First integrals and gradient flow for a generalized Darboux-Halphen system

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Abstract. First integrals are explicitly constructed for a third-order system of ODEs that arises as a reduction of the self-dual Yang-Mills equations and in the theory of hypercomplex manifolds. These first integrals are branched functions of the phase space variables, even in cases for which the general solution is single-valued. This branching is characterized in terms of the monodromy of the hypergeometric equations. The first integrals are then used to formulate a Nambu-Poisson structure of the system. A representation of the generalized Darboux-Halphen system as a gradient flow is also given.

1. Introduction

The system

\[
\begin{aligned}
\dot{\omega}_1 &= \omega_2 \omega_3 - \omega_1 (\omega_2 + \omega_3) + \tau^2, \\
\dot{\omega}_2 &= \omega_3 \omega_1 - \omega_2 (\omega_3 + \omega_1) + \tau^2, \\
\dot{\omega}_3 &= \omega_1 \omega_2 - \omega_3 (\omega_1 + \omega_2) + \tau^2,
\end{aligned}
\]

(1.1)

where

\[\tau^2 = \alpha^2 (\omega_1 - \omega_2)(\omega_3 - \omega_1) + \beta^2 (\omega_2 - \omega_3)(\omega_1 - \omega_2) + \gamma^2 (\omega_3 - \omega_1)(\omega_2 - \omega_3),\]

was first studied by Halphen [14] as a natural generalization of the classical Darboux-Halphen (DH) system, which corresponds to setting \(\tau = 0\) in equation (1.1). The classical Darboux-Halphen system first arose in Darboux’s study of triply orthogonal surfaces [11] and was later solved by Halphen [15]. The classical DH system has also appeared in studies of self-dual Bianchi-IX metrics with Euclidean signature [4, 13] and in reductions of the associativity equations on a three-dimensional Frobenius manifold [12]. Furthermore, if \((\omega_1, \omega_2, \omega_3)\) is a solution of the classical Darboux-Halphen system, then

\[y := -2(\omega_1 + \omega_2 + \omega_3)\]
satisfies the Chazy equation,

\[
\frac{d^3y}{dt^3} = 2y \frac{d^2y}{dt^2} - \left( \frac{dy}{dt} \right)^2.
\]

In [3] it was shown that \( y \) defined by equation (1.2) solves the generalized Chazy equation,

\[
\frac{d^3y}{dt^3} - 2y \frac{d^2y}{dt^2} + \left( \frac{dy}{dt} \right)^2 = \frac{4}{36 - n^2} \left( 6 \frac{dy}{dt} - y^2 \right)^2,
\]

where the \( \omega_i \)'s solve the generalized Darboux-Halphen system for the special choices of parameters \( (\alpha, \beta, \gamma) \) given by \( (2/n, 2/n, 2/n) \) and \( (1/3, 1/3, 2/n) \), et ctc. Note that equation (1.3) corresponds to the limit \( n \to \infty \) in equation (1.4). Equations (1.3) and (1.4) were first studied by Chazy in [8, 9, 10].

The system (1.1) arises in the study of the equation

\[
\dot{M} = (\text{adj} \ M)^T + M^T M - (\text{Tr} \ M) M,
\]

for a \( 3 \times 3 \) matrix valued function \( M(t) \) where \( \text{adj} \ M \) is the adjoint of \( M \) satisfying \( (\text{adj} \ M) M = (\det M) I \), \( M^T \) is the transpose of \( M \) and the dot denotes differentiation with respect to \( t \). Equation (1.5) was obtained in [6] as a reduction of the self-dual Yang-Mills equations with an infinite dimensional gauge group of diffeomorphisms \( \text{Diff}(S^3) \) of the three-sphere. Equation (1.5) also describes a class of self-dual Weyl Bianchi IX space-times with Euclidean signature [5]. More recently, equation (1.5) was used to describe SU(2) invariant hypercomplex 4-manifolds [16].

In section 2 we will review the reduction of equation (1.5) to the generalized DH system (1.1). The general solution will be constructed and a special case will be studied. In section 3, first integrals and “action-angle” variables are given for equation (1.1). The first integrals involve hypergeometric functions and are non-meromorphic, even in cases where the general solution is single-valued.

