the space D. To start with, we have $D \subset \mathcal{P}_{2\pi/r}$, since $D_w \subset \mathcal{P}_{2\pi/r}$ and $c(x) \in \mathcal{P}_{\pi/r}$. But since $c(x)$ is not even, functions in D are not even either. Accordingly, the decomposition

$$(2.43) \qquad D = ID_w^{(+)} \oplus ID_w^{(-)},$$

in invariant subspaces for

$$(2.44) \qquad \hat{\mathcal{A}}_\delta^{(p)} \equiv I\hat{A}_\delta^{(p)} I^{-1}, \quad \delta = +, -, \quad p = +, -,$$

is present, but not readily recognizable from the definition (1.31) of the AΔOs $\mathcal{A}_\pm$.

Turning to properties pertaining to the critical strip $\mathcal{S}$ (1.46), we begin by noting that the multiplication of $F(x) \in D_w$ by $1/c(x)$ takes out the poles (2.16) in the lower half plane, due to zeros of $1/c(x)$ at these locations. The only other zeros of $1/c(x)$ in $\mathcal{S}$ occur at

$$(2.45) \qquad x \equiv ia_l, ika_s, \quad k \in \mathbb{N}, \quad ka_s \leq a_l, \qquad (\text{mod } \pi/r).$$

But $1/c(x)$ also yields new poles in $\mathcal{S}$ at

$$(2.46) \qquad x \equiv ib + ija_s, \quad j \in \mathbb{N}, \quad b + ja_s \leq a_l, \qquad (\text{mod } \pi/r).$$

(More precisely, for $b \in (0, a_l]$ functions in D can have at most simple poles at these points.) Together with the upper half plane poles (2.16), i.e.,

$$(2.47) \qquad x \equiv i(a_l - b + na_s), \quad n \in \mathbb{N}^*, \quad na_s \leq b, \qquad (\text{mod } \pi/r),$$

(relevant for $b \in [a_s, a_s + a_l)$), these are the (at most simple) poles allowed for functions in D, whereas all functions in D have zeros at (2.45) with multiplicity at least one. Observe that the zeros at $ia_{-\delta}$ and $ia_{-\delta} + \pi/r$ are essential for $\mathcal{A}_\delta F, F \in D$, to yield a function in $\mathcal{H}$, cf. (1.29) and (1.31).

It is particularly clear in the setting just worked out that the two summands of the generalized Lamé operators are not *separately* symmetric on the pertinent domains. Indeed, the summand $T_{ia_{-\delta}}$ of $\mathcal{A}_\delta$ shifts functions in D without encountering poles (since $F(x) \in D$ has no poles for $\text{Im}\, x \in [-a_l, 0]$), but the vertical parts of the relevant contour integral only cancel when F happens to be π/r-periodic or π/r-antiperiodic. (On the other hand, when F belongs to $ID_w^{(+)}$ or $ID_w^{(-)}$, resp., then this is indeed the case.)

Unfortunately, it is false in general that two π/r-periodic and π/r-antiperiodic meromorphic functions that are regular for real x are orthogonal in $\mathcal{H}$. (Take for instance $F_1(x) = 1, F_2(x) = \sin rx$.) In view of this state of affairs, a symmetry analysis for the large class of commuting AΔOs of type (1.19) remains elusive.

3. JOINT EIGENFUNCTIONS: FIRST STEPS

Throughout this section the parameters belong to $\mathcal{D}_{\mathrm{irr}}$ (1.48). It is convenient to set

$$b_{+-} \equiv (N_+ + 1)a_+ - N_- a_-, \quad N_+, N_- \in \mathbb{N}. \tag{3.1}$$

For these b-values the c-function (1.34) specializes to

$$c(b_{+-}; x) = \mathcal{N} \frac{\prod_{k=1}^{N_-} s_+(x + ika_-)}{\prod_{j=0}^{N_+} s_-(x - ija_+)} \exp[(2N_+N_- + N_+ + N_- + 1)irx], \tag{3.2}$$

where $\mathcal{N}$ is a constant. (This readily follows from the AΔEs satisfied by the elliptic gamma function [6].)

We proceed to sketch how joint A_δ-eigenfunctions of the form

$$\Psi(b_{+-}; x, y) = \mathcal{N} \prod_{j=-N_+}^{N_+} \frac{1}{s_-(x - ija_+)} \cdot \mathcal{H}(b_{+-}; x, y), \tag{3.3}$$

$$\begin{aligned} &\mathcal{H}(b_{+-}; x, y) \\ &\equiv \prod_{\delta=+,-} \prod_{j=1}^{N_\delta} s_{-\delta}(x + z_j^\delta(y)) \cdot \exp[(2N_+N_- + N_+ + N_- + 1)irx + iyx], \end{aligned} \tag{3.4}$$

arise [4]. Viewing (3.3)–(3.4) as an Ansatz for the eigenvalue equations $A_\pm \Psi = E_\pm \Psi$, it follows that the zero functions $z_j^\pm(y)$ must satisfy a rather involved constraint system. Using the implicit and inverse function theorems, it can then be shown that for y real and sufficiently large, real-analytic solutions exist with asymptotics

$$z_l^\delta(y) = ila_\delta + O(\exp(-2la_\delta y)), \quad l = 1, \ldots, N_\delta, \quad \delta = +, -, \quad y \to \infty. \tag{3.5}$$

The associated eigenvalues $E_\delta(y)$ are real and have asymptotics

$$E_\delta(y) = \exp(a_{-\delta} y)(1 + O(\exp(-2a_s y))), \quad y \to \infty. \tag{3.6}$$

Moreover, all of the joint eigenfunctions corresponding to the eigenvalue pair $(E_+(y), E_-(y))$ with y sufficiently large are of the form $\lambda\Psi(x,y) + \mu\Psi(-x,y)$, with $\lambda, \mu \in \mathbb{C}$, cf. Appendix B in [4].

