

First integrals of a generalized Darboux–Halphen system

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(Received 14 November 2002; accepted 20 December 2002)

A third-order system of nonlinear, ordinary differential equations depending on three arbitrary parameters is analyzed. The system arises in the study of SU(2)-invariant hypercomplex manifolds and is a dimensional reduction of the self-dual Yang–Mills equation. The general solution, first integrals, and the Nambu–Poisson structure of the system are explicitly derived. It is shown that the first integrals are multi-valued on the phase space even though the general solution of the system is single-valued for special choices of parameters. © 2003 American Institute of Physics. [DOI: 10.1063/1.1556194]

I. INTRODUCTION

The study of integrable or solvable nonlinear systems dates back to the fundamental works of Euler, Liouville, Riemann, Poincaré, and many others. Surprisingly (perhaps), there is still no single adequate definition of “integrability.” Certainly, nonlinear systems which can be explicitly solved by quadratures in the real domain should be considered as integrable, as should the Hamiltonian systems with action-angle variables (integrability in the Liouville sense). In contrast, the notion of integrability in the complex plane is still in its early stages of development. For example, if the general solution of a nonlinear ordinary differential equation is everywhere single-valued in its domain of existence, then we consider the equation to be integrable in the complex plane. Fundamental contributions of Kovalevskaya,¹⁷ Painlevé,²⁴ and more recent work^{27,28} have led to some progress toward the understanding of complex integrability (or nonintegrability). But the complex behavior of large classes of physically important nonlinear equations still remains to be completely understood. Some of these equations can be “solved” in terms of linear equations but are not single-valued in the complex plane.

In this article we consider the system of nonlinear ordinary differential equations

$$\dot{M} = (\text{adj } M)^T + M^T M - (\text{Tr } M)M, \quad (1)$$

for a 3×3 matrix valued function $M(t)$ where $\text{adj } M$ is the adjoint matrix of M satisfying $(\text{adj } M)M = (\det M)I$, M^T is the transpose of M and the dot denotes differentiation with respect to t . The system (1) was obtained as a dimensional reduction of the self-dual Yang–Mills (SDYM) equations corresponding to an infinite-dimensional gauge group of diffeomorphisms $\text{Diff}(S^3)$ of a three-sphere.⁷ These equations were also derived in Ref. 16 where they were shown to represent an SU(2) invariant hypercomplex four-manifold. Since the Weyl curvature of a hypercomplex four-manifold is self-dual, Eq. (1) describes a class of self-dual Weyl Bianchi IX space-times with Euclidean signature.⁶

In the next section we will review the fact that Eq. (1) reduces to the system

$$\dot{\omega}_1 = \omega_2 \omega_3 - \omega_1(\omega_2 + \omega_3) + \tau^2,$$

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$$\begin{aligned}
\dot{\omega}_2 &= \omega_3 \omega_1 - \omega_2(\omega_3 + \omega_1) + \tau^2, \\
\dot{\omega}_3 &= \omega_1 \omega_2 - \omega_3(\omega_1 + \omega_2) + \tau^2, \\
\tau^2 &= \alpha_1^2(\omega_1 - \omega_2)(\omega_3 - \omega_1) + \alpha_2^2(\omega_2 - \omega_3)(\omega_1 - \omega_2) + \alpha_3^2(\omega_3 - \omega_1)(\omega_2 - \omega_3),
\end{aligned}
\tag{2}$$

for the functions $\omega_i(t)$, $i=1,2,3$, and where α_1 , α_2 , and α_3 are constants. We will refer to system (2) as the generalized Darboux–Halphen (DH) system, which will be the subject of our discussion for the remainder of this article. Equation (2) with $\tau \equiv 0$, becomes the classical DH system which first appeared in Darboux’s work on triply orthogonal surfaces⁸ and was later solved by Halphen.¹⁵ In subsequent studies, the classical DH system has arisen as the vacuum Einstein equations for hyperkähler Bianchi-IX metrics^{11,5} and in the similarity reductions of associativity equations on a three-dimensional Frobenius manifold.⁹ Halphen showed that the general system (2) can be solved in terms of hypergeometric functions.¹⁴ Special solutions have also been given in terms of theta functions and automorphic forms.^{7,23,1} Special cases of Eq. (2) arise in the study of solvable models of spherically symmetric shear-free fluids in general relativity¹³ as well.

