JOURNAL OF MATHEMATICAL PHYSICS VOLUME 43, NUMBER 4 APRIL 2002

Shear-free relativistic fluids and the absence of movable branch points

R. G. Halburd[a)]
Department of Mathematical Sciences, Loughborough University, Loughborough, Leics, LE11 3TU, United Kingdom

(Received 19 November 2001; accepted for publication 10 January 2002)

The problem of determining the metric for a nonstatic shear-free spherically symmetric fluid (either charged or neutral) reduces to the problem of determining a one-parameter family of solutions to a second-order ordinary differential equation (ODE) containing two arbitrary functions f and g. Choices for f and g are determined such that this ODE admits a one-parameter family of solutions that have poles as their only movable singularities. This property is strictly weaker than the Painlevé property and it is used to identify classes of solvable models. It is shown that this procedure systematically generates many exact solutions including the Vaidya metric, which does not arise from the standard Painlevé analysis of the second-order ODE. Interior solutions are matched to exterior Reissner–Nordstrøm metrics. Some solutions given in terms of second Painlevé transcendents are described. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1455688]

I. INTRODUCTION

Several authors have shown that the problem of finding a nonstatic solution of the Einstein–Maxwell equations for a shear-free spherically symmetric charged fluid is equivalent to the problem of finding a t-dependent solution to the equation

$$\frac{\partial^2 y(x,t)}{\partial x^2} = f(x)y^2(x,t) + g(x)y^3(x,t), \tag{1}$$

where f and g are arbitrary functions of x only[1,2] (see also Ref. 3 and the references therein). Given a solution y of Eq. (1), define $r=\sqrt{x}$, $Y(r,t)=1/y(x,t)$, and

$$T(r,t) = h(t)\frac{\partial}{\partial t}\ln y(r^2,t), \tag{2}$$

where h is an arbitrary nonvanishing function of t. In terms of these variables, the metric for the fluid is given by

$$ds^2 = T^2(r,t)dt^2 - Y^2(r,t)\{dr^2 + r^2 d\Omega^2\}, \tag{3}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the standard metric on the two-sphere. The density ρ and pressure p are given by

$$8\pi\rho = 3h^{-2} - 12xy_x^2 + 12yy_x + 8xfy^3 + 6xgy^4, \tag{4}$$

$$8\pi p = 4y(y-2xy_x)\frac{y_{xt}}{y_t} + 12xy_x^2 - 8yy_x + 2xgy^4 - 2h^{-3}h_t\frac{y}{y_t} - 3h^{-2}. \tag{5}$$

a)Electronic mail: r.g.halburd@lboro.ac.uk

The only nonvanishing components of the electromagnetic field are

$$F_{01} = -F_{10} = -h(t)E(r)\frac{\partial y}{\partial t},$$

where $E^2(r) = 2xg(x)$.

Although y is a function of two variables x and t, Eq. (1) is essentially an ODE for y as a function of x. As an ODE, the general solution of Eq. (1) contains two arbitrary constants. The general solution of Eq. (1) viewed as a partial differential equation (PDE) is obtained by replacing these arbitrary constants with arbitrary functions of t. Since T is a metric coefficient, it cannot be identically zero, so from Eq. (2) we see that y must have a nonconstant t-dependence. This leads us to the problem of finding families of solutions to Eq. (1) viewed as an ODE which depend on (at least) one parameter.

The connection between integrable systems (equations that are solvable, either explicitly or via a related linear problem) was first used by Kowlevskaya in her work on spinning tops.[4,5] She considered the equations of motion for a spinning top which depend on six parameters (the center of mass and the moments of inertia). Kowalevskaya noticed that in the known cases for which the equations could be integrated, the general solution was a meromorphic function of time when extended to the complex plane. She used local series analysis to determine all choices of the parameters for which the general solution was a meromorphic function of time and found a new set of values for the parameters for which she was then able to solve the equations in terms of ratios of hyper-elliptic functions.

The requirement that all solutions are meromorphic throughout the complex plane may be replaced with the requirement that all solutions be meromorphic on the covering space of **C** with a discrete set of points removed. In this way, branching of solutions is allowed at fixed singularities (singularities of the solutions that cannot occur at arbitrary locations in the complex plane but only at locations at which the equation itself is in some sense singular). An ODE is said to possess the Painlevé property if all movable singularities of all solutions are poles. This property is closely connected with the integrability (solvability) of the ODE. All ODEs that are known to possess the Painlevé property are integrable, either explicitly in terms of classically known functions, or via an associated linear problem. In particular, Painlevé, Gambier, and Fuchs classified all equations of the form

$$\frac{d^2y}{dx^2} = F\left(x;y,\frac{dy}{dx}\right), \tag{6}$$

where F is rational in y and dy/dx and analytic in x, that have the Painlevé property. They showed that each such equation could be transformed via a change of independent variable and an x-dependent Möbius transformation of y to 1 of 50 canonical equations. With the exception of six equations (the Painlevé equations P_I–P_{VI}) each of these canonical equations were solved in terms of classically known functions (see, e.g., Refs. 6 and 7). The first two Painlevé equations are

$$\frac{d^2\eta}{d\zeta^2} = 6\eta^2 + \zeta, \tag{7}$$

$$\frac{d^2\eta}{d\zeta^2} = 2\eta^3 + \zeta\eta + \alpha, \tag{8}$$

where α is an arbitrary complex constant. It was later shown that each Painlevé equation is the compatibility condition for a (linear) spectral problem. The Painlevé equations are considered to be integrable because of the underlying structure that emerges from these isomonodromy problems.[7,8]

It is important to note that the transformation of one the equations of the form (6) that possesses the Painlevé property to one of the canonical forms is itself determined by the solutions of a system of differential equations. A weaker definition of the Painlevé property is that all solutions are single-valued about all movable singularities. However, for equations of the form (6) this definition yields the same class of equations.

