First integrals and gradient flow for a generalized Darboux-Halphen system

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Abstract. First integrals are explicitly constructed for a third-order system of ODEs that arises as a reduction of the self-dual Yang-Mills equations and in the theory of hypercomplex manifolds. These first integrals are branched functions of the phase space variables, even in cases for which the general solution is single-valued. This branching is characterized in terms of the monodromy of the hypergeometric equations. The first integrals are then used to formulate a Nambu-Poisson structure of the system. A representation of the generalized Darboux-Halphen system as a gradient flow is also given.

1. Introduction

The system

$$\begin{aligned} \dot{\omega}_1 &= \omega_2\omega_3 - \omega_1(\omega_2+\omega_3) + \tau^2, \\ \dot{\omega}_2 &= \omega_3\omega_1 - \omega_2(\omega_3+\omega_1) + \tau^2, \\ \dot{\omega}_3 &= \omega_1\omega_2 - \omega_3(\omega_1+\omega_2) + \tau^2, \end{aligned} \tag{1.1}$$

where

$$\tau^2 = \alpha^2(\omega_1-\omega_2)(\omega_3-\omega_1) + \beta^2(\omega_2-\omega_3)(\omega_1-\omega_2) + \gamma^2(\omega_3-\omega_1)(\omega_2-\omega_3),$$

was first studied by Halphen [**14**] as a natural generalization of the classical Darboux-Halphen (DH) system, which corresponds to setting $\tau = 0$ in equation (1.1). The classical Darboux-Halphen system first arose in Darboux's study of triply orthogonal surfaces [**11**] and was later solved by Halphen [**15**]. The classical DH system has also appeared in studies of self-dual Bianchi-IX metrics with Euclidean signature [**4, 13**] and in reductions of the associativity equations on a three-dimensional Frobenius manifold [**12**]. Furthermore, if $(\omega_1, \omega_2, \omega_3)$ is a solution of the classical Darboux-Halphen system, then

$$y := -2(\omega_1+\omega_2+\omega_3) \tag{1.2}$$

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satisfies the Chazy equation,

$$\frac{d^3y}{dt^3} = 2y\frac{d^2y}{dt^2} - \left(\frac{dy}{dt}\right)^2 . \tag{1.3}$$

In [**3**] it was shown that y defined by equation (1.2) solves the generalized Chazy equation,

$$\frac{d^3y}{dt^3} - 2y\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 = \frac{4}{36-n^2}\left(6\frac{dy}{dt} - y^2\right)^2 , \tag{1.4}$$

where the ω_i's solve the generalized Darboux-Halphen system for the special choices of parameters (α, β, γ) given by $(2/n, 2/n, 2/n)$ and $(1/3, 1/3, 2/n)$, et cyc. Note that equation (1.3) corresponds to the limit $n \to \infty$ in equation (1.4). Equations (1.3) and (1.4) were first studied by Chazy in [**8, 9, 10**].

The system (1.1) arises in the study of the equation

$$\dot{M} = (\operatorname{adj} M)^T + M^T M - (\operatorname{Tr} M)M, \tag{1.5}$$

for a 3×3 matrix valued function $M(t)$ where $\operatorname{adj} M$ is the adjoint of M satisfying $(\operatorname{adj} M)\, M = (\det M)I$, M^T is the transpose of M and the dot denotes differentiation with respect to t. Equation (1.5) was obtained in [**6**] as a reduction of the self-dual Yang-Mills equations with an infinite dimensional gauge group of diffeomorphisms $\mathrm{Diff}(S^3)$ of the three-sphere. Equation (1.5) also describes a class of self-dual Weyl Bianchi IX space-times with Euclidean signature [**5**]. More recently, equation (1.5) was used to describe SU(2) invariant hypercomplex 4-manifolds [**16**].

In section 2 we will review the reduction of equation (1.5) to the generalized DH system (1.1). The general solution will be constructed and a special case will be studied. In section 3, first integrals and "action-angle" variables are given for equation (1.1). The first integrals involve hypergeometric functions and are non-meromorphic, even in cases where the general solution is single-valued.

