Special functions defined by analytic difference equations

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Abstract

We survey our work on a number of special functions that can be viewed as solutions to analytic difference equations. In the infinite-dimensional solution spaces of the pertinent equations, these functions are singled out by various distinctive features. In particular, starting from certain first order difference equations, we consider generalized gamma and zeta functions, as well as Barnes' multiple zeta and gamma functions. Likewise, we review the generalized hypergeometric function we introduced in recent years, emphasizing the four second order Askey-Wilson type difference equations it satisfies. Our results on trigonometric, elliptic and hyperbolic generalizations of the Hurwitz zeta function are presented here for the first time.

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1 Introduction

In the following we survey various special functions whose common feature is that they can be viewed as analytic solutions to (ordinary linear) analytic difference equations with analytic coefficients. Some of these functions have been known and studied for centuries, whereas a few others are of quite recent vintage.

The difference equations that will be relevant for our account are of three types. The first type is the first order equation

$$\frac{F(z+ia/2)}{F(z-ia/2)} = \Phi(z), \quad a > 0, \tag{1.1}$$

and the second type its logarithmic version,

$$f(z+ia/2) - f(z-ia/2) = \phi(z). \tag{1.2}$$

The third type is the second order equation

$$C^{(+)}(z)F(z+ia) + C^{(-)}(z)F(z-ia) + C^{(0)}(z)F(z) = 0, \quad a > 0. \tag{1.3}$$

Introducing the spaces

$$\mathcal{M} \equiv \{F(z) \mid F \text{ meromorphic}\}, \tag{1.4}$$

$$\mathcal{M}^* \equiv \mathcal{M} \setminus \{F(z) = 0, \forall z \in \mathbb{C}\}, \tag{1.5}$$

we require

$$\Phi, C^{(\pm)}, C^{(0)} \in \mathcal{M}^*, \tag{1.6}$$

and we only consider solutions $F \in \mathcal{M}$ to the equations (1.1) and (1.3). The right-hand side functions $\phi(z)$ in (1.2) are allowed to have branch points, giving rise to solutions $f(z)$ that have branch points, as well.

We are primarily interested in analytic difference equations (henceforth AΔEs) admitting solutions with various properties that render them unique. In this connection a crucial point to be emphasized at the outset is that the solutions form an infinite-dimensional space (assuming at least one non-trivial solution exists). More specifically, introducing spaces of α-periodic multipliers,

$$\mathcal{P}_\alpha^{(*)} \equiv \{\mu(z) \in \mathcal{M}^{(*)} \mid \mu(z+\alpha) = \mu(z)\}, \quad \alpha \in \mathbb{C}^*, \tag{1.7}$$

and assuming $F \in \mathcal{M}^*$ solves (1.1) or (1.3), it is obvious that $\mu(z)F(z), \mu \in \mathcal{P}_{ia}^*$, is also a solution. Likewise, adding any $\mu \in \mathcal{P}_{ia}^*$ to a given solution $f(z)$ of (1.2), one obtains another solution.

In view of this infinite-dimensional ambiguity, it is natural to try and single out solutions by requiring additional properties. In a nutshell, this is how the special functions at issue in this contribution will arise. The extra properties alluded to will be made explicit in Section 2, with Subsection 2.1 being devoted to the first order equations (1.1) and (1.2), and Subsection 2.2 to the second order one (1.3).

Section 2 is the only section of a general nature. In the remaining sections we focus on special AΔEs, giving rise to generalized gamma functions (Section 3), generalized zeta functions (Section 4), the multiple zeta and gamma functions introduced and studied by Barnes (Section 5), and a novel generalization of the hypergeometric function ${}_2F_1$ (Section 6). This involves special AΔEs of the multiplicative type (1.1) for the various gamma functions, of the additive type (1.2) for the zeta functions, and of the second order type (1.3) for our generalized hypergeometric function.

The special functions surveyed in Sections 3 and 6 play an important role in the context of quantum Calogero-Moser systems of the relativistic variety. Here we will not address such applications. We refer the interested reader to our lecture notes Refs. [1, 2] for an overview of Calogero-Moser type integrable systems and the special functions arising in that setting. The present survey overlaps to some extent with Sections 2 and 3 of Ref. [2], but for lack of space we do not consider generalized Lamé functions. The latter can also be regarded as special solutions to second order AΔEs. They are surveyed in Section 4 of Ref. [2] and in Ref. [3].

2 Ordinary linear A$\triangle$Es

In most of the older literature, nth order ordinary linear AΔEs are written in the form

$$\sum_{k=0}^{n} c_k(w)u(w+k) = 0, \quad w \in \mathbb{C}. \tag{2.1}$$

Here, the coefficients $c_0(w), \ldots, c_n(w)$ have some specified analyticity properties, and one is looking for solutions $u(w)$ with corresponding properties. We have adopted a slightly different form for the AΔEs (1.1), (1.3), since this is advantageous for most of the special cases at issue. (Note that these AΔEs can be written in the form (2.1) by taking $z \to iw$, shifting and scaling.)

To provide some historical perspective, we mention that the subject of analytic difference equations burgeoned in the late 18th century and was vigorously pursued in the 19th century. In the course of the 20th century, this activity subsided considerably. Nörlund was one of the few early 20th century authors still focussing on (ordinary, linear) AΔEs. His well-known 1924 monograph Ref. [4] summarized the state of the art, in particular as regards his own extensive work. Later monographs from which further developments can be traced include Milne-Thompson (1933) [5], Meschkowski (1959) [6], and Immink (1984) [7].

Toward the end of the 20th century, it became increasingly clear that a special class of partial analytic difference equations plays an important role in the two related areas of quantum integrable N-particle systems and quantum groups. An early appraisal of this state of affairs (mostly from the perspective of integrable systems) can be found in our survey Ref. [8]. The most general class of AΔEs involved can be characterized by the coefficient functions that occur: They are combinations of Weierstrass σ-functions (equivalently, Jacobi theta functions).

Even for the case of ordinary AΔEs (to which we restrict attention in this contribution), this class of coefficients, referred to as 'elliptic' for brevity, had not been studied in previous literature. Our outlook on the general first order case, which we present in Subsection 2.1, arose from the need to handle the elliptic case and its specializations. In the same vein, the aspects of the second order case dealt with in Subsection 2.2 are anticipating later specializations. In particular, we point out some questions of a general nature that, to our knowledge, have not been addressed in the literature, and whose answers would have an important bearing on the special second order AΔEs for which explicit solutions are known.

2.1 First order AΔEs

In this subsection we summarize some results concerning AΔEs of the forms (1.1) and (1.2), which we will use for the special cases in Sections 3–5. The pertinent results are taken from our paper Ref. [9], where proofs and further details can be found.

To start with, let us point out that all solutions to (1.1) satisfy

$$F(z+ika) \equiv \prod_{j=1}^{k} \Phi(z+(j-1/2)ia) \cdot F(z), \tag{2.2}$$

$$F(z-ika) \equiv \prod_{j=1}^{k} \frac{1}{\Phi(z-(j-1/2)ia)} \cdot F(z). \tag{2.3}$$

Whenever $\Phi(x+iy), x, y \in \mathbb{R}$, converges to 1 for $y \to \infty$, uniformly for x varying over arbitrary compact subsets of $\mathbb{R}$ and sufficiently fast, the infinite product

$$F_+(z) \equiv \prod_{j=1}^{\infty} \frac{1}{\Phi(z+(j-1/2)ia)} \tag{2.4}$$

defines a solution referred to as the upward iteration solution. Similarly, the downward iteration solution

$$F_-(z) \equiv \prod_{j=1}^{\infty} \Phi(z-(j-1/2)ia) \tag{2.5}$$

exists provided $\Phi(x+iy) \to 1$ for $y \to -\infty$ (uniformly on x-compacts and sufficiently fast).

The restrictions on $\Phi(z)$ for iteration solutions to exist are clearly quite strong. Indeed, they are violated in most of the special cases considered below. It is however possible to modify the iteration procedure, so as to handle larger classes of right-hand side functions $\Phi(z)$. This is the approach taken by Nörlund [4], which gives rise to the special solution he refers to as the 'Hauptlösung' (cf. also Ref. [10]).

Nörlund's methods do not apply to the case where $\Phi(z)$ is an elliptic or hyperbolic function, whereas the solution methods we consider next do not apply to all of Nörlund's class of $\Phi(z)$. The latter methods, however, can be used in particular for elliptic and hyperbolic right-hand sides $\Phi(z)$. (Specifically, by exploiting the elliptic and hyperbolic gamma functions of Subsections 3.3 and 3.4.)

We begin by imposing a rather weak requirement: We restrict attention to functions $\Phi(z)$ without poles and zeros in a strip $|\mathrm{Im}\, z| < c, c > 0$. (Usually, this can be achieved by a suitable shift of z.) Then we can take logarithms to trade the multiplicative AΔE (1.1) for the additive one (1.2), with z restricted to the strip $|\mathrm{Im}\, z| < c$. Let us next require that $\phi(z)$ have at worst polynomial increase on the real axis. Then we may and will restrict attention to solutions $f(z)$ that are polynomially bounded in the strip $|\mathrm{Im}\, z| \le a/2$ and that are moreover analytic for $|\mathrm{Im}\, z| < c + a/2$.

We call solutions $f(z)$ of the form just delineated *minimal* solutions: Their analytic behavior and asymptotics for $|\mathrm{Re}\, z| \to \infty$ are optimal. The polynomial boundedness is critical in proving that minimal solutions are unique up to an additive constant, whenever they exist. (Note that entire ia-periodic functions such as $\mathrm{ch}(2\pi z/a)$ are not polynomially bounded.)

It is readily verified that the resulting function $F(z) = \exp(f(z))$ extends to a meromorphic solution of (1.1) that has no poles and zeros for $|\mathrm{Im}\, z| < c + a/2$, and whose logarithm is polynomially bounded for $|\mathrm{Im}\, z| \leq a/2$. Once more, solutions to (1.1) with the latter properties are termed *minimal*, and now they are unique up to a multiplier $\alpha \exp(2\pi kz/a), \alpha \in \mathbb{C}^*, k \in \mathbb{Z}$. (The exponential ambiguity reflects the ambiguity in the branch choice for $\ln \Phi(z)$.)

A large class of right-hand sides arises as follows. Assume that $\phi(z)$ satisfies (in addition to the above)

$$\phi(x) \in L^1(\mathbb{R}), \quad \hat{\phi}(y) \in L^1(\mathbb{R}), \quad \hat{\phi}(y) = O(y), \quad y \to 0. \tag{2.6}$$

Here, $\hat{\phi}(y)$ denotes the Fourier transform, normalized by

$$\hat{\phi}(y) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \phi(x) e^{ixy}. \tag{2.7}$$

Then the function

$$f(z) = \int_{-\infty}^{\infty} dy \frac{\hat{\phi}(2y)}{\mathrm{sh} ay} e^{-2iyz}, \quad |\mathrm{Im}\, z| \leq a/2, \tag{2.8}$$

is clearly well defined, and analytic for $|\mathrm{Im}\, z| < a/2$. Proceeding formally, it is also obvious that $f(z)$ satisfies (1.2) (by virtue of the Fourier inversion formula).

As a matter of fact, it can be shown that the function $f(z)$ defined by (2.8) analytically continues to $|\mathrm{Im}\, z| < c + a/2$ and indeed obeys the AΔE (1.2). Moreover, $f(z)$ is bounded in the strip $|\mathrm{Im}\, z| \leq a/2$ and goes to 0 for $|\mathrm{Re}\, z| \to \infty$ in the latter strip. (Therefore, it is a *minimal* solution.) The solution $f(z)$ is uniquely determined by these properties. A useful alternative representation reads

$$f(z) = \frac{1}{2ia} \int_{-\infty}^{\infty} du \phi(u) \mathrm{th} \frac{\pi}{a}(z - u), \quad |\mathrm{Im}\, z| < a/2. \tag{2.9}$$

(See Theorem II.2 in Ref. [9] for proofs of the facts just mentioned.)

Another extensive class of right-hand sides $\Phi(z)$ admitting minimal solutions arises by assuming that $\Phi(z)$ has a real period, in addition to the standing assumption of absence of zeros and poles in a strip $|\mathrm{Im}\, z| < c$. To anticipate our later needs, we denote this period by $\pi/r, r > 0$. Now $\phi(x) = \ln \Phi(x), x \in \mathbb{R}$, is well defined up to a multiple of $2\pi i$. We assume first that $\Phi(x)$ has zero winding number around 0 in the period interval $[-\pi/2r, \pi/2r]$. Then $\phi(x)$ is a smooth π/r-periodic function, too. Defining its Fourier coefficients by

$$\hat{\phi}_n \equiv \frac{r}{\pi} \int_{-\pi/2r}^{\pi/2r} dx \phi(x) e^{2inrx}, \quad n \in \mathbb{Z}, \tag{2.10}$$

we also assume at first $\hat{\phi}_0 = 0$.

With these assumptions in effect, it is clear that the function

$$f(z) = \frac{1}{2} \sum_{n \in \mathbb{Z}^*} \frac{\hat{\phi}_n e^{-2inrz}}{\mathrm{sh} nra}, \quad |\mathrm{Im}\, z| \leq a/2, \tag{2.11}$$

is well defined, and analytic for $|\mathrm{Im}\, z| < a/2$. Again, it is formally obvious (by Fourier inversion) that $f(z)$ satisfies the AΔE (1.2). Once more, it can be shown that $f(z)$ (2.11) has an analytic continuation to $|\mathrm{Im}\, z| < c + a/2$ and solves (1.2) (cf. Theorem II.5 in Ref. [9]). Thus one obtains minimal solutions to the AΔE (1.1).

Next, we point out that the assumption $\hat{\phi}_0 = 0$ is not critical. Indeed, when $\hat{\phi}_0 \neq 0$, one need only add the function $\hat{\phi}_0 z/ia$ to $f(z)$ (2.11) to obtain a solution. Of course, this entails that $f(z)$ is no longer π/r-periodic.

The obvious generalization detailed in the previous paragraph is relevant to the case in which $\Phi(z)$ has winding number $l \in \mathbb{Z}^*$ on $[-\pi/2r, \pi/2r]$. Then one needs to take the z-derivative of the AΔE (1.2) to obtain a rhs $\phi'(z)$ that is π/r-periodic, but for which the zeroth Fourier coefficient of $\phi'(x)$ equals $2irl$, cf. (2.10). Thus the solution $f'(z)$ obtained as just sketched has a term linear in z, and so $f(z)$ is the sum of a π/r-periodic function and a quadratic function. Clearly, this still gives rise to a minimal solution $F(z) = \exp f(z)$ to the AΔE (1.1).

The idea underlying this construction is easily generalized to functions $\Phi(z)$ for which a suitable derivative of $\ln \Phi(z)$ has period π/r. Likewise, the Fourier transform method yielding the minimal solution (2.8) can be generalized to $\Phi(z)$ for which $\phi(z) \equiv \partial_z^k \ln \Phi(z)$ has the three properties (2.6) for a suitable $k \in \mathbb{N}^*$. (See Theorems II.3 and II.6 in Ref. [9] for the details.)

We have thus far focussed on the AΔE (1.1) with meromorphic rhs $\Phi(z)$ and its logarithmic version (1.2). But we may also consider (1.2) in its own right, relaxing the requirement that $\exp \phi(z)$ be meromorphic to the requirement that $\phi(z)$ be analytic for $|\mathrm{Im}\, z| < c$. In that case the above —inasmuch as it deals with (1.2)—is still valid. Moreover, the analytic continuation behavior of the minimal solution $f(z)$ *follows* from the AΔE (1.2), in the sense that one can extend $f(z)$ beyond the strip $|\mathrm{Im}\, z| < c + a/2$, whenever $\phi(z)$ can be extended beyond the strip $|\mathrm{Im}\, z| < c$. For example, a function $\phi(z)$ that has branch points at the edges of the latter strip gives rise to a minimal solution $f(z)$ that has the same type of branch points at the edges of the former strip.

This more general version of (1.2) is relevant for the various zeta functions encountered below. In those cases the pertinent functions $\phi(z)$ are analytic in a half plane $\mathrm{Im}\, z > -c$ or in a strip $|\mathrm{Im}\, z| < c$, and they extend to multi-valued functions on all of $\mathbb{C}$ due to certain branch points for $|\mathrm{Im}\, z| \geq c$. Using the AΔE, the zeta functions admit a corresponding extension. We will not spell that out, however, but focus on properties pertaining to the analyticity half plane or strip.