Shah and Vaidya,[1] Wyman,[9,10] Chatterjee,[11] Maharaj, Leach, and Maartens[12] and Srivastava[13] have studied Eq. (1) to determine choices of f and g for which the general solution has no movable critical points. In particular, Wyman[9] determined all choices of f in the uncharged ($g=0$) case. In Ref. 14 the author found all choices of f and g such that Eq. (1) possesses the Painlevé property. In particular, we have the following.

Proposition 1.1: Equation (1) possesses the Painlevé property (as an ODE in x) if and only if either

(1)

$$f(x)=6w^5(z), \quad g(x)=0, \tag{9}$$

where $w\neq 0$ and v are any solutions of

$$\frac{d^2v}{dz^2}=6v^2+az+b/2, \tag{10}$$

$$\frac{d^2w}{dz^2}=12vw, \tag{11}$$

where a, b are constants and z is given by

$$x=\int w^{-2}(z)dz, \tag{12}$$

or

(2)

$$f(x)=6v(z)w^5(z), \quad g(x)=2w^6(z), \tag{13}$$

where $w\neq 0$ and v are any solutions of

$$\frac{d^2v}{dz^2}=2v^3+(az+b)v+c/2, \tag{14}$$

$$\frac{d^2w}{dz^2}=(6v^2+az+b)w, \tag{15}$$

where a, b, c are constants and z is given by Eq. (12).

Furthermore, in both the above cases, the general solution of Eq. (1) is given by

$$y(x,t)=\frac{u(z,t)-v(z)}{w(z)}, \tag{16}$$

where u (in which t is treated as a parameter) is the general solution of the same second-order equation as v [i.e., in case 1, $u(z,t_0)$, where t_0 is a constant, solves Eq. (10) and in case 2 it solves Eq. (14)].

Note that Eq. (11) [resp. (15)] is the linearization of Eq. (10) [resp. (14)]. So if $v(z)=V(z;\epsilon)$ is a one-parameter family of solutions to Eq. (10) [resp. (14)], then $w(z):=V_\epsilon(z;\epsilon)$ is a solution to Eq. (11) [resp. (15)]. A second independent solution to Eq. (11) [resp. (15)] then follows by reduction of order. Equation (12) shows that $x=\hat{w}(z)/w(z)$, where $\hat{w}$ is a second solution of Eq. (11) [resp. (15)] satisfying the Wronskian condition $W(w,\hat{w})=w\hat{w}_z-\hat{w}w_z=1$.

If $a=0$, then the general solution to Eq. (10) [resp. (14)] can be given explicitly in terms of elliptic functions. In particular, the case in which $a=0$ and the fixed solution v of Eq. (14) is a constant corresponds to the large class of solutions found by Sussman.[15] In fact, most of the solutions that have appeared in the literature to date are special cases of Sussman's solutions. The case in which v is not constant is solved explicitly in Ref. 14. If $a\neq 0$, then Eq. (10) [resp. (14)] can be mapped to Eq. (7) [resp. (8)]. A class of solutions to Eq. (1) corresponding to the Airy function solutions to Eq. (8) is also described in Ref. 14.

Recall that we wish to find one-parameter families of solutions to Eq. (1). When f and g are chosen so that Eq. (1) possesses the Painlevé property then the equation is integrable and we can find a two-parameter family of solutions. In the present article a property, weaker than the Painlevé property but still complex-analytic in nature, is considered. Namely, we wish to find all one-parameter families of solutions $\mathcal{F}$ to Eq. (1) such that all movable singularities of all solutions in $\mathcal{F}$ are poles. In Sec. II we will find all solutions to Eq. (1) that are simultaneously solutions of a Riccati equation. This class of solutions contains the well-known solutions due to Shah and Vaidya,[16] which does not arise in a regular Painlevé analysis of Eq. (1). A class of solutions that generalizes that due to Shah and Vaidya which is given in terms of solutions to linear equations is also derived.

Sections III and IV address the question of whether the solutions found in Sec. II exhaust the set of all one-parameter families of solutions $\mathcal{F}$ described above. In Sec. V, boundary conditions are determined such that the Riccati solutions can be matched to the Reissner–Nordstrøm external solution. In Sec. VI we find solutions to Eq. (1) corresponding to $a\neq 0$ but $v\equiv 0$ in Eq. (14). In this case the general solution to Eq. (15) is given in terms of Airy functions. From this solution, families of solutions are obtained using the Bäcklund transformation of the second Painlevé equation.