As mentioned earlier, it was shown in Ref. 7 that Eq. (1) arises as a reduction of the SDYM equations. From the Lax pair for SDYM, it is possible to derive a linear problem (see, e.g., Ref. 2) which can be employed to solve the initial value problem for Eq. (1). This linear problem is related to the monodromy preserving deformations corresponding to the Riccati reduction of the Painlevé VI equation. Analysis of Eq. (1) using the associated linear problem was given in Refs. 6 and 16.

In Sec. II we outline the reduction of Eq. (1) to the generalized DH system (2) and derive its general solution. In Sec. III we discuss the first integrals and a set of “action-angle” variables for the DH system in terms of hypergeometric functions. We then analyze the behavior of the first integrals as functions of the dependent variables. In particular we find that the first integrals are transcendental and nonmeromorphic even though in certain cases, the general solution is single-valued in the complex t -plane. Indeed, the nonexistence of meromorphic first integrals for the classical DH equations was proved in Ref. 19. Finally, in Sec. IV we consider the dynamics of the DH system as a Nambu–Poisson flow in a three-dimensional manifold and investigate the algebraic properties of the underlying Nambu–Poisson structures.

II. SOLUTION OF THE DH SYSTEM

In this section we outline the procedure of constructing the general solution of Eq. (1) following the method discussed in Ref. 3. The matrix M in Eq. (1) is a complex-valued function of the (complex) independent variable t . In this article, we study the case where the symmetric part M_s of M has *distinct* eigenvalues. The degenerate cases corresponding to eigenvalues with higher multiplicities have been studied in Ref. 3.

The matrix M is first decomposed into symmetric and skew-symmetric parts and then the symmetric part M_s is diagonalized by a complex orthogonal matrix. (This is possible because of our assumption that the eigenvalues of M_s are distinct.) Thus we have

$$M = M_s + M_a = P(d + a)P^{-1},$$

$P \in \text{SO}(3, \mathbb{C})$, $d := \text{diag}(\omega_1, \omega_2, \omega_3)$ where the ω_i , $i=1,2,3$, are distinct, and the elements of the skew-symmetric matrix a are denoted as $a_{12} := \tau_3$, $a_{23} := \tau_1$, and $a_{31} := \tau_2$. Using the above factorization of M , Eq. (1) can be transformed into Eq. (2) with $\tau^2 := \tau_1^2 + \tau_2^2 + \tau_3^2$, together with the linear equation: $\dot{P} = -Pa$ for the matrix P . The equations for the skew-symmetric part,

$$\dot{\tau}_1 = -\tau_1(\omega_2 + \omega_3), \quad \dot{\tau}_2 = -\tau_2(\omega_3 + \omega_1), \quad \dot{\tau}_3 = -\tau_3(\omega_1 + \omega_2),$$

can be integrated to obtain

$$\tau_1^2 = \alpha_1^2(\omega_1 - \omega_2)(\omega_3 - \omega_1), \quad \tau_2^2 = \alpha_2^2(\omega_2 - \omega_3)(\omega_1 - \omega_2), \quad \tau_3^2 = \alpha_3^2(\omega_3 - \omega_1)(\omega_2 - \omega_3),$$

where α_1 , α_2 , and α_3 are arbitrary constants. This defines τ^2 in terms of the ω_i in Eq. (2). Once a solution of the DH system (2) has been found, the matrix M can be reconstructed after solving the linear equation ($\dot{P} = -Pa$) for P .

In order to solve Eq. (2), we set

$$\omega_1 = -\frac{1}{2} \frac{d}{dt} \ln \frac{s}{s-1}, \quad \omega_2 = -\frac{1}{2} \frac{d}{dt} \ln \frac{s}{s-1}, \quad \omega_3 = -\frac{1}{2} \frac{d}{dt} \ln \frac{s}{s}, \quad (3)$$

where the function $s(t)$ is given by the cross-ratio

$$s = \frac{\omega_1 - \omega_3}{\omega_2 - \omega_3}, \quad (4)$$

$\omega_i \neq \omega_j$ when $i \neq j$. Then it follows from Eq. (2) that $s(t)$ satisfies the Schwarzian equation

$$\frac{d}{dt} \left(\frac{\dot{s}}{s} \right) - \frac{1}{2} \left(\frac{\dot{s}}{s} \right)^2 + \frac{s^2}{2} V(s) = 0, \quad (5)$$

with

$$V(s) = \frac{1 - \alpha_2^2}{s^2} + \frac{1 - \alpha_3^2}{(s-1)^2} + \frac{\alpha_2^2 + \alpha_3^2 - \alpha_1^2 - 1}{s(s-1)}.$$