2. The solution of the generalized Darboux-Halphen system

In this section the solution of equation (1.5) is given by a factorization method which first appeared in [**1**]. The solution can also be obtained via associated linear problems. In [**2**], the solution was obtained via an evolving monodromy problem that arises as a reduction of the isospectral problem for the self-dual Yang-Mills equations. In [**16**], the solution was obtained via an isomonodromy problem which describes the Riccati solutions of the sixth Painlevé equation. Degenerate cases were discussed in [**3**].

We begin by decomposing the matrix M into its symmetric (M_s) and antisymmetric (M_a) parts. Furthermore, the eigenvalues of M_s are assumed to be distinct, so that it can be diagonalized using a complex orthogonal matrix. Thus we have

$$M = M_s + M_a = P(d + a)P^{-1}, \tag{2.1}$$

where $P \in \mathrm{SO}(3, \mathbf{C})$, $d = \operatorname{diag}(\omega_1, \omega_2, \omega_3)$ with $\omega_i \neq \omega_j$, $i \neq j$, and the elements of the skew-symmetric matrix a are given by $a_{ij} = \sum_{k=1}^{3} \varepsilon_{ijk}\tau_k$, where ε_{ijk} is totally skew-symmetric in its indices and $\varepsilon_{123} = 1$. Using the transformation (2.1), the diagonal part of equation (1.5) yields the system (1.1), where $\tau^2 = \tau_1^2 + \tau_2^2 + \tau_3^2$. The skew-symmetric part gives

$$\dot{\tau}_1 = -\tau_1(\omega_2 + \omega_3), \qquad \dot{\tau}_2 = -\tau_2(\omega_3 + \omega_1), \qquad \dot{\tau}_3 = -\tau_3(\omega_1 + \omega_2)\,, \tag{2.2}$$

and the off-diagonal symmetric part gives

$$\dot{P} = -Pa, \tag{2.3}$$

which is a linear equation for P.

Taking differences of equations in system (1.1) gives

$$\frac{d}{dt}(\omega_1 - \omega_2) = -2\omega_3(\omega_1 - \omega_2), \qquad \text{et cyc.} \tag{2.4}$$

Using equations (2.2) and (2.4), we can solve the τ_i's in terms of the ω_i's as

$$\tau_1^2 = \alpha^2(\omega_1 - \omega_2)(\omega_3 - \omega_1), \tag{2.5}$$
$$\tau_2^2 = \beta^2(\omega_2 - \omega_3)(\omega_1 - \omega_2), \qquad \tau_3^2 = \gamma^2(\omega_3 - \omega_1)(\omega_2 - \omega_3)\,,$$

where α, β, and γ are integration constants.

In terms of the cross ratio

$$s := \frac{\omega_1 - \omega_3}{\omega_2 - \omega_3}\,, \tag{2.6}$$

it follows from equation (1.1) that the ω_i's can be parameterized as

$$\omega_1 = -\frac{1}{2}\frac{d}{dt}\ln\frac{\dot{s}}{s(s-1)}\,, \quad \omega_2 = -\frac{1}{2}\frac{d}{dt}\ln\frac{\dot{s}}{s-1}\,, \quad \omega_3 = -\frac{1}{2}\frac{d}{dt}\ln\frac{\dot{s}}{s}\,. \tag{2.7}$$

where s satisfies

$$\frac{d}{dt}\left(\frac{\ddot{s}}{\dot{s}}\right) - \frac{1}{2}\left(\frac{\ddot{s}}{\dot{s}}\right)^2 + \frac{\dot{s}^2}{2}V(s) = 0\,, \tag{2.8}$$

with

$$\{s,t\} := \frac{d}{dt}\left(\frac{\ddot{s}}{\dot{s}}\right) - \frac{1}{2}\left(\frac{\ddot{s}}{\dot{s}}\right)^2$$

being the Schwarzian derivative and V is given by

$$V(s) = \frac{1-\beta^2}{s^2} + \frac{1-\gamma^2}{(s-1)^2} + \frac{\beta^2+\gamma^2-\alpha^2-1}{s(s-1)}. \tag{2.9}$$

Equation (2.8) is the Schwarzian equation, which describes the conformal mappings of the upper-half s-plane to the interior of a region of the complex sphere bounded by three regular circular arcs. If α, β, and γ are non-negative real numbers such that $\alpha+\beta+\gamma < 1$, then the angles subtended at the vertices $s = 0$, $s = 1$, and $s = \infty$ of this triangle are $\alpha\pi$, $\beta\pi$, and $\gamma\pi$. Furthermore, if α, β, and γ are chosen to be either reciprocals of integers or zero, then s is analytic on the interior of a circle on the complex sphere and cannot be analytically extended across this circle, which is a natural barrier [**19**].