II. RICCATI SOLUTIONS

One way of finding a one-parameter family of solutions to Eq. (1) such that the only movable singularities are poles is to find a family of solutions that are also solutions of a first-order equation of Painlevé type. In this section, solutions to Eq. (1) are found that are also solutions to a first-order differential equation of the form

$$\frac{dy}{dx}=R(x,y), \tag{17}$$

where R is rational in y and locally analytic in x. Fuchs[17] showed that the only equation of the form (17) with the Painlevé property is the Riccati equation,

$$\frac{dy}{dx}=\alpha(x)y^2+\beta(x)y+\gamma(x), \tag{18}$$

where α, β, and γ are (locally) analytic functions of x. The general solution of Eq. (18) is given by

$$y(x)=-\frac{1}{\alpha(x)}\frac{d}{dx}\ln\Phi(x),$$

where Φ is the general solution of the linear equation

$$\frac{d^2\Phi}{dx^2}-\left(\beta+\frac{\alpha_x}{\alpha}\right)\frac{d\Phi}{dx}+\alpha\gamma\Phi=0. \tag{19}$$

Differentiating Eq. (18) with respect to x and again using Eq. (18) to eliminate dy/dx in the resulting expression gives

$$\frac{d^2y}{dx^2}=2\alpha^2y^3+(\alpha_x+3\alpha\beta)y^2+(\beta_x+\beta^2+2\alpha\gamma)y+(\gamma_x+\beta\gamma). \tag{20}$$

It follows that every solution of Eq. (18) is a solution of Eq. (1) if and only if the equations

$$\gamma_x+\beta\gamma=0, \tag{21}$$

$$\beta_x+\beta^2+2\alpha\gamma=0, \tag{22}$$

$$\alpha_x+3\alpha\beta=f, \tag{23}$$

$$2\alpha^2=g, \tag{24}$$

are satisfied.

Solving Eqs. (21)–(24) gives three classes of Riccati equations.

Case 1: $\beta\equiv 0$, $\gamma\equiv 0$. The Riccati equation (18) becomes

$$\frac{dy}{dx}=\alpha y^2,$$

which has the general solution

$$y(x,t)=\frac{1}{H(x)+G(t)}, \tag{25}$$

where $H'(x)=-\alpha(x)$ and G is an arbitrary function of t.

Case 2: $\beta\not\equiv 0$, $\gamma\equiv 0$. The Riccati equation (18) becomes

$$\frac{dy}{dx}=\alpha y^2+\frac{1}{x+C}y,$$

where C is an arbitrary constant, which has the general solution

$$y(x,t)=\begin{cases}\dfrac{x}{H(x)+G(t)}, & H'(x)=-x\alpha(x), \quad \text{if } C=0,\\[2ex] \dfrac{1+kx/4}{H(x)+G(t)}, & H'(x)=-(1+kx/4)\alpha(x), \quad \text{if } C=4/k\neq 0,\end{cases} \tag{26}$$

where G is an arbitrary function of t.

Case 3: $\gamma\not\equiv 0$. The Riccati equation (18) becomes

$$\frac{dy}{dx}+\frac{1}{2}(\gamma^{-1})_{xx}y^2+\frac{\gamma'}{\gamma}y-\gamma=0. \tag{27}$$

In Sec. V, these Riccati solutions will be matched to an external Reissner–Nordstrøm metric.

Note that Eq. (25) corresponds to setting $k=0$ in Eq. (26b). Under the transformation

$$\tilde{r}=\frac{r}{1+kr^2/4}$$

(recall $x=r^2$) the solutions corresponding to Eq. (26b) give rise to the metric

$$ds^2=[F(\tilde{r})+G(t)]^{-2}dt^2-[F(\tilde{r})+G(t)]^2\left[\frac{d\tilde{r}^2}{1-k\tilde{r}^2}+\tilde{r}^2d\Omega^2\right], \tag{28}$$

where $F(\tilde{r})=H(r^2)$ and we have set $h(t)=1/\dot{G}(t)$. The metric (28) was obtained by Shah and Vaidya.[16] This metric does not arise from the standard Painlevé analysis of Eq. (1).

Solutions found in this section will be referred to as Riccati solutions.

III. LOCAL SERIES ANALYSIS

In this section we will analyze Eq. (1) as an ODE in the complex domain. In particular, we will determine necessary conditions that Eq. (1) possesses a one-parameter family of Laurent series solutions. We begin by considering the case in which $g=0$. Under the transformation (16) in which w is given by Eq. (9), v is given by Eq. (11), and z is given implicitly by Eq. (12), Eq. (1) becomes

$$\frac{d^2u}{dz^2}=6u^2+A(z), \tag{29}$$

where

$$A(z)=\frac{d^2v}{dz^2}-6v^2.$$

We will only consider a one-parameter family of solutions $\mathcal{G}$ such that there exists an open connected bounded set $\Omega\in\mathbf{C}$ such that at each point $z_0\in\Omega$ there is a function $u\in\mathcal{G}$ with a pole at $z=z_0$. We will now find a necessary condition on the function A such that Eq. (29) admits a formal Laurent series solution with a pole at a point $z_0\in\Omega$, where A is analytic. Substituting the Laurent series

$$u(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^{n-p}, \tag{30}$$

where p is a positive integer and $a_0\neq 0$, into Eq. (29) gives, to leading order,

$$p(p+1)a_0(z-z_0)^{-(p+2)}+\ldots=6a_0^2(z-z_0)^{-2p}+\ldots .$$

Equating the powers and coefficients of these leading-order terms gives

$$p=2,\quad a_0=1. \tag{31}$$

Using Eqs. (30) and (31) in Eq. (29) and equating coefficients of like powers of $z-z_0$ gives

$$(n+1)(n-6)a_n=P_n(a_0,a_1,\ldots,a_{n-1}), \tag{32}$$

where

$$P_n(a_0,a_1,\ldots,a_{n-1})=6\sum_{m=1}^{n-1}a_ma_{n-m}+\alpha_n(z_0),$$

and

$$\alpha_n(z_0)=\begin{cases}0, & n<4,\\ \dfrac{A^{(n-4)}(z_0)}{(n-4)!}, & n\geqslant 4,\end{cases}$$