The solution $s(t)$ of Eq. (5) is obtained implicitly by setting

$$t(s) = \frac{u_1(s)}{u_2(s)}, \quad (6)$$

where $u_1(s)$ and $u_2(s)$ are two independent solutions of the Fuchsian differential equation

$$\frac{d^2 u}{ds^2} + \frac{1}{4} V(s) u = 0 \quad (7)$$

with three regular singular points at 0, 1, and ∞ . The transformation

$$u(s) = s^{c/2} (1-s)^{(a+b-c+1)/2} \chi(s) \quad (8)$$

maps Eq. (7) to the Gauss hypergeometric equation

$$s(1-s) \frac{d^2 \chi}{ds^2} + [c - (a+b+1)s] \frac{d\chi}{ds} - ab\chi = 0, \quad (9)$$

where $a = (1 + \alpha_1 - \alpha_2 - \alpha_3)/2$, $b = (1 - \alpha_1 - \alpha_2 - \alpha_3)/2$, and $c = 1 - \alpha_2$. Thus we have the following.

Proposition 1: The general solution of the DH system (2) is given by Eq. (3) where the function $s(t)$ is defined by the inverse of the ratio $t(s) = \chi_1(s)/\chi_2(s)$ of two linearly independent solutions of the hypergeometric equation (9).

Equation (6) describes the conformal mapping of the upper (or lower) half s -plane onto the interior of a triangular region T bounded by three circular arcs in the complex t -plane (see, e.g., Ref. 22). When the parameters α_1 , α_2 , α_3 are non-negative real numbers satisfying $\alpha_1 + \alpha_2 + \alpha_3 < 1$, the circular arcs of T form angles $\pi\alpha_1$, $\pi\alpha_2$, $\pi\alpha_3$ at the vertices which are the images of the singular points $s=0$, $s=1$, and $s=\infty$ of Eq. (7). The inverse map $s(t)$, which solves Eq. (5),

is analytic in the interior of T and can be analytically extended by inversions across its boundary. If the parameters assume the values $\alpha_1 = 1/p_1$, $\alpha_2 = 1/p_2$, $\alpha_3 = 1/p_3$, where p_1, p_2, p_3 are positive integers or ∞ , then $s(t)$ can be extended to a single-valued, meromorphic function in a region D which is the uniform covering of an infinite number of nonoverlapping circular triangles obtained by inversions across the boundaries of T and its images. The boundary ∂D of D contains a dense set of essential singularities and forms a movable natural boundary. However, for general values of the parameters $\alpha_1, \alpha_2, \alpha_3$ the function $s(t)$ is densely branched about the movable singularities at the vertices of T . The solutions $\omega_i(t)$ to the DH system given by Eq. (3) inherit the same singularity structure as $s(t)$ and are also branched in the complex t -plane for generic choices of $\alpha_1, \alpha_2, \alpha_3$.

III. FIRST INTEGRALS AND ACTION-ANGLE VARIABLES

In the previous section we outlined a mechanism for expressing the general solution of the DH system via the solutions of a second-order, linear equation (7). This linearization scheme given by Eqs. (3)–(7) is implicit since the Schwarzian function $s(t)$ is the inverse of the ratio of the solutions of the linear equation. The first integrals of the DH system are determined by the arbitrary constants parametrizing the space of general solutions for the linear equation (7). However, these integrals do not have a simple dependence on the DH variables ω_i due to the implicit nature of the linearization process. In this section, we will discuss the properties of the first integrals as functions of the DH variables.

