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Physica D 218 (2006) 191–203
PHYSICA D
www.elsevier.com/locate/physd

Existence of finite-order meromorphic solutions as a detector of integrability in difference equations

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Received 15 April 2004; received in revised form 12 March 2006; accepted 10 May 2006
Available online 16 June 2006
Communicated by A.C. Newell

Abstract

The existence of sufficiently many finite-order (in the sense of Nevanlinna) meromorphic solutions of a difference equation appears to be a good indicator of integrability. It is shown that, out of a large class of second-order difference equations, the only equation that can admit a sufficiently general finite-order meromorphic solution is the difference Painlevé *II* equation. The proof given relies on estimates obtained by arguments related to singularity confinement. The existence of meromorphic solutions of a general class of first-order difference equations is also proven by a simple method based on Banach's fixed point theorem.

Keywords: Integrable; Discrete; Painlevé; Nevanlinna; Meromorphic

1. Introduction

An ordinary differential equation is said to possess the Painlevé property if all of its solutions are single-valued about all movable singularities (see, e.g., [1]). This property is a powerful indicator of integrability of a differential equation. In the early twentieth century, Painlevé [2,3], Fuchs [4] and Gambier [5] showed that, out of a general class of second-order ordinary differential equations, there were six equations possessing the Painlevé property which could not be integrated in terms of known functions. These equations are now known as the six Painlevé equations. The proof that these equations are indeed integrable had to wait until the latter part of the twentieth century when they were solved by inverse scattering techniques based on an associated isomonodromy problem; see, e.g., [6].

Several analogues of the Painlevé property for discrete equations have been discussed in the literature. In particular, Ablowitz et al. [7] considered discrete equations as delay equations in the complex plane which enabled them to utilize complex analytic methods. The equations they consider to be of "Painlevé type" (i) are of finite order in Nevanlinna theoretic sense, and (ii) have no digamma functions in their series expansions. They looked at, for instance, difference equations of the type

$$\overline{y} + \underline{y} = R(z; y), \tag{1.1}$$

where R is rational in both of its arguments and we have suppressed the z-dependence by writing $y \equiv y(z)$, $\overline{y} \equiv y(z+1)$ and $\underline{y} \equiv y(z-1)$. Ablowitz, Halburd and Herbst showed that if (1.1) has at least *one* non-rational finite-order meromorphic solution, then $\deg_y(R) \leq 2$. Indeed, the difference Painlevé *II* (dP_{II}) equation (3.21) lies within this class of equations. The assumption is considerably weaker than the continuous Painlevé property where *all* solutions are required to be single-valued about movable singularities. On the other hand, the class (1.1) with $\deg_y(R) \leq 2$ also includes many equations generally considered to be non-integrable.

We will show that the existence of a sufficiently general finite-order meromorphic solution is enough to single out the difference Painlevé *II* equation from a general class (1.1) of difference equations; see Theorem 3.1 below. We will also give examples of non-Painlevé type difference equations having

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0167-2789/$ - see front matter
doi:10.1016/j.physd.2006.05.005

special finite-order Riccati solutions. Therefore demanding the existence of a finite number (or even a one-parameter family) of finite-order solutions is not always enough to single out the dP_{II} from (1.1).

Costin and Kruskal [8] also applied complex analytic methods to detect integrability in discrete equations. Their approach is based on embedding solutions of a discrete equation into analyzable functions and is quite different from the property described in the present paper.

The most widely used detector of integrable discrete analogues of the Painlevé equations is the singularity confinement test of Grammaticos et al. [9]. The basic idea is to consider finite initial conditions leading to iterates which become infinite at a certain point. Such a singularity is said to be confined if the iterates are all finite after a finite number of steps and contain sufficient information about the initial conditions. The singularity confinement test has been successfully applied to discover many important discrete equations, which are widely believed to be integrable [10].

We will illustrate singularity confinement using the standard example [10]

$$y_{n+1} + y_{n-1} = \frac{a_n y_n + b_n}{1 - y_n^2}, \tag{1.2}$$

where (a_n) and (b_n) are given sequences. To go from finite values of y_n for $n \le k$ to $y_{k+1} = \infty$, we must have $y_k = \pm 1$. In order to analyze future iterates we let y_{k-1} have an arbitrary finite value c and we set $y_k = \pm 1 + \epsilon$ and then take the limit $\epsilon \to 0$. The next few iterates are

$$\begin{aligned} y_{k+1} &= \frac{1}{2}(-a_k \mp b_k)\epsilon^{-1} + O(1), \\ y_{k+2} &= \mp 1 + \frac{2a_{k+1} - a_k \mp b_k}{a_k \pm b_k}\epsilon + O(\epsilon^2), \\ y_{k+3} &= \pm \frac{(a_k \pm b_k)\{(b_{k+2} - b_k) \mp (a_{k+2} - 2a_{k+1} + a_k)\}}{2(2a_{k+1} - a_k \mp b_k)}\epsilon^{-1} \\ &\quad + O(1). \end{aligned}$$

In order for y_{k+3} to be finite (i.e., if the singularity is confined) we must have $(b_{k+2} - b_k) \mp (a_{k+2} - 2a_{k+1} + a_k) = 0$. If all singularities are confined then this condition must be true with both the "+" and "−" signs and for all k, in which case $a_k = \alpha n + \beta$ and $b_n = \gamma + \delta(-1)^n$ and Eq. (1.2) is the discrete Painlevé *II* equation.

Despite the demonstrated power and apparent simplicity of the singularity confinement test, its implementation is not without difficulty. In general the iterates of a difference equation oscillate between finite and infinite values. Therefore it is not always clear when the iterates truly leave the singularity. To be absolutely sure that the singularity is confined we would need to know information about infinitely many iterates. Another difficulty is that there exists a discrete equation discovered by Hietarinta and Viallet [11], which passes the singularity confinement test, but numerical studies suggest that it is chaotic. Their suggestion to avoid this problem is to demand that the iteration sequence has zero algebraic entropy. This approach is related to a number of techniques which use the slow growth of the degree of the nth iterate of the considered map (as a rational function of the initial conditions) to detect integrability [12–15].

In this paper we consider Eq. (1.1) using the notion of singularity confinement involving only five iterates. Nevanlinna theory is used to demonstrate that generic non-confinement of poles implies that the order of any corresponding meromorphic solution is infinite. This helps to clarify a link between the complex analytic approach [7] and the singularity confinement test [9]. The latter is used as a tool in the proof of Theorem 3.1 to show that the only difference equation of the type (1.1) having a sufficiently general finite-order solution is the difference Painlevé *II* equation.

The autonomous form of the difference Painlevé *II* equation, known as the McMillan map (3.25), admits a two-parameter family of finite-order meromorphic solutions. In the non-autonomous case, however, the question of existence of meromorphic solutions remains open.

In general, very little is known about the singularity structure of solutions of difference equations in the complex domain. We conclude this paper by bringing together a number of existence results for meromorphic solutions of first-order autonomous difference equations. We prove these results using Banach's fixed point theorem.

2. Nevanlinna theory

The Nevanlinna theory of meromorphic functions studies the density of the points in the complex plane at which a meromorphic function takes a prescribed value. It also provides a natural way to describe the growth of a meromorphic function. In this section we will first present some of the basic definitions and elementary facts from Nevanlinna theory. Then we will go on to prove a technical lemma, which will be applied in Section 3 to single out the difference Painlevé *II* equation from a general class of nonlinear second-order equations.