is polynomial in its arguments and we have expanded A as a power series about $z=z_0$. The recurrence relation (32) shows that a_n is uniquely determined in terms of $\{a_0,\ldots,a_{n-1}\}$, except in the case when $n=6$. In this case, the left side of Eq. (32) vanishes while the right side is a known function of $(a_0,\ldots,a_5)$. If the right side does not vanish, then there is no solution to Eq. (29) with a pole of any order at $z=z_0$. If the right side of Eq. (32) does vanish, then a formal Laurent series solution exists in which z_0 and a_6 are arbitrary constants. A direct calculation shows that the right side of Eq. (32) vanishes if and only if

$$A''(z_0)=0. \tag{33}$$

Now since Eq. (33) must be satisfied for all z_0 in the open set Ω, this implies that $A(z)=az+b/2$, for some constants a and b. So, we reproduce precisely those solutions given in case 1 of Proposition 1.1.

Now we consider the local series analysis of Eq. (1) when g is not identically zero. In particular, g does not vanish identically on Ω. From Sec. 2 we see that the requirement that there is a one-parameter family of solutions such that all movable singularities are poles yields more solutions than requiring that Eq. (1) possesses the Painlevé property.

Let v and w be defined by Eq. (13) where z is given by Eq. (12). [Note that we are not assuming that Eqs. (14) and (15) hold.] The transformation (16) gives

$$\frac{d^2u}{dz^2}=2u^3+B(z)u+C(z), \tag{34}$$

where

$$B(z)=\frac{w_{zz}}{w}-6v^2, \quad C(z)=v_{zz}-vB(z)-2v^3.$$

We now look for a local Laurent series solution to Eq. (34) with a pole at $z=z_0\in\Omega$. Leading order analysis shows that any such solution u must have a simple pole at $z=z_0$ with residue ±1. Hence we substitute the series

$$u(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^{n-1}, \quad a_0=\varepsilon=\pm1,$$

into Eq. (34) and equate coefficients of like powers of $z-z_0$ to obtain the recurrence relation

$$(n+1)(n-4)a_n=Q_n(a_0,\ldots,a_{n-1}), \tag{35}$$

where

$$Q_n(a_0,\ldots,a_{n-1})=2\left[\sum_{m=0}^{n}\sum_{m'=0}^{n-m} a_m a_{m'} a_{n-m-m'}-3a_n\right]+\beta_n(z_0)+\gamma_n(z_0),$$

and

$$\beta_n(z_0)=\begin{cases}0, & n<2,\\ \displaystyle\sum_{m=0}^{n-2}\frac{B^{(n-m-2)}(z_0)}{(n-m-2)!}a_m, & n\geqslant 2,\end{cases} \qquad \gamma(z_0)=\begin{cases}0, & n<3,\\ \displaystyle\frac{C^{(n-3)}(z_0)}{(n-3)!}, & n\geqslant 3.\end{cases}$$

Note that Q is polynomial in its arguments. The left side of Eq. (35) shows that a necessary and sufficient condition for the existence of a formal Laurent series solution with a pole at $z=z_0$ is $Q_4(a_0,a_1,a_2,a_3)=0$, which is equivalent to $B''(z_0)=-2\varepsilon C'(z_0)$, where $\varepsilon=\pm1=a_0$.

Now the general solution of Eq. (34) will have movable singularities with leading order behaviors that include both $+1$ and -1 residue poles (although, in general, these solutions will not be meromorphic and the Laurent series will have to be augmented by logarithm terms). So if we demand that all movable singularities of all solutions are poles [i.e., if we demand that Eq. (34) possess the Painlevé property], then $B''(z_0)=2'C(z_0)$ and $B''(z_0)=-2C'(z_0)$, for all $z_0\in\Omega$, leading to $B(z)=az+b$, and $C(z)=c/2$, where a, b, and c are arbitrary constants. So u satisfies Eq. (14).

Rather than demand that all movable singularities of all solutions of Eq. (34) are poles, we restrict our consideration to a subset of solutions $\mathcal{G}$ such that given any $z_0\in\Omega$, there is a solution in $\mathcal{G}$ with a pole at $z=z_0$. The above analysis shows that either we are left with Eq. (14) or we must consider the class of solutions where all movable singularities are poles and all but a finite number of these poles in ω have the same residue $\varepsilon=\pm1$. A necessary condition in this case is the differential equation $B''(z)=-2\varepsilon C'(z)$. In terms of $q(z):=B(z)/2$, we now restrict ourselves to the study of the subset of solutions to the equation

$$\frac{d^2u}{dz^2}=2u^3+2qu+(\kappa-\varepsilon q_z),\quad \varepsilon=\pm1, \tag{36}$$

where κ is an arbitrary constant, that admit only poles with residue ε in Ω.