2.2 Second order AΔEs

We begin by recalling some well-known facts concerning the general second order AΔE (1.3), cf. Ref. [4]. Let $F_1, F_2 \in \mathcal{M}^*$ be two solutions. Then their Casorati determinant,

$$\mathcal{D}(F_1, F_2; z) \equiv F_1(z + ia/2)F_2(z - ia/2) - F_1(z - ia/2)F_2(z + ia/2), \qquad (2.12)$$

vanishes identically if and only if $F_1/F_2 \in \mathcal{P}_{ia}$. Assuming from now on $F_1/F_2 \notin \mathcal{P}_{ia}$, it is readily verified that the function (2.12) solves the first order AΔE

$$\frac{\mathcal{D}(z + ia/2)}{\mathcal{D}(z - ia/2)} = \frac{C^{(-)}(z)}{C^{(+)}(z)}. \qquad (2.13)$$

Next, assume $F_3(z)$ is a third solution to (1.3). Then the functions

$$\mu_j(z) \equiv \mathcal{D}(F_j, F_3; z + ia/2)/\mathcal{D}(F_1, F_2; z + ia/2), \quad j = 1, 2, \tag{2.14}$$

belong to $\mathcal{P}_{ia}$ (1.7). (Indeed, quotients of Casorati determinants are ia-periodic in view of the AΔE (2.13).) It is routine to check that one has

$$F_3(z) = \mu_1(z)F_2(z) - \mu_2(z)F_1(z). \tag{2.15}$$

Conversely, any function of this form with $\mu_1, \mu_2 \in \mathcal{P}_{ia}$ solves (1.3). Therefore, whenever two solutions F_1, F_2 exist with $\mathcal{D}(F_1, F_2; z) \in \mathcal{M}^*$, the solution space may be viewed as a two-dimensional vector space over the field $\mathcal{P}_{ia}$ of ia-periodic meromorphic functions.

In contrast to ordinary second order differential and discrete difference equations, various natural existence questions have apparently not been answered in the literature. As a first example, the existence of a solution basis as just considered seems not to be known in general.

Further questions arise in the following situation. Assume that two AΔEs of the above form,

$$C_\delta^{(+)}(z)F(z + ia_{-\delta}) + C_\delta^{(-)}(z)F(z - ia_{-\delta}) + C_\delta^{(0)}(z)F(z) = 0, \quad a_\delta > 0, \quad \delta = +, -, \tag{2.16}$$

are given. Then one may ask for conditions on the two sets of coefficients such that joint solutions $F \in \mathcal{M}^*$ exist. Next, assume there do exist two joint solutions to (2.16) whose Casorati determinants w.r.t. a_+ and a_- belong to $\mathcal{M}^*$. When a_+/a_- is rational, this entails that the joint solution space is infinite-dimensional. (Indeed, letting $a_+ = pa$ and $a_- = qa$ with p, q coprime integers, one can allow arbitrary multipliers in $\mathcal{P}_{ia}$.) Now it is not hard to see that one has

$$\mathcal{P}_{ia_+} \cap \mathcal{P}_{ia_-} = \mathbb{C}, \qquad a_+/a_- \notin \mathbb{Q}. \tag{2.17}$$

But it is not clear whether this entails that the joint solution space is two-dimensional for irrational a_+/a_-.

On the other hand, assuming $F_1, F_2 \in \mathcal{M}^*$ are two joint solutions to (2.16) with a_+/a_- irrational, there is a quite useful extra assumption guaranteeing that the solution space is two-dimensional with basis $\{F_1, F_2\}$. Specifically, one need only assume

$$\lim_{\mathrm{Im}\, z \to \infty} F_1(z)/F_2(z) = 0, \tag{2.18}$$

for all $\mathrm{Re}\, z$ in some open interval. Indeed, the sufficiency of this condition can be easily gleaned from the proof of Theorem B.1 in Ref. [11], where certain special cases are considered.

Questions about joint solutions arise from commuting operator pairs, not only in the elliptic context of Ref. [11], but also in the hyperbolic one of Ref. [12] and Section 6. In order to detail the latter setting in general terms, consider the eigenvalue problems for the analytic difference operators

$$A_\delta \equiv C_\delta(z)T_{ia_{-\delta}} + C_\delta(-z)T_{-ia_{-\delta}} + V_\delta(z), \quad \delta = +, -, \tag{2.19}$$

on $\mathcal{M}$, with $C_\delta \in \mathcal{P}^*_{ia_\delta}$, $V_\delta \in \mathcal{P}_{ia_\delta}$, and $T_{\pm ia_\delta}$ given by

$$(T_\alpha F)(z) \equiv F(z - \alpha), \quad \alpha \in \mathbb{C}^*. \tag{2.20}$$

Thus we are interested in solutions to the AΔEs

$$A_\delta F = E_\delta F, \quad E_\delta \in \mathbb{C}, \quad \delta = +, -. \tag{2.21}$$

Fixing E_+ and E_-, these are of the form considered before. Moreover, since A_+ and A_- clearly commute, it is reasonable to ask for *joint* solutions.

In Section 6 we encounter an operator pair with this structure. Furthermore, we have one joint solution $R(z)$ available for a family of joint eigenvalues $(E_+(p), E_-(p)), p \in \mathbb{C}$. Fixing p, the three solutions $R(z+ia_+), R(z-ia_+)$ and $R(z)$ to the AΔE $A_+F = E_+(p)F$ are related via the AΔE $(A_-R)(z) = E_-(p)R(z)$ with coefficients in $\mathcal{P}_{ia_-}$, in agreement with the general theory. Put differently, when one fixes attention on one of the AΔEs, the other AΔE can be viewed as an extra requirement of monodromy type.

The coefficients occurring in Section 6 are analytic in the shift parameters a_+, a_- for $a_+, a_- \in \mathbb{C}^*$, and entire in four extra parameters. The joint solution $R(z)$ is real-analytic in a_+, a_- for $a_+, a_- \in (0, \infty)$ and meromorphic in the extra parameters. But we do not know whether another joint solution exists for general parameters. On the other hand, upon *specializing* the extra parameters, we do obtain two joint solutions for all $a_+, a_- \in (0, \infty)$ and $p \in \mathbb{C}$, which moreover satisfy the extra condition (2.18) for all $\mathrm{Re}\, z > 0$ and p in the right half plane, cf. Ref. [12].

In the latter article we also arrive at a commuting analytic difference operator pair $B_\delta(b), b \in \mathbb{C}$, as considered above, which depends analytically on the extra parameter b, and which admits joint eigenfunctions for a set of (a_+, a_-, b) that is *dense* in $(0, \infty)^2 \times \mathbb{R}$. But from the properties of these eigenfunctions one can deduce that the pair does not admit joint eigenfunctions depending continuously on a_+, a_-, b. (See the paragraphs between Eqs. (3.11) and (3.12) in Ref. [12].) From such concrete examples one sees that expectations based on experience with analytic differential equations need not be borne out for analytic difference equations.

3 Generalized gamma functions

When $F(z)$ solves the first order AΔE (1.1), it is evident that $1/F(z)$ solves the AΔE with rhs $1/\Phi(z)$. Likewise, when $F_1(z), F_2(z)$ are solutions to the AΔEs with right-hand sides $\Phi_1(z), \Phi_2(z)$, then $F_1(z)F_2(z)$ clearly solves the AΔE with rhs $\Phi_1(z)\Phi_2(z)$. Since elliptic right-hand sides $\Phi(z)$ and their various degenerations admit a factorization in terms of the Weierstrass σ-function and its degenerations, it is natural to construct 'building block' solutions corresponding to the σ-function and its trigonometric, hyperbolic and rational specializations.

We refer to the building blocks reviewed below as generalized gamma functions. Indeed, in the rational case considered in Subsection 3.1, the pertinent gamma function amounts to Euler's gamma function. Trigonometric, elliptic and hyperbolic gamma functions are surveyed in Subsections 3.2–3.4. As we will explain, these functions can all be viewed as minimal solutions, rendered unique by normalization requirements. They were introduced and studied from this viewpoint in Ref. [9]. The hyperbolic gamma function will be a key ingredient in Section 6, where we need a great many of its properties sketched in Subsection 3.4.

3.1 The rational gamma function

We define the rational gamma function by

$$G_{\mathrm{rat}}(a;z) \equiv (2\pi)^{-1/2} \exp(-\frac{iz}{a} \ln a)\Gamma(-\frac{iz}{a} + \frac{1}{2}). \tag{3.1}$$

(This definition deviates from our previous one in Ref. [2] by the first two factors on the rhs. We feel that the present definition is more natural from the perspective of the zeta functions in Sections 4 and 5.) The Γ-function AΔE $\Gamma(x+1) = x\Gamma(x)$ entails that G_{rat} solves the AΔE

$$\frac{G(z+ia/2)}{G(z-ia/2)} = -iz. \tag{3.2}$$

More generally, when $\Phi(z)$ is an arbitrary rational function, we can solve the AΔE (1.1) by a function $F(z)$ of the form

$$F(z) = \exp(c_0 + c_1 z) \frac{\prod_{j=1}^{M} G_{\mathrm{rat}}(a; z-\alpha_j)}{\prod_{k=1}^{N} G_{\mathrm{rat}}(a; z-\beta_k)}. \tag{3.3}$$

Indeed, when we let the integers N, M vary over $\mathbb{N}$ and c_1, α_j, β_k over $\mathbb{C}$, then we obtain *all* rational functions on the rhs of (1.1) (save for the zero function, of course).

We now explain the relation of G_{rat} to the notion of 'minimal solution'. To this end we first of all need a shift of z on the rhs of (3.2) in order to get a function $\Phi(z)$ that is analytic and free of zeros in a strip around the real axis. Taking

$$\Phi(z) \equiv -i(z + ia/2), \tag{3.4}$$

a simple contour integration yields

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ixy} \partial_x^2 \ln \Phi(x) = \begin{cases} -ay\exp(ay/2), & y < 0, \\ 0, & y \geq 0. \end{cases} \tag{3.5}$$

Therefore $\phi(z) \equiv \partial_z^2 \ln \Phi(z)$ has the three properties (2.6), and the corresponding minimal solution (2.8) is then given by

$$f(z) = -2 \int_{-\infty}^{0} dy \frac{ye^{ay}}{\mathrm{sh} ay} e^{-2iyz}. \tag{3.6}$$

The point is now that the rhs of (3.6) equals $\partial_z^2 \ln G_{\mathrm{rat}}(a; z + ia/2)$, so that $G_{\mathrm{rat}}(a; z + ia/2)$ may be characterized as a *minimal* solution to the AΔE with rhs (3.4). It is not hard to check the asserted equality via Gauss' formula for the psi function (the logarithmic derivative of $\Gamma(x)$). But one may in fact reobtain various results on the gamma function (such as Gauss' formula) by taking the function (3.6) as a starting point. Put differently, if one would not have been familiar with the gamma function beforehand, one would have been led to it (and to a substantial part of its theory) via the 'minimal solution' approach sketched above.

We have worked out the details of this useful perspective on Euler's gamma function in Appendix A of our paper Ref. [9]. We cite in particular one Γ-function representation derived there, which we have occasion to invoke in Subsection 4.1. It reads

$$\Gamma(w) = (2\pi)^{1/2} \exp\left(\int_0^{\infty} \frac{dt}{t} \left((w - \frac{1}{2})e^{-t} - \frac{1}{t} + \frac{e^{-wt}}{1-e^{-t}} \right) \right), \quad \mathrm{Re}\, w > 0, \tag{3.7}$$

cf. Eq. (A37) in Ref. [9]. Here we only add a few more remarks on several features that are important with an eye on the generalized gamma functions introduced and studied in later subsections.

First, it should be observed that the shift of z by $ia/2$ is arbitrary; any shift $z \to z+ic, c>0$, would yield substantially the same conclusions.

Second, a shift $z \to z-ic, c>0$, on the rhs of (3.2) yields in the same way as sketched above solutions to (3.2) of the form $\alpha \exp(-iza^{-1}\ln a)\exp((2k+1)\pi z/a)/\Gamma(iz/a+1/2)$, with $\alpha \in \mathbb{C}^*, k \in \mathbb{Z}$. Hence the quotient of such a solution and $G_{\rm rat}(a;z)$ is ia-periodic. This is in agreement with the well-known reflection equation, which becomes here

$$\Gamma(\frac{iz}{a}+\frac{1}{2})\Gamma(-\frac{iz}{a}+\frac{1}{2}) = \pi \mathrm{ch}(\frac{\pi z}{a})^{-1}. \tag{3.8}$$

(This identity can also be derived from the AΔE-viewpoint, cf. Appendix A in Ref. [9].)

Third, we recall that the minimal solutions to the AΔE (1.1) with rhs (3.4) (obtained via twofold integration of $f(z)$ (3.6)) are analytic and zero-free for $|\mathrm{Im}\, z| < a$. From the AΔE one then sees that they are analytic and zero-free for all z not in $-ia\mathbb{N}^*$, whereas they have simple poles for $z=-iak, k \in \mathbb{N}^*$. (Similarly, the shift $z \to z-ic, c>0$, in (3.2) yields minimal solutions without poles and with a zero sequence in the upper half plane.) Of course, this is once again well known for the special function $G_{\rm rat}(a;z)$ (3.1). It is illuminating to compare its pole sequence

$$z = -i(k+1/2)a, \quad k \in \mathbb{N}, \qquad \text{(poles)}, \tag{3.9}$$

to those of the generalized gamma functions introduced below.

3.2 The trigonometric gamma function

In the same way as $G_{\rm rat}(a;z)$ (3.1) serves as a building block to solve AΔEs with rational right-hand sides, the trigonometric gamma function can be used to handle trigonometric functions $\Phi(z)$ with period π/r. (More precisely, functions in the field $\mathbb{C}(\exp(2irz))$.) Specifically, letting

$$F(z) = \exp(c_0 + c_1 z + c_2 z^2)\frac{\prod_{j=1}^{M} G_{\rm trig}(r,a;z-\alpha_j)}{\prod_{k=1}^{N} G_{\rm trig}(r,a;z-\beta_k)}, \tag{3.10}$$

with $G_{\rm trig}(r,a;z)$ solving

$$\frac{G(z+ia/2)}{G(z-ia/2)} = 1-\exp(2irz), \tag{3.11}$$

one obtains all trigonometric functions by letting N, M vary over $\mathbb{N}$ and $c_1, c_2, \alpha_j, \beta_k$ over $\mathbb{C}$ in the quotient $F(z-ia/2)/F(z+ia/2)$.

The obvious solution to (3.11) is the upward iteration solution

$$G_{\rm trig}(r,a;z) \equiv \prod_{k=1}^{\infty}(1-q^{2k-1}e^{2irz})^{-1}, \qquad q \equiv e^{-ar}. \tag{3.12}$$

Indeed, the infinite product clearly converges, yielding a meromorphic solution without zeros and with poles for

$$z = j\pi/r - i(k+1/2)a, \quad j \in \mathbb{Z}, \quad k \in \mathbb{N}, \quad \text{(poles)}. \tag{3.13}$$

Another representation for $G_{\rm trig}$ arises by writing

$$\prod_{k=1}^{\infty}(1-q^{2k-1}e^{2irz})^{-1}=\exp(-\sum_{k=1}^{\infty}\ln(1-q^{2k-1}e^{2irz})), \tag{3.14}$$

and using the elementary Fourier series

$$\ln(1-e^{2ir(z+ic)})=-\sum_{n=1}^{\infty}\frac{e^{2inr(z+ic)}}{n},\quad c>0. \tag{3.15}$$

Indeed, this gives rise to the formula

$$G_{\rm trig}(r,a;z)=\exp\left(\sum_{n=1}^{\infty}\frac{e^{2inrz}}{2n{\rm sh}nra}\right),\quad {\rm Im}\, z>-a/2, \tag{3.16}$$

as is easily checked.

Taking $z\to z+ic, c>0$, in the AΔE (3.11), it is of the form discussed in Subsection 2.1. Taking next logarithms and using (3.15), one deduces that (3.16) amounts to the special solution (2.11). That is, $G_{\rm trig}(r,a;z)$ is once more a *minimal* solution to the (shifted) AΔE (3.11). As such, it is uniquely determined by the asymptotics

$$G_{\rm trig}(r,a;z)\sim 1,\quad {\rm Im}\, z\to\infty, \tag{3.17}$$

which can be read off from (3.16).

Our trigonometric gamma function is closely related to Thomae's q-gamma function $\Gamma_q(x)$ [13]: One has

$$G_{\rm trig}(r,a;z)=\exp(c_0+c_1z)\Gamma_{\exp(-2ar)}(-\frac{iz}{a}+\frac{1}{2}), \tag{3.18}$$

for suitable constants c_0,c_1. The AΔE-perspective from which $G_{\rm trig}$ arises leads to various other features that are detailed in Ref. [9]. A useful limit that is not mentioned explicitly there reads

$$\lim_{r\downarrow 0}\exp(-\frac{\pi^2}{12ar})G_{\rm trig}(r,a;0)=2^{-1/2}. \tag{3.19}$$

(This follows by combining Eqs. (3.128), (3.129) and (3.154) in *loc. cit.*, taking $z=0$.) Using Proposition III.20 in *loc. cit.*, we then deduce

$$\lim_{r\downarrow 0}\exp\left(-\frac{\pi^2}{12ar}+\frac{iz}{a}\ln(2r)\right)G_{\rm trig}(r,a;z)=G_{\rm rat}(a;z). \tag{3.20}$$

Finally, we point out that it is evident from the infinite product representation (3.12) that one can allow r to vary over the (open) right half plane, whereas one cannot take $r\in i\mathbb{R}$. This is why one needs another building block function to handle hyperbolic right-hand side functions $\Phi(z)$ in the AΔE (1.1), cf. Subsection 3.4.