The general solution of equation (2.8) is given implicitly by

$$t(s) = \frac{u_1(s)}{u_2(s)}, \tag{2.10}$$

where $u_1(s)$ and $u_2(s)$ are independent solutions to the Fuchsian equation

$$\frac{d^2u}{ds^2} + \frac{1}{4}V(s)u = 0. \tag{2.11}$$

Equation (2.11) is equivalent to the hypergeometric equation,

$$s(1-s)\frac{d^2\chi}{ds^2}+[c-(a+b+1)s]\frac{d\chi}{ds}-ab\chi=0, \tag{2.12}$$

where $a=(1+\alpha-\beta-\gamma)/2$, $b=(1-\alpha-\beta-\gamma)/2$, $c=1-\beta$, and

$$u(s)=s^{c/2}(1-s)^{(a+b-c+1)/2}\chi(s). \tag{2.13}$$

A fifth order reduction

We will now consider the case in which M has the special form

$$M=\begin{pmatrix} M_{11} & M_{12} & 0\\ M_{21} & M_{22} & 0\\ 0 & 0 & M_{33}\end{pmatrix}. \tag{2.14}$$

This special form of M was considered in [**5, 2**], where equation (1.5) was analyzed using an associated evolving monodromy problem. Here we will show that quantities that arise naturally from this monodromy analysis can be obtained in a straightforward manner from the factorization method described above.

Consider the factorization of M given by equation (2.1) where M is given by equation (2.14). Due to special block structure of M, its symmetric part can be diagonalized by an orthogonal matrix of the form

$$P=\begin{pmatrix} \cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{2.15}$$

where ψ is a (generally complex) function of t to be determined. That is, $M_s = PdP^{-1}$, where $d=\mathrm{diag}(\omega_1,\omega_2,\omega_3)$. Furthermore, the skew-symmetric part of M is unchanged by the adjoint action of P. So

$$a=P^{-1}M_aP=M_a=\frac{1}{2}(M-M^T)=\begin{pmatrix} 0 & \tau_3(t) & 0\\ -\tau_3(t) & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{2.16}$$

where $\tau_3(t)=\frac{1}{2}(M_{12}-M_{21})$. Since $\tau_1=\tau_2=0$ in this case, the ω_i's are given by equation (2.7) where s solves equation (2.8) with $\alpha=\beta=0$. From equation (2.5) and equation (2.7), we have

$$\tau_3(t)=\frac{i\gamma}{2}\frac{\dot{s}}{\sqrt{s}(s-1)},$$

where γ is a constant. With the $\tau_3(t)$ given above, equation (2.3) can be readily integrated to give

$$\psi=\frac{i\gamma}{2}\log\left(\frac{\sqrt{s}-1}{\sqrt{s}+1}\right)+\psi_0,$$

where ψ_0 is a constant. Finally, the matrix M in (2.14) is reconstructed from the various components P, d and a according to equation (2.1). Note that in order to obtain any solution of equation (1.5) where M is given by (2.14) we must fix the two constants γ and ψ_0 and choose a solution to equation (2.8) with $\alpha=\beta=0$ and the fixed value of γ.