The even joint eigenfunctions

$$\chi_n(b_{+-}; x) \equiv \Psi(b_{+-}; x, nr) + \Psi(b_{+-}; -x, nr), \quad n \in \mathbb{N}, \quad nr > K(b_{+-}), \tag{3.7}$$

are the ones relevant for our reinterpretation of the AΔOs A_δ as operators on the Hilbert space $\mathcal{H}_w$ (1.45). Here, K is chosen large enough so that various conditions are met. In particular, the solutions to the constraint system exist and take values in $i(0,\infty)$, the eigenvalue pair $(E_+(y), E_-(y))$ separates points on (K,∞), and $\chi_n(x)$ does not vanish identically. Introducing the integer

$$(3.8)\qquad M(b_{+-}) \equiv \max(0, [K(b_{+-})/r] + 1),$$

we therefore obtain an infinity of distinct joint A_δ-eigenfunctions $\chi_n, n \geq M$.

It is clear by inspection that the functions $\Psi(x,y)$ are real-analytic in x for $x \in \mathbb{R}$ (recall $a_+/a_- \notin \mathbb{Q}$) and that they obey the monodromy relation

$$(3.9)\qquad \Psi(x+\pi/r, y) = \exp(i\pi y/r)\Psi(x,y).$$

Accordingly, the functions $\chi_n(x)$ (3.7) are real-analytic for real x as well, and they belong to $\mathcal{P}^{(+)}/\mathcal{P}^{(-)}$ for n even/odd, cf. (2.1)–(2.5). It is not obvious, but true that for parameters in $\mathcal{C}_{\rm irr}$ we also have $\chi_n \in D_w^{(+)}/D_w^{(-)}$ for n even/odd. The crux is that the holomorphic functions (3.4) satisfy

$$(3.10)\qquad \begin{aligned} &\mathcal{H}(ik_+a_+ + ik_-a_-, y) = \mathcal{H}(-ik_+a_+ - ik_-a_-, y),\\ &\pm k_\delta \in \{0,\ldots,N_\delta\},\quad \delta = +,-, \end{aligned}$$

cf. [4] (3.44). The resulting zeros of $\mathcal{H}(x,nr) - \mathcal{H}(-x,nr)$ now cancel poles of the product in (3.3), leaving only poles in the strip $\mathcal{S}$ (1.46) given by (2.16). (This easily follows from (3.49)–(3.50) in [4]; note that all of the product poles in (3.3) are simple, since a_+/a_- is irrational.)

Since the joint eigenvalue pair $E_n \equiv (E_+(nr), E_-(nr))$ is real and $E_n \neq E_m$ for $M \leq n < m$, it now follows from Theorem 2.1 that for $b_{+-} \in (0, a_+ + a_-)$ the functions χ_n are pairwise orthogonal:

$$(3.11)\qquad (\chi_n, \chi_m)_w = 0, \qquad M \leq n < m.$$

Thus we have rederived one of the principal results of Section IV in [4] from Theorem 2.1. In the next section we will show that in case $M(b_{+-}) > 0$, there exist M independent additional joint A_δ-eigenfunctions $\chi_n, n = 0,\ldots,M-1$, belonging to $D_w^{(+)}/D_w^{(-)}$ for n even/odd, and having positive A_δ-eigenvalues $E_{\delta,n}, \delta = +,-$. Moreover, (3.11) holds for $0 \leq n < m$.

The results we are now heading for conclude this section, and will not be used in the next one. Fixing $r, a_+, a_- \in (0,\infty)$ with $a_+/a_- \notin \mathbb{Q}$, they have a bearing on the eventual existence of joint A_δ-eigenfunctions for arbitrary real b, which are continuous in b and reduce to a (possibly parameter- and y-dependent) multiple of $\Psi(b_{+-}; x, y)$ for the dense set of b-values b_{+-} (3.1).

Our better understanding of this interpolation problem stems from new representations for the Casorati determinants

$$(3.12)\qquad \begin{aligned} &\mathcal{D}_\delta(b_{+-}; x, y)\\ &\equiv \Psi(b_{+-}; x + ia_\delta/2, y)\Psi(b_{+-}; -x + ia_\delta/2, y) - (i \to -i),\quad \delta = +,-, \end{aligned}$$

whose derivations we now sketch. The key point is that the special Casorati determinants given by (2.43) in [4] can be exploited to get rid of the dependence of $\mathcal{D}_\delta$ on the zero functions $z_j^\delta, j = 1,\ldots,N_\delta$. To this end the plane wave factor in (3.4) should first be traded for dependence on z_j^δ and $z_j^{-\delta}$ via [4] (3.5) and (3.10). The $z_j^{-\delta}$-sum in [4] (3.10) can then be disposed of by using (cf. (1.27))

$$(3.13)\qquad s_\delta(x + ia_\delta/2 \pm z_j^{-\delta}) = -\exp[-2ir(x \pm z_j^{-\delta})]s_\delta(x - ia_\delta/2 \pm z_j^{-\delta}).$$

Finally, various $s_{\pm}$-factors not involving the z_j^{δ} can be combined in terms of the c-function $c(b_{+-};x)$ given by (3.2).

As a result of these calculations (whose details we skip), we obtain

$$(3.14)\quad \mathcal{D}_{\delta}(b_{+-};x,y) \sim c(b_{+-};x-ia_{\delta}/2)c(b_{+-};-x-ia_{\delta}/2)Q_{\delta}(b_{+-};x-ia_{\delta}/2,y).$$

Here and below, $\sim$ denotes equality up to a multiplicative factor that may depend on the parameters and y. The quotients Q_{δ} are given by

$$(3.15)\qquad Q_{\delta}(b_{+-};x,y) \equiv \prod_{j=1}^{N_{-\delta}} \frac{s_{\delta}(x+z_j^{-\delta})s_{\delta}(x-z_j^{-\delta})}{s_{\delta}(x+ija_{-\delta})s_{\delta}(x-ija_{-\delta})}.$$

The advantage of these formulas is that the existence of interpolations to arbitrary b can now be studied in terms of the factors Q_{δ}. (Indeed, the b-interpolation of the c-factors is immediate: one need only replace b_{+-} by b.)