IV. THE UNIQUENESS OF THE RICCATI SOLUTIONS

The only Riccati equation for which all solutions u are also solutions of Eq. (36) is

$$\frac{du}{dz}+\varepsilon(u^2+q)=0,\quad \kappa=0. \tag{37}$$

The general solution of equation (37) is given by

$$u=\varepsilon\frac{d}{dz}\ln\Phi, \tag{38}$$

where Φ is the general solution of the linear equation

$$\frac{d^2\Phi}{dz^2}+q\,\Phi=0. \tag{39}$$

We will show that these Riccati type solutions are identical to those found in Sec. II. All movable singularities of any solution to Eq. (37) are simple poles with residue ε. So the general solution to Eq. (37) is a one-parameter family of solutions to Eq. (36) of the kind considered at the end of the previous section. The perturbation argument described below suggests that this is the only such one-parameter family. We will then provide a proof based on Wiman–Valiron theory for the case in which q is a polynomial. Wiman–Valiron theory is particularly useful for finding entire solutions of analytic differential equations.[18]

We will now show how the Riccati equations derived in Sec. II are related to the solutions of Eq. (36) described at the end of Sec. III. It may be verified that the identity

$$w(u_z+\varepsilon[u^2+q])=y_x+\varepsilon w^3y^2+w(w_z+2\varepsilon vw)y+w(v_z+\varepsilon[v^2+q]) \tag{40}$$

follows from Eqs. (12) and (16). Furthermore, given Eqs. (12) and (13), where v and w satisfy

$$v_{zz}=2v^3+2qv-\varepsilon q_z \tag{41}$$

and

$$w_{zz}=(6v^2+2q)w, \tag{42}$$

respectively, it can be shown that Eqs. (21)–(24) are equivalent to

$$\alpha=-\varepsilon w^3, \tag{43}$$

$$\beta=-w(w_z+2\varepsilon v w), \tag{44}$$

$$\gamma=-w(v_z+\varepsilon[v^2+q]). \tag{45}$$

Equations (40) and (43)-(45) show that the Riccati solutions found in Sec. II are the same as those constructed using (16) where u is the general solution of Eq. (37) and v, w satisfy Eqs. (41) and (42), respectively. It is interesting to note that the solution by Shah and Vaidya discussed in Sec. II corresponds to the case in which v also satisfies the Ricatti equation (37). The $\gamma\neq 0$ case (case 3 in Sec. II) corresponds to a non-Ricatti solution v of Eq. (41) (although u still satisfies a Riccati equation).

Next we address the question of whether the class $\mathcal{G}$ consists only of Riccati solutions. Consider Eq. (36) with $q(z)=q_0+hQ(z)$, where q_0 is a complex constant and h is a small complex parameter. To leading order in h, Eq. (36) is

$$\frac{d^2u}{dz^2}=2u^3+2q_0u+\kappa.$$

If u is not constant, then

$$\left(\frac{du}{dz}\right)^2=u^4+2q_0u^2+2\kappa u+C, \tag{46}$$

where C is an integration constant. The nonconstant solutions of Eq. (46) are elliptic functions with simple poles of residue ± 1. The only solutions with poles of residue $\varepsilon=\pm 1$ but no poles of residue $-\varepsilon=\mp 1$ correspond to the case in which $\kappa=0$ and $C=q_0^2$ in which case Eq. (46) factors into two Riccati equations and u satisfies

$$\frac{du}{dz}+\varepsilon(u^2+q_0)=0.$$

The arguments given above assume that the one-parameter family of solutions $\mathcal{G}$ have poles in an open set Ω. In the following we show rigorously that we have found all one-parameter families of solutions that have only poles as their movable singularities under the assumption that q is a polynomial.

Consider the system of first-order equations

$$\frac{du}{dz}=\tilde{u}-\varepsilon u^2-\varepsilon q, \tag{47}$$

$$\frac{d\tilde{u}}{dz}=\kappa+2\varepsilon u\tilde{u}. \tag{48}$$

Differentiating Eq. (47) with respect to z and using Eq. (48) to eliminate $d\tilde{u}/dz$ gives Eq. (36). We wish to show that if there is a one-parameter family of solutions u having only movable poles with residue ε, then $\tilde{u}$ is identically zero. Note that if $\tilde{u}$ is identically zero, then Eq. (47) becomes Eq. (37) and Eq. (48) implies that $\kappa=0$. If $\tilde{u}$ does not vanish identically, then we can solve Eq. (48) for u and substitute it into Eq. (47) to give

$$2\tilde{u}\frac{d^2\tilde{u}}{dz^2}=\left(\frac{d\tilde{u}}{dz}\right)^2+4\tilde{u}^2(\varepsilon\tilde{u}-q)-\kappa^2. \tag{49}$$

We will prove the following.

Proposition 4.1: If q is a polynomial, then either any entire solution of Eq. (49) is a constant or

$$q=q_0+q_1z+q_2z^2,$$

where

$$q_1^2-4q_0q_2=\kappa^2 \tag{50}$$

and

$$\tilde{u}=\varepsilon q. \tag{51}$$

Note that if $\tilde{u}$ is one of the solutions given in Proposition 4.1 but $\tilde{u}\not\equiv 0$, then since $\tilde{u}$ contains no free parameters (i.e., no parameters other than those in the equation itself) u is given by Eq. (48) and so does not represent a one-parameter family of solutions to Eq. (36).

Proof: We will begin by showing that any polynomial solution of Eq. (49) is either a constant or the solution (51). We will then use the central index from Wiman–Valiron theory to show that there are no transcendental (i.e., nonpolynomial) solutions.