2. The solution of the generalized Darboux-Halphen system

In this section the solution of equation (1.5) is given by a factorization method which first appeared in [1]. The solution can also be obtained via associated linear problems. In [2], the solution was obtained via an evolving monodromy problem that arises as a reduction of the isospectral problem for the self-dual Yang-Mills equations. In [16], the solution was obtained via an isomonodromy problem which describes the Riccati solutions of the sixth Painlevé equation. Degenerate cases were discussed in [3].

We begin by decomposing the matrix \( M \) into its symmetric \( (M_s) \) and antisymmetric \( (M_a) \) parts. Furthermore, the eigenvalues of \( M_s \) are assumed to be distinct, so that it can be diagonalized using a complex orthogonal matrix. Thus we have

\[
M = M_s + M_a = P(d + a)P^{-1},
\]

where \( P \in \text{SO}(3, \mathbb{C}) \), \( d = \text{diag}(\omega_1, \omega_2, \omega_3) \) with \( \omega_i \neq \omega_j, \ i \neq j \), and the elements of the skew-symmetric matrix \( a \) are given by \( a_{ij} = \sum_{k=1}^{3} \varepsilon_{ijk} \tau_k \), where \( \varepsilon_{ijk} \) is totally skew-symmetric in its indices and \( \varepsilon_{123} = 1 \). Using the transformation (2.1), the
diagonal part of equation (1.5) yields the system (1.1), where \( \tau^2 = \tau_1^2 + \tau_2^2 + \tau_3^2 \).

The skew-symmetric part gives

\[
(2.2) \quad \dot{\tau}_1 = -\tau_1 (\omega_2 + \omega_3), \quad \dot{\tau}_2 = -\tau_2 (\omega_3 + \omega_1), \quad \dot{\tau}_3 = -\tau_3 (\omega_1 + \omega_2),
\]

and the off-diagonal symmetric part gives

\[
(2.3) \quad \dot{P} = -Pa,
\]

which is a linear equation for \( P \).

Taking differences of equations in system (1.1) gives

\[
(2.4) \quad \frac{d}{dt} (\omega_1 - \omega_2) = -2\omega_3 (\omega_1 - \omega_2), \quad \text{et cyc.}
\]

Using equations (2.2) and (2.4), we can solve the \( \tau_i \)'s in terms of the \( \omega_i \)'s as

\[
(2.5) \quad \tau_2^1 = \alpha_2 (\omega_1 - \omega_2)(\omega_3 - \omega_1), \quad \tau_2^2 = \beta_2 (\omega_2 - \omega_3)(\omega_1 - \omega_2), \quad \tau_2^3 = \gamma_2 (\omega_3 - \omega_1)(\omega_2 - \omega_3),
\]

where \( \alpha, \beta, \) and \( \gamma \) are integration constants.

In terms of the cross ratio

\[
(2.6) \quad s := \frac{\omega_1 - \omega_3}{\omega_2 - \omega_3},
\]

it follows from equation (1.1) that the \( \omega_i \)'s can be parameterized as

\[
(2.7) \quad \omega_1 = -\frac{1}{2} \frac{d}{dt} \ln \frac{\dot{s}}{s(s-1)}, \quad \omega_2 = -\frac{1}{2} \frac{d}{dt} \ln \frac{\dot{s}}{s-1}, \quad \omega_3 = -\frac{1}{2} \frac{d}{dt} \ln \frac{1}{s}.
\]

where \( s \) satisfies

\[
(2.8) \quad \frac{d}{dt} \left( \frac{\dot{s}}{s} \right) - \frac{1}{2} \left( \frac{\dot{s}}{s} \right)^2 + \frac{\dot{s}^2}{2} V(s) = 0,
\]

with

\[
\{s, t\} := \frac{d}{dt} \left( \frac{\dot{s}}{s} \right) - \frac{1}{2} \left( \frac{\dot{s}}{s} \right)^2
\]

being the Schwarzian derivative and \( V \) is given by

\[
(2.9) \quad V(s) = \frac{1 - \beta^2}{s^2} + \frac{1 - \gamma^2}{(s-1)^2} + \frac{\beta^2 + \gamma^2 - \alpha^2 - 1}{s(s-1)}.
\]

Equation (2.8) is the Schwarzian equation, which describes the conformal mappings of the upper-half \( s \)-plane to the interior of a region of the complex sphere bounded by three regular circular arcs. If \( \alpha, \beta, \) and \( \gamma \) are non-negative real numbers such that \( \alpha + \beta + \gamma < 1 \), then the angles subtended at the vertices \( s = 0, s = 1, \) and \( s = \infty \) of this triangle are \( \alpha \pi, \beta \pi, \) and \( \gamma \pi \). Furthermore, if \( \alpha, \beta, \) and \( \gamma \) are chosen to be either reciprocals of integers or zero, then \( s \) is analytic on the interior of a circle on the complex sphere and cannot be analytically extended across this circle, which is a natural barrier [19].