In [**5**] the general solution of equation (1.5) with M of the form (2.14) was found via a different method which involved the analysis of a certain evolving monodromy problem. This led to the following combination of the matrix elements of M

$$\alpha_\pm=(M_{11}-M_{22})\mp i(M_{12}+M_{21}),\qquad R^2=\alpha_+\alpha_-,$$

$$\beta_\pm = \omega \pm i(M_{21} - M_{12}), \qquad \omega = M_{11} + M_{22} - 2M_{33},$$

together with the conserved quantity

$$a^2 = \frac{(M_{12} - M_{21})^2}{R^2 - \omega^2}.$$

In the factorization method outlined above, these variables arise naturally from the component matrices P, d and a as follows

$$R = \omega_1 - \omega_2, \quad \alpha_\pm = R e^{\mp 2i\psi}, \quad \omega = (\omega_1 - \omega_3) + (\omega_2 - \omega_3), \quad M_{12} - M_{21} = 2\tau_3,$$

and $a = \gamma$.

3. First Integrals

In this section, following [**7**], we will use the method of solution given in section 2 to construct first integrals for equation (1.1). We begin by constructing explicit first integrals for the Schwarzian equation (2.8), as the formulas are much simpler in this case. Let u_1 and u_2 be two linearly independent solutions of equation (2.11) satisfying the Wronskian condition $W(u_1, u_2) = u_1 u_2' - u_2 u_1' = 1$. Then any solution to equation (2.8) is given implicitly by

$$t(s) = \frac{J_2 u_1(s) - J_1 u_2(s)}{I_2 u_1(s) - I_1 u_2(s)}, \tag{3.1}$$

where I_k and J_k, $k = 1, 2$, are constants satisfying

$$I_1 J_2 - I_2 J_1 = 1. \tag{3.2}$$

Hence any three of the constants I_1, I_2, J_1, J_2 can be taken as independent first integrals.

Differentiating equation (3.1) with respect to t gives

$$I_2 u_1 - I_1 u_2 = \dot{s}^{1/2}. \tag{3.3}$$

Differentiation of equation (3.3) gives

$$I_2 u_2' - I_1 u_2' = \frac{1}{2} \dot{s}^{-3/2} \ddot{s}. \tag{3.4}$$

Solving the system (3.3–3.4) for I_1 and I_2 gives

$$I_k = \frac{d\phi_k}{dt}, \qquad \phi_k = \dot{s}^{-1/2} u_k(s), \qquad k = 1, 2. \tag{3.5}$$

The constants J_1 and J_2 are given by the solution of equations (3.1), (3.5) and the normalization condition (3.2). This yields $J_k = tI_k - \phi_k$, $k = 1, 2$.

In terms of the gDH variables, we have

$$\begin{aligned}
\phi_k &= \sqrt{2} \left(\frac{(\omega_2 - \omega_3)}{(\omega_1 - \omega_2)(\omega_1 - \omega_3)} \right)^{1/2} u_k \left(\frac{\omega_1 - \omega_3}{\omega_2 - \omega_3} \right), \\
I_k &= \sqrt{2} \left(\frac{(\omega_1 - \omega_3)(\omega_1 - \omega_2)}{(\omega_2 - \omega_3)} \right)^{1/2} u_k' \left(\frac{\omega_1 - \omega_3}{\omega_2 - \omega_3} \right) \\
&\quad - (\omega_1 - \omega_2 - \omega_3) \left(\frac{(\omega_2 - \omega_3)}{2(\omega_1 - \omega_3)(\omega_1 - \omega_2)} \right)^{1/2} u_k \left(\frac{\omega_1 - \omega_3}{\omega_2 - \omega_3} \right).
\end{aligned} \tag{3.6}$$

Using the new variables ϕ_k and I_k instead of the gDH variables ω_i's, equation (1.1) can be formulated as a Hamiltonian system

$$\dot{\phi}_k = \frac{\partial H}{\partial I_k} = I_k\,, \qquad \dot{I}_k = -\frac{\partial H}{\partial \phi_k} = 0\,, \qquad H = \frac{I_1^2 + I_2^2}{2}\,, \quad k = 1, 2, \tag{3.7}$$

together with the constraint

$$\phi_1 I_2 - \phi_2 I_1 = W(u_1, u_2) = 1. \tag{3.8}$$

Although I_1 and I_2 are constant functions of t, they are multivalued functions of $\{\omega_1, \omega_2, \omega_3\}$ and of the Schwarzian variables $\{s, \dot{s}, \ddot{s}\}$. In terms of the solutions χ_1, χ_2 of the hypergeometric equation (2.12), the first integrals are given by