3.3 The elliptic gamma function

The elliptic gamma function is the minimal solution to the AΔE

$$\frac{G(z+ia/2)}{G(z-ia/2)} = \exp\left(-\sum_{n=1}^{\infty}\frac{\cos(2nrz)}{n\mathrm{sh}(nrb)}\right), \quad |\mathrm{Im}\, z| < b, \tag{3.21}$$

obtained via the formula (2.11):

$$G_{\mathrm{ell}}(r,a,b;z) \equiv \exp\left(i\sum_{n=1}^{\infty}\frac{\sin(2nrz)}{2n\mathrm{sh}(nra)\mathrm{sh}(nrb)}\right), \quad |\mathrm{Im}\, z| < (a+b)/2. \tag{3.22}$$

Indeed, our definition entails that the functions

$$F(z) = \exp(c_0 + c_1 z + c_2 z^2 + c_3 z^3)\frac{\prod_{j=1}^{M} G_{\mathrm{ell}}(r,a,b;z-\alpha_j)}{\prod_{k=1}^{N} G_{\mathrm{ell}}(r,a,b;z-\beta_k)}, \tag{3.23}$$

give rise to all elliptic right-hand side functions $\Phi(z)$ with periods π/r and ib.

To explain why this is true, we recall first that any elliptic function with periods π/r and ib admits a representation as

$$\alpha\prod_{j=1}^{N}\frac{\sigma(z-\gamma_j;\frac{\pi}{2r},\frac{ib}{2})}{\sigma(z-\delta_j;\frac{\pi}{2r},\frac{ib}{2})}, \tag{3.24}$$

where $\alpha, \gamma_j, \delta_j \in \mathbb{C}$. The crux is now that the function on the rhs of (3.21) is of the form

$$\exp(d_0 + d_1 z + d_2 z^2)\sigma(z+ib/2;\frac{\pi}{2r},\frac{ib}{2}). \tag{3.25}$$

Hence the functions $F(z+ia/2)/F(z-ia/2)$, with $F(z)$ given by (3.23), yield all functions (3.24) (with $\alpha \neq 0$) by choosing

$$N = M, \quad \alpha_j = \gamma_j + ib/2, \quad \beta_j = \delta_j + ib/2, \quad j = 1,\ldots,N, \tag{3.26}$$

and appropriate constants c_1, c_2, c_3 determined by the constants α, d_0, d_1 and d_2.

Next, we mention that the elliptic gamma function can also be written as an infinite product

$$G_{\mathrm{ell}}(r,a,b;z) = \prod_{m,n=1}^{\infty}\frac{1-q_a^{2m-1}q_b^{2n-1}e^{-2irz}}{1-q_a^{2m-1}q_b^{2n-1}e^{2irz}}, \quad q_a \equiv e^{-ar}, \quad q_b \equiv e^{-br}. \tag{3.27}$$

To check this, one need only proceed as in the trigonometric case: The infinite product can be written as the exponential of a series; using the Fourier series (3.15) and summing the resulting geometric series yields (3.22).

From (3.27) one can read off meromorphy in z and the locations of poles and zeros. Specifically, one obtains the doubly-infinite sequences

$$z = j\pi/r - i(k+1/2)a - i(l+1/2)b, \quad j \in \mathbb{Z}, \quad k,l \in \mathbb{N}, \quad \text{(poles)}, \tag{3.28}$$

$$z = j\pi/r + i(k+1/2)a + i(l+1/2)b, \quad j \in \mathbb{Z}, \quad k,l \in \mathbb{N}, \quad \text{(zeros)}. \tag{3.29}$$

It is immediate from the product representations (3.12) and (3.27) that one has

$$G_{\rm trig}(r,a;z) = \lim_{b\uparrow\infty} G_{\rm ell}(r,a,b;z-ib/2). \tag{3.30}$$

It is also not difficult to see that for the renormalized function

$$\tilde{G}_{\rm ell}(r,a,b;z) \equiv \exp(\frac{\pi^2 z}{6irab})G_{\rm ell}(r,a,b;z), \tag{3.31}$$

the $r\downarrow 0$ limit exists. This yields the function

$$G_{\rm hyp}(a,b;z) = \lim_{r\downarrow 0}\tilde{G}_{\rm ell}(r,a,b;z), \tag{3.32}$$

studied in the next subsection. For further properties of the elliptic gamma function we refer to Subsection III.B in Ref. [9].

3.4 The hyperbolic gamma function

The defining AΔE of the hyperbolic gamma function reads

$$\frac{G(z+ia/2)}{G(z-ia/2)} = 2{\rm ch}(\pi z/b). \tag{3.33}$$

Clearly, any solution $G_{\rm hyp}(a,b;z)$ of (3.33) can be used to solve (1.1) with $\Phi(z)$ an arbitrary hyperbolic function with period ib. Indeed, all functions $\Phi\in\mathbb{C}(\exp(2\pi z/b))$ arise via functions of the form

$$F(z) = \exp(c_0+c_1z+c_2z^2)\frac{\prod_{j=1}^M G_{\rm hyp}(a,b;z-\alpha_j)}{\prod_{k=1}^N G_{\rm hyp}(a,b;z-\beta_k)}. \tag{3.34}$$

As before, there is a certain arbitrariness in the choice of AΔE for the building block function. Our choice $2{\rm ch}(\pi z/b)$ together with the requirement that the solution be minimal will lead us to a function $G_{\rm hyp}(a,b;z)$ with various features that would be spoiled by any other choice, however. (In particular, the constant 2 cannot be changed without losing the remarkable $(a\leftrightarrow b)$-invariance of the hyperbolic gamma function (3.35).)

Following the method to construct minimal solutions sketched in Subsection 2.1, one readily verifies that $\phi(z)\equiv\partial_z^3\ln(2{\rm ch}(\pi z/b))$ has the three properties (2.6). The Fourier transform can be done explicitly, and integrating up three times then yields our hyperbolic gamma function,

$$G_{\rm hyp}(a,b;z) \equiv \exp\left(i\int_0^\infty \frac{dy}{y}\left(\frac{\sin(2yz)}{2{\rm sh}(ay){\rm sh}(by)} - \frac{z}{aby}\right)\right), \quad |{\rm Im}\, z| < (a+b)/2. \tag{3.35}$$

Since $G_{\rm hyp}(a,b;z)$ has no poles and zeros in the strip $|{\rm Im}\, z| < (a+b)/2$, one readily deduces from the defining AΔE (3.33) that $G_{\rm hyp}$ extends to a meromorphic function with poles and zeros given by

$$z = -i(k+1/2)a - i(l+1/2)b, \qquad k,l\in\mathbb{N}, \qquad \text{(poles)}, \tag{3.36}$$

$$z = i(k+1/2)a + i(l+1/2)b, \qquad k,l\in\mathbb{N}, \qquad \text{(zeros)}. \tag{3.37}$$

In Section 6 we need various other properties of $G_{\rm hyp}(a,b;z)$. Some automorphy properties are immediate from (3.35): One has

$$G_{\rm hyp}(a,b;-z) = 1/G_{\rm hyp}(a,b;z), \tag{3.38}$$

$$G_{\rm hyp}(a,b;z) = G_{\rm hyp}(b,a;z), \tag{3.39}$$

$$G_{\rm hyp}(\lambda a,\lambda b;\lambda z) = G_{\rm hyp}(a,b;z), \quad \lambda > 0. \tag{3.40}$$

It is also easy to establish from the defining AΔE that the multiplicity of a pole or zero z_0 equals the number of distinct pairs $(k,l) \in \mathbb{N}^2$ giving rise to z_0, cf. (3.36), (3.37). In particular, the pole at $-i(a+b)/2$ and zero at $i(a+b)/2$ are simple, and for a/b irrational *all* poles and zeros are simple.

The remaining properties we have occasion to use are not clear by inspection, and we refer to Subsection IIIA in Ref. [9] for a detailed account. First, we need to know the residue at the simple pole $-i(a+b)/2$. It is given by

$$\mathrm{Res}(-i(a+b)/2) = \frac{i}{2\pi}(ab)^{1/2}. \tag{3.41}$$

(Here and below, we take positive square roots of positive quantities.)

Second, we need two distinct zero step size limits of the hyperbolic gamma function. The first one reads

$$\lim_{b\downarrow 0} \frac{G_{\rm hyp}(\pi,b;z+i\lambda b)}{G_{\rm hyp}(\pi,b;z+i\mu b)} = \exp((\lambda-\mu)\ln(2\mathrm{ch}z)), \tag{3.42}$$

where the limit is uniform on compact subsets of the cut plane

$$\mathbb{C} \setminus \{\pm iz \in [\pi/2,\infty)\}. \tag{3.43}$$

When $\lambda-\mu$ is an integer, this limit is immediate from the $(a \leftrightarrow b)$-invariance of $G_{\rm hyp}$ and the AΔE (3.33). For $\lambda-\mu \notin \mathbb{Z}$, the emergence of the logarithmic branch cuts on the imaginary axis may be viewed as a consequence of the coalescence of an infinite number of zeros and poles on the cuts.

The second zero step size limit yields the connection to the Γ-function. Consider the function

$$H(\rho;z) \equiv G_{\rm hyp}(1,\rho;\rho z+i/2)\exp(iz\ln(2\pi\rho))/(2\pi)^{1/2}. \tag{3.44}$$

From the AΔE (3.33) and its $(a \leftrightarrow b)$-counterpart one sees that H satisfies the AΔE

$$\frac{H(\rho;z+i/2)}{H(\rho;z-i/2)} = \frac{i\mathrm{sh}(\pi\rho z)}{\pi\rho}, \tag{3.45}$$

and reflection equation

$$H(\rho;z)H(\rho;-z) = \pi^{-1}\mathrm{ch}\pi z. \tag{3.46}$$

(Recall also (3.38) to check (3.46).) Therefore, it should not come as a surprise that one has

$$\lim_{\rho\downarrow 0} H(\rho;z) = 1/\Gamma(iz+1/2), \tag{3.47}$$

uniformly for z in $\mathbb{C}$-compacts.

We have made a (convenient) choice for the first parameter a of the hyperbolic gamma function, but it should be pointed out that the scale invariance (3.40) can be used to handle arbitrary a. In particular, recalling the definition (3.1), this yields the limit

$$\lim_{b\uparrow\infty} \exp\left(\frac{iz}{a}\ln(2\pi/b)\right) G_{\mathrm{hyp}}(a,b;z-ib/2) = G_{\mathrm{rat}}(a;z), \tag{3.48}$$

which should be compared to the trigonometric limit (3.20).

Third, we need the $|\mathrm{Re}\, z| \to \infty$ asymptotics of G_{hyp}. To detail this, we set

$$c \equiv \max(a,b), \quad \sigma = 1-\epsilon, \quad \epsilon > 0. \tag{3.49}$$

Then we have

$$\mp i \ln G_{\mathrm{hyp}}(a,b;z) = -\frac{\pi z^2}{2ab} - \frac{\pi}{24}\left(\frac{a}{b}+\frac{b}{a}\right) + O(\exp(\mp 2\pi\sigma \mathrm{Re}\, z/c), \quad \mathrm{Re}\, z \to \pm\infty. \tag{3.50}$$

Here, the bound is uniform for $\mathrm{Im}\, z$ in $\mathbb{R}$-compacts, but it is not uniform as $\epsilon \downarrow 0$, since it is false for $\sigma = 1$ and $a = b$.

Finally, in Subsection 4.4 we need the representation

$$\partial_z \ln G_{\mathrm{hyp}}(a,b;z) = \frac{\pi}{2ia^2}\int_{-\infty}^{\infty} du \frac{\ln(2\mathrm{ch}(\pi u/b))}{\mathrm{ch}^2(\pi(z-u)/a)}, \quad |\mathrm{Im}\, z| < a/2, \tag{3.51}$$

which cannot be found in Ref. [9]. To prove its validity, we first note that when we take two more z-derivatives, then the resulting formula amounts to a special case of (2.9). Indeed, this follows upon trading the z-derivatives for u-derivatives, and then integrating by parts three times.

As a consequence, $\partial_z \ln G_{\mathrm{hyp}}$ is given by the rhs of (3.51) plus a term of the form $Az+B$. Now since $\partial_z G_{\mathrm{hyp}}/G_{\mathrm{hyp}}$ is even, we must have $A = 0$. To show $B = 0$, we consider the $\mathrm{Re}\, z \to \infty$ asymptotics of the rhs of (3.51). Changing variables $u \to z - x$, it is routine to check it reads $-i\pi z/ab + o(1)$. Comparing to the derivative of (3.50) for $\mathrm{Re}\, z \to \infty$, we deduce $B = 0$. (By virtue of Cauchy's formula, it is legitimate to differentiate the bound (3.50).)

To conclude this subsection we add some further information on the hyperbolic gamma function. First of all, this function was introduced in another guise and from a quite different perspective in previous literature—a fact we were not aware of at the time we wrote Ref. [9]. Indeed, our hyperbolic gamma function is related to a function that is nowadays referred to as Kurokawa's double sine function [14]. The latter is usually denoted $S_2(x|\omega_1,\omega_2)$, and the relation reads

$$G_{\mathrm{hyp}}(a,b;z) = S_2(iz+(a+b)/2|a,b). \tag{3.52}$$

The first occurrence of the double sine function is however in a series of papers by Barnes, published a century ago. He generalized the gamma function from another viewpoint to his so-called multiple gamma functions; the double sine function is then a quotient of two double gamma functions. We return to Barnes' multiple gamma functions in Subsection 5.2, where they will be tied in with the minimal solution ideas of Subsection 2.1.

Multiple gamma functions show up in particular in number theory. The double gamma and sine functions were studied from this viewpoint in a paper by Shintani [15]. He derived a product formula that can be used to tie in the hyperbolic and trigonometric gamma functions in a quite explicit way, and we continue by presenting the pertinent formulas.

First, we should mention that $G_{\rm hyp}(a,b;z)$ admits a representation as an infinite product of Γ-functions, from which meromorphy properties in a,b and z follow by inspection. To be specific, using the scale-invariant variables

$$\lambda \equiv -iz/a, \qquad \rho \equiv b/a, \tag{3.53}$$

one has

$$\begin{aligned} G_{\rm hyp}(a,b;z+\frac{ia}{2})^2 &= 2\cos(\pi\lambda/\rho)e^{2\lambda\ln 2} \\ &\cdot \prod_{j=0}^{\infty} \frac{\Gamma((j+\frac{1}{2})\rho+\lambda)}{\Gamma((j+\frac{1}{2})\rho-\lambda)} \frac{\Gamma(1+(j+\frac{1}{2})\rho+\lambda)}{\Gamma(1+(j+\frac{1}{2})\rho-\lambda)} e^{-4\lambda\ln(j+\frac{1}{2})\rho}. \end{aligned} \tag{3.54}$$

Here, one needs $\rho \in \mathbb{C} \setminus (-\infty, 0]$ for the infinite product to converge, cf. Prop. III.5 in Ref. [9]. (Observe that for non-real ρ the poles and zeros on the rhs are double, as should be the case.)

It follows in particular from this representation that $G_{\rm hyp}(a,b;z)$ has an analytic continuation to $b \in i(0,\infty)$. Now for $b = i\pi/r, r > 0$, the rhs of (3.33) can be rewritten as

$$\exp(-irz)[1-\exp(2ir[z+\pi/2r])]. \tag{3.55}$$

Comparing to the rhs of (3.11), we deduce that the quotient function

$$\exp(-rz^2/2a)G_{\rm trig}(r,a;z+\pi/2r)/G_{\rm hyp}(a,i\pi/r;z) \tag{3.56}$$

is ia-periodic.

Using Shintani's formula, we can determine this ia-periodic quotient explicitly. For the case at hand his product formula amounts to

$$G_{\rm hyp}(a,i\pi/r;z) = \exp\left(-\frac{rz^2}{2a} - \frac{1}{24}(ra - \frac{\pi^2}{ra})\right) \prod_{k=1}^{\infty} \frac{1+\exp(2i\pi a^{-1}[iz+i(k-1/2)\pi r^{-1}])}{1+\exp(2ir[z+i(k-1/2)a])}. \tag{3.57}$$

From the definition (3.12) of the trigonometric gamma function we then get the remarkable relation

$$G_{\rm hyp}(a,i\pi/r;z) = \exp\left(-\frac{rz^2}{2a} - \frac{1}{24}(ra - \frac{\pi^2}{ra})\right) \frac{G_{\rm trig}(r,a;z+\pi/2r)}{G_{\rm trig}(\pi/a,\pi/r;iz+a/2)}, \tag{3.58}$$

from which the ia-periodic function (3.56) can be read off. (The reader who is familiar with modular transformation properties may find more information on this angle below Eq. (A.27) in Ref. [16], where we rederived Shintani's product formula.)

4 Generalized zeta functions

This section is concerned with the Hurwitz zeta function $\zeta(s,w)$, and with 'trigonometric', 'elliptic' and 'hyperbolic' generalizations thereof. These generalizations are arrived at by combining the perspective of Subsection 2.1 with the trigonometric, elliptic and hyperbolic gamma functions of Section 3, respectively. To our knowledge, the generalized zeta functions studied in Subsections 4.2–4.4 have not appeared in previous literature. (In any event, our results reported there are hitherto unpublished.) A different generalization of the Hurwitz zeta function has been introduced and studied by Ueno and Nishizawa [17].