The general solution of equation (2.8) is given implicitly by

\[
(2.10) \quad t(s) = \frac{u_1(s)}{u_2(s)},
\]

where \( u_1(s) \) and \( u_2(s) \) are independent solutions to the Fuchsian equation

\[
(2.11) \quad \frac{d^2u}{ds^2} + \frac{1}{4} V(s) u = 0.
\]
Equation (2.11) is equivalent to the hypergeometric equation,

\begin{equation}
(2.12) \quad s(1-s) \frac{d^2 \chi}{ds^2} + [c - (a + b + 1)s] \frac{d \chi}{ds} - ab \chi = 0,
\end{equation}

where \( a = (1 + \alpha - \beta - \gamma)/2 \), \( b = (1 - \alpha - \beta - \gamma)/2 \), \( c = 1 - \beta \), and

\begin{equation}
(2.13) \quad u(s) = s^{c/2} (1-s)^{(a+b-c+1)/2} \chi(s).
\end{equation}

A fifth order reduction

We will now consider the case in which \( M \) has the special form

\begin{equation}
(2.14) \quad M = \begin{pmatrix}
M_{11} & M_{12} & 0 \\
M_{21} & M_{22} & 0 \\
0 & 0 & M_{33}
\end{pmatrix}.
\end{equation}

This special form of \( M \) was considered in \([5, 2]\), where equation (1.5) was analyzed using an associated evolving monodromy problem. Here we will show that quantities that arise naturally from this monodromy analysis can be obtained in a straightforward manner from the factorization method described above.

Consider the factorization of \( M \) given by equation (2.1) where \( M \) is given by equation (2.14). Due to special block structure of \( M \), its symmetric part can be diagonalized by an orthogonal matrix of the form

\begin{equation}
(2.15) \quad P = \begin{pmatrix}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{pmatrix},
\end{equation}

where \( \psi \) is a (generally complex) function of \( t \) to be determined. That is, \( M_s = P d P^{-1} \), where \( d = \text{diag}(\omega_1, \omega_2, \omega_3) \). Furthermore, the skew-symmetric part of \( M \) is unchanged by the adjoint action of \( P \). So

\begin{equation}
(2.16) \quad a = P^{-1} M_a P = M_a = \frac{1}{2} (M - M^T) = \begin{pmatrix}
0 & \tau_3(t) & 0 \\
-\tau_3(t) & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\end{equation}

where \( \tau_3(t) = \frac{i}{2} (M_{12} - M_{21}) \). Since \( \tau_1 = \tau_2 = 0 \) in this case, the \( \omega_i \)'s are given by equation (2.7) where \( s \) solves equation (2.8) with \( \alpha = \beta = 0 \). From equation (2.5) and equation (2.7), we have

\[ \tau_3(t) = \frac{i \gamma}{2} \frac{s}{\sqrt{s(s-1)}} \]

where \( \gamma \) is a constant. With the \( \tau_3(t) \) given above, equation (2.3) can be readily integrated to give

\[ \psi = \frac{i \gamma}{2} \log \left( \frac{\sqrt{s} - 1}{\sqrt{s} + 1} \right) + \psi_0, \]

where \( \psi_0 \) is a constant. Finally, the matrix \( M \) in (2.14) is reconstructed from the various components \( P \), \( d \) and \( a \) according to equation (2.1). Note that in order to obtain any solution of equation (1.5) where \( M \) is given by (2.14) we must fix the two constants \( \gamma \) and \( \psi_0 \) and choose a solution to equation (2.8) with \( \alpha = \beta = 0 \) and the fixed value of \( \gamma \).