2.1. A brief review of value distribution theory

The *Nevanlinna characteristic*

$$T(r, y) := N(r, y) + m(r, y)$$

of a non-constant meromorphic function y is the sum of the *counting function* $N(r, y)$ and the *proximity function* $m(r, y)$. The counting function is defined by

$$N(r, \infty) \equiv N(r, y) := \int_0^r \frac{n(t, y) - n(0, y)}{t}\mathrm{d}t + n(0, y)\log r,$$

where $n(r, y)$ is the number of poles (counting multiplicities) of y in the disc $\{z : |z| \le r\}$. It is a measure of the number of poles in the disc of radius r centered at the origin. Similarly,

$$\overline{N}(r, \infty) \equiv \overline{N}(r, y) := \int_0^r \frac{\overline{n}(t, y) - \overline{n}(0, y)}{t}\mathrm{d}t + \overline{n}(0, y)\log r,$$

where $\overline{n}(r, y)$ counts each pole only once, ignoring multiplicity. The proximity function

$$m(r, \infty) \equiv m(r, y) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ |y(r\mathrm{e}^{\mathrm{i}\theta})|\mathrm{d}\theta,$$

where

$$\log^+ x = \max(0, \log x),$$

describes the average "closeness" of y to ∞ on a circle of radius r. Similarly we define

$$m(r,a) \equiv m\left(r, \frac{1}{y-a}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{1}{y(re^{i\theta})-a}\right| d\theta,$$

and

$$N(r,a) \equiv N\left(r, \frac{1}{y-a}\right) = \int_0^r \frac{n(t,a)-n(0,a)}{t} dt + n(0,a)\log r,$$

where $n(r,a)$ counts the number of a-points (i.e., the points $z \in \mathbb{C}$ such that $f(z) = a$) of y in the disc of radius r centered in the origin.

The characteristic function $T(r, y)$ has many properties which make it very useful in the analysis of meromorphic functions. First, given meromorphic functions f and g,

$$T(r, f+g) \le T(r, f) + T(r, g) + \log 2 \tag{2.1}$$

and

$$T(r, fg) \le T(r, f) + T(r, g) \tag{2.2}$$

by the definition of $T(r, y)$. These inequalities hold also for the proximity function $m(r, \cdot)$ and for the counting function $N(r, \cdot)$, and they are applied frequently below. Second, the function $T(r, y)$ is an increasing function of r and a convex increasing function of $\log r$. This fact is crucial when analyzing the growth of a meromorphic function y in a neighborhood of infinity. This is described by the *order of growth* defined by

$$\rho(y) = \limsup_{r\longrightarrow\infty} \frac{\log T(r, y)}{\log r},$$

which, in the case when y is entire, is equivalent to the classical growth order

$$\sigma(y) = \limsup_{r\longrightarrow\infty} \frac{\log\log M(r, y)}{\log r},$$

where $M(r, y)$ is the maximum modulus of y in the disc of radius r.

Another useful fact is that the characteristic function $T(r, y)$ offers a way of classifying some of the natural subfields of the field of meromorphic functions. For instance,

A meromorphic function y *is* $\begin{cases}\text{rational}\\ \text{constant}\end{cases}$

if and only if $T(r, y) = \begin{cases} O(\log r)\\ O(1),\end{cases}$

as $r \to \infty$. These classifications are convenient when distinguishing between constant, rational and non-rational functions, for example, when they appear as coefficients in differential or difference equations.

The First Main Theorem of Nevanlinna theory states that

$$T(r, y) = T\left(r, \frac{1}{y-a}\right) + O(1) \tag{2.3}$$

for all complex numbers a. This implies that if y takes the value a less often than average, i.e., $N(r, a)$ is relatively small, there must be a compensation in $m(r, a)$. For instance, if a meromorphic function has few poles, then, loosely speaking, the values of that function must stay relatively "close" to infinity. To quote Rolf Nevanlinna himself, *the total affinity of a meromorphic function* y *towards each value* a *is the same, independent of* a [16]. In a sense this behavior is analogous to rational functions. If a rational function R is understood as a self-mapping of the extended plane, then the equation

$$R(z) = a$$

always has the same number of roots, counting multiplicities, independently of a. We will now consider the exponential function in the context of the First Main Theorem. Since $e^z \ne 0, \infty$ for all $z \in \mathbb{C}$, the counting functions $N(r, 0)$ and $N(r, \infty)$ are identically zero. Therefore, by the First Main Theorem, $m(r, 0)$ and $m(r, \infty)$ are large for $r \gg 1$, which means that on a large part of the circle $|z| = r$ the exponential function must be close to 0 and on another large part it must be close to ∞. This is reflected in the fact that the exponential function is small in the negative half plane $\Re(z) < 0$, and large in the positive half plane $\Re(z) > 0$. The fact that the proximity functions for 0 and ∞ are indeed large can be verified by a direct computation, which results in $m(r, 0) = m(r, \infty) = r/\pi$. On the other hand, by solving the equation $e^z = a$ for z, we have $N(r, a) = r/\pi + O(1)$ for all $a \ne 0, \infty$. The First Main Theorem then implies that $m(r, a) = O(1)$ for all finite non-zero a, which means that e^z is mostly far from any such values.

Finally, the Second Main Theorem of Nevanlinna theory addresses the question of the relative size of the components $m(r, a)$ and $N(r, a)$ in the sum (2.3). If $a_1, a_2, \ldots, a_q$ are distinct complex numbers, $q \ge 2$, then

$$(q-1)T(r, y) \le \overline{N}(r, y) + \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{y-a_j}\right) + S(r, y) \tag{2.4}$$

where

$$S(r, y) = O\left(\log(rT(r, y))\right), \tag{2.5}$$

except possibly for a finite length of r-values. In particular, if y is of finite order, the error term $S(r, y)$ grows at most like $O(\log r)$ without an exceptional set. Inequality (2.4) holds also when y is a non-constant rational function, with the error term $S(r, y)$ replaced by $O(1)$, but since there are other more efficient methods to deal with rational functions it is often assumed in applications that y is non-rational. In general we will use the notation $S(r, f)$ to denote any quantity which is $o(T(r, f))$ outside of a set of finite linear measure.

The importance of the Second Main Theorem arises from the fact that it is a profound generalization of Picard's theorem.

In fact, Picard's theorem follows easily from (2.4). If a non-constant meromorphic function g does not assume three values a_1, a_2 and ∞, it follows by (2.4) that $T(r, g) \leq S(r, g)$, which is an obvious contradiction. Therefore g must be constant. Choosing three finite values does not affect the reasoning significantly.

A number of generalizations of inequality (2.4) have been proposed (see, e.g., [17–19]). Nevanlinna himself gave an extension of (2.4) to three *small* target functions. A meromorphic function $a(z)$ is said to be small with respect to y when $T(r, a) = S(r, y)$. The following theorem is a slight refinement of Nevanlinna's original result; see [20, p. 47].

Theorem 2.1. *Suppose that y is a non-rational meromorphic function. If a and b are small meromorphic functions with respect to y such that $a^2 \not\equiv 4b$, then*

$$T(r, y) \leq \overline{N}(r, y) + \overline{N}\left(r, \frac{1}{y^2 + ay + b}\right) + S(r, y). \tag{2.6}$$

Obviously, simply factorizing $y^2 + ay + b$ takes the right-hand side of (2.6) in the same form as in Nevanlinna's theorem, except that then the target functions may have some square root type branching instead of being meromorphic. To deal with this we use Nevanlinna theory for *algebroid* functions. Roughly speaking an algebroid function is a ν-valued function in the complex plane, which is single-valued and meromorphic on a ν-sheeted covering surface of the complex plane. In other words, algebroid functions may have some isolated νth root type branch points, but otherwise they are meromorphic on a suitable domain. For a more precise definition, and elementary results in algebroid Nevanlinna theory, we refer to the original papers due to Selberg [21–23], Ullrich [24] and Valiron [25], who created the algebroid version of Nevanlinna theory in the late 1920's and early 1930's. Otherwise the proof is just a slight modification of Nevanlinna's original proof.