A suitable specialization also leads to a 'trigonometric' and 'hyperbolic' generalization of Riemann's zeta function $\zeta(s) = \zeta(s, 1)$. Via the latter we are led to some integral representations for $\zeta(s)$ that seem to be new. See also Suslov's contribution to these proceedings [18], where—among other things—different generalizations of $\zeta(s)$ are discussed.

4.1 The rational zeta function

The Hurwitz zeta function is defined by

$$\zeta(s, w) \equiv \sum_{m=0}^{\infty} (w + m)^{-s}, \quad \mathrm{Re}\, w > 0, \quad \mathrm{Re}\, s > 1. \tag{4.1}$$

From this definition it is immediate that $\zeta(s, w)$ is a solution to the (additive) first order AΔE

$$\zeta(w + 1) - \zeta(w) = -w^{-s}. \tag{4.2}$$

Let us now introduce

$$\phi_s(z) \equiv -(c - iz)^{-s}, \quad c > 0, \tag{4.3}$$

and consider the AΔE

$$f(z + i/2) - f(z - i/2) = \phi_s(z). \tag{4.4}$$

Using Euler's integral

$$\Gamma(x) = \int_0^{\infty} u^{x-1} e^{-u} du, \quad \mathrm{Re}\, x > 0, \tag{4.5}$$

we may write

$$\phi_s(z) = -\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-ct} e^{itz} dt, \quad \mathrm{Re}\, s > 0, \quad \mathrm{Im}\, z > -c. \tag{4.6}$$

Thus we have

$$\hat{\phi}_s(y) = \begin{cases} -(-y)^{s-1} e^{cy} / \Gamma(s), & y < 0, \\ 0, & y \geq 0, \end{cases} \tag{4.7}$$

cf. (2.7). Taking $\mathrm{Re}\, s \geq 2$, we deduce that $\phi_s(z)$ satisfies the conditions (2.6). The corrresponding minimal solution (2.8) to the AΔE (4.4) reads

$$f_s(z) = \frac{1}{\Gamma(s)} \int_0^{\infty} dy (2y)^{s-1} e^{-2cy} e^{2iyz} / \mathrm{sh} y, \quad \mathrm{Re}\, s \geq 2, \tag{4.8}$$

where we may take $\mathrm{Im}\, z > -c - 1/2$. Writing

$$1/\mathrm{sh} y = 2e^{-y} \sum_{m=0}^{\infty} e^{-2my}, \quad y > 0, \tag{4.9}$$

this yields (using (4.5) once more)

$$f_s(z) = \sum_{m=0}^{\infty} (\frac{1}{2} + m + c - iz)^{-s} = \zeta(s, \frac{1}{2} + c - iz), \quad \mathrm{Re}\, s \geq 2. \tag{4.10}$$

The upshot is that for $\mathrm{Re}\, s \geq 2$ we may view the function $\zeta(s, 1/2 + c - iz)$ as the minimal solution (2.8) to the AΔE (4.4). Note that it manifestly has the properties mentioned above the alternative representation (2.9). For the case at hand the latter specializes to

$$\zeta(s, \frac{1}{2} + c - iz) = \frac{i}{2} \int_{-\infty}^{\infty} (c - iu)^{-s} \tanh \pi(z - u) du. \tag{4.11}$$

Next, we integrate by parts in (4.11) and change variables to obtain

$$\zeta(s, w) = \frac{\pi}{2(s-1)} \int_{-\infty}^{\infty} \frac{(w - 1/2 + ix)^{1-s}}{\mathrm{ch}^2(\pi x)} dx, \quad \mathrm{Re}\, w > 1/2. \tag{4.12}$$

Clearly, one can shift the contour up by $i(1-\epsilon)/2, \epsilon > 0$, so as to handle more generally w in the right half plane. From these formulas (which we have not found in the literature) one can easily deduce some well-known features of $\zeta(s, w)$.

Specifically, one infers that for $\mathrm{Re}\, w > 0$ the function $s \mapsto \zeta(s, w)$ has a meromorphic continuation to $\mathbb{C}$, yielding a simple pole at $s = 1$ with residue 1. Moreover, one obtains

$$\zeta(0, w) = 1/2 - w, \tag{4.13}$$

and

$$\partial_s \zeta(s, w)|_{s=0} = -\frac{1}{2} + w + \frac{\pi}{2} \int_{-\infty}^{\infty} \frac{(w - 1/2 + ix) \ln(w - 1/2 + ix)}{\mathrm{ch}^2(\pi x)} dx, \quad \mathrm{Re}\, w > 1/2. \tag{4.14}$$

The latter formula gives rise to a representation for $\ln \Gamma(w)$ that seems to be new. Indeed, one also has

$$\partial_s \zeta(s, w)|_{s=0} = \ln \Gamma(w) - \frac{1}{2} \ln(2\pi), \quad \mathrm{Re}\, w > 0. \tag{4.15}$$

This well-known relation can be more easily derived via the representation

$$\zeta(s, w) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-wt}}{1 - e^{-t}} dt, \quad \mathrm{Re}\, s > 1, \tag{4.16}$$

which follows from (4.8) and (4.10). Indeed, noting

$$\frac{t}{1 - e^{-t}} = 1 + \frac{t}{2} + O(t^2), \quad t \to 0, \tag{4.17}$$

we can write (using (4.5))

$$\zeta(s, w) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{dt}{t^2} t^s e^{-wt} \left(\frac{t}{1 - e^{-t}} - 1 - \frac{t}{2} \right) + w^{-s} \left(\frac{w}{s-1} + \frac{1}{2} \right). \tag{4.18}$$

It is clear by inspection that this representation continues analytically to $\mathrm{Re}\, s > -1$, and it yields (using $1/\Gamma(s) = s + O(s^2)$ for $s \to 0$)

$$\partial_s \zeta(s,w)|_{s=0} = \int_0^\infty \frac{dt}{t} e^{-wt} \left(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2} \right) - w + w \ln w - \frac{\ln w}{2}. \tag{4.19}$$

Using the Γ-representation (3.7), it is now straightforward to check (4.15).

The above properties of the Hurwitz zeta function have been known for a long time, cf. e.g. Ref. [19]. But the minimal solution interpretation of $\zeta(s,w)$ seems to be new. From the AΔE viewpoint it can also be understood why the relation (4.15) between the zeta and gamma functions holds true. Indeed, due to the (analytic continuation of the) AΔE (4.2), the lhs of (4.15) satisfies the AΔE

$$f(w+1) - f(w) = \ln w, \tag{4.20}$$

just as the rhs. (Of course, the constant does not follow from this reasoning.)

We now introduce a 'rational zeta function'

$$Z_{\mathrm{rat}}(a;s,z) \equiv a^{-s} \zeta(s, -\frac{iz}{a} + \frac{1}{2}), \quad a > 0, \quad \mathrm{Im}\, z > -a/2. \tag{4.21}$$

In view of (4.11) and (4.12), it admits integral representations

$$Z_{\mathrm{rat}}(a;s,z) = \frac{i}{2a} \int_{-\infty}^{\infty} (-iz + ix)^{-s} \mathrm{th}(\pi x/a) dx, \quad \mathrm{Re}\, s > 1, \quad \mathrm{Im}\, z > 0, \tag{4.22}$$

$$Z_{\mathrm{rat}}(a;s,z) = \frac{\pi}{2a^2(s-1)} \int_{-\infty}^{\infty} \frac{(-iz+ix)^{1-s}}{\mathrm{ch}^2(\pi x/a)} dx, \quad \mathrm{Im}\, z > 0. \tag{4.23}$$

Furthermore, it satisfies the AΔE

$$Z(z + ia/2) - Z(z - ia/2) = -(-iz)^{-s}. \tag{4.24}$$

(As before, it is understood that in (4.22)–(4.24) the logarithm branch is fixed by choosing $\ln(-iz)$ real for $z \in i(0,\infty)$.) Finally, upon combining (4.15), (4.13) and (3.1), one obtains

$$\partial_s Z_{\mathrm{rat}}(a;s,z)|_{s=0} = \ln(G_{\mathrm{rat}}(a;z)), \quad \mathrm{Im}\, z > -a/2. \tag{4.25}$$

In the following subsections we take the above state of affairs as a lead to introduce 'zeta functions' that are minimal solutions to trigonometric, elliptic and hyperbolic generalizations of (4.24). These generalizations turn out to be such that analogs of (4.25) are valid, with the rational gamma function replaced by the trigonometric, elliptic and hyperbolic gamma functions from Subsections 3.2–3.4, resp.

4.2 The trigonometric zeta function

Following the ideas explained at the end of the previous subsection, we start from the AΔE

$$Z(z + ia/2) - Z(z - ia/2) = -(1 - e^{2irz})^{-s}. \tag{4.26}$$

Indeed, the s-derivative of the rhs at $s = 0$ reads $\ln(1-\exp(2irz))$, which equals $\ln G_{\mathrm{trig}}(z+ia/2) - \ln G_{\mathrm{trig}}(z-ia/2)$, cf. Subsection 3.2. Thus we expect to obtain a minimal solution Z_{trig} to the (shifted) AΔE (4.26), satisfying

$$\partial_s Z_{\mathrm{trig}}(r,a;s,z)|_{s=0} = \ln(G_{\mathrm{trig}}(r,a;z)). \tag{4.27}$$

We proceed by validating this expectation. Taking $z \to z + ia/2$, we obtain an AΔE

$$\begin{aligned} f(z+ia/2) - f(z-ia/2) &= -(1-qe^{2irz})^{-s} \\ &= -\exp(s\sum_{n=1}^{\infty} n^{-1}q^n e^{2inrz}), \end{aligned} \tag{4.28}$$

to which the theory sketched in Subsection 2.1 applies. Indeed, for all $s \in \mathbb{C}$ the rhs is analytic in the half plane $\mathrm{Im}\, z > -a/2$ and π/r-periodic.

As a consequence, a minimal solution to (4.28) is given by

$$\frac{iz}{a} + \frac{1}{2}\sum_{k=1}^{\infty} \frac{t_k(r,a;s)}{\mathrm{sh}(kra)} e^{2ikrz}, \tag{4.29}$$

with

$$t_k(r,a;s) = \frac{r}{\pi}\int_{-\pi/2r}^{\pi/2r} \exp(s\sum_{n=1}^{\infty} n^{-1}q^n e^{2inrx})e^{-2ikrx}dx, \quad k \in \mathbb{N}^*. \tag{4.30}$$

Due to analyticity and π/r-periodicity of the integrand, the contour can be shifted down by $i(a-\epsilon)/2$ for any $\epsilon > 0$. From this we obtain a majorization

$$|t_k(r,a;s)| \le \exp(C(\epsilon)|s|)\exp(-kr(a-\epsilon)), \quad k \in \mathbb{N}^*, \tag{4.31}$$

where $C(\epsilon) > 0$ diverges as $\epsilon \downarrow 0$. Now $t_k(s)$ is manifestly entire. Hence it readily follows that the solution (4.29) is analytic in the half plane $\mathrm{Im}\, z > -a$ for fixed $s \in \mathbb{C}$, and entire in s for z in the latter half plane.

We now define the 'trigonometric zeta function' by taking $z \to z - ia/2$ in this solution. Expanding the first exponential in (4.30), one readily obtains

$$t_k(r,a;s) = q^k p_k(s), \tag{4.32}$$

with $p_k(s)$ the polynomial

$$p_k(s) \equiv \sum_{m=1}^{k} \frac{s^m}{m!} \sum_{\substack{n_1,\ldots,n_m=1 \\ n_1+\cdots+n_m=k}}^{k} \frac{1}{n_1}\cdots\frac{1}{n_m}. \tag{4.33}$$

Thus our definition amounts to

$$Z_{\mathrm{trig}}(r,a;s,z) \equiv \frac{i(z-ia/2)}{a} + \sum_{k=1}^{\infty} \frac{p_k(s)}{2\mathrm{sh}(kra)} e^{2ikrz}, \quad \mathrm{Im}\, z > -a/2. \tag{4.34}$$

Next, we observe that (4.33) entails

$$\partial_s p_k(s)|_{s=0} = 1/k. \tag{4.35}$$

From this we deduce

$$\partial_s Z_{\rm trig}(r,a;s,z)|_{s=0} = \sum_{k=1}^{\infty} \frac{e^{2ikrz}}{2k{\rm sh}(kra)}. \tag{4.36}$$

Comparing this to (3.16), we see that (4.27) holds true, as announced.

An alternative representation for $Z_{\rm trig}$ can be obtained by noting that when one adds 1 to the rhs of (4.26), one obtains an AΔE that admits an upward iteration solution. A uniqueness argument then yields

$$Z_{\rm trig}(r,a;s,z) = \frac{i(z-ia/2)}{a} + \sum_{l=1}^{\infty}\left((1-q^{2l-1}e^{2irz})^{-s}-1\right), \quad {\rm Im}\, z > -a/2. \tag{4.37}$$

Observe that the relation (4.27) to $G_{\rm trig}$ is also clear from this formula and (3.12).

The relation to the Hurwitz zeta function can be readily established from (4.37), as well. Indeed, one has

$$\begin{aligned}
\lim_{r\downarrow 0}(2r)^s Z_{\rm trig}(r,a;s,z) &= \lim_{r\downarrow 0}\sum_{n=0}^{\infty}\left(\left(\frac{2r}{1-\exp(-2r[(n+1/2)a-iz])}\right)^s-(2r)^s\right)\\
&= \sum_{n=0}^{\infty}[(n+1/2)a-iz]^{-s}\\
&= a^{-s}\zeta(s,-iz/a+1/2), \quad {\rm Re}\, s>1, \quad {\rm Im}\, z>-a/2.
\end{aligned}\tag{4.38}$$

This can be abbreviated as

$$\lim_{\alpha\downarrow 0}\alpha^s Z_{\rm trig}(\alpha/2,a;s,z) = Z_{\rm rat}(a;s,z), \quad {\rm Re}\, s>1, \quad {\rm Im}\, z>-a/2, \tag{4.39}$$

cf. (4.21).

The Riemann zeta function $\zeta(s)$ is obtained from $Z_{\rm rat}(a;s,z)$ by choosing $a=1$ and $z=i/2$, cf. (4.21). Thus, setting

$$\zeta_{\rm trig}(\alpha;s) \equiv \alpha^s Z_{\rm trig}(\alpha/2,1;s,i/2), \tag{4.40}$$

one obtains a 'trigonometric' generalization of $\zeta(s)$. In view of (4.34) and (4.37) it admits the representations

$$\zeta_{\rm trig}(\alpha;s) = \alpha^s \sum_{k=1}^{\infty}\frac{e^{-k\alpha}}{1-e^{-k\alpha}}p_k(s), \tag{4.41}$$

$$\zeta_{\rm trig}(\alpha;s) = \sum_{n=1}^{\infty}\left(\left(\frac{\alpha}{1-e^{-n\alpha}}\right)^s-\alpha^s\right). \tag{4.42}$$

These formulas entail in particular that $\zeta_{\rm trig}(\alpha;s)$ is positive for all $(\alpha,s)\in(0,\infty)^2$.

4.3 The elliptic zeta function

Proceeding as before, we should start from the AΔE

$$Z(z+ia/2)-Z(z-ia/2) = -\exp\left(s\sum_{n=1}^{\infty}\frac{\cos(2nrz)}{n{\rm sh}(nrb)}\right), \tag{4.43}$$

cf. (3.21). The rhs is manifestly analytic for $|\mathrm{Im}\, z| < b/2$, so we need not shift z. Thus we define Z_{ell} as the minimal solution

$$Z_{\mathrm{ell}}(r,a,b;s,z) \equiv \frac{iz}{a} e_0(r,b;s) - \sum_{k\in\mathbb{Z}^*} \frac{e_k(r,b;s)}{2\mathrm{sh}(kra)} e^{-2ikrz}, \tag{4.44}$$

where $e_k(s)$ are the Fourier coefficients

$$e_k(r,b;s) = \frac{r}{\pi} \int_{-\pi/2r}^{\pi/2r} \exp\left(s \sum_{n=1}^{\infty} \frac{\cos(2nrx)}{n\mathrm{sh}(nrb)} \right) e^{2ikrx} dx, \quad k \in \mathbb{Z}. \tag{4.45}$$

Next, we note that by analyticity and π/r-periodicity of the integrand, we may shift the contour by ic, with $2c \in (-b, b)$. Therefore, choosing any $\epsilon \in (0, b]$, we obtain

$$|e_k(r,b;s)| \le \exp(C(\epsilon)|s|) \exp(-|k|r(b-\epsilon)), \quad k \in \mathbb{Z}, \tag{4.46}$$

with $C(\epsilon) > 0$ diverging as $\epsilon \downarrow 0$. Thus Z_{ell} is analytic in the strip $|\mathrm{Im}\, z| < (a+b)/2$ for fixed $s \in \mathbb{C}$, and entire in s for fixed z in the latter strip.