In \([5]\) the general solution of equation (1.5) with \( M \) of the form (2.14) was found via a different method which involved the analysis of a certain evolving monodromy problem. This led to the following combination of the matrix elements of \( M \)

\[ \alpha \pm = (M_{11} - M_{22}) \mp i(M_{12} + M_{21}), \quad R^2 = \alpha_+ \alpha_- \]
\[ \beta_{\pm} = \omega \pm i(M_{21} - M_{12}), \quad \omega = M_{11} + M_{22} - 2M_{33}, \]

together with the conserved quantity
\[ a^2 = \frac{(M_{12} - M_{21})^2}{R^2 - \omega^2}. \]

In the factorization method outlined above, these variables arise naturally from the component matrices \( P, d \), and \( a \) as follows
\[ R = \omega_1 - \omega_2, \quad \alpha_{\pm} = \pm Re^{\mp 2i\psi}, \quad \omega = (\omega_1 - \omega_3) + (\omega_2 - \omega_3), \quad M_{12} - M_{21} = 2\tau_3, \]
and \( a = \gamma \).

### 3. First Integrals

In this section, following [7], we will use the method of solution given in section 2 to construct first integrals for equation (1.1). We begin by constructing explicit first integrals for the Schwarzian equation (2.8), as the formulas are much simpler in this case. Let \( u_1 \) and \( u_2 \) be two linearly independent solutions of equation (2.11) satisfying the Wronskian condition
\[ W(u_1, u_2) = u_1u_2' - u_2u_1' = 1. \]
Then any solution to equation (2.8) is given implicitly by
\[ t(s) = \frac{J_2u_1(s) - J_1u_2(s)}{I_2u_1(s) - I_1u_2(s)}, \]
where \( I_k \) and \( J_k \), \( k = 1, 2 \), are constants satisfying
\[ I_1J_2 - I_2J_1 = 1. \]
Hence any three of the constants \( I_1, I_2, J_1, J_2 \) can be taken as independent first integrals.

Differentiating equation (3.1) with respect to \( t \) gives
\[ I_2u_1 - I_1u_2 = \dot{s}^{1/2}. \]
Differentiation of equation (3.3) gives
\[ I_2u_1' - I_1u_2' = \frac{1}{2} \dot{s}^{-3/2} \ddot{s}. \]
Solving the system (3.3–3.4) for \( I_1 \) and \( I_2 \) gives
\[ I_k = \frac{d\phi_k}{dt}, \quad \phi_k = \dot{s}^{-1/2}u_k(s), \quad k = 1, 2. \]
The constants \( J_1 \) and \( J_2 \) are given by the solution of equations (3.1), (3.5) and the normalization condition (3.2). This yields \( J_k = tI_k - \phi_k, k = 1, 2 \).

In terms of the gDH variables, we have
\[ \phi_k = \sqrt{2} \left( \frac{(\omega_2 - \omega_3)}{(\omega_1 - \omega_2)(\omega_1 - \omega_3)} \right)^{1/2} \frac{u_k}{u_k} \left( \frac{\omega_1 - \omega_3}{\omega_2 - \omega_3} \right), \]
(3.6)
\[ I_k = \sqrt{2} \left( \frac{(\omega_2 - \omega_3)}{(\omega_1 - \omega_2)(\omega_1 - \omega_3)} \right)^{1/2} \frac{u_k'}{u_k} \left( \frac{\omega_1 - \omega_3}{\omega_2 - \omega_3} \right) \]
\[ - (\omega_1 - \omega_2 - \omega_3) \left( \frac{(\omega_2 - \omega_3)}{2(\omega_1 - \omega_3)(\omega_1 - \omega_2)} \right)^{1/2} \frac{u_k}{u_k} \left( \frac{\omega_1 - \omega_3}{\omega_2 - \omega_3} \right). \]
Using the new variables $\phi_k$ and $I_k$ instead of the gDH variables $\omega_i$’s, equation (1.1) can be formulated as a Hamiltonian system

\begin{equation}
\dot{\phi}_k = \frac{\partial H}{\partial I_k} = I_k, \quad \dot{I}_k = -\frac{\partial H}{\partial \phi_k} = 0, \quad H = \frac{I_1^2 + I_2^2}{2}, \quad k = 1, 2,
\end{equation}