For the hyperbolic specialization, where $r=0$ and $s_{\delta}(x)$ reduces to $\sinh(\pi x/a_{\delta})a_{\delta}/\pi$, the interpolation question can now be answered negatively by using (3.14). We detail this case first, since it renders clear what remains to be shown in the elliptic case. In the hyperbolic case we need not work with the zero system, since a far more accessible representation of the joint eigenfunctions and their eigenvalues exists for $b=b_{+-}$ [7]. From the latter it is clear that the constraint system admits solution vectors z^{+} and z^{-} for arbitrary real y, and that $\Psi(x,-y), y>0$, amounts to $\Psi(-x,y)$. Moreover, the constraint system decouples into separate systems for z^{+} and z^{-}.

The latter feature is critical: it entails that for a given $y>0$ the zero vector z^{+} is independent of N_{-}. To see how this can be exploited, let us first be more precise about the assumption that the functions $\Psi(b_{+-};x,\pm y), y>0$, admit arbitrary-$b$ interpolations. Specifically, we assume that for $y\in(0,\infty)$ and $b\in\mathbb{R}$ joint A_{δ}-eigenfunctions $\mathcal{I}(b;x,y)$ and $\mathcal{J}(b;x,y)$ exist with the following properties:

(i) for fixed $(b,y)\in\mathbb{R}\times(0,\infty)$ they are analytic in the region $\mathcal{R}\equiv\{\mathrm{Re}\, x>0\}$;

(ii) for fixed $(x,y)\in\mathcal{R}\times(0,\infty)$ they are continuous in b on $\mathbb{R}$;

(iii) for $b=b_{+-}$ they reduce to multiples of $\Psi(b_{+-};x,\pm y)$.

We stress that we are deviating from our requirement that eigenfunctions be meromorphic in x. This is because the poles of $\Psi(b_{+-};x,\pm y)$ get dense on the imaginary x-axis as $N_{+},N_{-}\to\infty$, so that the imaginary axis might be a natural boundary for b not of the form b_{+-}.

We now use (3.14) with $\delta=-$ to exclude the existence of such interpolating functions. Specifically, (iii) entails that the Casorati determinant $C_{-}(b;x,y)$ of $\mathcal{I}(b;x,y)$ and $\mathcal{J}(b;x,y)$ reduces to a multiple of $\mathcal{D}_{-}(b_{+-};x,y)$ for $b=b_{+-}$. Fixing $b\in\mathbb{R}$ not of the form b_{+-}, and choosing a sequence $b_{+-}^{(n)}$ converging to b as $n\to\infty$, the numbers $N_{+}^{(n)}$ and $N_{-}^{(n)}$ must go to ∞. By the assumed continuity in b, this entails that for a suitable multiple $\lambda_{-}^{(n)}$ the $n\to\infty$ limit $L_{-}(b;x,y)$ of $\lambda_{-}^{(n)}Q_{-}(b_{+-}^{(n)};x,y)$ exists. Since $Q_{-}(b_{+-};x,y)$ does not depend on N_{-}, however, L_{-} can only depend on b via a multiplicative constant. Since $L_{-}(b;x,y)$ is also continuous in b and reduces to a multiple of $Q_-(b_{+-};x,y)$ for $b=b_{+-}$, we arrive at the desired contradiction. (Indeed, two distinct choices of N_+ yield two distinct functions Q_- of x, cf. (3.15).)

Returning to the elliptic case, we encounter several snags when we try to emulate the above hyperbolic reasoning. The first one is that we do not know whether there exists a y-interval (C,∞), with C *independent* of $(N_+,N_-)\in\mathbb{N}^2$, on which the constraint system admits solutions. Let us assume this is the case, however. Then the natural requirements for a joint A_δ-eigenfunction $\mathcal{I}(b;x,y)$ that interpolates $\Psi(b_{+-};x,y)$ consist again of the above items (i)–(iii), but now with the region $\mathcal{R}$ equal to the strip $\mathrm{Re}\, x\in(0,\pi/r)$ and y varying over (C,∞). Moreover, in view of the monodromy relation (3.9), the function

$$\mathcal{J}(b;x,y)\equiv\mathcal{I}(b;\pi/r-x,y),\quad (b,x,y)\in\mathbb{R}\times\mathcal{R}\times(C,\infty), \tag{3.16}$$

is then an interpolation of $\Psi(b_{+-};-x,y)$. Comparing the Casorati determinants of $\mathcal{I}$ and $\mathcal{J}$ to (3.14), it follows again that suitable multiples of $Q_\pm(b_{+-};x,y)$ have finite limits $L_\pm(b;x,y)$ for b_{+-} converging to any $b\in\mathbb{R}$.

Unfortunately, the coupling of the two systems for z^+ and z^- via the spectral variable y (cf. (3.16)–(3.18) in [4]) now prevents us from concluding that the limit function $L_-(b;x,y)$ can only depend on b via a multiplicative x-independent factor. (This would again yield a contradiction to $L_-(b;x,y)$ being proportional to $Q_-(b_{+-};x,y)$ when b equals b_{+-}.)

Even so, it seems difficult to believe that the N_--dependence of z^+ (which only makes itself felt in variations of the curve parameter t_+ for a fixed y [4]) could lead to such a variety of distinct limit functions $L_-(b;x,y)$. Moreover, since we have already shown non-existence of interpolations for $r=0$, an elliptic interpolation in the above sense cannot have well-behaved $r\downarrow 0$ limits for arbitrary b, whereas it does have $r\downarrow 0$ limits for the dense set of b-values b_{+-}. In sum, the above renders the existence of an elliptic interpolation extremely unlikely.

To conclude this section, we observe that the above reasoning has a 'local' character, in the following sense. When we restrict the region $\mathcal{R}$ by decreasing the interval over which $\mathrm{Re}\, x$ is allowed to vary, then the notion of $\mathcal{A}_\delta$-eigenfunction still makes sense, since the shifts are in the imaginary x-direction. This observation is relevant in relation to previous results on the existence of *formal* interpolating joint $\mathcal{A}_\pm$-eigenfunctions [15]. Indeed, in view of the above hyperbolic 'no-interpolation' result, the explicit formal power series (2.57)–(2.58) in [15] cannot converge in a half plane $\mathrm{Re}\, x>R$, no matter how large R and y are chosen.