Proof of Theorem 2.1. Let

$$\phi(z) = \frac{y(z) - \alpha_+(z)}{\alpha_-(z) - \alpha_+(z)}, \tag{2.7}$$

where

$$\alpha_\pm(z) = -\frac{1}{2}a(z) \pm \frac{1}{2}\sqrt{a(z)^2 - 4b(z)}$$

are algebroid functions. Then by the algebroid version of the Second Main Theorem (2.4), we have

$$T(r, \phi) \leq \overline{N}(r, \phi) + \overline{N}\left(r, \frac{1}{\phi}\right) + \overline{N}\left(r, \frac{1}{\phi - 1}\right) + N_\xi(r, \phi) + S(r, \phi),$$

where $N_\xi(r, \phi)$ is a measure of the branch points of ϕ. However, from (2.7) it follows that $N_\xi(r, \phi) = S(r, \phi) = S(r, y)$. Therefore,

$$\begin{aligned} T(r, y) &= T(r, \phi) + S(r, y) \\ &\leq \overline{N}(r, \phi) + \overline{N}\left(r, \frac{1}{\phi}\right) + \overline{N}\left(r, \frac{1}{\phi - 1}\right) + S(r, y) \\ &\leq \overline{N}(r, y) + \overline{N}\left(r, \frac{1}{y - \alpha_+}\right) + \overline{N}\left(r, \frac{1}{y - \alpha_-}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{\alpha_- - \alpha_+}\right) \\ &\quad + \overline{N}(r, \alpha_+) + \overline{N}(r, \alpha_-) + S(r, y) \\ &= \overline{N}(r, y) + \overline{N}\left(r, \frac{1}{y^2 + ay + b}\right) + S(r, y), \end{aligned}$$

which completes the proof. □

2.2. Applications to difference equations

Consider the equation

$$\overline{y} = R(z; y), \tag{2.8}$$

where R is rational in both arguments. Yanagihara [26] showed that if Eq. (2.8) admits a finite-order non-rational meromorphic solution, then the degree of R as a function of y, $\deg_y(R)$, is at most one. In this case, Eq. (2.8) is either linear or a linearizable discrete analogue of the Riccati equation. Using similar arguments it was shown in [7] that if Eq. (1.1) admits a finite-order non-rational meromorphic solution then $\deg_y(R) \leq 2$. In both cases the proof relies heavily on the Valiron–Mohon'ko identity; see, e.g., [27], which states that

$$T(r, R(z; y)) = \deg_y(R) T(r, y) + O(\log r) \tag{2.9}$$

assuming that y is non-rational. In our study of Eq. (1.1), we will seek more precise information by considering the pole distribution of y. In the following lemma, we will find a lower bound for the number of poles of y which is enough to imply, for most equations of the type (1.1), that y has infinite order of growth. If we additionally assume that the denominator of $R(z; y)$ has two distinct roots, and that certain two types of singularity appear generically, then the only remaining equation will be the dP_{II}.

Lemma 2.2. *Let y be a non-rational meromorphic solution of the equation*

$$\overline{y} + \underline{y} = \frac{c_2 y^2 + c_1 y + c_0}{y^2 + ay + b} =: R(z; y), \tag{2.10}$$

where $R(z; y)$ has degree two as a function of y and where a, b and c_j's are rational functions such that $a^2 \not\equiv 4b$. If there exist $r_0 > 0$ and $\alpha < 2$ such that

$$\overline{N}(r, \overline{y} + \underline{y}) \leq \alpha \overline{N}(r + 3, y) \tag{2.11}$$

for all $r > r_0$, then y has infinite order.

We will see in Section 3 that inequality (2.11) is closely related to singularity confinement.

Proof of Lemma 2.2. Assuming that y is of finite order, Theorem 2.1 implies

$$T(r, y) - \overline{N}(r, y) \leq \overline{N}\left(r, \frac{1}{y^2 + ay + b}\right) + O(\log r). \tag{2.12}$$

Hence, by (2.10) and (2.11),

$$\begin{aligned} T(r,y)-\overline{N}(r,y) &\le \overline{N}\left(r,\frac{c_2y^2+c_1y+c_0}{y^2+ay+b}\right)+O(\log r)\\ &= \overline{N}(r,\overline{y}+\underline{y})+O(\log r)\\ &\le \alpha\overline{N}(r+3,y)+O(\log r). \end{aligned} \tag{2.13}$$

Also, defining $N_1(r,y):=N(r,y)-\overline{N}(r,y)$, we have

$$\begin{aligned} T(r,\overline{y}+\underline{y})-\overline{N}(r,\overline{y}+\underline{y}) &= m(r,\overline{y}+\underline{y})+N_1(r,\overline{y}+\underline{y})\\ &\le m(r,\overline{y})+m(r,\underline{y})+N_1(r,\overline{y})\\ &\quad +N_1(r,\underline{y})+O(1)\\ &= T(r,\overline{y})-\overline{N}(r,\overline{y})+T(r,\overline{y})\\ &\quad -\overline{N}(r,\overline{y})+O(1). \end{aligned}$$

Thus, by the Valiron–Mo'honko theorem (see, e.g., [27]) and Eq. (2.13), we have

$$\begin{aligned} 2T(r,y) &= T(r,\overline{y}+\underline{y})+O(\log r)\\ &= \overline{N}(r,\overline{y}+\underline{y})+T(r,\overline{y}+\underline{y})-\overline{N}(r,\overline{y}+\underline{y})\\ &\quad +O(\log r)\\ &\le \alpha\overline{N}(r+3,y)+T(r,\overline{y})-\overline{N}(r,\overline{y})+T(r,\underline{y})\\ &\quad -\overline{N}(r,\underline{y})+O(\log r)\\ &\le \alpha\overline{N}(r+3,y)+\alpha\overline{N}(r+3,\overline{y})+\alpha\overline{N}(r+3,\underline{y})\\ &\quad +O(\log r)\\ &\le 3\alpha\overline{N}(r+4,y)+O(\log r), \end{aligned}$$

and so

$$T(r,y)-\overline{N}(r,y)\le \frac{3\alpha}{2}\overline{N}(r+4,y)-\overline{N}(r,y)+O(\log r).$$

Now assume that

$$\begin{aligned} T(r,y)-\overline{N}(r,y) &\le \frac{(j+2)\,\alpha}{2}\overline{N}(r+2j+2,y)\\ &\quad -j\overline{N}(r,y)+O(\log r) \end{aligned} \tag{2.14}$$

for all $j=1,\ldots,n$. Then we have

$$\begin{aligned} 2T(r,y) &\le \alpha\overline{N}(r+3,y)+\frac{(n+2)\,\alpha}{2}\overline{N}(r+2n+2,\overline{y})\\ &\quad -n\overline{N}(r,\overline{y})+\frac{(n+2)\,\alpha}{2}\overline{N}(r+2n+2,\underline{y})\\ &\quad -n\overline{N}(r,\underline{y})+O(\log r)\\ &\le (n+3)\alpha\overline{N}(r+2n+3,y)-2n\overline{N}(r-1,y)\\ &\quad +O(\log r). \end{aligned}$$

Hence,

$$\begin{aligned} T(r,y)-\overline{N}(r,y) &\le \frac{(n+3)\,\alpha}{2}\overline{N}(r+2n+4,y)\\ &\quad -(n+1)\overline{N}(r,y)+O(\log r) \end{aligned}$$

and so (2.14) holds for all $j\in\mathbb{N}$ by the induction principle. Thus,

$$\overline{N}(r,y)\le \frac{(n+2)\,\alpha}{2n}\overline{N}(r+2n+2,y)+O(\log r) \tag{2.15}$$

for all $n\in\mathbb{N}$. Now, since $\alpha<2$, we can choose $n_\alpha\in\mathbb{N}$ such that

$$\frac{(n_\alpha+2)\,\alpha}{2n_\alpha}<1.$$

Therefore, by Eq. (2.15), $\overline{N}(r,y)$ has exponential growth, and so y is of infinite order.

Note that $\overline{N}(r,y)$ cannot grow like the error term $O(\log r)$ since y has infinitely many poles. To see this, combine Eq. (2.10) and the Second Main Theorem to obtain

$$T(r,y)\le \overline{N}(r,y)+\overline{N}(r,\overline{y}+\underline{y})+O(\log r).$$

Therefore, assuming that y has finitely many poles leads to a contradiction, since then $T(r,y)=O(\log r)$, which is only possible when y is a rational function.

3. Singularity confinement and value distribution

The aim of this section is to prove Theorem 3.1, which uses value distribution theory to single out the difference Painlevé *II* equation from a large class of difference equations. The main idea is to re-interpret singularity confinement in terms of the value distribution of meromorphic solutions of difference equations. Solutions with sufficiently many non-confined singularities are shown to satisfy an inequality of the form Lemma 2.2 and hence they have infinite order.

3.1. Reduction to the difference Painlevé II equation

We will now study the second-order difference equation

$$\overline{y}+\underline{y}=R(z;y), \tag{3.1}$$

where R is rational in both arguments, and the denominator of R has at least two distinct roots as a polynomial in y. In particular, we will show that if the difference Eq. (3.1) has a sufficiently general finite-order meromorphic solution, then it must reduce to the difference Painlevé *II* equation. For our purpose, it is sufficient to use a notion of singularity confinement involving only five iterates. More precisely, we say that for Eq. (3.1) the singularity at a point $z=z_0$ is confined if $y(z_0)=\infty$ but $y(z_0\pm 1)$ and $y(z_0\pm 2)$ are finite.