together with the constraint

\begin{equation}
\phi_1 I_2 - \phi_2 I_1 = W(u_1, u_2) = 1.
\end{equation}

Although $I_1$ and $I_2$ are constant functions of $t$, they are multivalued functions of $\{\omega_1, \omega_2, \omega_3\}$ and of the Schwarzian variables $\{s, \bar{s}, \ddot{s}\}$. In terms of the solutions $\chi_1, \chi_2$ of the hypergeometric equation (2.12), the first integrals are given by

\begin{equation}
[I_1 \ I_2] = \sigma [\lambda \ 1] \begin{bmatrix} \chi_1(s) & \chi_2(s) \\ \chi_1'(s) & \chi_2'(s) \end{bmatrix},
\end{equation}

where

$$
\sigma(s, \ddot{s}) = s^{c/2}(1 - s)^{(a + b - c + 1)/2}, \quad \lambda(s, \bar{s}, \ddot{s}) = \frac{a + b + 1 - cs}{2s(1 - s)} - \frac{\ddot{s}}{\bar{s}}^{2/3}.
$$

Next we will discuss the dependence of $I_1$ and $I_2$ on $s$, $\bar{s}$, and $\ddot{s}$. Clearly $I_k$, $k = 1, 2$, is single-valued as a function of $\ddot{s}$ and has square-root branch points as a function of $\ddot{s}$ about $\ddot{s} = 0$ and $\ddot{s} = \infty$. In fact, the conserved quantities $I_1^2$ and $I_2^2$ are single-valued as functions of $\ddot{s}$. Holding $s$ and $\bar{s}$ fixed, $I_\mu$ can only admit branch points at $s = 0$, $s = 1$, and $s = \infty$. Let $\gamma_0$ and $\gamma_1$ be two closed curves with a common base point in the finite complex $s$-plane enclosing the points $s = 0$ and $s = 1$ respectively, and traversed once in the positive direction. Analytic continuation of $\sigma$ along $\gamma_0$ and $\gamma_1$ gives

$$
\gamma_0 : \sigma \mapsto e^{i\pi c} \sigma, \quad \gamma_1 : \sigma \mapsto e^{i\pi(a + b - c)} \sigma.
$$

Analytic continuation along $\gamma_0$ and $\gamma_1$ transforms the fundamental matrix of solutions of equation (2.12) according to

$$
\gamma_\mu : \left( \begin{array}{cc} \chi_1(s) & \chi_2(s) \\ \chi_1'(s) & \chi_2'(s) \end{array} \right) \mapsto \left( \begin{array}{cc} \chi_1(s) & \chi_2(s) \\ \chi_1'(s) & \chi_2'(s) \end{array} \right) M_{\mu}, \quad \mu = 0, 1.
$$

For generic values of $a, b, c$, and for the choice of basis solutions: $\chi_1 = F(a, b, c; s), \chi_2 = F(a, b, a + b - c + 1; 1 - s)$ of the hypergeometric equation, the monodromy matrices $M_{\mu}$ are given by [20]

\begin{align*}
M_0 &= \begin{pmatrix} 1 & e^{-2\pi ic} - e^{-2\pi ic} \\ 0 & e^{-2\pi ic} \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} e^{-2\pi i(a + b - c)} & 0 \\ 1 - e^{-2\pi i(a - c)} & 1 \end{pmatrix}.
\end{align*}

So under analytic continuation, the first integrals $I_1, I_2$ transform as

$$
\gamma_0 : [I_1 \ I_2] \mapsto [I_1 \ I_2] M_0 e^{i\pi c}, \quad \gamma_1 : [I_1 \ I_2] \mapsto [I_1 \ I_2] M_1 e^{i\pi(a + b - c)}.
$$

Analytic continuation around $s = \infty$ is equivalent to a loop around $s = 0$ and $s = 1$. Hence the branching of the first integrals $I_1$ and $I_2$ is characterized in terms of the monodromy group for the hypergeometric equation.