4. HILBERT SPACE RESULTS FOR PARAMETERS IN $\mathcal{C}_{\mathrm{irr}}$

Throughout this section we assume that the parameters belong to $\mathcal{C}$ (1.44). We begin by obtaining an auxiliary result (Lemma 4.1) that is valid for all of $\mathcal{C}$, but which involves an assumption. This assumption is satisfied for parameters in $\mathcal{C}_{\mathrm{irr}}$, so in that case we can proceed much further. In detail, the assumption is that there exist an integer $M \in \mathbb{N}$ (which may depend on the parameters) and pairwise orthogonal vectors

$$\chi_n \in \mathcal{H}_w, \quad n \geq M, \quad n \in \mathbb{N}, \tag{4.1}$$

with

$$\chi_n \in \mathcal{H}_w^{(+)}, \ \ n \text{ even}, \qquad \chi_n \in \mathcal{H}_w^{(-)}, \ \ n \text{ odd}, \tag{4.2}$$

and $n \to \infty$ asymptotics

$$\chi_n(x) = c(b;x)e^{inrx} + c(b;-x)e^{-inrx} + \rho_n(x), \quad x \in (0,\pi/r), \quad n \to \infty, \tag{4.3}$$

where $c(b;x)$ is given by (1.34) and the remainder vectors satisfy

$$\sum_{n>M} (\rho_n, \rho_n)_w < \infty. \tag{4.4}$$

Observe that we are not assuming that the vectors χ_n are $\hat{A}_\delta$-eigenfunctions, even though we aim to apply the result following from the above assumption to that case. The result in question is of a geometric nature: it yields a precise description of the orthogonal complement $\mathcal{L}_w$ of the vectors $\chi_M, \chi_{M+1}, \ldots$

Lemma 4.1. *The $\mathcal{H}_w$-subspace $\mathcal{L}_w$ is M-dimensional. More specifically, we have*

$$\dim(\mathcal{L}_w^{(+)}) = [(M+1)/2], \quad \dim(\mathcal{L}_w^{(-)}) = [M/2], \tag{4.5}$$

where

$$\mathcal{L}_w^{(p)} \equiv \mathcal{L}_w \cap \mathcal{H}_w^{(p)}, \quad p = +, -. \tag{4.6}$$

To prove this lemma, we need some new ingredients. First, it is convenient to switch to the Hilbert space $\mathcal{H}$ (1.14) by a similarity transformation that differs from (2.40), namely,

$$U : \mathcal{H}_w \to \mathcal{H}, \quad f(x) \mapsto w(x)^{1/2} f(x). \tag{4.7}$$

(Here and below, the positive square root of positive functions is taken.) The U-images of the subspaces $\mathcal{H}_w^{(p)}$ are then equal to the $\mathcal{H}$-subspaces

$$\mathcal{H}^{(p)} \equiv \{f \in \mathcal{H} \mid f(\pi/r - x) = pf(x)\}, \quad p = +, -. \tag{4.8}$$

(Recall (2.6).) The asymptotics (4.3) entails

$$U\chi_n = \alpha_n + U\rho_n, \tag{4.9}$$

$$\alpha_n(x) \equiv (-e^{2irx}u(x))^{1/2} e^{inrx} + \text{c.c.}, \quad x \in (0, \pi/r), \tag{4.10}$$

where c.c. stands for complex conjugate, and where the square root sign is determined by requiring

$$(-e^{2irx}u(x))^{1/2} = c(x)/w(x)^{1/2}, \quad x \in (0,\pi/r), \tag{4.11}$$

cf. (1.36)–(1.37).

Next, we recall that for parameters in C the u-function admits the representation

$$u(b;x) = \exp\left(2i\sum_{n=1}^{\infty} s_n(b)\sin(2nrx)\right), \tag{4.12}$$

$$s_n(b) \equiv \frac{\sinh(nr(a_+ - b))\sinh(nr(a_- - b))}{n\sinh(nra_+)\sinh(nra_-)}, \tag{4.13}$$

cf. [6] (4.87). We now define a new c-function

$$c_P(b;x) \equiv (1 - e^{-2irx})^{-1}\exp\left(-\sum_{n=1}^{\infty} s_n(b)e^{-2inrx}\right), \tag{4.14}$$

and w-function

$$\begin{aligned} w_P(b;x) &\equiv 1/c_P(b;x)c_P(b;-x) \\ &= 4\sin^2(rx)\exp\left(2\sum_{n=1}^{\infty} s_n(b)\cos(2nrx)\right). \end{aligned} \tag{4.15}$$

Note that the u-function

$$u_P(b;x) \equiv -e^{-2irx}c_P(b;x)/c_P(b;-x), \tag{4.16}$$

satisfies

$$u_P(b;x) = u(b;x), \tag{4.17}$$

an equality that is crucial in the sequel.

The weight function $w_P(b;x)$ is positive on $(0,\pi/r)$, so by the Gram-Schmidt procedure applied to the functions $1, \cos rx, \cos 2rx, \ldots$ we obtain an orthonormal base

$$P_n(x) \equiv p_n(\cos rx), \qquad n \in \mathbb{N}, \tag{4.18}$$

in the Hilbert space

$$\mathcal{H}_P \equiv L^2((0,\pi/r), w_P(x)dx), \tag{4.19}$$

where $p_n(y)$ are polynomials of degree n. Since we have

$$w_P(\pi/r - x) = w_P(x), \tag{4.20}$$

we obtain an orthogonal decomposition

$$\mathcal{H}_P = \mathcal{H}_P^{(+)} \oplus \mathcal{H}_P^{(-)}, \quad \mathcal{H}_P^{(p)} \equiv \{f \in \mathcal{H}_P \mid f(\pi/r - x) = pf(x)\}, \quad p = +,-, \tag{4.21}$$

with $P_n \in \mathcal{H}_P^{(+)}/\mathcal{H}_P^{(-)}$ for n even/odd. (Indeed, (4.20) implies the polynomials $p_n(y)$ have parity $(-)^n$.)