Assume that y is a non-rational meromorphic solution of (3.1). Using the argument by Ablowitz et al. [7], which combines the Valiron–Mohon'ko identity (2.9) and the fact that

$$T(r,y(z\pm 1))\le (1+\epsilon)T(r+1,y)+O(1)$$

holds for all $\epsilon>0$, when r is sufficiently large, we have

$$T(r+1,y)\ge \frac{\deg_y(R)}{2(1+\epsilon)}T(r,y)+O(\log r).$$

This implies that y is of infinite order unless the degree of $R(z;y)$ is at most two. Therefore (3.1) reduces into

$$\overline{y}+\underline{y}=\frac{\kappa_2y^2+\kappa_1y+\kappa_0}{y^2+ay+b}, \tag{3.2}$$

where the coefficients of the right-hand side are rational functions, and $a^2\not\equiv 4b$ since we demanded that the denominator of $R(z; y)$ has at least two distinct roots. The transformation $y \to y - a/2$ transforms (3.2) into

$$\overline{y} + \underline{y} = \frac{c_2 y^2 + c_1 y + c_0}{y^2 - p^2} =: \frac{P(z; y)}{Q(z; y)}, \tag{3.3}$$

where c_j's are rational functions and $p^2 = a^2/4 - b \not\equiv 0$.

Suppose that $Q(z_0 - 1; y(z_0 - 1)) = 0$ and $P(z_0 - 1; y(z_0 - 1)) \neq 0$ for some $z_0 \in \mathbb{C}$, which is not a pole or zero of any of the coefficients of R. Then y has a pole at either z_0 or $z_0 - 2$. We assume, without loss of generality, that $y(z_0) = \infty$. Eq. (3.3) now shows that

$$y(z_0 + 1) + y(z_0 - 1) = c_2(z_0). \tag{3.4}$$

Next, we determine the condition for $y(z_0 + 2)$ to be finite. If $y(z_0) = \infty$ and $y(z_0 + 2)$ is finite, then by (3.3),

$$Q(z_0 + 1; y(z_0 + 1)) = 0.$$

Therefore, by (3.4),

$$(c_2(z_0) - y(z_0 - 1))^2 - p(z_0 + 1)^2 = 0. \tag{3.5}$$

Since also

$$Q(z_0 - 1; y(z_0 - 1)) = y(z_0 - 1)^2 - p(z_0 - 1)^2 = 0, \tag{3.6}$$

we have

$$2c_2(z_0) y(z_0 - 1) + p(z_0 + 1)^2 - p(z_0 - 1)^2 - c_2(z_0)^2 = 0. \tag{3.7}$$

We will now divide our consideration into different cases depending on how many confined and non-confined singularities a meromorphic solution of (3.3) has. In general, there are some singularities that arise from the poles of the coefficients. A single pole of a coefficient may force the solutions to have an infinite string of poles regardless of the initial conditions. We divide these fixed singularities similarly into confined and non-confined singularities, with the exception of poles which are next to a pole of a coefficient. These exceptional singularities, which we call *adjoining*, are problematic because they may appear to be confined but they do not give sufficient information about the coefficients. However, since there are only finitely many adjoining singularities, their existence does not affect our reasoning significantly. Also, there can only be finitely many points z_0 such that $Q(z_0; y(z_0)) = P(z_0; y(z_0)) = 0$, since otherwise it would follow that $c_2 p^2 \pm c_1 p + c_0 = 0$, which is impossible due to the fact that $R(z; y)$ has degree 2 as a rational function of y.

We begin by assuming that there are infinitely many confined singularities. In particular, suppose first that there are infinitely many confined singularities such that c_2 vanishes. Then $c_2(z) \equiv 0$ and we arrive at the equation

$$p(z - 1)^2 - p(z + 1)^2 = 0, \tag{3.8}$$

which can be integrated to obtain

$$p(z) = A, \tag{3.9}$$

where, since p is a polynomial, $A \neq 0$ is an arbitrary complex constant. Therefore, making the transformation $y = Aw$, Eq. (3.3) takes the form

$$\overline{w} + \underline{w} = \frac{a_1 w + a_0}{1 - w^2}, \tag{3.10}$$

where a_0, a_1 are certain rational functions depending on A and c_j's.

Suppose now that there are infinitely many confined singularities where c_2 does not vanish. Then by (3.6) and (3.7), we have

$$c_2(z)^4 - 2(p(z+1)^2 + p(z-1)^2) c_2(z)^2 + (p(z+1)^2 - p(z-1)^2)^2 = 0,$$

which can be solved for c_2 to obtain

$$c_2(z) = \pm(p(z+1) \pm p(z-1)), \tag{3.11}$$

where $p(z)$ is meromorphic on a suitable Riemann surface. Since c_2 is rational by assumption, the right-hand side of (3.11) cannot have any branching. But because p has at most finitely many branch points by definition, they cannot all cancel each other in $\pm(p(z+1) \pm p(z-1))$ unless there are none to begin with. Hence p must be a rational function. So, finally, Eq. (3.3) becomes

$$\overline{y} + \underline{y} = \frac{(\mu_1 \overline{p} + \mu_2 \underline{p}) y^2 + c_1 y + c_0}{y^2 - p^2}, \tag{3.12}$$

where $\mu_j^2 = 1$ for $j = 1, 2$.

Now consider the case in which y has only finitely many confined singularities but infinitely many non-confined singularities. Since Eq. (3.3) is of the form (2.10), we will see below that Lemma 2.2 may be applied to show that such solutions are of infinite order. This will be done by obtaining an inequality of the type (2.11). Since $\overline{y} + \underline{y}$ has a pole if and only if the denominator of the right-hand side of (3.2) vanishes, we need to look at singularity sequences of y containing as many zeros of $y^2 - p^2$ compared to the number of ∞'s as possible. Assume therefore that $y = \pm p =: \varepsilon$ for some $z_0 \in \mathbb{C}$. Then either $y(z_0 + 1) = \infty$, or $y(z_0 - 1) = \infty$. Strictly speaking there is also a third possibility, namely that $c_2 y^2 + c_1 y + c_0$ vanishes at z_0. However, since we have already shown that there can be only finitely many points where both the denominator and the numerator of right-hand side of (3.3) have zeros, and therefore these points cannot affect the order of the considered solution, we do not consider this case any further.

Suppose now that $y(z_0 \pm 1) = \infty$ for both choices of the sign. Then $y(z_0 \pm 2) = c_2(z_0 \pm 1) - \varepsilon(z_0)$. This is a finite value which, in general, is not equal to $\varepsilon(z_0 \pm 2)$. If this is indeed the case, $y(z_0 \pm 3) = \infty$, which implies that $y(z_0 \pm 4) = c_2(z_0 \pm 3) - c_2(z_0 \pm 1) + \varepsilon(z_0)$. By continuing in this manner we have a sequence of poles propagating to infinity in both directions, unless at some point the sequence returns to a zero of $y^2 - p^2$. This type of sequence looks like

$$\begin{array}{c|l} y: & \ldots, \ \infty \ , \ k_{-2} \ , \ \infty \ , \varepsilon, \ \infty \ , \ k_2 \ , \ \infty \ , \ldots \\ \hline z: & \ldots, z_0 - 3, z_0 - 2, z_0 - 1, z_0, z_0 + 1, z_0 + 2, z_0 + 3, \ldots, \end{array} \tag{3.13}$$

where $k_{\pm 2} \neq \varepsilon(z_0 \pm 2)$ are finite values. It is also possible that no zeros of $y^2 - p^2$ appear in (3.13) at all, in which case we have a sequence of the type

$$\ldots, \infty, k_{-2}, \infty, k_0, \infty, k_2, \infty, \ldots. \tag{3.14}$$

In sequence (3.13) there is only one pole of $\overline{y} + \underline{y}$, while in (3.14) there is none. On the other hand, both sequences have infinitely many poles of y, and so, for these sequences, (2.11) holds with any $\alpha > 0$. It is therefore clear that if most poles of a meromorphic solution of (3.3) appear as part of sequences (3.13) or (3.14), then the solution must be of infinite order.