We proceed by demonstrating the expected relation

$$\partial_s Z_{\mathrm{ell}}(r,a,b;s,z)|_{s=0} = \ln(G_{\mathrm{ell}}(r,a,b;z)), \quad |\mathrm{Im}\, z| < (a+b)/2. \tag{4.47}$$

To this end we observe that (4.45) entails

$$\partial_s e_k(s)|_{s=0} = \begin{cases} 0, & k = 0, \\ 1/2k\mathrm{sh}(krb), & k \in \mathbb{Z}^*. \end{cases} \tag{4.48}$$

Hence we have

$$\partial_s Z_{\mathrm{ell}}(r,a,b;s,z)|_{s=0} = -\sum_{k\in\mathbb{Z}^*} \frac{e^{-2ikrz}}{4k\mathrm{sh}(krb)\mathrm{sh}(kra)}. \tag{4.49}$$

Comparing this to (3.22), we deduce (4.47).

4.4 The hyperbolic zeta function

In the hyperbolic context the pertinent AΔE reads

$$Z(z + ia/2) - Z(z - ia/2) = -[2\mathrm{ch}(\pi z/b)]^{-s}, \tag{4.50}$$

cf. (3.33). From the known Fourier transform

$$\int_{-\infty}^{\infty} \frac{e^{ixy}}{[2\mathrm{ch}(\pi x/b)]^s} dx = \frac{b}{2\pi} \Gamma(\frac{s}{2} + \frac{iby}{2\pi}) \Gamma(\frac{s}{2} - \frac{iby}{2\pi}) \Gamma(s)^{-1}, \quad \mathrm{Re}\, s > 0, \tag{4.51}$$

and the Γ-function asymptotics, we readily deduce that the conditions (2.6) apply to the function

$$\phi(z) = -\partial_z [2\mathrm{ch}(\pi z/b)]^{-s}, \quad \mathrm{Re}\, s > 0. \tag{4.52}$$

The minimal solution (2.9) corresponding to the rhs (4.52) can be rewritten as

$$f(z) = \frac{i\pi}{2a^2} \int_{-\infty}^{\infty} du [2\mathrm{ch}(\pi u/b)]^{-s} / \mathrm{ch}^2(\pi(z-u)/a), \quad \mathrm{Re}\, s > 0. \tag{4.53}$$

Integrating once with respect to z, we obtain a minimal solution

$$Z_{\rm hyp}(a,b;s,z)\equiv \frac{i}{2a}\int_{-\infty}^{\infty}du[2{\rm ch}(\pi u/b)]^{-s}{\rm th}(\pi(z-u)/a),\ \ {\rm Re}\, s>0,\ \ |{\rm Im}\, z|<a/2, \quad (4.54)$$

to (4.50). More precisely, (4.54) is the unique minimal solution that is odd in z. (The requirement of oddness fixes the arbitrary constant.)

Of course, we can also start from (2.8) and (4.51) to obtain a second representation

$$Z_{\rm hyp}(a,b;s,z)=\frac{ib}{2\pi^2}\int_0^{\infty}dy\frac{\sin(2yz)}{{\rm sh}(ay)}\Gamma(\frac{s}{2}+\frac{iby}{\pi})\Gamma(\frac{s}{2}-\frac{iby}{\pi})\Gamma(s)^{-1},\ \ {\rm Re}\, s>0,\ \ |{\rm Im}\, z|<a/2. \quad (4.55)$$

This formula appears less useful than (4.54), however.

Performing suitable contour shifts in (4.54), one easily checks that for fixed s with ${\rm Re}\, s>0$ the function $Z_{\rm hyp}$ is analytic in the strip $|{\rm Im}\, z|<(a+b)/2$, and that for fixed z in the latter strip $Z_{\rm hyp}$ is analytic in the right half s-plane. Alternatively, these features readily follow from (4.55). But the representations (4.54) and (4.55) are ill defined already for s on the imaginary axis.

This different behavior in s (as compared to the trigonometric and elliptic cases) can be understood from the AΔE (4.50). For ${\rm Re}\, s<0$ the function on the rhs is no longer polynomially bounded in the strip $|{\rm Im}\, z|<a/2$, so the theory summarized in Subsection 2.1 does not apply. Even so, $Z_{\rm hyp}(a,b;s,z)$ (with $|{\rm Im}\, z|<(a+b)/2$) admits a meromorphic continuation to the half plane ${\rm Re}\, s>-2b/a$.

We proceed by proving the assertion just made. To this end we begin by noting that the assertion holds true for $\partial_z^k Z_{\rm hyp}$ with $k\geq 1$ (cf. (4.53)). Now we exploit the identity

$${\rm ch}(\pi u/b)^{-s}=\frac{b^2}{\pi^2 s^2}\partial_u^2({\rm ch}(\pi u/b)^{-s})+(1+\frac{1}{s}){\rm ch}(\pi u/b)^{-s-2}, \quad (4.56)$$

which is easily checked. Inserting it in (4.54) and integrating by parts twice, we deduce the functional equation

$$Z_{\rm hyp}(a,b;s,z)=\frac{b^2}{\pi^2 s^2}\partial_z^2 Z_{\rm hyp}(a,b;s,z)+4(1+\frac{1}{s})Z_{\rm hyp}(a,b;s+2,z). \quad (4.57)$$

Consider now the two terms on the rhs of (4.57). The first one has a meromorphic extension to ${\rm Re}\, s>-2b/a$, and the second one to ${\rm Re}\, s>-2$. In case $b>a$, one can iterate the functional equation, so as to continue the second term further to the left. In this way one finally obtains a meromorphic continuation to the half plane ${\rm Re}\, s>-2b/a$, as asserted.

At face value, (4.57) seems to entail that $Z_{\rm hyp}$ has a pole for $s=0$. This is not the case, however. Indeed, we may write (4.57) as

$$\begin{aligned} Z_{\rm hyp}(a,b;s,z) &= \frac{i}{2a}\int_{-\infty}^{\infty}du\Bigg(-\frac{b}{as}\frac{1}{{\rm ch}^2(\pi(z-u)/a)}\cdot\frac{2{\rm sh}(\pi u/b)}{[2{\rm ch}(\pi u/b)]^{s+1}} \\ &\quad +4\,(1+\frac{1}{s})\frac{{\rm th}(\pi(z-u)/a)}{[2{\rm ch}(\pi u/b)]^{s+2}}\Bigg). \end{aligned} \quad (4.58)$$

From this it is clear that there can be at most a simple pole at $s=0$. But in fact we have

$$\begin{aligned}\lim_{s\to 0} sZ_{\rm hyp}(a,b;s,z) &= \frac{i}{2a}\int_{-\infty}^{\infty} du \left(-\frac{b}{a}\frac{\mathrm{th}(\pi u/b)}{\mathrm{ch}^2(\pi(z-u)/a)}+\frac{\mathrm{th}(\pi(z-u)/a)}{\mathrm{ch}^2(\pi u/b)}\right)\\ &= \frac{i}{2a}\int_{-\infty}^{\infty} du \frac{b}{\pi}\partial_u[\mathrm{th}(\pi u/b)\mathrm{th}(\pi(z-u)/a)]\\ &= 0. \end{aligned}\tag{4.59}$$

Hence $Z_{\rm hyp}(a,b;s,z)$ is regular at $s=0$, as claimed. (The same reasoning shows that $Z_{\rm hyp}$ has no poles in the half plane $\mathrm{Re}\, s>-2b/a$.)

Next, we study the function

$$L(a,b;z)\equiv \partial_s Z_{\rm hyp}(a,b;s,z)|_{s=0}. \tag{4.60}$$

Its z-derivative is given by

$$-\frac{i\pi}{2a^2}\int_{-\infty}^{\infty} du \ln[2\mathrm{ch}(\pi u/b)]/\mathrm{ch}^2(\pi(z-u)/a), \tag{4.61}$$

cf. (4.53). Comparing this to (3.51), we infer

$$\partial_z L(a,b;z)=\partial_z \ln G_{\rm hyp}(a,b;z). \tag{4.62}$$

Thus we have $L=G_{\rm hyp}+C$. Finally, because both L and $\ln G_{\rm hyp}$ are odd, we obtain $C=0$. As a result, we have proved

$$\partial_s Z_{\rm hyp}(a,b;s,z)|_{s=0}=\ln(G_{\rm hyp}(a,b;z)), \quad |\mathrm{Im}\, z|<(a+b)/2. \tag{4.63}$$

We continue by considering some special cases. First, let us observe that one has

$$Z_{\rm hyp}(a,b;2,z)=-\frac{b^2}{4\pi^2}\partial_z^2 \ln G_{\rm hyp}(a,b;z). \tag{4.64}$$

Indeed, when we take the z-derivative of (3.51), we can integrate by parts twice to obtain this formula, cf. (4.54) with $s=2$. Alternatively, (4.64) follows by combining equality of the AΔEs satisfied by both functions, their minimal solution character, and oddness in z.

Second, we point out the explicit specialization

$$Z_{\rm hyp}(a,a;1,z)=\frac{i}{4}\mathrm{th}(\pi z/2a). \tag{4.65}$$

To check this, one need only verify that the rhs satisfies (4.50) with $b=a$, $s=1$. (Indeed, (4.65) then follows from minimality and oddness.)

Third, we claim

$$Z_{\rm hyp}(a,b;0,z)=iz/a. \tag{4.66}$$

To show this, we use (the continuation to $s=0$ of) (4.53) to infer that the z-derivative of the lhs is a constant. The rhs is odd and satisfies (4.50) with $s=0$, so (4.66) follows.

Fourth, we note that $Z_{\rm hyp}(a,b;s,z)$ is not polynomially bounded for $|\mathrm{Re}\, z|\to\infty$ whenever $\mathrm{Re}\, s$ is negative. Indeed, this feature follows from the analytic continuation of the AΔE (4.50). As an illuminating special case, we use (4.53) to calculate

$$\partial_z Z_{\rm hyp}(\pi,\pi;-1,z)=\frac{i}{\pi}\int_{-\infty}^{\infty} dx \frac{\mathrm{ch}(z-x)}{\mathrm{ch}^2 x}=i\mathrm{ch}(z), \tag{4.67}$$

so that (by oddness)

$$Z_{\rm hyp}(\pi,\pi;-1,z) = i{\rm sh}(z). \tag{4.68}$$

Next, we obtain the relation to the rational zeta function (4.21). Changing variables in (4.54), we get

$$Z_{\rm hyp}(a,b;s,z-ib/2) = \frac{i}{2a}\int_{-\infty}^{\infty}[-2i{\rm sh}(\pi(z-x)/b)]^{-s}{\rm th}(\pi x/a)dx, \quad {\rm Im}\, z\in(0,b). \tag{4.69}$$

(The logarithm implied here is real-valued for $x=0, z\in i(0,b)$.) Recalling (4.22), we now deduce

$$\lim_{b\uparrow\infty}\left(\frac{2\pi}{b}\right)^{s} Z_{\rm hyp}(a,b;s,z-ib/2) = Z_{\rm rat}(a;s,z), \quad {\rm Re}\, s>1, \quad {\rm Im}\, z>0. \tag{4.70}$$

Via suitable contour shifts, this limiting relation can be readily extended to the half plane ${\rm Im}\, z > -a/2$.

To conclude this section, we introduce and study a natural 'hyperbolic' generalization of the Riemann zeta function, viz.,

$$\zeta_{\rm hyp}(\rho;s) \equiv (2\pi\rho)^{s} Z_{\rm hyp}(1,1/\rho;s,i/2-i/2\rho), \quad \rho>0. \tag{4.71}$$

Indeed, from (4.70) we have

$$\lim_{\rho\downarrow 0}\zeta_{\rm hyp}(\rho;s) = \zeta(s), \quad {\rm Re}\, s>1. \tag{4.72}$$

From (4.71) we read off that $\zeta_{\rm hyp}(\rho;s)$ is analytic in the half plane ${\rm Re}\, s > -2/\rho$ and that

$$\zeta_{\rm hyp}(1;s) = 0. \tag{4.73}$$

Moreover, from (4.69) we deduce the representation

$$\zeta_{\rm hyp}(\rho;s) = \frac{i}{2}\int_{-\infty}^{\infty}{\rm th}(\pi x)\left(\frac{\pi\rho}{i{\rm sh}(\pi\rho(x-i/2))}\right)^{s}dx, \quad \rho\in(0,2), \quad {\rm Re}\, s>0. \tag{4.74}$$

We proceed by turning (4.74) into a more telling formula. To this end we first write

$$\zeta_{\rm hyp}(\rho;s) = \frac{i}{2}\int_{-\infty}^{\infty}{\rm th}(\pi x)\left(\frac{1}{ix+1/2}\right)^{s}\left(\left(\frac{\pi\rho(x-i/2)}{{\rm sh}(\pi\rho(x-i/2))}\right)^{s} - \left(\frac{\pi(x-i/2)}{{\rm sh}(\pi(x-i/2))}\right)^{s}\right)dx, \tag{4.75}$$

where the three logarithms are real for $x=0$. The point is now that for ${\rm Re}\, s\in(0,2)$ we may shift the contour up by $i/2$ and take $x\to u/\pi$ to obtain the representation

$$\zeta_{\rm hyp}(\rho;s) = \pi^{s-1}\sin(\pi s/2)\int_{0}^{\infty}\frac{du}{u^{s}}\coth(u)h(\rho;s,u), \quad \rho\in(0,2), \quad {\rm Re}\, s\in(0,2), \tag{4.76}$$

where we have introduced

$$h(\rho;s,u) \equiv \left(\frac{\rho u}{{\rm sh}(\rho u)}\right)^{s} - \left(\frac{u}{{\rm sh}(u)}\right)^{s}, \quad \rho\geq 0. \tag{4.77}$$

From the latter representation we read off in particular that $\zeta_{\rm hyp}(\rho; s)$ is positive for $(\rho, s) \in (0,1) \times (0,2)$. It can also be exploited to obtain representations for the Riemann zeta function that have not appeared in previous literature, to our knowledge.

In order to derive the latter, we observe that (4.77) entails uniform bounds

$$|h(\rho; s, u)| \le C_1 u^2, \quad \rho \in [0,1), \quad {\rm Re}\, s \in (0,2), \quad u \in \mathbb{R}, \tag{4.78}$$

$$|h(\rho; s, u)| \le C_2, \quad \rho \in [0,1), \quad {\rm Re}\, s \in (0,2), \quad u \in \mathbb{R}. \tag{4.79}$$

Therefore, choosing ${\rm Re}\, s \in (1,2)$, we may interchange the $\rho \downarrow 0$ limit and the integration, yielding

$$\zeta(s) = \pi^{s-1} \sin(\pi s/2) \int_0^\infty \coth(u) \left(\frac{1}{u^s} - \frac{1}{{\rm sh}(u)^s} \right) du, \quad {\rm Re}\, s \in (1,2), \tag{4.80}$$

cf. (4.72). Writing $u^{-s} = (1-s)^{-1} \partial_u u^{1-s}$, we may integrate by parts to get

$$\zeta(s) = \frac{1}{s-1} \pi^{s-1} \sin(\pi s/2) \int_0^\infty u^{1-s} \partial_u (\coth u [1 - (u/{\rm sh}\, u)^s]) du. \tag{4.81}$$

Evidently, the latter representation makes sense and is valid for ${\rm Re}\, s \in (0,2)$. Using the functional equation, we also deduce

$$\zeta(s) = -\frac{2^{s-1}}{\Gamma(1+s)} \int_0^\infty u^s \partial_u (\coth u [1 - (u/{\rm sh}\, u)^{1-s}]) du, \quad {\rm Re}\, s \in (-1,1). \tag{4.82}$$

5 Barnes' multiple zeta and gamma functions

In this section we present a summary of some results from our forthcoming paper Ref. [20]. Specifically, we focus on those results that hinge on interpreting the Barnes functions from our AΔE perspective.

Barnes' multiple zeta function may be defined by the series

$$\zeta_N(s, w | a_1, \ldots, a_N) = \sum_{m_1, \ldots, m_N = 0}^{\infty} (w + m_1 a_1 + \cdots + m_N a_N)^{-s}, \tag{5.1}$$

where we have

$$a_1, \ldots, a_N > 0, \quad {\rm Re}\, w > 0, \quad {\rm Re}\, s > N. \tag{5.2}$$

It is immediate that these functions are related by the recurrence

$$\zeta_{M+1}(s, w + a_{M+1} | a_1, \ldots, a_{M+1}) - \zeta_{M+1}(s, w | a_1, \ldots, a_{M+1}) = -\zeta_M(s, w | a_1, \ldots, a_M), \tag{5.3}$$

with

$$\zeta_0(s, w) \equiv w^{-s}. \tag{5.4}$$

As Barnes shows [21], ζ_N has a meromorphic continuation in s, with simple poles only at $s = 1, \ldots, N$. He defined his multiple gamma function $\Gamma_N^B(w)$ in terms of the s-derivative at $s = 0$,

$$\Psi_N(w | a_1, \ldots, a_N) \equiv \partial_s \zeta_N(s, w | a_1, \ldots, a_N)|_{s=0}, \quad N \in \mathbb{N}. \tag{5.5}$$

Analytically continuing (5.3), it follows that the functions Ψ_N satisfy the recurrence

$$\Psi_{M+1}(w + a_{M+1}|a_1, \dots, a_{M+1}) - \Psi_{M+1}(w|a_1, \dots, a_{M+1}) = -\Psi_M(w|a_1, \dots, a_M). \quad (5.6)$$

Comparing (5.1) with $N = 1$ to (4.21), we infer

$$\zeta_1(s, w|a) = Z_{\mathrm{rat}}(a; s, i[w - a/2]). \quad (5.7)$$

Also, comparing (5.5) and (4.25), we get

$$\begin{aligned} \Psi_1(w|a) &= \ln(G_{\mathrm{rat}}(a; i[w - a/2])) \\ &= \ln((2\pi)^{-1/2} \exp([w/a - 1/2] \ln a)\Gamma(w/a)), \end{aligned} \quad (5.8)$$

cf. (3.1). Now we have already seen that G_{rat} and Z_{rat} can be viewed as minimal solutions to AΔEs of the form (1.1) and (1.2), respectively (cf. Subsections 3.1 and 4.1). In Subsections 5.1 and 5.2 we sketch how, more generally, the functions ζ_N and Ψ_N can be tied in with the AΔE lore reviewed in Subsection 2.1.