Next, we explain the rationale of this construction: we have defined $c_P(x)$ and $w_P(x)$ such that the Hilbert space asymptotics of $P_n(x)$ as $n \to \infty$ is explicitly known. Specifically, $w_P(rx)$ belongs to the class $\mathcal{W}_{1,1}$ defined in [10], so from Theorem 2.4 in [10] we deduce

$$P_n(x) = \left(\frac{r}{2\pi}\right)^{1/2} \left(c_P(x)e^{inrx} + c_P(-x)e^{-inrx} + \rho_{n,P}(x)\right), \tag{4.22}$$
$$x \in (0, \pi/r), \quad n \to \infty.$$

Here the remainder vectors $\rho_{n,P}$ satisfy

$$\|\rho_{n,P}\|_P = O(e^{-\kappa n}), \quad \kappa > 0, \quad n \to \infty, \tag{4.23}$$

with $\|\cdot\|_P$ the norm in $\mathcal{H}_P$ (4.19).

Transforming now to $\mathcal{H}$ (1.14) via

$$U_P : \mathcal{H}_P \to \mathcal{H}, \quad f(x) \mapsto w_P(x)^{1/2} f(x), \tag{4.24}$$

we have due to (4.17)

$$U_P P_n = \left(\frac{r}{2\pi}\right)^{1/2} \alpha_n + R_n^{(0)}, \quad R_n^{(0)} \equiv \left(\frac{r}{2\pi}\right)^{1/2} U_P \rho_{n,P}, \tag{4.25}$$

where α_n is given by (4.10). We are now prepared to prove Lemma 4.1.

Proof of Lemma 4.1. Setting

$$\mathcal{L} \equiv U\mathcal{L}_w, \quad \mathcal{L}^{(p)} \equiv U\mathcal{L}_w^{(p)}, \quad p = +, -, \tag{4.26}$$

with U given by (4.7), we need only show that the dimension of $\mathcal{L}$ equals M and that

$$\dim(\mathcal{L}^{(+)}) = [(M+1)/2], \quad \dim(\mathcal{L}^{(-)}) = [M/2]. \tag{4.27}$$

To this end we introduce the vectors

$$\eta_n \equiv \left(\frac{r}{2\pi}\right)^{1/2} U\chi_n, \quad n \geq M, \tag{4.28}$$

and unit vectors

$$\phi_n \equiv \eta_n / \|\eta_n\|, \quad n \geq M \tag{4.29}$$

$$\phi_n^{(0)} \equiv U_P P_n, \quad n \in \mathbb{N}. \tag{4.30}$$

Due to (4.9) and (4.25), we have

$$\phi_n^{(0)} - \eta_n = R_n^{(0)} - R_n, \quad R_n \equiv \left(\frac{r}{2\pi}\right)^{1/2} U\rho_n, \quad n \geq M, \tag{4.31}$$

where (cf. (4.23) and (4.4))

$$\|R_n^{(0)}\| = O(e^{-\kappa n}), \quad n \to \infty, \tag{4.32}$$

$$\sum_{n>M} (R_n, R_n) < \infty. \tag{4.33}$$

Using (4.31), we now estimate the norm of $\phi_n^{(0)} - \phi_n$ in terms of the norms of the remainder vectors:

$$
\begin{aligned}
\|\phi_n^{(0)} - \phi_n\| &\le \|\phi_n^{(0)} - \eta_n\| + \|\eta_n - \phi_n\| \\
&= \|\phi_n^{(0)} - \eta_n\| + |1 - \|\eta_n\|| \\
&\le 2\|\phi_n^{(0)} - \eta_n\| \\
&\le 2(\|R_n^{(0)}\| + \|R_n\|), \qquad n \ge M.
\end{aligned} \tag{4.34}
$$

Thanks to the bounds (4.32) and (4.33), this entails that there exists an integer $I > M$ such that

$$\sum_{n=I}^{\infty} \|\phi_n^{(0)} - \phi_n\|^2 < 1. \tag{4.35}$$

We denote the orthogonal projection on the subspace spanned by $\phi_n, n \ge I$, by P_b, and introduce the complementary projection and subspace

$$P_s \equiv \mathbf{1} - P_b, \quad \mathcal{H}_s \equiv P_s\mathcal{H}. \tag{4.36}$$

(Here, the subscripts stand for 'big' and 'small'.) Clearly, to prove $\dim(\mathcal{L}) = M$ it suffices to show

$$\dim(\mathcal{H}_s) = I. \tag{4.37}$$

We proceed to prove (4.37). Assume that h is a unit vector that is orthogonal to all of the unit vectors

$$\phi_n^{(0)}, \quad n = 0, \ldots, I-1, \quad \phi_n, \quad n \ge I. \tag{4.38}$$

Since $\phi_n^{(0)}, n \in \mathbb{N}$, is an orthonormal base of $\mathcal{H}$, we have

$$1 = \|h\|^2 = \sum_{n=0}^{\infty} |(h, \phi_n^{(0)})|^2 = \sum_{n=I}^{\infty} |(h, \phi_n^{(0)})|^2 = \sum_{n=I}^{\infty} |(h, \phi_n^{(0)} - \phi_n)|^2. \tag{4.39}$$

On the other hand, by the Schwarz inequality and (4.35) we can majorize the rhs by

$$\|h\|^2 \sum_{n=I}^{\infty} \|\phi_n^{(0)} - \phi_n\|^2 < 1, \tag{4.40}$$

so we have arrived at a contradiction.

As a consequence, the span of the unit vectors (4.38) is dense in $\mathcal{H}$. In particular, $\mathcal{H}_s$ must be spanned by the vectors $P_s\phi_n^{(0)}, n = 0, \ldots, I-1$. From this we deduce $\dim(\mathcal{H}_s) \le I$.