Let u_1 and u_2 be any two linearly independent solutions of Eq. (7) with Wronskian $W(u_1, u_2) = u_1 u_2' - u_2 u_1' = 1$, where prime denotes differentiation with respect to s . The general solution of the Schwarzian equation (5) is given implicitly by [cf. Eq. (6)]

$$t(s) = \frac{J_2 u_1(s) - J_1 u_2(s)}{I_2 u_1(s) - I_1 u_2(s)}, \quad (10)$$

where I_α and J_α , $\alpha = 1, 2$, are constants satisfying $I_1 J_2 - I_2 J_1 \neq 0$. Only three of the four constants can be chosen independently because it is evident from Eq. (10) that only their ratios are related to $s(t)$ and its first two t -derivatives. Therefore, without loss of generality we take them to satisfy $I_1 J_2 - I_2 J_1 = 1$. Differentiating Eq. (10) twice with respect to s we obtain two linear equations for I_1 and I_2 :

$$I_2 u_1 - I_1 u_2 = s^{1/2}, \quad I_2 u_1' - I_1 u_2' = \frac{1}{2} s^{-3/2} \ddot{s},$$

whose solutions are

$$I_\alpha = \frac{d\phi_\alpha}{dt}, \quad \phi_\alpha = s^{-1/2} u_\alpha(s), \quad \alpha = 1, 2. \quad (11)$$

The remaining two constants are then obtained from Eqs. (10) and (11) and the normalization $I_1 J_2 - I_2 J_1 = 1$. They are given by

$$J_\alpha = t I_\alpha - \phi_\alpha, \quad \alpha = 1, 2.$$

Viewed as functions of t, s, \dot{s} and \ddot{s} , the I_α and J_α are first integrals for the Schwarzian equation. This fact can be verified directly by differentiating the expressions for I_α and J_α with respect to t , and using Eq. (5). Moreover, by solving the functions s, \dot{s} and \ddot{s} from Eqs. (3) and (4), the I_α and J_α can be expressed in terms of the DH variables ω_i and t . Hence, they are also integrals of motion for the DH system. The explicit expressions for ϕ_α and I_α in terms of the DH variables are as follows:

$$\phi_\alpha = \sqrt{2r(\omega_i)} u_\alpha(s(\omega_i)), \quad I_\alpha = \sqrt{\frac{2}{r(\omega_i)}} u'_\alpha(s(\omega_i)) - (\omega_1 - \omega_2 - \omega_3) \sqrt{\frac{r(\omega_i)}{2}} u_\alpha(s(\omega_i)), \quad (12)$$

where $r(\omega_i) = \sqrt{(\omega_2 - \omega_3)/(\omega_1 - \omega_2)(\omega_1 - \omega_3)}$ and $s(\omega_i)$ is given by Eq. (4). Equation (12) [equivalently, Eq. (11)] represents a nonalgebraic, transcendental transformation defined via the solution u_α of the Fuchsian equation (7), between the ω_i (or s, \dot{s}, \ddot{s}) and the variables $\{\phi_\alpha, I_\alpha\}$. In terms of these new variables, the nonlinear DH system (2) can be reformulated as a linear Hamiltonian system [cf. Eq. (11)]

$$\dot{\phi}_\alpha = \frac{\partial H}{\partial I_\alpha} = I_\alpha, \quad \dot{I}_\alpha = -\frac{\partial H}{\partial \phi_\alpha} = 0, \quad H = \frac{I_1^2 + I_2^2}{2}, \quad \alpha = 1, 2, \quad (13)$$

together with the algebraic constraint

$$\phi_1 I_2 - \phi_2 I_1 = W(u_1, u_2) = 1 \quad (14)$$

among the coordinates ϕ_α and the canonically conjugate “momenta” I_α . Since the latter system (13) can be integrated by quadratures, the canonical coordinates $\{I_\alpha, \phi_\alpha\}$ can be regarded as playing the role of the action-angle variables for the DH system. The dynamics in the four-dimensional phase space is restricted to the constraint subspace defined by Eq. (14). This represents an indefinite quadric which is a connected but noncompact, three-dimensional submanifold of the phase space. The flow is determined by a one-dimensional linear subspace: $c_1 \phi_1 - c_2 \phi_2 = 1$, obtained as the intersection of the constraint submanifold with the level sets of the first integrals $I_1 = c_1$, $I_2 = c_2$, where c_1, c_2 are constants determined by the initial conditions in (2).

The above results lead to the next proposition.

Proposition 2: Let ω_i , $i=1,2,3$, be a solution of the generalized DH system (2) and let u_1, u_2 be any two solutions of Eq. (7) with unit Wronskian. Then I_α and $J_\alpha = tI_\alpha - \phi_\alpha$, $\alpha=1,2$, are first integrals of the DH system, where ϕ_α and I_α are given by Eq. (12). Furthermore, the DH system are equivalent to a constrained Hamiltonian system given by Eqs. (13) and (14) with $\{\phi_\alpha, I_\alpha\}$ as the canonical variables. The associated Hamilton’s equations (13) are linear and can be solved by quadratures.