Let q and $\tilde{u}$ be polynomials of degree M and N, respectively. Furthermore, we assume that $\tilde{u}$ is not constant (i.e., $N\geqslant 1$ and $\tilde{a}_N\neq 0$). Then q and $\tilde{u}$ have expansions of the form

$$q(z)=\sum_{m=0}^{M} q_m z^m, \quad \tilde{u}(z)=\sum_{n=0}^{N} \tilde{a}_n z^n. \tag{52}$$

Substituting the expansions (52) into Eq. (49) and balancing the dominant terms for large z gives $M=N$. Equation (49) then becomes

$$\sum_{i,j=0}^{N} i(2i-j-2)\tilde{a}_i\tilde{a}_j z^{i+j-2}+\kappa^2=\sum_{i,j,k=0}^{N} 4\tilde{a}_i\tilde{a}_j(\varepsilon\tilde{a}_k-q_k)z^{i+j+k}. \tag{53}$$

Now the polynomial on the left side of Eq. (53) is of degree at most $2N-2$ while the degree of the polynomial on the right side is of degree at most $3N$. Since $\tilde{a}_N\neq 0$, then the coefficient of z^{3N} in Eq. (49) gives $\tilde{a}_N=\varepsilon q_N$. Arguing by induction, equating the coefficients of $z^{3M-1},z^{3M-2},\ldots,z^{2N+1}$ to zero gives $\tilde{a}_{N-n}=\varepsilon q_{N-n}$, $n=1,\ldots,N$. Hence $\tilde{u}=\varepsilon q$ and the right side of Eq. (53) vanishes identically. On equating all coefficients of powers of z to zero on the left side of Eq. (49) we find that $N=2$ and q_0, q_1, and q_2 satisfy Eq. (50).

Now we will use Wiman–Valiron theory to show that all entire solutions to Eq. (49) are polynomials. Since $\tilde{u}$ is entire it has an expansion of the form

$$\tilde{u}(z)=\sum_{n=0}^{\infty} \tilde{a}_n z^n.$$

The *central index* $\nu(r,\tilde{u})$ is the greatest non-negative integer m such that

$$|\tilde{a}_m|r^m=\max_{n\geqslant 0}|\tilde{a}_n|r^n.$$

If $\tilde{u}$ is nonpolynomial, then $\nu(r,\tilde{u})$ is increasing, piecewise constant, right-continuous, and tends to $+\infty$ as $r\rightarrow+\infty$.

In terms of the central index we have the following lemma (see, e.g., Ref. 19).

Lemma 4.2: Let $\tilde{u}$ *be a nonpolynomial entire function, and* $\nu=\nu(r,\tilde{u})$ *be its central index. Let* $0<\delta<1/4$ *and* z *be such that* $|z|=r$ *and*

$$|\tilde{u}(z)|>\nu(r,\tilde{u})^{-(1/4)+\delta}\max_{|z|=r}|\tilde{u}(z)| \tag{54}$$

holds. Then there exists a set $F\subset\mathbf{R}$ *of finite logarithmic measure, i.e.,* $\int_F dt/t<+\infty$ *such that*

$$\tilde{u}^{(m)}(z)=\left(\frac{\nu(r,\tilde{u})}{z}\right)^m(1+o(1))\tilde{u}(z) \tag{55}$$

holds for all $m\geqslant 0$ *and* $r\notin F$.

Lemma 4.2 says that for all positive r outside of the set F (which has finite logarithmic measure), the estimate (55) holds near the maximum of $|\tilde{u}|$ on the circle $|z|=r$ [where "near the maximum" means the set of z satisfying Eq. (54)].

Assume that there is a nonpolynomial solution $\tilde{u}$ of Eq. (49). Applying the estimate (55) to Eq. (49) gives

$$\left(\frac{\nu(r,\tilde{u})}{z}\right)^2\tilde{u}^2\sim 4\varepsilon\tilde{u}^3. \tag{56}$$

Since $\nu(r,\tilde{u})$ grows much slower than $\tilde{u}$,[20] it follows that Eq. (56) cannot be balanced. Thus the only entire solutions to Eq. (49) are polynomials. ■

V. BOUNDARY CONDITIONS FOR THE RICCATI SOLUTIONS

In this section we will match the Riccati solutions introduced in Sec. II to an external Reissner–Nordstrøm metric

$$ds^2=\hat{\Gamma}d\hat{t}^2-\hat{\Gamma}^{-1}d\hat{r}^2-\hat{r}^2d\Omega^2, \tag{57}$$

where $d\Omega^2$ is the standard metric on the two-sphere and

$$\hat{\Gamma}=1-\frac{2m}{\hat{r}}+\frac{4\pi e^2}{\hat{r}^2},$$

and m and e are constants. Let Σ_0 be the interface $r=r_0$ between the two solutions. The two metrics (3) and (57) can be matched across Σ_0 provided

$$p(r_0,t)=0, \tag{58}$$

$$g(r_0^2)=2\pi\left(\frac{e}{r_0^3}\right)^2, \tag{59}$$

$$2m=\left[\frac{4\pi e^2}{r}y+\frac{r^3}{h^2y^3}+2\frac{r^2}{y^2}y_r-\frac{r^3}{y^3}y_r^2\right]_{r=r_0}, \tag{60}$$

for all t.[2]

Equation (59) is equivalent to

$$\alpha^2(r_0^2)=\pi\left(\frac{e}{r_0^3}\right)^2. \tag{61}$$

Using Eq. (18) to eliminate y_x and $y_{xt}=(2\alpha y+\beta)y_t$ from Eq. (5) gives

$$8\pi p=4\{(\beta^2+2\alpha\gamma)x-\beta)y^2+2\gamma(2\beta x-1)y+3x\gamma^2\}-3h^{-2}-2h^{-3}h_t\frac{y}{y_t}. \tag{62}$$