$$\begin{bmatrix} I_1 & I_2 \end{bmatrix} \;=\; \sigma \,[\lambda \quad 1] \begin{bmatrix} \chi_1(s) & \chi_2(s) \\ \chi_1'(s) & \chi_2'(s) \end{bmatrix}, \tag{3.9}$$

where

$$\sigma\,(s, \dot{s}) = s^{c/2}(1-s)^{(a+b-c+1)/2}\dot{s}^{1/2}, \text{ and } \quad \lambda(s, \dot{s}, \ddot{s}) = \frac{a+b+1-cs}{2s(1-s)} - \frac{\ddot{s}}{2\dot{s}^2}\,.$$

Next we will discuss the dependence of I_1 and I_2 on s, $\dot{s}$, and $\ddot{s}$. Clearly I_k, $k = 1, 2$, is single-valued as a function of $\ddot{s}$ and has square-root branch points as a function of $\dot{s}$ about $\dot{s} = 0$ and $\dot{s} = \infty$. In fact, the conserved quantities I_1^2 and I_2^2 are single-valued as functions of $\dot{s}$. Holding $\dot{s}$ and $\ddot{s}$ fixed, I_μ can only admit branch points at $s = 0$, $s = 1$, and $s = \infty$. Let γ_0 and γ_1 be two closed curves with a common base point in the finite complex s-plane enclosing the points $s = 0$ and $s = 1$ respectively, and traversed once in the positive direction. Analytic continuation of σ along γ_0 and γ_1 gives

$$\gamma_0 :\; \sigma \mapsto \mathrm{e}^{i\pi c}\sigma, \qquad \gamma_1 :\; \sigma \mapsto \mathrm{e}^{i\pi(a+b-c)}\sigma.$$

Analytic continuation along γ_0 and γ_1 transforms the fundamental matrix of solutions of equation (2.12) according to

$$\gamma_\mu : \qquad \begin{pmatrix} \chi_1(s) & \chi_2(s) \\ \chi_1'(s) & \chi_2'(s) \end{pmatrix} \mapsto \begin{pmatrix} \chi_1(s) & \chi_2(s) \\ \chi_1'(s) & \chi_2'(s) \end{pmatrix} M_\mu, \qquad \mu = 0, 1.$$

For generic values of a, b, c, and for the choice of basis solutions: $\chi_1 = F(a, b, c; s)$, $\chi_2 = F(a, b, a+b-c+1; 1-s)$ of the hypergeometric equation, the monodromy matrices M_μ are given by [**20**]

$$M_0 = \begin{pmatrix} 1 & \mathrm{e}^{-2\pi i b} - \mathrm{e}^{-2\pi i c} \\ 0 & \mathrm{e}^{-2\pi i c} \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} \mathrm{e}^{-2\pi i(a+b-c)} & 0 \\ 1 - \mathrm{e}^{-2\pi i(a-c)} & 1 \end{pmatrix}.$$

So under analytic continuation, the first integrals I_1, I_2 transform as

$$\gamma_0 :\; \begin{bmatrix} I_1 & I_2 \end{bmatrix} \mapsto \begin{bmatrix} I_1 & I_2 \end{bmatrix} M_0 \mathrm{e}^{i\pi c}, \qquad \gamma_1 :\; \begin{bmatrix} I_1 & I_2 \end{bmatrix} \mapsto \begin{bmatrix} I_1 & I_2 \end{bmatrix} M_1 \mathrm{e}^{i\pi(a+b-c)}\,.$$

Analyic continuation around $s = \infty$ is equivalent to a loop around $s = 0$ and $s = 1$. Hence the branching of the first integrals I_1 and I_2 is characterized in terms of the monodromy group for the hypergeometric equation.

The first integrals in equation (3.9) for the classical DH system ($\alpha = \beta = \gamma = 0$) are expressed in terms of the special hypergeometric equation (2.12) with $a = b = 1/2$, $c = 1$. In this case, the monodromy matrices are given by

$$M_0 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

relative to the choice of basis $\chi_1 = F(1/2, 1/2, 1; s)$ and $\chi_2 = iF(1/2, 1/2, 1; 1-s)$. Note that in this case, I_1 and I_2 are still branched, even though the solution itself is single-valued. The non-existence of meromorphic first integrals for the classical Darboux-Halphen system was proved in [**17**]. We show by explicit construction that the first integrals do indeed exist but they are non-algebraic and multi-valued.