In brief, ζ_{M+1} and Ψ_{M+1} may be viewed as *minimal* solutions to the equations (5.3) and (5.6), reinterpreted as AΔEs of the form (1.2), with the right-hand sides viewed as the given function $\phi(z)$. In this way we arrive at a precise version of Barnes' expression 'simplest solution' [21], and we are led to new and illuminating integral representations for the Barnes functions.

5.1 Multiple zeta functions

Using the identity

$$\prod_{j=1}^{N}(1 - e^{-a_j t})^{-1} = \sum_{m_1, \dots, m_N = 0}^{\infty} \exp(-t(m_1 a_1 + \cdots + m_N a_N)), \quad a_1, \dots, a_N, t > 0, \quad (5.9)$$

and Euler's integral (4.5), we can rewrite $\zeta_N(s, w)$ (5.1) as

$$\zeta_N(s, w) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s e^{-wt} \prod_{j=1}^{N}(1 - e^{-a_j t})^{-1}, \ \mathrm{Re}\, s > N, \ \mathrm{Re}\, w > 0. \quad (5.10)$$

Consider now the function

$$\phi_{M,s}(z) \equiv -\zeta_M(s, A_M + d - iz), \quad d > -A_M, \quad \mathrm{Re}\, s \geq M + 2, \quad (5.11)$$

where we have introduced

$$A_N \equiv \frac{1}{2} \sum_{j=1}^{N} a_j, \qquad N \in \mathbb{N}. \quad (5.12)$$

Since we choose the displacement parameter d greater than $-A_M$, we obtain a non-empty strip $|\mathrm{Im}\, z| < A_M + d$ in which $\phi_{M,s}(z)$ is defined and analytic. Thus we can use (5.10) to write

$$\phi_{M,s}(z) = -\frac{2^{1-M}}{\Gamma(s)} \int_0^\infty dy \frac{(2y)^{s-1} e^{-2dy}}{\prod_{j=1}^{M} \mathrm{sh}(a_j y)} \cdot e^{2iyz}, \quad \mathrm{Re}\, s \geq M+2, \quad \mathrm{Im}\, z > -A_M - d. \quad (5.13)$$

Now the Fourier transform $\hat{\phi}_{M,s}(y)$ (2.7) can be read off from (5.13). Clearly, it belongs to $L^1(\mathbb{R})$, and it satisfies $\hat{\phi}_{M,s}(y) = O(y)$ for $y \to 0$. Moreover, using the series representation (5.1) for $\phi_{M,s}(x), x \in \mathbb{R}$, it is easily seen that $\phi_{M,s}(x)$ belongs to $L^1(\mathbb{R})$, too.

As a consequence, the conditions (2.6) are satisfied. Therefore, we obtain a minimal solution

$$\begin{aligned} f_{M,s}(z) &= \frac{2^{-M}}{\Gamma(s)} \int_0^\infty dy \frac{(2y)^{s-1} e^{-2dy}}{\prod_{j=1}^{M+1} \mathrm{sh}(a_j y)} \cdot e^{2iyz} \\ &= \zeta_{M+1}(s, A_{M+1} + d - iz), \end{aligned} \tag{5.14}$$

to the AΔE

$$f(z + ia_{M+1}/2) - f(z - ia_{M+1}/2) = \phi_{M,s}(z), \quad \mathrm{Re}\, s \geq M+2, \quad \mathrm{Im}\, z > -A_M - d. \tag{5.15}$$

In summary, for $\mathrm{Re}\, s \geq M+2$ we may view ζ_{M+1} as the unique minimal solution (2.8).

Next, we exploit the formula (2.9) with $\phi(z)$ given by $\phi_{M,s}(z)$ (5.11). Changing variables, it yields

$$\zeta_{M+1}(s, A_{M+1} + d - iz) = \frac{i}{2a_{M+1}} \int_{-\infty}^{\infty} dx \zeta_M(s, c - iz + ix) \mathrm{th} \frac{\pi}{a_{M+1}} x, \tag{5.16}$$

where $c = A_M + d$ and where we may take $\mathrm{Im}\, z > -c$. This relation can now be iterated, but first we integrate by parts, using the relation

$$\partial_w \zeta_N(s, w) = -s \zeta_N(s+1, w), \quad N \in \mathbb{N}. \tag{5.17}$$

(This formula can be read off from (5.1).) Doing so, we obtain

$$\begin{aligned} \zeta_N(s, A_N + d - iz) &= \frac{\pi}{2a_N^2} \frac{1}{s-1} \int_{-\infty}^{\infty} dx \zeta_{N-1}(s-1, A_{N-1} + d - iz + ix)/\mathrm{ch}^2(\pi x/a_N) \\ &= \int_{\mathbb{R}^N} \left(\prod_{n=1}^{N} \frac{\pi \mathrm{ch}^{-2}(\pi x_n/a_n)}{2a_n^2(s-n)} \right) \left(d - iz + i \sum_{n=1}^{N} x_n \right)^{N-s} d^N x. \end{aligned} \tag{5.18}$$

(Recall (5.4) to check the last iteration step.)

Now we derived this new representation for $\mathrm{Re}\, s > N$ and $\mathrm{Im}\, z > -d$. But it is evident that it yields a meromorphic s-continuation of ζ_N to all of $\mathbb{C}$, with simple poles occurring only at $s = 1, \ldots, N$. Moreover, suitable contour shifts yield analyticity in z for $\mathrm{Im}\, z > -A_N - d$. (Observe that we are generalizing arguments we already presented in Subsection 4.1, where we are dealing with ζ_1, cf. (5.7).)

The above integral representation (5.18) was derived for the first time in Ref. [20]. For $\mathrm{Re}\, s \leq N+1$ one can also view $\zeta_N(s, w)$ as a minimal solution, but now in the more general sense of Theorem II.3 in Ref. [9]. (This hinges on (5.17) and its iterates.) The details can be found in Section 4 of Ref. [20].

In Ref. [20] we also approach $\zeta_N(s, w)$ from yet another, quite elementary angle. The latter does not involve AΔEs, but rather a certain class of Laplace-Mellin transforms. In this way we can easily rederive various explicit formulas and properties established in other ways by Barnes. For brevity we refrain from sketching this other approach, which is complementary to the AΔE perspective.

5.2 Multiple gamma functions

As we have just shown, $\zeta_N(s,w)$ has a meromorphic continuation to $\mathbb{C}$, which is analytic in $s=0$. From (5.18) we can also obtain a representation for the function $\Psi_N(w|a_1,\ldots,a_N)$ (5.5), namely,

$$\begin{aligned}\Psi_N(A_N+d-iz) &= \sum_{l=1}^{N}\frac{1}{l}\cdot\zeta_N(0,A_N+d-iz)\\ &+ (-)^{N+1}\left(\prod_{n=1}^{N}\frac{\pi}{2na_n^2}\int_{-\infty}^{\infty}\frac{dx_n}{\mathrm{ch}^2(\pi x_n/a_n)}\right)I_N(x),\end{aligned}\tag{5.19}$$

where the integrand reads

$$I_N(x)=\left(d-iz+i\sum_{n=1}^{N}x_n\right)^N\ln\left(d-iz+i\sum_{n=1}^{N}x_n\right).\tag{5.20}$$

The $s=0$ value of ζ_N appearing here follows from (5.18), too, yielding an Nth degree polynomial in z. (The coefficients of the latter can be expressed in terms of the Bernoulli numbers [20].)

As before, the restriction $\mathrm{Im}\,z>-d$ can be relaxed to $\mathrm{Im}\,z>-A_N-d$ by contour shifts, revealing that $\Psi_N(w)$ is analytic for $\mathrm{Re}\,w>0$. Now a second continuation of the logarithm in the shifted integrand to $z\in\mathbb{C}\setminus i(-\infty,-A_N-d]$ reveals that $\Psi_N(w)$ admits an analytic continuation to the cut plane $\mathbb{C}\setminus(-\infty,0]$. Defining

$$\Gamma_N(w|a_1,\ldots,a_N)\equiv\exp\Psi_N(w|a_1,\ldots,a_N),\tag{5.21}$$

it follows that $\Gamma_N(w)$ is analytic and zero-free in this cut plane. Furthermore, the recurrence (5.6) entails

$$\Gamma_{M+1}(w|a_1,\ldots,a_{M+1})=\Gamma_M(w|a_1,\ldots,a_M)\Gamma_{M+1}(w+a_{M+1}|a_1,\ldots,a_{M+1}),\quad M\in\mathbb{N}.\tag{5.22}$$

We can now determine the analytic character of $\Gamma_N(w)$ on the cut by exploiting (5.22). Specifically, taking first $M=0$ and noting $\Gamma_0(w)=1/w$, we can iterate (5.22) to get

$$\Gamma_1(w|a_1)=\prod_{k=0}^{l-1}\frac{1}{w+ka_1}\cdot\Gamma_1(w+la_1|a_1),\quad l\in\mathbb{N}^*.\tag{5.23}$$

From this we read off that $\Gamma_1(w|a_1)$ has a meromorphic extension without zeros and with simple poles for $w\in-a_1\mathbb{N}$. Writing next

$$\Gamma_2(w|a_1,a_2)=\prod_{k=0}^{l-1}\Gamma_1(w+ka_2|a_1)\cdot\Gamma_2(w+la_2|a_1,a_2),\quad l\in\mathbb{N}^*,\tag{5.24}$$

we deduce that $\Gamma_2(w|a_1,a_2)$ has a meromorphic extension without zeros and with poles for $w=-(k_1a_1+k_2a_2), k_1,k_2\in\mathbb{N}$. The multiplicity of a pole w_0 equals the number of distinct pairs (k_1,k_2) such that $w_0=-(k_1a_1+k_2a_2)$. In particular, all poles are simple when a_1/a_2 is irrational, and the pole at $w=0$ is always simple. Proceeding recursively, it is now clear that $\Gamma_N(w)$ has a meromorphic extension, without zeros and with poles for $w = -(k_1 a_1 + \cdots + k_N a_N), k_1, \ldots, k_N \in \mathbb{N}$, the pole at $w = 0$ being simple.

Before concluding this section with some remarks, we would like to point out that the above account of the functions $\zeta_N(s, w)$ and $\Gamma_N(w)$ has only involved Euler's formula (4.5) and the 'minimal solution' consequences of the properties (2.6). Taking the latter for granted, the arguments in this section are self-contained, leading quickly and simply to a significant part of Barnes' results.

At this point it should be remarked that Barnes used a different normalization for his multiple gamma function $\Gamma_N^B(w)$. Specifically, the relation to $\Gamma_N(w)$ reads

$$\Gamma_N^B(w) = \rho_N \Gamma_N(w), \tag{5.25}$$

where ρ_N is Barnes' modular constant. Our use of $\Gamma_N(w)$ is in accord with most of the later literature. The constant ρ_N in (5.25) is (by definition) equal to the reciprocal residue of $\Gamma_N(w)$ at its simple pole $w = 0$. Equivalently, one has

$$w\Gamma_N^B(w) \to 1, \quad w \to 0. \tag{5.26}$$

We further remark that Kurokawa's double sine function $S_2(x|\omega_1, \omega_2)$, which we encountered in Subsection 3.4 (see (3.52)), can be defined by

$$S_2(w|a_1, a_2) \equiv \Gamma_2(a_1 + a_2 - w|a_1, a_2)/\Gamma_2(w|a_1, a_2). \tag{5.27}$$

(Note that one may replace Γ_2 by Γ_2^B in this formula.)

Finally, the special function $\Gamma_N(w)$ can once again be interpreted as a minimal solution to an AΔE of the form (1.1). This is explained in detail in Section 4 of Ref. [20]. We also mention that the second (Laplace transform) approach alluded to at the end of Subsection 5.1 can be exploited to derive uniform large-w asymptotics away from the cut $(-\infty, 0]$, cf. Section 3 in Ref. [20].

6 A generalized hypergeometric function

The subject of this section is a function $R(a_+, a_-, \mathbf{c}; v, \hat{v})$, depending on parameters $a_+, a_- \in (0, \infty)$, couplings $\mathbf{c} = (c_0, c_1, c_2, c_3) \in \mathbb{R}^4$ and variables $v, \hat{v} \in \mathbb{C}$. It generalizes both the hypergeometric function ${}_2F_1(a, b, c; w)$ and the Askey-Wilson polynomials $p_n(q, \alpha, \beta, \gamma, \delta; \cos v)$. (For a complete account of the features of ${}_2F_1$ used below we refer to Refs. [19, 23]. For information on the Askey-Wilson polynomials, see Refs. [22, 13].) The R-function was introduced in Ref. [1]. Detailed proofs of several assertions made below can be found in our paper Ref. [16].

In Subsection 6.1 we review various features of the ${}_2F_1$-function that admit a generalization to the R-function. The latter is defined in Subsection 6.2, where we also obtain some automorphy properties. In Subsection 6.3 we introduce the four independent hyperbolic difference operators of which the R-function is an eigenfunction. In Subsection 6.4 we derive the specialization to the Askey-Wilson polynomials. Subsection 6.5 concerns the 'nonrelativistic limit' $R \to {}_2F_1$. In Subsection 6.6 we study how the R-function is related to the Ismail-Rahman functions [24, 25], which are eigenfunctions of the trigonometric Askey-Wilson difference operator. Though we shed some light on this issue, we leave several questions open.

6.1 Some reminders on ${}_2F_1$

The hypergeometric function was already known to Euler in terms of an integral representation. Our generalized hypergeometric function is defined in terms of an integral as well, but this integral representation does not generalize Euler's integral representation for ${}_2F_1$, but rather the much later one due to Barnes.

The latter representation can be readily understood from Gauss' series representation,

$$ {}_2F_1(a,b,c;w) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{\Gamma(b+n)}{\Gamma(b)} \frac{\Gamma(c)}{\Gamma(c+n)} \frac{w^n}{n!}. \tag{6.1} $$

Using for instance the ratio test, one sees that this power series converges for $|w| < 1$. The Barnes representation makes it possible to analytically continue ${}_2F_1$ to the cut plane $|\mathrm{Arg}(-w)| < \pi$. It reads

$$ \int_{\mathcal{C}} dz \exp(-iz\ln(-w)) \cdot \frac{\Gamma(iz)\Gamma(c)}{2\pi\Gamma(c-iz)} \cdot \frac{\Gamma(a-iz)\Gamma(b-iz)}{\Gamma(a)\Gamma(b)}. \tag{6.2} $$

Here, the logarithm branch is fixed by choosing $\ln(-w) \in \mathbb{R}$ for $w \in (-\infty, 0)$. Taking first $\mathrm{Re}\, a, \mathrm{Re}\, b > 0$, the contour $\mathcal{C}$ runs along the real axis from $-\infty$ to ∞, with a downward indentation at the origin to avoid the pole due to $\Gamma(iz)$. Thus it separates the downward pole sequences starting at $-ia$ and $-ib$ from the upward sequence starting at 0.

Invoking the asymptotics of the Γ-function, one sees that the integrand has exponential decay for $\mathrm{Re}\, z \to \pm\infty$, provided $|\mathrm{Arg}(-w)| < \pi$. Thus the integral yields an analytic function of w in the cut plane. After multiplication by $2\pi i$, the residues at the simple poles $z = in$ of the integrand are equal to the terms in the Gauss series (6.1). A second somewhat subtle application of the Γ-function asymptotics now shows that when one moves the contour $\mathcal{C}$ up across the poles $0, i, \ldots, in$, picking up $2\pi i$ times the residues in the process, then the integral over the shifted contour converges to 0 for $n \to \infty$, provided $|w| < 1$. Thus the integral (6.2) yields an analytic continuation to the cut plane $|\mathrm{Arg}(-w)| < \pi$, as advertised.

The analyticity region cannnot be much improved, since the function ${}_2F_1(a,b,c;w)$ has a logarithmic branch point at $w = 1$ for generic $a, b, c \in \mathbb{C}$. In this connection we should add that the representation (6.2) can be modified to handle arbitrary $a, b \in \mathbb{C}$: For $\mathrm{Re}\, a \leq 0$ and/or $\mathrm{Re}\, b \leq 0$ one need only shift the contour $\mathcal{C}$ up, so that the downward pole sequences starting at $-ia$ and $-ib$ stay below it. In this way one can demonstrate that for fixed w in the cut plane one obtains a meromorphic function of a, b and c.