Using Eq. (18) to eliminate $y_r=2ry_x$ from Eq. (60) and using Eq. (62) in Eq. (58), we see that Eqs. (58) and (60) are equivalent to

$$h^{-2}(t)=4\left[r^2\gamma+\gamma(2r^2\beta-1)y+\{(\beta^2+2\alpha\gamma)r^2-\beta\}y^2+\left(\frac{m}{2r^3}+2r^2\alpha\beta-\alpha\right)y^3\right]_{r=r_0}. \tag{63}$$

A. Dust solutions

Setting p identically zero in Eq. (62) and solving for h^{-2} gives

$$h^{-2}=4[x\gamma+\gamma(2x\beta-1)y+\{(\beta^2+2\alpha\gamma)x-\beta\}y^2+\delta y^3], \tag{64}$$

where δ is a function of x only. Recall that h is a function of t only. Differentiating Eq. (64) with respect to x and using Eqs. (21) and (22) gives

$$3\alpha\delta y^2+\{\delta_x+3\beta\delta+2\alpha[(\beta^2+2\alpha\gamma)x-\beta]\}y+\gamma\{2x\alpha_x+3[\alpha+\delta]\}=0, \tag{65}$$

for all t. Since y must have nonconstant t-dependence, the coefficients of different powers of y in Eq. (65) must vanish identically. If $\alpha=0$, then $g=0$ and there are no Riccati solutions. Therefore the coefficient of y^2 in Eq. (65) shows that δ is identically zero. The constant term in Eq. (65) shows that either $\gamma\equiv 0$ or $\alpha(x)=\kappa x^{-3/2}$, where κ is a constant.

If $\gamma=0$, then the coefficient of y in Eq. (65) shows that either $\beta=0$ or $\beta=1/x$. These solutions correspond to the solutions (25) and (26), respectively. Finally, if $\gamma\not\equiv 0$ and $\alpha\not\equiv 0$, then recall from Sec. II (case 3) that for any Riccati solution we must have $\alpha=-(\gamma^{-1})_{xx}$ and $\beta=-\gamma_x/\gamma$. It follows that the coefficient of y and the constant term in Eq. (65) cannot both vanish identically.

VI. BÄCKLUND TRANSFORMATIONS AND SPECIAL SOLUTIONS

In this section we will construct what is perhaps the simplest solution of Eq. (1) involving a genuine transcendent of the second Painlevé equation. It is simple in the sense that we have an explicit formula for the dependence of x on z. We will then use the well-known Bäcklund transformation of the second Painlevé equation to construct a countable family of equations of the form (1) together with their general solutions in terms of second Painlevé transcendents.

If $a\neq 0$, then, after rescaling z and v, Eqs. (14) and (15) become

$$\frac{d^2v}{dz^2}=2v^3+zv+\alpha, \tag{66}$$

$$\frac{d^2w}{dz^2}=(6v^2+z)w, \tag{67}$$

where α is an arbitrary constant. Equation (66) is the standard form of the second Painlevé equation. We will denote the general solution of Eq. (66) by $v(z)=P_{II}(z;\alpha;c_1,c_2)$, where c_1 and c_2 are independent parameters (e.g., $c_1=v(0)$ and $c_2=v'(0)$).

Recall that, apart from the solution due to Shah and Vaidya [(Eq. (28)], many of the solutions that appear in the literature are special cases of the solutions of Sussman,[15] which correspond to the special case of Proposition 1.1 in which $a=0$ and v is a constant. Note that if $a\neq 0$, then Eq. (66) [which is a rescaled version of Eq. (14)] admits a constant solution if and only if $\alpha=0$. In this case the constant solution is $v\equiv 0$, which is equivalent to the case $f\equiv 0$.

If $v\equiv 0$, then Eq. (67) has the general solution

$$w(z)=\mu \mathrm{Ai}(z)+\nu \mathrm{Bi}(z),$$

where Ai and Bi are the Airy functions and μ and ν are arbitrary constants which are not both zero. From Eq. (12) we have

$$x=\pi \frac{\rho \mathrm{Ai}(z)+\sigma \mathrm{Bi}(z)}{\mu \mathrm{Ai}(z)+\nu \mathrm{Bi}(z)},$$

where ρ and σ are arbitrary constants satisfying $\mu\sigma-\nu\rho=1$. [Note the identity $\mathrm{Ai}(z)\mathrm{Bi}'(z)-\mathrm{Bi}(z)\mathrm{Ai}'(z)=\pi^{-1}$.] In particular, choosing $\mu=\sigma=1$ and $\nu=\rho=0$, we see that the general solution of $y_{xx}=\mathrm{Bi}^6(z)y^3$ is

$$y(x)=\frac{P_{II}(z;0;c_1,c_2)}{\mathrm{Bi}(z)},$$

where c_1 and c_2 are arbitrary constants (or functions of t, viewing the equation as a PDE) and z is given by

$$\frac{\mathrm{Bi}(z)}{\mathrm{Ai}(z)}=\frac{x}{\pi}.$$

Now we will see how to generate other solutions from the $v=0$ case just described. Let v be a solution of Eq. (66) where $\alpha\neq-\frac{1}{2}$. Then it is well known[21] that

$$\tilde{v}:=-v-\frac{1+2\alpha}{2v_z+2v^2+z} \tag{68}$$

satisfies Eq. (66) with α replaced by $\alpha+1$. Equation (68) is the standard Bäcklund transformation of Eq. (66). Let $V(z;\epsilon)$ be a one-parameter (i.e., ϵ) family of solutions to Eq. (66). Since Eq. (67) is the linearization of Eq. (66), it follows that $W(z;\epsilon):=V_\epsilon(z;\epsilon)$ is a solution to Eq. (67). Substituting $v=V(z;\epsilon)$ into Eq. (68) and differentiating with respect to ϵ shows that

$$\tilde{w}:=-w+2(1+2\alpha)\frac{w_z+2vw}{(2v_z+2v^2+z)^2} \tag{69}$$

satisfies Eq. (67) with v replaced by $\tilde{v}$, whenever w satisfies Eq. (67).