Next, we assume $\dim(\mathcal{H}_s) < I$. Then there exists a unit vector

$$\psi = \sum_{n=0}^{I-1} \alpha_n \phi_n^{(0)}, \tag{4.41}$$

such that $P_s\psi = 0$. Hence $P_b\psi = \psi$, so that

$$\psi = \sum_{n=I}^{\infty} \beta_n \phi_n. \tag{4.42}$$

Moreover, since ψ is a unit vector, we have

$$(4.43)\qquad \sum_{n=0}^{I-1} |\alpha_n|^2 = \sum_{n=I}^{\infty} |\beta_n|^2 = 1.$$

Consider now the unit vector

$$(4.44)\qquad \psi^{(0)} \equiv \sum_{n=I}^{\infty} \beta_n \phi_n^{(0)}.$$

On the one hand, we have

$$(4.45)\qquad \|\psi - \psi^{(0)}\|^2 = \|\sum_{n=0}^{I-1} \alpha_n \phi_n^{(0)} - \sum_{n=I}^{\infty} \beta_n \phi_n^{(0)}\|^2 = \sum_{n=0}^{I-1} |\alpha_n|^2 + \sum_{n=I}^{\infty} |\beta_n|^2 = 2.$$

On the other hand, from the Schwarz inequality and (4.35) we obtain

$$(4.46)\qquad \begin{aligned} \|\psi - \psi^{(0)}\|^2 &= \|\sum_{n=I}^{\infty} \beta_n(\phi_n - \phi_n^{(0)})\|^2 \leq \left(\sum_{n=I}^{\infty} |\beta_n| \|\phi_n - \phi_n^{(0)}\|\right)^2 \\ &\leq \sum_{n=I}^{\infty} |\beta_n|^2 \cdot \sum_{n=I}^{\infty} \|\phi_n - \phi_n^{(0)}\|^2 < 1, \end{aligned}$$

contradicting (4,45). Hence we have proved (4.37).

To obtain the stronger result (4,27), we need only repeat the above reasoning for the Hilbert spaces $\mathcal{H}^{(+)}$ and $\mathcal{H}^{(-)}$, recalling that ϕ_n and $\phi_n^{(0)}$ belong to $\mathcal{H}^{(+)}$ for n even and to $\mathcal{H}^{(-)}$ for n odd. As this is merely a matter of introducing suitable notation, we skip the details. □

As already mentioned in the introduction, a large part of the reasoning in this proof can be found on pp. 72–73 of Higgins' monograph [11]. In the latter setting, however, stronger assumptions are made: $\{\phi_n^{(0)}\}_{n\in\mathbb{N}}$ and $\{\phi_n\}_{n\in\mathbb{N}}$ are given sets of pairwise othogonal unit vectors, with the first set assumed complete. Completeness of the second set is then shown to follow from the assumption

$$(4.47)\qquad \sum_{n=0}^{\infty} \|\phi_n^{(0)} - \phi_n\|^2 < \infty.$$

From now on we assume that the parameters belong to C_{irr}. As detailed in Section 3, this restriction ensures the existence of joint $\hat{A}_\delta$-eigenfunctions χ_M, χ_{M+1}, that are pairwise orthogonal and satisfy (4.1)–(4.2). Furthermore, from (3.2)–(3.7) we see that (4.3) holds true, with

$$(4.48)\qquad \rho_n(x) = O(\exp(-2nra_s)), \quad x \in \mathbb{R}, \quad n \to \infty,$$

and the bound uniform on compact subsets of $\mathbb{R}$. Thus (4.4) holds true as well.

The upshot is that all assumptions of Lemma 4.1 are met. The resulting finite-dimensionality of the orthocomplement $\mathcal{L}_w$ of $\chi_M, \chi_{M+1}, \ldots$ (as explicitly expressed in (4.5)) plays a pivotal role in the sequel. Skipping technicalities, we now preview a first crucial consequence of $\dim(\mathcal{L}_w) < \infty$.

We are going to encode the salient features of $\chi_M, \chi_{M+1}, \ldots$ in a subspace D_χ

(4.58) of the domain D_w (2.17), which in turn contains a subspace D_0 (4.63) that is still dense in $\mathcal{H}_w$:

(4.49) $$D_0 \subset D_\chi \subset D_w \subset \mathcal{H}_w.$$

Denoting the orthogonal projection onto the span of $\chi_n, n \geq M$, by P_B and letting $\phi \in D_0$, we then prove

(4.50) $$\phi_B \equiv P_B \phi \in D_\chi$$

in Lemma 4.2 below. The vector $\phi - \phi_B$, which belongs to $\mathcal{L}_w$ by construction, therefore belongs to D_χ as well. The crux is now that the finite dimension of $\mathcal{L}_w$ allows us to deduce that $\mathcal{L}_w$ is a *subspace* of D_χ, cf. the paragraph containing (4.77). (If $\mathcal{L}_w$ were infinite-dimensional, this would not follow, so that we would be left in the dark as concerns the character of vectors in $\mathcal{L}_w$.)

Turning to the details, we begin by defining the products

(4.51) $$\Pi_\delta(x) \equiv \prod_{j=-N_{-\delta}}^{N_{-\delta}} s_\delta(x - ija_{-\delta}), \quad \delta = +, -.$$

Recalling (3.2) and (1.37), we see that the w-function can be written

(4.52) $$w(b_{+-}; x) = (-)^{N_+ + N_- + 1} \mathcal{N}^{-2} s_-(x) s_+(x) \Pi_-(x)/\Pi_+(x).$$

Furthermore, (3.3) and (3.7) entail

(4.53) $$\chi_n(x) = \mathcal{N} \psi_n(x)/\Pi_-(x),$$

where we have introduced

(4.54) $$\psi_n(x) \equiv \mathcal{H}(x, nr) - \mathcal{H}(-x, nr), \quad n \geq M.$$

Next, we define the vector space

(4.55) $$\begin{aligned} \mathcal{O} \equiv \{F \in \mathcal{P}_{2\pi/r} \mid F(x) \text{ entire, odd,} \\ F(z_{kl}) = F(z_{kl} + \pi/r) = 0, \quad |k| \leq N_+, \ |l| \leq N_-\}, \end{aligned}$$

where

(4.56) $$z_{kl} \equiv ika_+ + ila_-, \quad k, l \in \mathbb{Z}.$$

(This space is the same space as $\mathcal{O}_1$ (4.11) in [6].) Due to the identities (3.10), we have