It may happen, however, that after a pole the sequence returns to a zero of $y^2 - p^2$. For instance, we may have $y(z_0) = \varepsilon(z_0)$, $y(z_0 \pm 1) = \infty$, and $y(z_0 \pm 2) = \varepsilon(z_0 \pm 2)$. Then, depending on multiplicities, either $y(z_0 \pm 3) = \infty$, or $y(z_0 \pm 3) = k_{\pm 3} \in \mathbb{C}$. Therefore it is possible to have a sequence which looks like (3.13), except that some or all finite values k_j have been replaced by zeros of the denominator $y^2 - p^2$. For instance,

$$\ldots, \infty, k_{-4}, \infty, \varepsilon(z_0 - 2), \infty, \varepsilon(z_0), \infty, k_2, \infty, \varepsilon(z_0 + 4), k_5, \ldots.$$

In any case, for unconfined singularities, a sequence of the form

$$\ldots, \varepsilon(z_0 - 2), \infty, \varepsilon(z_0), \infty, \varepsilon(z_0 + 2), \ldots \tag{3.15}$$

contains the most poles of $\overline{y} + \underline{y}$ compared to the number of poles of y. Note that $\overline{y} + \underline{y}$ might have two adjacent poles, but they always arise from a sequence of the type

$$\ldots, \infty, \varepsilon(z_0 + 1), \varepsilon(z_0 + 2), \infty, \ldots.$$

Hence for any meromorphic solution of Eq. (3.3) with infinitely many unconfined singularities and no confined or adjoining singularities, we have the estimate (2.11) with $\alpha = 3/2$. Taking the possible finitely many adjoining and confined singularities into account, inequality (2.11) still holds with

$$\alpha = \frac{3}{2} + \epsilon < 2,$$

where $\epsilon > 0$ may be chosen arbitrarily small. Therefore, by Lemma 2.2, y is of infinite order. Note also that, at least in principle, the sequence

$$\ldots, \varepsilon, \varepsilon, \infty, \varepsilon, \varepsilon, \infty, \varepsilon, \varepsilon, \ldots$$

may appear in a finite-order solution of Eq. (3.3). These poles, however, are confined since we defined confinement using five iterates.

Finally, if there are only finitely many points at which $y^2 - p^2$ vanishes, then by Theorem 2.1 and (3.3),

$$T(r, y) \leq \overline{N}(r, \overline{y} + \underline{y}) + \overline{N}(r, y) + S(r, y). \tag{3.16}$$

But if y has only finitely many poles, then so has $\overline{y} + \underline{y}$, which means by (3.16) that $T(r, y) = S(r, y)$. This is a contradiction, since obviously for non-rational function y the error term (2.5) cannot be as big as the characteristic function $T(r, y)$.

It follows that if y is a non-rational meromorphic solution of Eq. (3.3) then either y has infinite order or y has infinitely many confined singularities. We have already shown that in the latter case Eq. (3.3) reduces to either (3.10) or (3.12). We will now explore further equations (3.10) and (3.12) under the assumption that they admit meromorphic solutions with infinitely many confined singularities.

We will first look at Eq. (3.10). Note that in order to achieve confinement it is necessary to reduce Eq. (3.1) into (3.10), but this reduction alone is not sufficient. Namely, after some manipulation, Eq. (3.10) implies

$$\begin{aligned}(1 - \overline{w}^2)(\overline{\overline{w}} - \underline{\underline{w}}) &= \overline{a}_0 + \overline{a}_1\overline{w} - \underline{a}_0 - \underline{a}_1\underline{w} \\ &\quad - (w + \underline{\underline{w}})\left[\frac{2\underline{w}(a_0 + a_1 w)}{1 - w^2} - \left(\frac{a_0 + a_1 w}{1 - w^2}\right)^2\right].\end{aligned} \tag{3.17}$$

If w is of finite order then it must have infinitely many confined singularities. All confined singularities of Eq. (3.10) appear as a part of the sequence

$$\ldots, l_{-2}, \varepsilon, \infty, -\varepsilon, l_2, \ldots \tag{3.18}$$

where $\varepsilon = \pm 1$ and $l_{\pm 2}$ are finite values, which may or may not be equal to $\pm\varepsilon$. In other words, at least for one choice of ε there must be infinitely many points z_0 such that when $z \mapsto z_0$, we have $\underline{w} \mapsto \varepsilon$, $w \mapsto \infty$, $\overline{w} \mapsto -\varepsilon$ and $\overline{\overline{w}}$ and $\underline{\underline{w}}$ have finite limits. In this limit, Eq. (3.17) becomes

$$\begin{aligned}&a_1(z_0 + 1) - 2a_1(z_0) + a_1(z_0 - 1) - \varepsilon[a_0(z_0 + 1) \\ &\quad - a_0(z_0 - 1)] = 0.\end{aligned} \tag{3.19}$$

Since Eq. (3.19) holds at infinitely many points and a_0 and a_1 are rational functions, it follows that (3.19) holds for all z_0. Integrating Eq. (3.19), we obtain

$$a_1(z + 1) - a_1(z) = \varepsilon\,[a_0(z + 1) + a_0(z)] + \kappa_\varepsilon, \tag{3.20}$$

where κ_ε is a constant.

Note that, if Eq. (3.20) holds for *both* choices of $\varepsilon = \pm 1$, then Eq. (3.3) is precisely the difference Painlevé *II* equation,

$$\overline{w} + \underline{w} = \frac{(\lambda z + \mu)w + \nu}{1 - w^2}, \tag{3.21}$$

where λ, μ, and ν are constant parameters. However, the existence of a non-rational meromorphic solution of finite order is not sufficient to show that (3.20) holds for both $\varepsilon = 1$ *and* $\varepsilon = -1$. In fact, any solution of the Riccati equation

$$\overline{w} = \frac{A + \varepsilon w}{\varepsilon - w}$$

satisfies Eq. (3.3) with $a_0 = \varepsilon(A - \underline{A})$ and $a_1 = A + \underline{A} + 2$. In this case, Eq. (3.20) is satisfied with $\kappa_\varepsilon = 0$. Also, the singularity sequences of meromorphic solutions of such equation are like (3.18), but with only one fixed value of ε.

To eliminate these Riccati solutions we distinguish between two types of singularity. If w has a pole at $z = z_0$, we will say the singularity at z_0 is of type I if $w(z_0 \pm 1) = \pm\varepsilon$ and of type II if $w(z_0 \pm 1) = \mp\varepsilon$. Note that even though there may be poles which are neither type I nor type II, for instance those of the type (3.14), nevertheless all points where w is ± 1 will occur as part of one of these two types. We denote by $\overline{n}_{\mathrm{I}}(r, w)$

the number of type I poles (ignoring multiplicities) in the disc $\{z : |z| < r\}$. Similarly, the function $\overline{n}_{\rm II}(r, w)$ counts poles of type II.

The finite-order Riccati solutions described above are degenerate in that they possess only type I singularities and no type II singularities. In order to avoid these solutions we further demand that there is a non-rational finite-order meromorphic solution with "comparably many" poles of both types I and II. In particular, we assume that there is a finite real constant $c \geq 1$, such that

$$c^{-1}\overline{n}_{\rm I}(r, w) \leq \overline{n}_{\rm II}(r, w) \leq c\overline{n}_{\rm I}(r, w), \tag{3.22}$$

for sufficiently large r. Note that the (differential) Painlevé *II* equation also has infinitely many poles of two types, namely the ones with residue 1 and -1. Let us assume that all but finitely many of the type II singularities of a meromorphic solution of (3.10) are unconfined. Now, since confined type I singularities look like (3.18), and the "worst" non-confined sequence, i.e. singularity sequence with most ± 1-points compared to the number of poles, is

$$\ldots, \varepsilon, \infty, -\varepsilon, \infty, \varepsilon, \ldots$$

we can exhaust all the ± 1's by associating at most two of them with each type I singularity and at most one with each type II singularity, with only finitely many exceptions. Hence for sufficiently large r,

$$\begin{aligned}\overline{n}(r, \overline{w} + \underline{w}) &= \overline{n}\left(r, \frac{1}{w-1}\right) + \overline{n}\left(r, \frac{1}{w+1}\right) + O(\log r)\\ &\leq 2(1+\epsilon)\overline{n}_{\rm I}(r+3, w) + (1+\epsilon)\overline{n}_{\rm II}(r+3, w)\\ &\leq (1+\epsilon)\left(2 - (c+1)^{-1}\right)\overline{n}(r+3, w),\end{aligned}$$

which implies the inequality (2.11) with $\alpha < 2$ when we choose small enough $\epsilon > 0$. Thus w has infinite order by Lemma 2.2. Hence, if a solution w has finite order then it must have infinitely many confined singularities of both types and so Eq. (3.3) is dP_{II}, Eq. (3.21).