Applying the Bäcklund transformations (68) and (69) to $v(z)=0$, $w(z)=\mathrm{Ai}(z)$, described above, yields $\tilde{v}(z)=-z^{-1}$ and $\tilde{w}(z)=2z^{-2}\mathrm{Ai}'(z)-\mathrm{Ai}(z)$. It follows that

$$y(x)=\frac{z^2P_{II}(z;1;c_1,c_2)+z}{2\mathrm{Ai}'(z)-z^2\mathrm{Ai}(z)}$$

is the general solution of Eq. (1) with $f(x)=6\tilde{v}(z)\tilde{w}^5(z)$ and $g(x)=2\tilde{w}^6(z)$, where

$$\frac{2\mathrm{Bi}'(z)-z^2\mathrm{Bi}(z)}{2\mathrm{Ai}'(z)-z^2\mathrm{Ai}(z)}=\frac{x}{\pi}.$$

Repeated application of the Bäcklund transformations (68) and (69) will generate a countable family of equations of the form (1) and solutions in which v is a rational function of z and w is a rational function of z, the Airy functions Ai and Bi and their first derivatives.

VII. DISCUSSION

The search for metrics modeling nonstatic shear-free spherically symmetric charged fluids naturally leads to the problem of finding one-parameter families of solutions to Eq. (1). The

Painlevé property is a very powerful detector of the integrability of ODEs. Indeed, most of the solutions to Eq. (1) that have appeared in the literature to date arise naturally from the Painlevé analysis of Eq. (1) (see Ref. 14). However, since Eq. (1) is second-order while we only require a one-parameter family of solutions, it is not necessary for us to describe the general solution. From this point of view, requiring the Painlevé property is too restrictive.

In this article we have considered the problem of determining one-parameter families of solutions to Eq. (1) whose only movable singularities are poles. Besides the solutions covered by Proposition 1.1 [which corresponds to the cases in which Eq. (1) possesses the full Painlevé property] we found one-parameter families of solutions that satisfy Riccati equations. In particular, this class of solutions contains those of Shah and Vaidya, which do not arise in the standard Painlevé analysis of Eq. (1). The procedure for matching the Riccati solutions to an external Reissner–Nordstrøm metric was also described.

Finally, a special subclass of solutions that arise in Proposition 1.1 were described. In general, when $a \neq 0$, the transformation between x and z involves derivatives of a second Painlevé transcendent. In the class of solutions described in Sec. VI, v, w, and x are given explicitly in terms of Airy functions and there first derivatives—only u is a genuine Painlevé transcendent. Presumably this is the simplest class of solutions characterized by Proposition 2 that contains a genuine Painlevé transcendent.

ACKNOWLEDGMENT

This work was supported by the Royal Society (Research Grant No. 22204).

[1] Y. P. Shah and P. C. Vaidya, Tensor N. S. **19**, 191 (1968).
[2] M. C. Faulkes, Can. J. Phys. **47**, 1989 (1969).
[3] A. Krasiński. *Inhomogeneous Cosmological Models* (Cambridge University Press, Cambridge, 1997).
[4] S. Kowalevski, Acta Math. **12**, 177 (1889).
[5] S. Kowalevski, Acta Math. **14**, 81 (1889).
[6] E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956).
[7] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering, London Mathematical Society Lecture Note Series* (Cambridge University Press, Cambridge, 1991). Vol. 149
[8] H. Flaschka and A. C. Newell, Commun. Math. Phys. **76**, 65 (1980).
[9] M. Wyman, Canad. Math. Bull. **19**, 343 (1976).
[10] M. Wyman, Aust. J. Phys. **31**, 111 (1978).
[11] S. Chatterjee, Gen. Relativ. Gravit. **16**, 381 (1984).
[12] S. D. Maharaj, P. G. L. Leach, and R. Maartens, Gen. Relativ. Gravit. **28**, 35 (1996).
[13] D. C. Srivastava, Fortschr. Phys. **40**, 31 (1992).
[14] R. Halburd, Class. Quantum Grav. **18**, 11 (2001).
[15] R. Sussman, J. Math. Phys. **28**, 1118 (1987).
[16] Y. P. Shah and P. C. Vaidya, Ann. Inst. Henri Poincare, Sect. A **6**, 219 (1967).
[17] L. Fuchs, Sitz. Akad. Wiss. Berlin **32**, 669 (1884).
[18] Y. M. Chiang and R. G. Halburd, ''On the meromorphic solutions to an equation of Hayman,'' Dept. Math. Sciences, preprint no. 02/08, Loughborough University (2002).
[19] G. Jank and L. Volmann, *Einführung in die Theorie der ganzen und meromophen Funktionen mit Anwendungen auf Differentialgleichungen* (Birkháuser, Basel, 1985).
[20] E. Hille, *Ordinary Differential Equations in the Complex Domain* (Wiley, New York, 1976).
[21] B. Gambier, Acta Math. **33**, 1 (1909).