4. Nambu-Poisson Structure and Gradient flow

The gDH system (1.1) can also be viewed as a complex dynamical system given by $(\dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3) = \mathbf{X}$ where the vector field

$$\mathbf{X} = \left(\omega_2\omega_3 - \omega_1\omega_2 - \omega_3\omega_1 + \tau^2, \omega_3\omega_1 - \omega_2\omega_3 - \omega_1\omega_2 + \tau^2, \omega_1\omega_2 - \omega_3\omega_1 - \omega_2\omega_3 + \tau^2\right).$$

The generalized DH flow given by the integral curves of $\mathbf{X}$ lie on the intersection of the level sets of the first integrals ($I_1 =$ constant and $I_2 =$ constant) in a three-dimensional complex manifold $M^3 := \mathbf{C} \setminus \{\omega_i \neq \omega_j, i \neq j\}$. Since the I_k's are conserved under the gDH flow,

$$\frac{dI_k}{dt} = \mathbf{X} \cdot \nabla I_k = 0, \qquad k = 1, 2,$$

it follows that the vector field $\mathbf{X}$ is proportional to $\nabla I_1 \times \nabla I_2$. Explicit calculation shows that,

$$\mathbf{X} = \frac{1}{4\Delta} \nabla I_1 \times \nabla I_2$$

where

$$\Delta = (\omega_2 - \omega_3)(\omega_3 - \omega_1)(\omega_1 - \omega_2).$$

The gDH equations can be expressed as

$$\dot{\omega}_j = \mathbf{X} \cdot \nabla \omega_j = \begin{cases} (4\Delta)^{-1} \nabla I_2 \cdot (\nabla \omega_j \times \nabla I_1) =: \{\omega_j, I_1\}_1, \\ -(4\Delta)^{-1} \nabla I_1 \cdot (\nabla \omega_j \times \nabla I_2) =: \{\omega_j, I_2\}_2. \end{cases}$$

It can be verified that the brackets

$$\begin{aligned} B_1(g,h) = \{g,h\}_1 &= (4\Delta)^{-1} \nabla I_2 \cdot (\nabla g \times \nabla h), \\ B_2(g,h) = \{g,h\}_2 &= -(4\Delta)^{-1} \nabla I_1 \cdot (\nabla g \times \nabla h), \end{aligned}$$

are Poisson (i.e., they are bi-linear, anti-symmetric and satisfy the Jacobi identity). So $\mathbf{X}$ is a Hamiltonian vector field with respect to the two Poisson strucures B_1 and B_2 with Hamiltonians I_1 and I_2 respectively.

The Poisson structures B_1 and B_2 are degenerate (rank 2) and admit Casimir functions I_2 and I_1 respectively. This is easily verified by using the vector triple product identity

$$\begin{aligned} \{I_2, g\}_1 &= (4\Delta)^{-1} \nabla I_2 \cdot (\nabla I_2 \times \nabla g) = (4\Delta)^{-1} \nabla g \cdot (\nabla I_2 \times \nabla I_2) = 0, \\ \{I_1, g\}_2 &= (4\Delta)^{-1} \nabla I_2 \cdot (\nabla I_1 \times \nabla g) = (4\Delta)^{-1} \nabla g \cdot (\nabla I_1 \times \nabla I_1) = 0, \end{aligned}$$

for any smooth function g on M. Furthermore, since $\{I_1, I_2\}_1 = \{I_1, I_2\}_2 = 0$, the integrals of the gDH system are in involution. The Poisson structures B_1 and B_2 are compatible in the sense that

$$B := \lambda_1 B_1 + \lambda_2 B_2, \qquad \lambda_i = \lambda_i(I_1, I_2)$$

is also a Poisson structure for gDH with a Hamiltonian $H(I_1, I_2)$ satisfying

$$\lambda_1 \frac{\partial H}{\partial I_1} + \lambda_2 \frac{\partial H}{\partial I_2} = 0.$$

Thus the gDH system is *bi-Hamiltonian*.