(4.57) $$\psi_n \in \mathcal{O}, \quad n \geq M.$$

Therefore, setting

(4.58) $$D_\chi \equiv \Pi_-(x)^{-1} \mathcal{O},$$

we obtain from (4.53)

(4.59) $$\chi_n \in D_\chi, \quad n \geq M.$$

It is not hard to check the inclusion

$$(4.60) \qquad D_\chi \subset D_w,$$

announced above (cf. (4.49)), as well as the decomposition

$$(4.61) \qquad D_\chi = D_\chi^{(+)} \oplus D_\chi^{(-)}, \quad D_\chi^{(\pm)} \equiv D_\chi \cap \mathcal{H}_w^{(\pm)}.$$

We now introduce spaces

$$(4.62) \qquad D_0^{(p)} \equiv s_+(x)\Pi_+(x)\mathrm{Pol}^{(p)}, \quad p = +, -,$$

$$(4.63) \qquad D_0 \equiv D_0^{(+)} \oplus D_0^{(-)}.$$

(The space $\Pi_-(x)D_0$ equals the space $\mathcal{O}_-$ (4.18) in [6].) Since $s_+(x)\Pi_+(x)\Pi_-(x)p(\cos(rx)) \in \mathcal{O}$ for a polynomial $p(y)$, we have

$$(4.64) \qquad D_0^{(p)} \subset D_\chi^{(p)}, \quad p = +, -.$$

Moreover, $D_0^{(p)}$ is readily seen to be dense in $\mathcal{H}_w^{(p)}$, so the same is true for $D_\chi^{(p)}$. We are now prepared for our next lemma.

Lemma 4.2. *Denoting the projection onto the $\mathcal{H}_w$-subspace spanned by $\chi_n, n \geq M$, by P_B, we have*

$$(4.65) \qquad P_B D_0 \subset D_\chi.$$

Proof. Fixing a polynomial $p(y)$, we set

$$(4.66) \qquad \phi(x) \equiv s_+(x)\Pi_+(x)P(x), \quad P(x) \equiv p(\cos(rx)),$$

so that $\phi \in D_0$. Putting $\phi_B \equiv P_B\phi$, we should show (4.50) or, equivalently (cf. (4.58)),

$$(4.67) \qquad \Pi_-(x)\phi_B(x) \in \mathcal{O}.$$

In order to prove (4.67), we begin by noting

$$(4.68) \qquad \phi_B = \sum_{n \geq M} \chi_n(\chi_n, \phi)_w / (\chi_n, \chi_n)_w.$$

We now estimate the two inner products on the rhs. Using (4.52), (3.7) and (4.66), we have

$$(4.69) \qquad (\chi_n, \phi)_w = \int_0^{\pi/r} dx w(x)\chi_n^*(x)\phi(x) = C \int_{-\pi/r}^{\pi/r} dx s_-(x)s_+(x)^2 \mathcal{H}(x, nr)P(x),$$

where C is a constant. Recalling (3.4), we see that this is of the form

$$(4.70) \qquad (\chi_n, \phi)_w = \int_{-\pi/r}^{\pi/r} dx \left(e(x) \prod_{\delta=+,-} \prod_{j=1}^{N_\delta} s_{-\delta}(x + z_j^\delta(nr)) \right) e^{inrx},$$

with $e(x)$ entire and $2\pi/r$-periodic. Shifting the contour up by iR, we therefore obtain

$$(4.71)\qquad (\chi_n, \phi)_w = e^{-nrR} \int_{-\pi/r}^{\pi/r} dx \left(e(x+iR) \prod_{\delta=+,-} \prod_{j=1}^{N_\delta} s_{-\delta}(x + iR + z_j^\delta(nr)) \right) e^{inrx}.$$

Recalling (3.5), we readily deduce

$$(4.72)\qquad |(\chi_n, \phi)_w| \le C_R e^{-nrR}, \quad n \ge M,$$

where $R > 0$ is arbitrary.

Next, using (4.3) and (1.36) we obtain

$$(4.73)\qquad (\chi_n - \rho_n, \chi_n - \rho_n)_w = 2\pi/r - \int_{-\pi/r}^{\pi/r} dxu(x)e^{2irx}e^{2inrx}.$$

Since $b \in (0, a_+ + a_-)$, the u-function is analytic in a strip $|\mathrm{Im}\, x| \le \kappa$, $\kappa \in (0, a_s]$ (cf. the representation (4.12)–(4.13)). Since it is also $2\pi/r$-periodic, the Fourier coefficient in (4.73) is $O(\exp(-2nr\kappa))$ as $n \to \infty$. Recalling the bound (4.48), we infer

$$(4.74)\qquad (\chi_n, \chi_n)_w = 2\pi/r + O(e^{-2nr\kappa}), \quad n \to \infty.$$

On the other hand, in view of (3.4) and (3.5) the entire functions $\psi_n(x)$ (4.54) satisfy

$$(4.75)\qquad |\psi_n(x)| \le C_d e^{nrd}, \quad n \ge M, \quad |\mathrm{Im}\, x| \le d,$$

where $d > 0$ is arbitrary. Therefore the series

$$(4.76)\qquad \sum_{n \ge M} \psi_n(x)(\chi_n, \phi)_w/(\chi_n, \chi_n)_w$$

converges absolutely and uniformly on arbitrary strips $|\mathrm{Im}\, x| \le d$, yielding a function in $\mathcal{O}$ (4.55), cf. (4.57). Recalling (4.68) and (4.53), this entails (4.67). □

As already sketched in the paragraph containing (4.50), we can now combine Lemmas 4.1 and 4.2 to deduce the critical inclusion

$$(4.77)\qquad \mathcal{L}_w \subset D_\chi.$$

Indeed, with (4.50) proved in Lemma 4.2, we infer $\phi - \phi_B \in D_\chi$. Now we also have $\phi - \phi_B \in \mathcal{L}_w$ and we know that when ϕ ranges over D_0, the vectors $\phi - \phi_B$ are dense in $\mathcal{L}_w$. Since $\dim(\mathcal{L}_w) < \infty$ by Lemma 4.1, the vectors $\phi - \phi_B$ must range over *all* of $\mathcal{L}_w$, and so (4.77) follows.