The first integrals in equation (3.9) for the classical DH system ($\alpha = \beta = \gamma = 0$) are expressed in terms of the special hypergeometric equation (2.12) with $a = b = 1/2, c = 1$. In this case, the monodromy matrices are given by

\begin{align*}
M_0 &= \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.
\end{align*}
relative to the choice of basis $\chi_1 = F(1/2, 1/2, 1; s)$ and $\chi_2 = iF(1/2, 1/2, 1; 1 - s)$. Note that in this case, $I_1$ and $I_2$ are still branched, even though the solution itself is single-valued. The non-existence of meromorphic first integrals for the classical Darboux-Halphen system was proved in [17]. We show by explicit construction that the first integrals do indeed exist but they are non-algebraic and multi-valued.

4. Nambu-Poisson Structure and Gradient flow

The gDH system (1.1) can also be viewed as a complex dynamical system given by $(\dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3) = X$ where the vector field $X = (\omega_2\omega_3 - \omega_1\omega_2 + \tau^2, \omega_3\omega_1 - \omega_2\omega_3 - \omega_1\omega_2 + \tau^2, \omega_1\omega_2 - \omega_3\omega_1 - \omega_2\omega_3 + \tau^2)$. The generalized DH flow given by the integral curves of $X$ lie on the intersection of the level sets of the first integrals $(I_1 = \text{constant} \text{ and } I_2 = \text{constant})$ in a three-dimensional complex manifold $M^3 := \mathbb{C} \setminus \{\omega_i \neq \omega_j, i \neq j\}$. Since the $I_k$'s are conserved under the gDH flow,

$$\frac{dI_k}{dt} = X \cdot \nabla I_k = 0, \quad k = 1, 2,$$

it follows that the vector field $X$ is proportional to $\nabla I_1 \times \nabla I_2$. Explicit calculation shows that,

$$X = \frac{1}{4\Delta} \nabla I_1 \times \nabla I_2$$

where

$$\Delta = (\omega_2 - \omega_3)(\omega_3 - \omega_1)(\omega_1 - \omega_2).$$

The gDH equations can be expressed as

$$\dot{\omega}_j = X \cdot \nabla \omega_j = \begin{cases} (4\Delta)^{-1} \nabla I_2 \cdot (\nabla \omega_j \times \nabla I_1) =: \{\omega_j, I_1\}_1, \\ - (4\Delta)^{-1} \nabla I_1 \cdot (\nabla \omega_j \times \nabla I_2) =: \{\omega_j, I_2\}_2. \end{cases}$$

It can be verified that the brackets

$$B_1(g, h) = \{g, h\}_1 = (4\Delta)^{-1} \nabla I_2 \cdot (\nabla g \times \nabla h),$$

$$B_2(g, h) = \{g, h\}_2 = -(4\Delta)^{-1} \nabla I_1 \cdot (\nabla g \times \nabla h),$$

are Poisson (i.e., they are bi-linear, anti-symmetric and satisfy the Jacobi identity). So $X$ is a Hamiltonian vector field with respect to the two Poisson structures $B_1$ and $B_2$ with Hamiltonians $I_1$ and $I_2$ respectively.

The Poisson structures $B_1$ and $B_2$ are degenerate (rank 2) and admit Casimir functions $I_2$ and $I_1$ respectively. This is easily verified by using the vector triple product identity

$$\{I_2, g\}_1 = (4\Delta)^{-1} \nabla I_2 \cdot (\nabla I_2 \times \nabla g) = (4\Delta)^{-1} \nabla g \cdot (\nabla I_2 \times \nabla I_2) = 0,$$

$$\{I_1, g\}_2 = (4\Delta)^{-1} \nabla I_1 \cdot (\nabla I_1 \times \nabla g) = (4\Delta)^{-1} \nabla g \cdot (\nabla I_1 \times \nabla I_1) = 0,$$

for any smooth function $g$ on $M$. Furthermore, since $\{I_1, I_2\}_1 = \{I_1, I_2\}_2 = 0$, the integrals of the gDH system are in involution. The Poisson structures $B_1$ and $B_2$ are compatible in the sense that

$$B := \lambda_1 B_1 + \lambda_2 B_2, \quad \lambda_i = \lambda_i(I_1, I_2)$$

is also a Poisson structure for gDH with a Hamiltonian $H(I_1, I_2)$ satisfying

$$\lambda_1 \frac{\partial H}{\partial I_1} + \lambda_2 \frac{\partial H}{\partial I_2} = 0.$$
Thus the gDH system is bi-Hamiltonian.