Next, we recall that the hypergeometric function can be used to diagonalize the two-coupling family of Schrödinger operators

$$ H(g,\tilde{g}) \equiv -\frac{d^2}{dx^2} + \frac{g(g-1)\nu^2}{\mathrm{sh}^2 \nu x} - \frac{\tilde{g}(\tilde{g}-1)\nu^2}{\mathrm{ch}^2 \nu x}. \tag{6.3} $$

Specifically, one first performs the similarity transformation

$$ \tilde{H}(g,\tilde{g}) \equiv w(\nu x)^{-1/2} H(g,\tilde{g}) w(\nu x)^{1/2}, \tag{6.4} $$

where $w(y)$ is the 'weight function'

$$ w(y) \equiv \mathrm{sh} y^{2g} \mathrm{ch} y^{2\tilde{g}}. \tag{6.5} $$

A straightforward calculation yields

$$\tilde{H}(g,\tilde{g}) = -\frac{d^2}{dx^2} - 2\nu[g\coth(\nu x) + \tilde{g}\tanh(\nu x)]\frac{d}{dx} - \nu^2(g+\tilde{g})^2. \tag{6.6}$$

Then one has the eigenvalue equation

$$\tilde{H}(g,\tilde{g})\Psi_{\rm nr} = p^2\Psi_{\rm nr}, \tag{6.7}$$

where $\Psi_{\rm nr}$ is the nonrelativistic wave function

$$\Psi_{\rm nr}(\nu,g,\tilde{g};x,p) \equiv {}_2F_1(\frac{1}{2}(g+\tilde{g}-\frac{ip}{\nu}), \frac{1}{2}(g+\tilde{g}+\frac{ip}{\nu}), g+\frac{1}{2}; -\mathrm{sh}^2\nu x). \tag{6.8}$$

Indeed, (6.7) is simply the rational ODE satisfied by ${}_2F_1(a,b,c;w)$, transformed to hyperbolic form via the substitution $w = -\mathrm{sh}^2\nu x$.

The wave function $\Psi_{\rm nr}$ (6.8) is also an eigenfunction of two analytic difference operators, one of which acts on x, while the second one acts on the spectral variable p. We will obtain this fact (which cannot be found in the textbook literature) as a corollary of the nonrelativistic limit in Subsection 6.5.

6.2 The 'relativistic' hypergeometric function

The function $R(a_+,a_-,\mathbf{c};v,\hat{v})$ we are about to introduce can be used in particular to diagonalize an analytic difference operator (from now on AΔO) that arises in the context of the relativistic Calogero-Moser system, cf. Ref. [2], Subsection 3.3.

But just as the nonrelativistic wave function $\Psi_{\rm nr}$ (6.8) serves as an eigenfunction for a 2-coupling generalization of the (reduced, two-particle) nonrelativistic Calogero-Moser Hamiltonian $H(g,0)$ (6.3), we will find that the relativistic wave function $\Psi_{\rm rel}$ (6.46) we associate below with the R-function is in fact an eigenfunction of a 4-coupling generalization of the relativistic counterpart of $H(g,0)$. Moreover, it is an eigenfunction of three more independent AΔOs with a similar structure. (In the nonrelativistic limit two of these give rise to the AΔOs mentioned at the end of the previous subsection.)

We have split the integrand in (6.2) in three factors to anticipate a corresponding factorization of the integrand for the R-function. Setting

$$\hat{c}_0 \equiv (c_0+c_1+c_2+c_3)/2, \tag{6.9}$$

the latter reads

$$I(a_+,a_-,\mathbf{c};v,\hat{v},z) \equiv F(c_0;v,z)K(a_+,a_-,\mathbf{c};z)F(\hat{c}_0;\hat{v},z). \tag{6.10}$$

Here, the functions F and K involve the hyperbolic gamma function $G(z) \equiv G_{\rm hyp}(a_+,a_-;z)$ from Subsection 3.4, cf. (3.35).

Specifically, F and K are defined by

$$F(d;y,z) \equiv \left(\frac{G(z+y+id-ia)}{G(y+id-ia)}\right)(y\to -y), \tag{6.11}$$

$$K(a_+,a_-,\mathbf{c};z) \equiv \frac{1}{G(z+ia)}\prod_{j=1}^{3}\frac{G(is_j)}{G(z+is_j)}, \tag{6.12}$$

where we use the notation

$$s_1 \equiv c_0 + c_1 - a_-/2, \quad s_2 \equiv c_0 + c_2 - a_+/2, \quad s_3 \equiv c_0 + c_3, \quad a \equiv (a_+ + a_-)/2. \tag{6.13}$$

We have suppressed the dependence on a_+ and a_- in G and in F, since these functions are invariant under the interchange of a_+ and a_-. (Note K is not invariant, since s_1 and s_2 are not.)

Just as we first have chosen $\mathrm{Re}\, a, \mathrm{Re}\, b > 0$ so as to define the integration contour $\mathcal{C}$ in the Barnes representation (6.2), we begin by choosing

$$s_j \in (0, a), \quad j = 1, 2, 3, \quad c_0, \hat{c}_0, v, \hat{v} \in (0, \infty). \tag{6.14}$$

Then we choose once more the contour $\mathcal{C}$ going from $-\infty$ to ∞ in the z-plane, with a downward indentation at the origin. The choices just detailed ensure that $\mathcal{C}$ separates the four upward pole sequences coming from the four z-dependent G-functions in K (6.12) and the four downward pole sequences coming from the z-dependent G-functions in the two F-factors of the integrand (6.10). (At this point the reader should recall the pole-zero properties of the hyperbolic G-function, cf. (3.36), (3.37).)

Our R-function is now defined by the integral

$$R(a_+, a_-, \mathbf{c}; v, \hat{v}) \equiv \frac{1}{(a_+ a_-)^{1/2}} \int_{\mathcal{C}} dz I(a_+, a_-, \mathbf{c}; v, \hat{v}, z). \tag{6.15}$$

The asymptotics (3.50) of the G-function plays the same role as the Stirling formula for the Γ-function in showing that the integral converges. Indeed, using (3.50) one readily obtains

$$I(a_+, a_-, \mathbf{c}; v, \hat{v}, z) = O(\exp(\mp 2\pi(\frac{1}{a_+} + \frac{1}{a_-})\mathrm{Re}\, z)), \quad \mathrm{Re}\, z \to \pm\infty. \tag{6.16}$$

Therefore, R is well defined and analytic in v and $\hat{v}$ for $\mathrm{Re}\, v, \mathrm{Re}\, \hat{v} \neq 0$.

The R-function has in fact much stronger analyticity properties, but to demonstrate these in detail is well beyond our present scope. Thus we only summarize some results, referring to Ref. [16] for proofs. Briefly, the R-function extends to a function that is meromorphic in all of its eight arguments, as long as a_+ and a_- stay in the (open) right half plane. Moreover, the pole varieties and their associated orders are explicitly known. For the case of fixed positive a_+, a_- and (generic) real c_0, c_1, c_2, c_3, the R-function is meromorphic in v and $\hat{v}$, with poles that can (but need not) occur solely for certain points on the imaginary axis. These points correspond to collisions of v- and $\hat{v}$-dependent z-poles in the integrand with z-poles in the three upward s_j-pole sequences and points that are given by the poles of the factors $1/G(\pm v + ic_0 - ia)$ and $1/G(\pm \hat{v} + i\hat{c}_0 - ia)$ in the integrand.

We conclude this subsection by listing some automorphy properties of the R-function. To this end we introduce the 'dual couplings'

$$\hat{\mathbf{c}} \equiv J\mathbf{c}, \quad J \equiv \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \tag{6.17}$$

whence one has (cf. (6.13))

$$c_0 + c_j = \hat{c}_0 + \hat{c}_j, \quad s_j = \hat{s}_j, \quad j = 1,2,3. \tag{6.18}$$

We also define the transposition

$$I\mathbf{c} \equiv (c_0, c_2, c_1, c_3). \tag{6.19}$$

Then one has the symmetries

$$R(a_+, a_-, \mathbf{c}; v, \hat{v}) = R(a_+, a_-, \hat{\mathbf{c}}; \hat{v}, v), \quad (\mathrm{self-duality}), \tag{6.20}$$

$$R(a_+, a_-, \mathbf{c}; v, \hat{v}) = R(a_-, a_+, I\mathbf{c}; v, \hat{v}), \tag{6.21}$$

$$R(\lambda a_+, \lambda a_-, \lambda \mathbf{c}; \lambda v, \lambda \hat{v}) = R(a_+, a_-, \mathbf{c}; v, \hat{v}), \quad \lambda > 0, \quad (\text{scale invariance}). \tag{6.22}$$

These features can be quite easily checked directly from the definition (6.15). (Use the G-function properties (3.39) and (3.40) to check (6.21) and (6.22), resp.)

6.3 Eigenfunction properties

In order to detail the four AΔOs for which our R-function is a joint eigenfunction, we introduce the quantities

$$s_\delta(y) \equiv \mathrm{sh}(\pi y / a_\delta), \quad c_\delta(y) \equiv \mathrm{ch}(\pi y / a_\delta), \tag{6.23}$$

$$C_\delta(\mathbf{c}; z) \equiv \frac{s_\delta(z - ic_0)}{s_\delta(z)} \frac{c_\delta(z - ic_1)}{c_\delta(z)} \frac{s_\delta(z - ic_2 - ia_{-\delta}/2)}{s_\delta(z - ia_{-\delta}/2)} \frac{c_\delta(z - ic_3 - ia_{-\delta}/2)}{c_\delta(z - ia_{-\delta}/2)}, \tag{6.24}$$

$$A_\delta(\mathbf{c}; y) \equiv C_\delta(\mathbf{c}; y)\left(T^y_{ia_{-\delta}} - 1\right) + C_\delta(\mathbf{c}; -y)\left(T^y_{-ia_{-\delta}} - 1\right) + 2c_\delta(i(c_0 + c_1 + c_2 + c_3)), \tag{6.25}$$

where $\delta = +, -$, and where the superscript y on the shifts indicates the variable they act on. The eigenfunction properties of the R-function are now specified in the following proposition, whose proof we sketch.

Proposition 6.1 *The function $R(a_+, a_-, \mathbf{c}; v, \hat{v})$ is a joint eigenfunction of the AΔOs*

$$A_+(\mathbf{c}; v), \ A_-(I\mathbf{c}; v), \ A_+(\hat{\mathbf{c}}; \hat{v}), \ A_-(I\hat{\mathbf{c}}; \hat{v}), \tag{6.26}$$

with eigenvalues

$$2c_+(2\hat{v}), \ 2c_-(2\hat{v}), \ 2c_+(2v), \ 2c_-(2v). \tag{6.27}$$

Sketch of proof. In view of the symmetries (6.20) and (6.21) we need only prove the AΔE

$$A_+(\mathbf{c}; v) R(\mathbf{c}; v, \hat{v}) = 2c_+(2\hat{v}) R(\mathbf{c}; v, \hat{v}). \tag{6.28}$$

Also, due to the analyticity properties of the R-function already detailed, we may restrict the parameters and imaginary parts of v and $\hat{v}$ in such a way that R is given by

$$R(\mathbf{c}; v, \hat{v}) = \frac{1}{(a_+ a_-)^{1/2}} \int_{\mathcal{C}} F(c_0; v, z) K(\mathbf{c}; z) F(\hat{c}_0; \hat{v}, z) dz, \tag{6.29}$$

and that we may let the AΔO $A_+(\mathbf{c}; v)$ act on the integrand.

A main tool in proving the second-order AΔE (6.28) is now to exploit the first-order AΔE

$$\frac{G(z+ia_-/2)}{G(z-ia_-/2)} = 2c_+(z), \tag{6.30}$$

satisfied by the G-function. Indeed, using (6.30), one readily checks that the function F (6.11) solves the two AΔEs

$$\frac{F(d;y+ia_-/2,z)}{F(d;y-ia_-/2,z)} = \frac{s_+(y+z+id-ia_-/2)}{s_+(y-z-id+ia_-/2)}\frac{s_+(y-id+ia_-/2)}{s_+(y+id-ia_-/2)}, \tag{6.31}$$

$$\frac{F(d;y,z-ia_-)}{F(d;y,z)} = \frac{1}{4s_+(y+z+id-ia_-)s_+(y-z-id+ia_-)}. \tag{6.32}$$

Using (6.31) with $d=c_0$ and $y=v$, one can calculate the quotient

$$Q(\mathbf{c};v,z) \equiv (A_+(\mathbf{c};v)F)(c_0;v,z)/F(c_0;v,z). \tag{6.33}$$

A key point is now that Q can be rewritten as

$$2c_+(2z+2i\hat{c}_0) + \frac{4\prod_{j=1}^4 c_+(z-ia_-/2+is_j)}{s_+(v+z+ic_0-ia_-)s_+(v-z-ic_0+ia_-)}, \quad s_4 \equiv a. \tag{6.34}$$

This fact amounts to a functional equation that can be proved by comparing residues at simple poles and $|\mathrm{Re}\, v| \to \infty$ asymptotics. We now observe that the denominator in (6.34) appears in (6.32) with $d=c_0, y=v$. Thus we get

$$A_+(\mathbf{c};v)F(c_0;v,z) = 2c_+(2z+2i\hat{c}_0)F(c_0;v,z) + F(c_0;v,z-ia_-)\Pi(\mathbf{c};z-ia_-/2), \tag{6.35}$$

where we have introduced the product

$$\Pi(\mathbf{c};z) \equiv 16\prod_{j=1}^4 c_+(z+is_j). \tag{6.36}$$

The upshot of these calculations is the identity

$$\begin{aligned} A_+(\mathbf{c};v)R(\mathbf{c};v,\hat{v}) \;&=\; \frac{1}{(a_+a_-)^{1/2}}\int_{\mathcal{C}} dz[2c_+(2z+2i\hat{c}_0)I(\mathbf{c};v,\hat{v},z) \\ &\quad +F(c_0;v,z-ia_-)\Pi(\mathbf{c};z-ia_-/2)K(\mathbf{c};z)F(\hat{c}_0;\hat{v},z)]. \end{aligned} \tag{6.37}$$

To proceed, we now shift $\mathcal{C}$ down by ia_- in the second term and then take $z \to z+ia_-$. Then we are in the position to exploit a critical property of the function $K(\mathbf{c};z)$: Due to (6.30) it obeys the AΔE

$$K(\mathbf{c};z+ia_-/2) = K(\mathbf{c};z-ia_-/2)/\Pi(\mathbf{c};z). \tag{6.38}$$

Therefore we obtain

$$\begin{aligned} A_+(\mathbf{c};v)R(\mathbf{c};v,\hat{v}) \;&=\; \frac{1}{(a_+a_-)^{1/2}}\int_{\mathcal{C}} dz[2c_+(2z+2i\hat{c}_0)I(\mathbf{c};v,\hat{v},z) \\ &\quad +F(c_0;v,z)K(\mathbf{c};z)F(\hat{c}_0;\hat{v},z+ia_-)]. \end{aligned} \tag{6.39}$$

Finally, we use (6.32) with $d = \hat{c}_0, y = \hat{v}$ to get

$$\begin{aligned} F(\hat{c}_0; \hat{v}, z + ia_-) &= 4s_+(\hat{v} + z + i\hat{c}_0)s_+(\hat{v} - z - i\hat{c}_0)F(\hat{c}_0; \hat{v}, z) \\ &= 2[c_+(2\hat{v}) - c_+(2z + 2i\hat{c}_0)]F(\hat{c}_0; \hat{v}, z). \end{aligned} \tag{6.40}$$

Then substitution in (6.39) yields (6.28). □

The joint eigenfunction property just demonstrated shows that the R-function may be viewed as a solution to a so-called bispectral problem, cf. Grünbaum's contribution to these proceedings [26]. In this respect, it has however a much more restricted character. Indeed, it solves in fact a 'quadrispectral' problem. (Note that the latter problem can be posed more generally whenever one is dealing with a pair of commuting AΔOs of the form (2.19).)

6.4 The Askey-Wilson specialization

We continue by sketching how the Askey-Wilson polynomials arise as a specialization of the R-function. With the restriction (6.14) on the arguments in effect, we may and will use the representation (6.15). We are going to exploit the analyticity properties of the R-function in the variable $\hat{v}$ and the eigenvalue equation

$$A_+(\hat{\mathbf{c}}; \hat{v})R(\mathbf{c}; v, \hat{v}) = 2c_+(2v)R(\mathbf{c}; v, \hat{v}). \tag{6.41}$$

To prevent nongeneric singularities, we choose $\hat{c}_0$ rationally independent of $a_+, a_-, \hat{c}_1, \hat{c}_2$ and $\hat{c}_3$. Then R has no pole at the points $\hat{v} = i\hat{c}_0 + ina, n \in \mathbb{Z}$, so we may define

$$R_n(v) \equiv R(\mathbf{c}; v, i\hat{c}_0 + ina_-), \quad n \in \mathbb{Z}. \tag{6.42}$$

Proposition 6.2 *One has $R_n(v) = P_n(c_+(2v)), n \in \mathbb{N}$, with $P_n(u)$ a polynomial of degree n in u.*

Proof. The pole of $I(z)$ at $z = 0$ is simple and has residue $-i(a_+a_-)^{1/2}/2\pi$. (This follows from (3.41) and (3.38).) Thus we have

$$R(\mathbf{c}; v, \hat{v}) = 1 + \frac{1}{(a_+a_-)^{1/2}} \int_{\mathcal{C}^+} dz I(\mathbf{c}; v, \hat{v}, z), \tag{6.43}$$

where $\mathcal{C}^+$ denotes the contour $\mathcal{C}$ with an upward indentation at $z = 0$ (instead of downward).