We are at last prepared to derive further consequences of Lemma 4.1 and (4.77) for the Hilbert space operators $\hat{A}_\pm$ introduced in Section 2. In view of (4.77) and (4.60), they are well defined on $\mathcal{L}_w$. Since they are symmetric (by Theorem 2.1) and leave the linear hull of $\chi_n, n \ge M$, invariant, they map $\mathcal{L}_w$ into itself. Since their action amounts to the action of the AΔOs $A_\pm$, they commute on $\mathcal{L}_w$ and leave the subspaces $\mathcal{L}_w^{(\pm)}$ invariant. Finally, since they are symmetric, the dimension formula (4.5) entails there exist pairwise orthogonal joint eigenvectors $\chi_0, \chi_2, \ldots, \chi_{2[(M-1)/2]}$ spanning $\mathcal{L}_w^{(+)}$ and $\chi_1, \chi_3, \ldots, \chi_{2[M/2]-1}$ spanning $\mathcal{L}_w^{(-)}$, the eigenvalues being real.

In the following theorem we summarize and extend these results.

Theorem 4.3. *Assume that the parameters belong to $C_{\rm irr}$ (1.47). Then there exists an orthogonal base of joint $\hat{A}_\pm$-eigenvectors*

(4.78) $$\chi_n \in D_\chi, \quad n \in \mathbb{N},$$

with $\chi_n \in D_\chi^{(+)}$ for n even and $\chi_n \in D_\chi^{(-)}$ for n odd. The $\hat{A}_\pm$-eigenvalues on χ_n satisfy

(4.79) $$E_{n,\pm} \in (0, \infty), \quad n \in \mathbb{N}.$$

The operators $\hat{A}_\pm$ are essentially self-adjoint on their definition domain D_w (2.17) and any vector f in the domain of the self-adjoint closure of $\hat{A}_\delta$ has the following properties: f is the restriction to $(0, \pi/r)$ of a function $f(x)$ that is meromorphic in the strip $|\mathrm{Im}\, x| < a_{-\delta}$, its only poles occurring at the locations (2.16) and being of multiplicity at most one; moreover, $f(x)$ is even and $2\pi/r$-periodic.

Proof. We have already proved the first assertion. Recalling (2.37) and (2.36), we deduce (4.79). Since $\hat{A}_+$ and $\hat{A}_-$ are e.s.a. on the span of their eigenvectors $\chi_n, n \in \mathbb{N}$, they are a fortiori e.s.a. on D_χ and D_w.

It remains to prove the asserted properties of f. To this end we write f as

(4.80) $$f = \sum_{n=0}^{\infty} \chi_n (\chi_n, f)_w / (\chi_n, \chi_n)_w.$$

Then we have

(4.81) $$(\hat{A}_\delta)^{-} f = \sum_{n=0}^{\infty} E_{n,\delta} \chi_n (\chi_n, f)_w / (\chi_n, \chi_n)_w,$$

the series converging in the strong $\mathcal{H}_w$-topology. Recalling (4.74), we deduce

(4.82) $$E_{n,\delta} (\chi_n, f)_w = O(1), \quad n \to \infty.$$

Now $E_{n,\delta}$ equals $E_\delta(nr)$ for $n \geq M$, so (3.6) entails

(4.83) $$(\chi_n, f)_w = O(\exp(-nra_{-\delta})), \quad n \to \infty.$$

Combining (4.83) with (4.80), we readily obtain the pertinent f-properties from arguments already detailed in the proof of Lemma 4.2. More specifically, (4.83) plays the role of (4.72), and comparing it to the bound (4.75), we see that the series

(4.84) $$\sum_{n=0}^{\infty} \Pi_-(x) \chi_n(x) (\chi_n, f)_w / (\chi_n, \chi_n)_w,$$

whose terms are functions in $\mathcal{O}$ (4.55), converges absolutely and uniformly on any strip $|\mathrm{Im}\, x| \leq a_{-\delta} - \epsilon$, $\epsilon > 0$. The eventual simple poles then arise as before from multiplication by $\Pi_-(x)^{-1}$. (They are the zeros of $\Pi_-(x)$ in $|\mathrm{Im}\, x| < a_{-\delta}$ that are not matched by the zeros of functions in $\mathcal{O}$.) □

It seems not an easy task to characterize the boundary values of $f(x)$ (4.80) at the pertinent strip boundaries. In this connection it should be noted that the ranges of the self-adjoint operators $(\hat{A}_\pm)^-$ are equal to $\mathcal{H}_w$, since their inverses are bounded due to (4.79) and (3.6). (In fact, their inverses are even trace class, as follows from (3.6).)

To conclude, we state a conjecture.

Conjecture 4.4. *All of the joint eigenvectors $\chi_n, n \in \mathbb{N}$, are of the form*

$$\chi_n(b_{+-};x) = \mathcal{N} \prod_{j=-N_+}^{N_+} \frac{1}{s_-(x - ija_+)} (\mathcal{H}_n(b_{+-};x) - \mathcal{H}_n(b_{+-};-x)), \tag{4.85}$$

where $\mathcal{H}_n$ is given by

$$\mathcal{H}_n(b_{+-};x) = \prod_{\delta=+,-} \prod_{j=1}^{N_\delta} s_{-\delta}(x + z_{j,n}^\delta) \cdot \exp[(2N_+N_- + N_+ + N_- + 1 + n)irx], \tag{4.86}$$

for certain complex numbers $z_{j,n}^\delta$.

For $n \geq M(b_{+-})$ we already know this is true, cf. (3.3)–(3.8). We believe that this conjecture might be proved by a more refined analysis of the constraint system, yielding equality of $z_{j,n}^\delta$ to $z_j^\delta(nr)$ for *all* $n \in \mathbb{N}$. We stress that even if this could be pushed through, the arguments related to Lemma 4.1 would still be needed to prove *completeness* of $\{\chi_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}_w$.

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