Finally, we will deal with the Eq. (3.12). Writing it in the form

$$\overline{y} + \underline{y} = \frac{(\mu_1\overline{p} + \mu_2\underline{p})y^2 + c_1 y + c_0}{(y-p)(y+p)}, \tag{3.23}$$

where all coefficients are rational, we may again distinguish between two types of singularity. The sequence

$$\ldots, p(z_0 - 1), \infty, \pm p(z_0 + 1), \ldots$$

is associated with type I singularities, while singularities appearing in

$$\ldots, -p(z_0 - 1), \infty, \pm p(z_0 + 1), \ldots$$

are called type II singularities. The assumption (3.22) once again implies that if y is a non-rational finite-order meromorphic solution then there must be infinitely many singularities of both types which are confined. But then $\mu_1\overline{p} + \mu_2\underline{p} \equiv 0$, which contradicts the assumption $c_2 \not\equiv 0$.

We conclude that the only type of equation admitting a finite-order meromorphic solution in which both kinds of singularities are present in comparable numbers is (3.21), the difference Painlevé *II* equation. We summarize our findings with the following theorem.

Theorem 3.1. *Let $R(z; y)$ be rational in both of its arguments such that its denominator has at least two distinct roots. If the second-order difference equation*

$$\overline{y} + \underline{y} = R(z; y), \tag{3.24}$$

admits a non-rational meromorphic solution of finite order such that (3.22) holds, then (3.24) is the difference Painlevé II equation

$$\overline{y} + \underline{y} = \frac{(\lambda z + \mu)y + \nu}{1 - y^2},$$

where λ, μ and ν are constants.

3.2. Existence of meromorphic solutions

We will conclude this section by considering the existence of meromorphic solutions of a class of second-order difference equations. The autonomous form of the difference Painlevé *II* equation,

$$\overline{y} + \underline{y} = \frac{\mu y + \nu}{1 - y^2}, \tag{3.25}$$

is equivalent to a map introduced by McMillan in his analysis of the stability of periodic systems. In particular, the McMillan map gives rise to a family of closed invariant curves within the periodic system [28]. We will follow a known procedure for solving Eq. (3.25) in terms of elliptic functions. From the form of the solutions obtained it will be evident that they are finite-order meromorphic functions. The existence of these solutions shows that the reduction procedure used earlier in this section to single out the difference Painlevé *II* equation is non-vacuous, at least in the autonomous case.

Eq. (3.25) can be written as

$$\begin{aligned}&\overline{y}^2 + y^2 - (y^2 + \underline{y}^2) - (y^2\overline{y}^2 - \underline{y}^2 y^2)\\ &\quad = \mu(y\overline{y} - \underline{y}y) + \nu(\overline{y} + y - (y + \underline{y}))\end{aligned}$$

which may be integrated once to obtain

$$\overline{y}^2 + y^2 - \overline{y}^2 y^2 - \mu\overline{y}y - \nu(\overline{y} + y) + C_1 = 0, \tag{3.26}$$

where $C_1 \in \mathbb{C}$ is the first of the two required free parameters. Following Baxter [29], we will now use a suitable Möbius transformation to remove the $\overline{y} + y$ term from (3.26). To this end, let

$$y = \frac{aw + b}{w + c}, \tag{3.27}$$

where a is a root of

$$\begin{aligned}&-2\nu a^6 + \left(4C_1 + 4 - 4\mu + \mu^2\right)a^5 + (5\nu\mu - 10\nu)\,a^4\\ &\quad + 10\nu^2 a^3 - 10C_1\nu a^2\\ &\quad + \left(4C_1 + 4C_1^2 - 4\mu C_1 + C_1\mu^2 - 2\nu^2 + \nu^2\mu\right)a\\ &\quad + \nu C_1\mu + \nu^3 - 2\nu C_1\end{aligned}$$

such that

$$C_1 + 2a^2 - a^4 - 2\nu a - \mu a^2 \neq 0,$$

and the constants b and c satisfy

$$\frac{b}{c} = \frac{-\mu a^2 - 3\nu a + 2C_1 + 2a^2}{\mu a + \nu - 2a + 2a^3}$$

and

$$-\mu b^2 c^2 - 2\nu bc^3 + C_1 c^4 + 2b^2c^2 - b^4 = C_1 + 2a^2 - a^4 - 2\nu a - \mu a^2.$$

Substituting (3.27) into Eq. (3.26) reduces the latter equation into

$$w\overline{w}^2 + A(\overline{w}^2 + w^2) + 2B\overline{w}w + 1 = 0, \tag{3.28}$$

where

$$A = C_1c^2 - b^2a^2 - \nu bc - \mu abc + a^2c^2 - \nu ac^2 + b^2$$

and

$$2B = -\mu a^2c^2 - \mu b^2 - 4\nu ac^2 - 4b^2a^2 + 8abc + 4C_1c^2 - 2\mu abc - 4\nu bc.$$

In what follows we concentrate only on the generic Eq. (3.25), and ignore special cases which can arise, for instance, when one of the constants A or B vanishes. Regarding Eq. (3.28) as a quadratic equation for $\overline{w}$, we obtain

$$\overline{w} = \frac{-Bw \pm \sqrt{-Aw^4 + (B^2 - A^2 - 1)w^2 - A}}{w^2 + A}. \tag{3.29}$$

In order to express the argument of the square root as a perfect square, we define the parameters k and η by the equations

$$A = -\frac{1}{k\,\mathrm{sn}^2\eta},$$

and

$$B = \frac{\mathrm{cn}\,\eta\,\mathrm{dn}\,\eta}{k\,\mathrm{sn}^2\eta},$$

where sn, cn and dn are the standard Jacobi elliptic functions with modulus k. These choices of A and B imply that

$$k + k^{-1} = (B^2 - A^2 - 1)A^{-1},$$

and so, using the transformation

$$w = k^{\frac{1}{2}}\,\mathrm{sn}\,u,$$

where $\mathrm{sn}\,u$ denotes the Jacobian elliptic sn function with argument u and modulus k, Eq. (3.29) becomes

$$\mathrm{sn}\,\overline{u} = \frac{\mathrm{cn}\,\eta\,\mathrm{dn}\,\eta\,\mathrm{sn}\,u \pm \mathrm{sn}\,\eta\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{1 - k^2\,\mathrm{sn}^2\eta\,\mathrm{sn}^2 u}.$$

This is solved by $u = \eta z + C_2$, where $C_2 \in \mathbb{C}$ is the second required free parameter. Summarizing the above reasoning,

$$y = \frac{ak^{\frac{1}{2}}\mathrm{sn}(\eta z + C_2) + b}{k^{\frac{1}{2}}\mathrm{sn}(\eta z + C_2) + c} \tag{3.30}$$

is a meromorphic solution of (3.25), where $C_2 \in \mathbb{C}$, and a, b, c, η and k depend on the coefficients of (3.25) and on the free parameter $C_1 \in \mathbb{C}$. Clearly (3.30) has finite order of growth.

4. First-order difference equations

The existence of nontrivial meromorphic solutions of the first-order nonlinear difference equation

$$\overline{y} = R(y), \tag{4.1}$$

where R is a rational function with constant coefficients, is well established. However, the complete treatment of Eq. (4.1) is scattered in a number of papers; see, for instance Kimura [30], Shimomura [31] and Yanagihara [26]. Here we present a straightforward proof of the existence of meromorphic solutions of (4.1) by introducing suitable contraction mappings in appropriate Banach spaces.