The symmetric representation of the gDH system using both Hamiltonians \( I_1, I_2 \) is

\[
\dot{\omega}_j = (4\Delta)^{-1}\nabla\omega_j \cdot (\nabla I_1 \times \nabla I_2) = (4\Delta)^{-1} \left( \frac{\partial(\omega, I_1, I_2)}{\partial(\omega, I_1, I_2)} \right) =: \{\omega, I_1, I_2\}.
\]

This is an example of a Nambu-Poisson bracket similar to rigid body dynamics in three dimensions [18, 21].

The Darboux-Halphen system (1.1) is also a gradient flow. In terms of local coordinates \( \omega_i \), it can be written as

\[
\dot{\omega}_i = \sum_{j=1}^{3} g^{ij} \frac{\partial \Phi}{\partial \omega_j},
\]

where \( g^{ij} \) is a constant contravariant metric and \( \Phi \) is a potential function. The metric is given by

\[
(g^{ij}) = \begin{pmatrix}
\kappa + 4\gamma^2 & \kappa + 4\beta^2 & \kappa + 4\alpha^2 \\
\kappa + 4\gamma^2 & m(\beta, \gamma, \alpha) & \kappa + 4\alpha^2 \\
\kappa + 4\beta^2 & \kappa + 4\alpha^2 & m(\gamma, \alpha, \beta)
\end{pmatrix},
\]

where

\[
\kappa = (\alpha + \beta + \gamma)(\alpha + \beta - \gamma)(\alpha - \beta + \gamma)(\alpha - \beta - \gamma) - 1,
\]

and

\[
m(\alpha, \beta, \gamma) = (1 - \alpha^2 + (\beta + \gamma)^2)(1 - \alpha^2 + (\beta - \gamma)^2).
\]

The potential \( \Phi \) is a homogeneous polynomial of degree 3 in the \( \omega_i \)'s and is invariant under the simultaneous cyclic permutation of \( \{\omega_1, \omega_2, \omega_3\} \) and \( \{\alpha, \beta, \gamma\} \). In terms of the function

\[
F(\omega_1, \omega_2, \omega_3; \alpha, \beta, \gamma) = [(1 - \alpha^2)(3\alpha^2 - 2\beta^2 - 2\gamma^2 + 1) + (\beta^2 - \gamma^2)^2] \times \\
\omega_1 [\alpha^2\omega_1^2 + 3\beta^2\omega_2^2 + 3\gamma^2\omega_3^2 + (1 - \alpha^2 - 3\beta^2 - 3\gamma^2)\omega_2\omega_3],
\]

the potential is expressed as

\[
\Phi = -\frac{4}{3 \det(g^{ij})} [F(\omega_1, \omega_2, \omega_3; \alpha, \beta, \gamma) + F(\omega_2, \omega_3, \omega_1; \beta, \gamma, \alpha) + F(\omega_3, \omega_1, \omega_2; \gamma, \alpha, \beta)].
\]

With respect to the metric \( g^{ij} \), the constant \( \Phi \) surfaces are orthogonal to the curves obtained by the intersection of the constant \( I_1 \) and \( I_2 \) surfaces.

In the classical Darboux-Halphen case \( \alpha = \beta = \gamma = 0 \), we have

\[
g^{ij} = \begin{pmatrix}
1 & -1 & -1 \\
-1 & 1 & -1 \\
-1 & -1 & 1
\end{pmatrix}
\]

and \( \Phi = \omega_1\omega_2\omega_3 \).

For the fifth order reduction \( \alpha = \beta = 0 \) discussed in section 2,

\[
g^{ij} = \begin{pmatrix}
(1 + \gamma^2)^2 & \gamma^4 + 4\gamma^2 - 1 & \gamma^4 - 1 \\
\gamma^4 + 4\gamma^2 - 1 & (1 + \gamma^2)^2 & \gamma^4 - 1 \\
\gamma^4 - 1 & \gamma^4 - 1 & (1 - \gamma^2)^2
\end{pmatrix}
\]

and the corresponding potential function is given by

\[
\Phi = \frac{\gamma^2\omega_3^2[3(1 - \gamma^2)(\omega_1 + \omega_2) + (1 + 3\gamma^2)\omega_3] + 3(1 - \gamma^2)^2\omega_1\omega_2\omega_3}{3(1 - \gamma^2)^3}.
\]
References


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