We can now let $\hat{v}$ converge to $i\hat{c}_0$ without $\hat{v}$-dependent poles crossing $\mathcal{C}^+$. The factor $1/G(-\hat{v} + i\hat{c}_0 - ia)$ in $I(z)$ has a zero for $\hat{v} = i\hat{c}_0$, whereas the factor $1/G(\hat{v} + i\hat{c}_0 - ia)$ has no pole for $\hat{v} = i\hat{c}_0$ (due to the rational independence requirement). Hence we deduce

$$R_0(v) = 1. \tag{6.44}$$

Next, we write out the eigenvalue equation (6.41) for $\hat{v} = i\hat{c}_0 + ina_-$:

$$\begin{aligned} C_+(\hat{\mathbf{c}}; i\hat{c}_0 + ina_-)[R_{n-1}(v) - R_n(v)] + C_+(\hat{\mathbf{c}}; -i\hat{c}_0 - ina_-)[R_{n+1}(v) - R_n(v)] \\ +2c_+(2ic_0)R_n(v) = 2c_+(2v)R_n(v). \end{aligned} \tag{6.45}$$

The rational independence assumption entails that the coefficients are well defined, and that $C_+(\hat{\mathbf{c}}; -i\hat{c}_0 - ina_-)$ does not vanish for $n \in \mathbb{N}$. Since we have $C_+(\hat{\mathbf{c}}; i\hat{c}_0) = 0$ (cf. (6.24)), we may now use (6.44) as a starting point to prove the assertion recursively. $\Box$

Note that when one restricts attention to a *finite* number of the above functions $R_n(v)$, one may let $\hat{c}_0$ vary over suitable intervals without encountering singularities or zeros of the recurrence coefficients (save for $C_+(\hat{\mathbf{c}}; i\hat{c}_0)$, of course). We can now continue a_+ analytically to $-i\pi/r, r > 0$, to obtain polynomials $P_n(\cos(2rv))$ with recurrence coefficients that can be read off from (6.45). The a_+-continuation turns the hyperbolic AΔO $A_+(\mathbf{c}; v)$ into a trigonometric AΔO with eigenvalue $2\mathrm{ch}2r(\hat{c}_0 + na_-)$ on $P_n(\cos(2rv))$. In essence, the latter AΔO is the Askey-Wilson AΔO and the recurrence is the 3-term recurrence of the Askey-Wilson polynomials. More precisely, taking $r = 1/2$, the polynomials $P_n(\cos v)$ turn into the Askey-Wilson polynomials $p_n(q, \alpha, \beta, \gamma, \delta; \cos v)$ under a suitable parameter substitution and n-dependent renormalization, cf. Ref. [16].

6.5 The 'nonrelativistic' limit $R \to {}_2F_1$

We continue by clarifying the relation between the R- and ${}_2F_1$-functions. To this end we introduce the relativistic wave function

$$\Psi_{\mathrm{rel}}(\beta, \nu, (g_0, g_1, g_2, g_3); x, p) \equiv R(\pi, \beta\nu, \beta\nu(g_0, g_1, g_2, g_3); \nu x, \beta p/2). \tag{6.46}$$

Now we change variables $z \to \beta\nu z$ in the integral representation (6.15), and rewrite the result as

$$\Psi_{\mathrm{rel}} = \int_{\mathcal{C}} dz S_l M S_r, \tag{6.47}$$

where

$$S_l \equiv \exp(2iz \ln 2) F(\beta\nu g_0; \nu x, \beta\nu z), \tag{6.48}$$

$$S_r \equiv \exp(2iz \ln(2\beta\nu)) F(\beta\nu \hat{g}_0; \beta p/2, \beta\nu z), \tag{6.49}$$

$$M \equiv \left(\frac{\beta\nu}{\pi}\right)^{1/2} \exp(-2iz \ln(4\beta\nu)) K(\pi, \beta\nu, \beta\nu(g_0, g_1, g_2, g_3); \beta\nu z). \tag{6.50}$$

The factorization performed here ensures that the $\beta \downarrow 0$ limit of the three factors exists. Indeed, using the two zero step size limits (3.42) and (3.47), we obtain

$$\lim_{\beta \downarrow 0} S_l = \exp(-iz \ln(\mathrm{sh}^2 \nu x)), \tag{6.51}$$

$$\lim_{\beta \downarrow 0} S_r = \left(\frac{\Gamma(-\frac{ip}{2\nu} + \frac{1}{2}(g + \tilde{g}) - iz)}{\Gamma(-\frac{ip}{2\nu} + \frac{1}{2}(g + \tilde{g}))}\right)(p \to -p), \quad g \equiv g_0 + g_2, \quad \tilde{g} \equiv g_1 + g_3, \tag{6.52}$$

$$\lim_{\beta \downarrow 0} M = \frac{\Gamma(iz)\Gamma(g + \frac{1}{2})}{2\pi\Gamma(g + \frac{1}{2} - iz)}, \tag{6.53}$$

where the limits are uniform on sufficiently small discs around any point on the contour.

When we now interchange these $\beta \downarrow 0$ limits with the contour integration, we obviously get

$$\lim_{\beta \downarrow 0} \Psi_{\mathrm{rel}}(\beta, \nu, (g_0, g_1, g_2, g_3); x, p) = \Psi_{\mathrm{nr}}(\nu, g, \tilde{g}; x, p), \quad g \equiv g_0 + g_2, \quad \tilde{g} \equiv g_1 + g_3, \tag{6.54}$$

cf. (6.8), (6.2). To date, we have no justification for this interchange. A uniform L^1 tail bound as $\beta \downarrow 0$ would suffice (by dominated convergence), but it remains to supply such a bound. In any case, we conjecture that the limit (6.54) holds true uniformly on x-compacts in $\{\mathrm{Re}\, x > 0, |\mathrm{Im}\, x| < \pi/2\nu\}$ and p-compacts in $\mathbb{C}$.

Let us now consider the $\beta \downarrow 0$ limits of the above four AΔOs with parameters and variables

$$a_+ = \pi, \quad a_- = \beta\nu, \quad \mathbf{c} = \beta\nu(g_0, g_1, g_2, g_3), \quad v = \nu x, \quad \hat{v} = \beta p/2. \tag{6.55}$$

Clearly, $A_-(\hat{\mathbf{c}}; \hat{v})$ and its eigenvalue $\hat{E}_- = 2\mathrm{ch}(2\pi x/\beta)$ diverge for $\beta \downarrow 0$. For the remaining AΔOs and their eigenvalues one readily verifies the following limiting behavior:

$$A_+(\mathbf{c}; v) = 2 + \beta^2 \tilde{H}(g, \tilde{g}) + O(\beta^4), \quad \beta \downarrow 0, \tag{6.56}$$

$$E_+ = 2\mathrm{ch}(\beta p) = 2 + \beta^2 p^2 + O(\beta^4), \quad \beta \downarrow 0, \tag{6.57}$$

$$\lim_{\beta \downarrow 0} A_-(\mathbf{c}; v) = \exp(-i\pi(g + \tilde{g}))T^x_{i\pi/\nu} + (i \to -i), \quad (\mathrm{Re}\, x > 0), \tag{6.58}$$

$$E_- = 2\mathrm{ch}(\pi p/\nu), \tag{6.59}$$

$$\lim_{\beta \downarrow 0} A_+(\hat{\mathbf{c}}; \hat{v}) = \frac{[p - i\nu(g + \tilde{g})]}{p} \cdot \frac{[p - i\nu - i\nu(g - \tilde{g})]}{p - i\nu}(T^p_{2i\nu} - 1) + (i \to -i) + 2, \tag{6.60}$$

$$\hat{E}_+ = 2\mathrm{ch}(2\nu x). \tag{6.61}$$

It can be shown directly that the limiting operators do have the pertinent eigenvalues on Ψ_{nr} (6.8). Indeed, for the AΔO $A_+(\mathbf{c}; v)$ this amounts to (6.7). The limit AΔO on the rhs of (6.58) does have eigenvalue $2\mathrm{ch}(\pi p/\nu)$ by virtue of the known analytic continuation of ${}_2F_1(a, b, c; w)$ across the logarithmic branch cut $w \in [1, \infty)$. (To appreciate the role of the cut, it may help to recall that the rhs of (6.8) reduces to $\cos(xp)$ for $g = \tilde{g} = 0$.) Finally, the eigenvalue $2\mathrm{ch}(2\nu x)$ for the AΔO on the rhs of (6.60) can be verified by using the contiguous relations of the hypergeometric function, cf. Ref. [19].

6.6 The relation to the Ismail-Rahman functions

As we mentioned above (6.17), the R-function admits a meromorphic continuation in the shift parameters a_+, a_-, provided one requires they stay in the right half plane. Equivalently, scaling out a_+ via (6.22), we retain meromorphy as long as the scale-invariant quotient $\rho = a_-/a_+$ varies over the cut plane $\mathbb{C} \setminus (-\infty, 0]$. (Just as for the hyperbolic gamma function, where the meromorphic continuation is explicitly given by (3.54), we do not know what happens when ρ converges to the cut.)

Consider now the function

$$Q(r, a, \mathbf{c}; v, \hat{v}) \equiv R(\overline{\omega}\pi r^{-1}, \omega a, \omega\mathbf{c}; \omega v, \omega\hat{v}), \quad \omega \equiv e^{i\pi/4}, \quad r, a > 0. \tag{6.62}$$

It is meromorphic in $c_0, \ldots, c_3, v$ and $\hat{v}$, and satisfies

$$AQ = 2\cos(2r\hat{v})Q, \tag{6.63}$$

$$BQ = 2\mathrm{ch}(2\pi a^{-1}\hat{v})Q. \tag{6.64}$$

Here, A and B are the AΔOs (cf. (6.24) and (6.25))

$$A \equiv C_t(r, a, \mathbf{c}; v)[\exp(-iad/dv) - 1] + (v \to -v) + 2\mathrm{ch}(2r\hat{c}_0), \tag{6.65}$$

$$B \equiv C_h(r,a,\mathbf{c};v)[\exp(-\pi r^{-1}d/dv)-1]+(v\to -v)+2\cos(2\pi a^{-1}\hat{c}_0), \tag{6.66}$$

with

$$C_t(z)\equiv \frac{\sin r(z-ic_0)}{\sin rz}\frac{\cos r(z-ic_1)}{\cos rz}\frac{\sin r(z-ic_2-ia/2)}{\sin r(z-ia/2)}\frac{\cos r(z-ic_3-ia/2)}{\cos r(z-ia/2)}, \tag{6.67}$$

$$\begin{aligned} C_h(z) \quad \equiv \quad & \frac{\mathrm{sh}\pi a^{-1}(z-ic_0)}{\mathrm{sh}\pi a^{-1}z}\frac{\mathrm{ch}\pi a^{-1}(z-ic_1)}{\mathrm{ch}\pi a^{-1}z} \\ & \times\frac{\mathrm{sh}\pi a^{-1}(z-ic_2-i\pi r^{-1}/2)}{\mathrm{sh}\pi a^{-1}(z-i\pi r^{-1}/2)}\frac{\mathrm{ch}\pi a^{-1}(z-ic_3-i\pi r^{-1}/2)}{\mathrm{ch}\pi a^{-1}(z-i\pi r^{-1}/2)}. \end{aligned} \tag{6.68}$$

Moreover, one has

$$Q(r,a,\mathbf{c};v,i\hat{c}_0+ina)=P_n(r,a,\mathbf{c};\cos(2rv)), \quad n\in\mathbb{N}, \tag{6.69}$$

where $P_n(x)$ are the polynomials from Subsection 6.4.

It is immediate from (6.64) that we have the implication

$$Q(v+\pi/r,\hat{v})=Q(v,\hat{v})\Rightarrow \pm i\hat{v}=\hat{c}_0+ka, \quad k\in\mathbb{Z}. \tag{6.70}$$

Put differently, $Q(v,\hat{v})$ is not π/r-periodic in v for generic spectral parameter $\hat{v}$. Now as we mentioned at the end of Subsection 6.4, the trigonometric AΔO A is essentially the Askey-Wilson AΔO. Ismail and Rahman [24] construct independent solutions $F_j(r,a,\mathbf{c};v,\hat{v}), j=1,2$, to the Askey-Wilson AΔE (6.63) in terms of the ${}_8\phi_7$ basic hypergeometric function. Their solutions are manifestly π/r-periodic in v for *arbitrary* spectral parameters. But in contrast to our solution $Q(r,a,\mathbf{c};v,\hat{v})$, the functions $F_j(r,a,\mathbf{c};v,\hat{v})$ do not admit analytic continuation to the hyperbolic regime. This is for basically the same reason as G_{trig} (3.12) does not admit continuation to a hyperbolic gamma function: When one takes q from the open unit disc to the unit circle, one looses convergence. (Cf. the last paragraph of Subsection 3.2.)

On the other hand, the general theory sketched in Subsection 2.2 entails that we must have

$$Q(v,\hat{v})=\mu_1(v,\hat{v})F_2(v,\hat{v})-\mu_2(v,\hat{v})F_1(v,\hat{v}), \tag{6.71}$$

with μ_1,μ_2 ia-periodic in v, cf. (2.15). The open problem of finding these multipliers explicitly can be further illuminated by a reasoning that is of interest in its own right.

As a first step, let us note that the above AΔOs A and B commute. But in contrast to the situation considered in Subsection 2.2, the shifts in the complex plane are not in the same direction. Therefore, the space of joint eigenfunctions is left invariant by elliptic multipliers, with the period parallellogram corresponding to the two directions. But when one lets the two directions become equal, such elliptic multipliers generically will diverge. Accordingly, joint eigenfunctions in general do not converge in the pertinent limit.

The point is now that from the Ismail-Rahman solutions F_1,F_2 to (6.63) we can construct *joint* solutions J_1,J_2 to (6.63) and (6.64), as we will detail in a moment. It is quite plausible (but we could not prove) that J_1 and J_2 form a basis for the space of joint solutions, viewed as a vector space over the field of elliptic functions with periods π/r and ia. Assuming this, the problem of explicitly finding μ_1,μ_2 in (6.71) gets narrowed down to the problem of finding two elliptic functions e_j such that

$$Q(v,\hat{v})=e_1(v,\hat{v})J_2(v,\hat{v})-e_2(v,\hat{v})J_1(v,\hat{v}). \tag{6.72}$$

(In this connection it should be recalled that Q (6.62) is given in terms of a rather inaccessible integral.)

We continue by filling in the details concerning the functions J_1, J_2. They are defined by

$$J_j(r,a,\mathbf{c};v,\hat{v}) \equiv F_j(r,a,\mathbf{c};v,\hat{v})F_j(\pi/a,\pi/r,i\mathbf{c};iv,i\hat{v}), \quad j=1,2. \tag{6.73}$$

(This makes sense, since F_1 and F_2 are meromorphic in $c_0,\ldots,c_3,v$ and $\hat{v}$.) To see that they solve both (6.63) and (6.64), note first that since the first F_j-factor is π/r-periodic in v, the second one is ia-periodic in v. Therefore, J_j still solves (6.63). Next, observe that the substitution

$$r,a,\mathbf{c},v,\hat{v} \to \pi/a,\pi/r,i\mathbf{c},iv,i\hat{v}, \tag{6.74}$$

that turns the first factor into the second one, also turns A (and its dual) into B (and its dual). Hence the second F_j-factor solves (6.64), so that J_j solves (6.64), too.

The elliptic multiplier question is of particular interest in view of recent work by Suslov [27], and by Koelink and Stokman [28]. They study Hilbert space properties associated with an even and self-dual linear combination $\Phi(r,a,\mathbf{c};v,\hat{v})$ of the Ismail-Rahman functions F_1, F_2. Now the above 'doubling' argument (which was suggested by the relation (3.58) between the trigonometric and hyperbolic gamma functions) can be applied to Φ as well. It entails that the function

$$\Psi(r,a,\mathbf{c};v,\hat{v}) \equiv \Phi(r,a,\mathbf{c};v,\hat{v})\Phi(\pi/a,\pi/r,i\mathbf{c};iv,i\hat{v}), \tag{6.75}$$

is an even and self-dual solution to (6.63) and (6.64), just as $Q(r,a,\mathbf{c};v,\hat{v})$. Thus one would be inclined to expect that the even, self-dual function μ defined by

$$Q(r,a,\mathbf{c};v,\hat{v}) = \mu(r,a,\mathbf{c};v,\hat{v})\Psi(r,a,\mathbf{c};v,\hat{v}), \tag{6.76}$$

is already elliptic with periods $\pi/r, ia$.

We cannot rule out this hunch, but it does lead to consequences that seem startling. Indeed, assuming μ *is* elliptic, it has doubly-periodic poles and zeros. The zeros must adjust the poles of the two Φ-factors in (6.75) (which can be read off from their series representations, cf. Ref. [28]) to those of the Q-function, whose *eventual* locations follow from Ref. [16]. But this gives rise to zero patterns for the Q- and Φ-functions that appear quite unexpected—though we cannot exclude them for the time being.

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