Since the Riccati case $\deg_y(R) = 1$ can be solved explicitly, the existence of meromorphic solutions is guaranteed for this type of difference equation. When $\deg_y(R) \geq 2$, Julia showed [32] that there is always a fixed point γ of R such that either

$$|R'(\gamma)| > 1 \tag{4.2}$$

or

$$R'(\gamma) = 1. \tag{4.3}$$

In the case (4.2), Eq. (4.1) can be mapped into a Schröder functional equation, which is known to have meromorphic solutions. The asymptotic properties of such solutions are also well known. This case has been studied by Shimomura in the case when R is a polynomial [31], and by Yanagihara for rational R [26].

A treatment of the more complicated case (4.3) can be found in a paper due to Kimura [30]. His main focus, however, is in the iteration of analytic functions, and the existence result follows as a corollary from his other work. He solves the case (4.3) by applying a fixed point theorem from the theory of normal families due to Hukuhara, but the uniqueness of the solution is not immediately guaranteed, and must be proved separately. We will now present a simple proof based on Banach's fixed point theorem. This approach has a number of advantages. First, the basic idea is very direct and simple, although some technical calculations cannot be avoided. Second, our argument still works even if we allow some of the coefficients of (4.1) to have a certain type of z-dependence. In what follows we will, however, concentrate only on the autonomous equation for the sake of simplicity. Finally, the same idea can be used to deal with both cases (4.2) and (4.3). We start with case (4.2).

Theorem 4.1. *Choose $\alpha \in \mathbb{C}$ and let γ be a fixed point of $R(y)$ such that $\lambda := R'(\gamma)$ satisfies $|\lambda| > 1$. Then Eq. (4.1) has a unique meromorphic solution in the complex plane such that*

$$(y(z) - \gamma)\lambda^{-z} \longrightarrow \alpha, \quad \text{as } \Re(z) \longrightarrow -\infty. \tag{4.4}$$

We will find a contraction mapping which will have a unique fixed point in a suitable Banach space consisting of a certain class of analytic functions. This fixed point will then turn out to be the desired solution of the difference Eq. (4.1). We begin by expanding the right-hand side of (4.1) as a Taylor series,

$$y(z+1) = \sum_{j=0}^{\infty} \frac{1}{j!} R^{(j)}(\gamma)\,(y(z)-\gamma)^j . \tag{4.5}$$

Substituting $y(z) = \lambda^z w(z) + \gamma$, we then have

$$w(z+1) - w(z) = \sum_{j=2}^{\infty} \frac{1}{j!} R^{(j)}(\gamma) \lambda^{(j-1)z-1} w(z)^j . \tag{4.6}$$

To motivate the definition of the appropriate operator, we write

$$\begin{cases} w(z) - w(z-1) = \sum_{j=2}^{\infty} \frac{1}{j!} R^{(j)}(\gamma) \lambda^{(j-1)(z-1)-1} w(z-1)^j \\ w(z-1) - w(z-2) \\ \quad = \sum_{j=2}^{\infty} \frac{1}{j!} R^{(j)}(\gamma) \lambda^{(j-1)(z-2)-1} w(z-2)^j \\ \vdots \\ w(z-m+1) - w(z-m) \\ \quad = \sum_{j=2}^{\infty} \frac{1}{j!} R^{(j)}(\gamma) \lambda^{(j-1)(z-m)-1} w(z-m)^j . \end{cases} \tag{4.7}$$

By summing Eqs. (4.7), we obtain

$$w(z) - w(z-m) = \sum_{j=2}^{\infty} \sum_{k=1}^{m} \frac{1}{j!} R^{(j)}(\gamma) \lambda^{(j-1)(z-k)-1} w(z-k)^j ,$$

and letting $m \longrightarrow \infty$, assumption (4.4) yields the formal identity

$$w(z) = \alpha + \sum_{j=2}^{\infty} \sum_{k=1}^{\infty} \frac{1}{j!} R^{(j)}(\gamma) \lambda^{(j-1)(z-k)-1} w(z-k)^j . \tag{4.8}$$

Now define an operator T by

$$T[w](z) = \alpha + \sum_{j=2}^{\infty} \sum_{k=1}^{\infty} \frac{1}{j!} R^{(j)}(\gamma) \lambda^{(j-1)(z-k)-1} w(z-k)^j . \tag{4.9}$$

Let X be the set of all functions $z \longrightarrow g(z)$, analytic and bounded in

$$D(s,t) = \{z : \Re(z) < -s,\ s > 0,\ \Im(z) > -t,\ t > 0\} \tag{4.10}$$

for which $g(z) \longrightarrow \alpha$ as $\Re(z) \longrightarrow -\infty$ and $\|g-\alpha\| \le b$, where

$$\|g-\alpha\| = \sup_{z \in D(s,t)} |g(z) - \alpha| . \tag{4.11}$$

Now, choosing s sufficiently large, T is a contraction mapping in the Banach space X. Banach's fixed point theorem implies that T has a unique fixed point g in X. This fixed point is the sought after solution of (4.1) satisfying (4.4), analytic in $D(s,t)$. By analytic continuation the solution g is meromorphic in $D(+\infty, t)$. Since t is arbitrary, g is in fact meromorphic in the whole complex plane.

We will now deal with the remaining case $R'(\gamma) = 1$. The basic idea is the same as before.

Theorem 4.2. *Choose $\alpha \in \mathbb{C}$ and $\delta \in (0,1)$ and let γ be a fixed point of R satisfying $R'(\gamma) = 1$. Moreover, let $m \in \mathbb{N}$ be the smallest number such that $R^{(m+1)}(\gamma) \neq 0$. Then there exists a constant $s > 0$, and a unique solution $y(z)$ of (4.1), meromorphic in the complex plane, such that for all $z \in D(s) := \{z : \Re(z) < -s\}$,*

$$y(z) = \gamma - \frac{C_1}{z + \alpha + \beta \log z + W(z)}$$

if $m = 1$, and

$$y(z) = \gamma - \frac{C_m}{\left(z + \alpha + \beta z^{\frac{m-1}{m}} + W(z)\right)^{\frac{1}{m}}},$$

if $m \ge 2$. Here β is a fixed constant,

$$C_m = \left(\frac{m}{(m+1)!} R^{(m+1)}(\gamma)\right)^{-\frac{1}{m}}$$

for $m \in \mathbb{N}$, and

$$|W(z)| \le |z|^{-\frac{1}{m}+\delta}$$

for all $z \in D(s)$.

Once again we begin by expanding the right-hand side of (4.1) as a Taylor series,

$$w(z+1) = w(z) + \sum_{j=2}^{\infty} \frac{1}{j!} R^{(j)}(\gamma) w(z)^j , \tag{4.12}$$

where $w(z) = y(z) - \gamma$. By applying the transformation $w(z) = 1/g(z)$ to Eq. (4.12), we have

$$\begin{aligned} g(z+1) &= \frac{g(z)}{1 + \sum_{j=2}^{\infty} \frac{1}{j!} R^{(j)}(\gamma) g(z)^{-j+1}} \\ &= g(z) \left(1 - \frac{1}{2} R''(\gamma) g(z)^{-1} + \cdots \right). \end{aligned} \tag{4.13}$$

We suppose first that $R''(\gamma) \neq 0$, which implies that $m = 1$. By denoting

$$h(z) = -\frac{2g(z)}{R''(\gamma)},$$

Eq. (4.13) takes the form

$$h(z+1) = h(z) + 1 + \sum_{j=1}^{\infty} c_j h(z)^{-j},$$

where $c_j \in \mathbb{C}$ for all $j \in \mathbb{N}$. To summarize, we have transformed Eq. (4.1) into

$$y(z+1) = F(y(z)), \tag{4.14}$$

where

$$F(z) = z + 1 + \sum_{j=1}^{\infty} c_j z^{-j}. \tag{4.15}$$

Therefore, if $\widetilde{y}(z)$ is a solution of (4.14),

$$y(z) = \gamma - \frac{2}{R''(\gamma)} \frac{1}{\widetilde{y}(z)}$$

is a solution of the original Eq. (4.1).

We will next prove that Eq. (4.14) has an analytic solution in the set $D(s)$. This will be done by showing that there exists a function $W(z)$ such that the following three conditions are satisfied:

(1) $W(z)$ is analytic in the domain $D(s)$, where s is a sufficiently large number.
(2) $|W(z)| \le |z|^{-1+\delta}$ for all $z \in D(s)$, where $\delta \in (0, 1)$ is a fixed constant.
(3) $Y(z) = z + \alpha + \beta \log z + W(z)$, where $\alpha \in \mathbb{C}$ is arbitrary and β is a fixed constant, is a solution of (4.14).