The symmetric representation of the gDH system using *both* Hamiltonians I_1, I_2 is

$$\dot{\omega}_j = (4\Delta)^{-1}\nabla\omega_j \cdot (\nabla I_1 \times \nabla I_2) = (4\Delta)^{-1}\frac{\partial(\omega_i, I_1, I_2)}{\partial(\omega_1, \omega_2, \omega_3)} =: \{\omega_j, I_1, I_2\}.$$

This is an example of a Nambu-Poisson bracket similar to rigid body dynamics in three dimensions [**18, 21**].

The Darboux-Halphen system (1.1) is also a gradient flow. In terms of local coordinates ω_i, it can be written as

$$\dot{\omega}_i = \sum_{j=1}^{3} g^{ij}\frac{\partial\Phi}{\partial\omega_j},$$

where g^{ij} is a constant contravariant metric and Φ is a potential function. The metric is given by

$$(g^{ij}) = \begin{pmatrix} m(\alpha,\beta,\gamma) & \kappa+4\gamma^2 & \kappa+4\beta^2 \\ \kappa+4\gamma^2 & m(\beta,\gamma,\alpha) & \kappa+4\alpha^2 \\ \kappa+4\beta^2 & \kappa+4\alpha^2 & m(\gamma,\alpha,\beta) \end{pmatrix},$$

where

$$\kappa = (\alpha+\beta+\gamma)(\alpha+\beta-\gamma)(\alpha-\beta+\gamma)(\alpha-\beta-\gamma) - 1,$$

and

$$m(\alpha,\beta,\gamma) = (1-\alpha^2+(\beta+\gamma)^2)(1-\alpha^2+(\beta-\gamma)^2).$$

The potential Φ is a homogeneous polynomial of degree 3 in the ω_i's and is invariant under the *simultaneous* cyclic permutation of $\{\omega_1,\omega_2,\omega_3\}$ and $\{\alpha,\beta,\gamma\}$. In terms of the function

$$F(\omega_1,\omega_2,\omega_3;\alpha,\beta,\gamma) = \left[(1-\alpha^2)(3\alpha^2-2\beta^2-2\gamma^2+1)+(\beta^2-\gamma^2)^2\right]\times$$
$$\omega_1\left[\alpha^2\omega_1^2+3\beta^2\omega_2^2+3\gamma^2\omega_3^2+(1-\alpha^2-3\beta^2-3\gamma^2)\omega_2\omega_3\right],$$

the potential is expressed as

$$\Phi = -\frac{4}{3\ \det(g^{ij})}\left[F(\omega_1,\omega_2,\omega_3;\alpha,\beta,\gamma)+F(\omega_2,\omega_3,\omega_1;\beta,\gamma,\alpha)+F(\omega_3,\omega_1,\omega_2;\gamma,\alpha,\beta)\right].$$

With respect to the metric g^{ij}, the constant Φ surfaces are orthogonal to the curves obtained by the intersection of the constant I_1 and I_2 surfaces.

In the classical Darboux-Halphen case ($\alpha=\beta=\gamma=0$), we have

$$g^{ij} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \Phi = \omega_1\omega_2\omega_3.$$

For the fifth order reduction ($\alpha=\beta=0$) discussed in section 2,

$$g^{ij} = \begin{pmatrix} (1+\gamma^2)^2 & \gamma^4+4\gamma^2-1 & \gamma^4-1 \\ \gamma^4+4\gamma^2-1 & (1+\gamma^2)^2 & \gamma^4-1 \\ \gamma^4-1 & \gamma^4-1 & (1-\gamma^2)^2 \end{pmatrix}$$

and the corresponding potential function is given by

$$\Phi = \frac{\gamma^2\omega_3^2[3(1-\gamma^2)(\omega_1+\omega_2)+(1+3\gamma^2)\omega_3]+3(1-\gamma^2)^2\omega_1\omega_2\omega_3}{3(1-\gamma^2)^3}.$$

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