We will use Banach's fixed point theorem to find such a function. For this purpose we define a family X of analytic functions W such that

$$|W(z)| \le |z|^{-1+\delta} \tag{4.16}$$

holds in $D(s)$. It is easy to see that X is a complete metric space.

We will now find a suitable operator in the Banach space X. Condition (3) yields

$$\begin{aligned} &W(z+1) - W(z) \\ &\quad = -\beta \log\left(1 + \frac{1}{z}\right) + \frac{c_1}{z + \alpha + \beta \log z + W(z)} \\ &\qquad + \sum_{j=2}^{\infty} \frac{c_j}{(z + \alpha + \beta \log z + W(z))^j}. \end{aligned} \tag{4.17}$$

We want our operator also to satisfy condition (2). For this purpose we need the right-hand side of (4.17) to tend to zero sufficiently fast as $|z|$ tends to infinity. Therefore we fix $\beta = c_1$ to cancel out the z^{-1} term, which gives

$$\begin{aligned} &W(z+1) - W(z) \\ &\quad = \beta \sum_{j=2}^{\infty} (-1)^j \frac{1}{j z^j} + \frac{c_1}{z} \sum_{n=1}^{\infty} \left(-\frac{\alpha + \beta \log z + W(z)}{z}\right)^n \\ &\qquad + \sum_{j=2}^{\infty} \frac{c_j}{(z + \alpha + \beta \log z + W(z))^j}. \end{aligned} \tag{4.18}$$

Next we consider

$$\begin{aligned} &W(z) - W(z-m) \\ &\quad = \sum_{k=1}^{m} (W(z+1-k) - W(z-k))\, j(z-k)^j, \end{aligned}$$

and formally take the limit $m \longrightarrow \infty$. Then, by (4.18) and using the fact that $\lim_{\Re(z) \to -\infty} W(z) = 0$, we have

$$W(z) = T[W](z),$$

where

$$\begin{aligned} &T[W](z) \\ &\quad = \sum_{k=1}^{\infty} \frac{c_1}{z-k} \sum_{n=1}^{\infty} \left(-\frac{\alpha + \beta \log(z-k) + W(z-k)}{z-k}\right)^n \\ &\qquad + \sum_{k=1}^{\infty} \sum_{j=2}^{\infty} \frac{c_j}{(z - k + \alpha + \beta \log(z-k) + W(z-k))^j} \\ &\qquad + \beta \sum_{k=1}^{\infty} \sum_{j=2}^{\infty} (-1)^j \frac{1}{j(z-k)^j}. \end{aligned} \tag{4.19}$$

A straightforward yet tedious calculation shows that

$$|T[W](z)| \le |z|^{-1+\delta} \tag{4.20}$$

for all $z \in D(s)$, where δ is the constant introduced in the requirement (2). Therefore, the right-hand side of (4.19) is absolutely and uniformly convergent, and hence $T[W](z)$ is analytic in $D(s)$.

A similar calculation will show that $T[W]$ is also a contraction. Thus, by Banach's fixed point theorem, the mapping $T : X \longrightarrow X$ has a unique fixed point. Hence the existence of a function $W(z)$ satisfying (1), (2) and (3) is proved. In particular,

$$Y(z) = z + \alpha + \beta \log z + W(z)$$

is a solution of (4.14). The final step is to continue the analytic solution $Y(z)$ into a meromorphic solution in the whole complex plane by using Eq. (4.1).

We will finally outline the proof in the case $R''(\gamma) = 0$. Full details are not presented, since the reasoning is very similar to that of the case $R''(\gamma) \neq 0$. We will write down a suitable operator and specify an appropriate Banach space on which it acts.

Assume that

$$R^{(j)}(\gamma) = 0$$

for all $j = 2, \ldots, m$, and that $R^{(m+1)}(\gamma) \neq 0$. The transformation

$$h(z) = -\frac{(m+1)!\, g(z)^m}{m R^{(m+1)}(\gamma)}$$

maps Eq. (4.13) into

$$y(z+1) = \widetilde{F}(y(z)), \tag{4.21}$$

where

$$\widetilde{F}(z) = z + 1 + \sum_{j=m}^{\infty} c_j z^{1 - \frac{j+1}{m}}. \tag{4.22}$$

Let $\delta \in (0, \frac{1}{m})$. The required Banach space $\widetilde{X}$ is the family of analytic functions W such that

$$|W(z)| \le |z|^{-\frac{1}{m} + \delta} \tag{4.23}$$

holds in $D(s)$. By applying Banach's fixed point theorem with a similar reasoning as in the case $R''(\gamma) \neq 0$, but, instead of

(4.19), using the operator

$$T[W](z) = \sum_{k=1}^{\infty} \frac{c_m}{(z-k)^{\frac{1}{m}}} \times \sum_{j=1}^{\infty} a_j \sum_{n=1}^{\infty} \left(- \frac{\alpha + \beta (z-k)^{\frac{m-1}{m}} + W(z-k)}{z-k} \right)^n + \sum_{k=1}^{\infty} \sum_{j=m+1}^{\infty} \frac{c_j}{\left(z-k+\alpha+\beta(z-k)^{\frac{m-1}{m}} + W(z-k)\right)^{\frac{j+1}{m}-1}} - \beta \sum_{k=1}^{\infty}\sum_{j=2}^{\infty} \frac{b_j}{(z-k)^{j-\frac{m-1}{m}}},$$

where

$$a_j = \frac{1}{j!}\prod_{n=0}^{j-1}\left(\frac{1}{m} - n\right)$$

and

$$b_j = \frac{1}{j!}\prod_{n=0}^{j-1}\left(\frac{m-1}{m} - n\right)$$

for all $j \in \mathbb{N}$, we have that for any $\alpha \in \mathbb{C}$ there exists a constant $s > 0$, and a unique solution $\widetilde{y}(z)$ of (4.21), meromorphic in the complex plane, such that

$$\widetilde{y}(z) = z + \alpha + \beta z^{\frac{m-1}{m}} + W(z),$$

where

$$|W(z)| \le |z|^{-\frac{1}{m}+\delta}$$

for all $z \in D(s)$, and $\beta = \frac{mc_m}{m-1}$.

5. Discussion

In this paper we have shown that all non-rational meromorphic solutions of a class of difference equations are of infinite order. Out of the remaining equations within the class (1.1) the only one which may have finite-order meromorphic solutions having comparably many singularities of two types is the difference Painlevé *II* equation. This is in some way analogous to what happens with the (differential) Painlevé *II* equation, which also has meromorphic solutions with two types of singularities depending on whether the residue of each pole of the solution is either 1 or −1. The generic solution of the continuous P_{II} equation has infinitely many poles of both types, which is analogous to the assumption (3.22) we made concerning the singularities of the difference Painlevé *II* equation.

When applying singularity confinement to investigate the integrability of an equation there are infinitely many points in the iteration sequence where the confinement may occur. It may therefore be unclear at which stage the singularity confinement should be imposed. Hietarinta and Viallet [11] suggested, while studying the discrete Painlevé *I* equation, that it is crucial that the confinement occurs in the first possible instance. Our findings indicate that is indeed the case for the difference Painlevé *II* equation. If the singularity sequence is generically cut on some later point, this will still be enough to force any corresponding meromorphic solution to have infinite order of growth, which indicates the non-integrability of the considered equation. On the other hand, confining all poles of the solution from both sides with two finite values is just sufficient to break the singularity pattern leading to infinite order.

The existence of a finite-order meromorphic solution with two types of singularity appears to be a strong indicator of integrability. Indeed, this condition was sufficient to single out dP_{II} from a general class of difference equations. To finalize this claim it is important to know whether or not integrable difference equations do have meromorphic solutions. For the McMillan map, as well as the autonomous Riccati difference equation, the existence of such solutions is well known, but for the non-autonomous dP_{II} the question remains still open.

Acknowledgement

The research reported in this paper was supported by EPSRC grant number GR/R92141/01.

Note added in proof

Laine, Rieppo and Silvennoinen [33] have generalized Lemma 2.2 to a class of higher-order equations. Their proof avoids the use of algebroid functions.

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