The first integrals I_α , $\alpha=1,2$, are constant functions of t in the domain of analyticity of the $\omega_i(t)$, and their values are determined by the initial conditions. However, the I_α are not single-valued as functions of ω_i (or equivalently of the Schwarzian variables s, \dot{s}, \ddot{s}). The nonanalytic behavior is essentially due to the fact that in the complex s -plane, continuation along closed circuits around the branch points $s=0$, $s=1$, and $s=\infty$ transforms any two independent solutions of the Fuchsian equation (7) by the corresponding monodromy matrix. The branching properties of the I_α can be characterized explicitly by expressing them as functions of s, \dot{s} , and \ddot{s} and the fundamental matrix of solutions of the hypergeometric equation (9). If the u_α in Eq. (11) are replaced by the solutions of the hypergeometric equation (9) by using the transformation (8), then this yields

$$\begin{bmatrix} I_1 & I_2 \end{bmatrix} = \sigma \begin{bmatrix} \lambda & 1 \end{bmatrix} \begin{bmatrix} \chi_1(s) & \chi_2(s) \\ \chi'_1(s) & \chi'_2(s) \end{bmatrix}, \quad (15)$$

where

$$\sigma(s, \dot{s}) = s^{c/2} (1-s)^{(a+b-c+1)/2} \dot{s}^{1/2} \quad \text{and} \quad \lambda(s, \dot{s}, \ddot{s}) = \frac{a+b+1-cs}{2s(1-s)} - \frac{\ddot{s}}{2\dot{s}^2}.$$

It is clear from Eq. (15) that I_α are not branched as functions of \dot{s} and that they have square-root branch points as a function of \dot{s} at $\dot{s}=0$ and $\dot{s}=\infty$ (in fact, I_α^2 are single-valued as functions of both \dot{s} and \ddot{s}). When \dot{s} and \ddot{s} are held fixed, the only places where the I_α can be branched are at $s=0$, $s=1$, and $s=\infty$. Let γ_0 and γ_1 be two closed curves with a common base point in the finite

complex s -plane enclosing the points $s=0$ and $s=1$, respectively, and traversed once in the positive direction. Analytic continuation along γ_0 and γ_1 transforms the fundamental matrix of solutions of Eq. (9) according to

$$\gamma_\mu: \begin{pmatrix} \chi_1(s) & \chi_2(s) \\ \chi'_1(s) & \chi'_2(s) \end{pmatrix} \mapsto \begin{pmatrix} \chi_1(s) & \chi_2(s) \\ \chi'_1(s) & \chi'_2(s) \end{pmatrix} M_\mu, \quad \mu=0,1.$$

For generic values of a, b, c and for the choice of basis solutions, $\chi_1 = F(a, b, c; s)$, $\chi_2 = F(a, b, a+b-c+1; s)$ of the hypergeometric equation, the monodromy matrices M_μ are given by²⁵

$$M_0 = \begin{pmatrix} 1 & e^{-2\pi ib} - e^{-2\pi ic} \\ 0 & e^{-2\pi ic} \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} e^{-2\pi i(a+b-c)} & 0 \\ 1 - e^{-2\pi i(a-c)} & 1 \end{pmatrix}.$$

The only other source of branching in Eq. (15) arises from the analytic continuation of σ along γ_μ which yields

$$\gamma_0: \sigma \mapsto e^{i\pi c} \sigma, \quad \gamma_1: \sigma \mapsto e^{i\pi(a+b-c)} \sigma.$$

The branching at $s=\infty$ can be determined from the branching at $s=0$ and $s=1$. A closed circuit (defined in a similar way as for γ_0 and γ_1 above) around the point $s=\infty \in \mathcal{CP}^1$ is homotopic to $\gamma_0^{-1} \circ \gamma_1^{-1}$. The corresponding monodromy matrix is given by $M_\infty = (M_1 M_0)^{-1}$. The monodromy matrix M for any closed circuit γ can be expressed in terms of the fundamental monodromy matrices M_0 and M_1 associated with γ_0 and γ_1 , respectively. Finally, taking all the sources of branching into account in (15), we obtain the following result.

Proposition 3: The first integrals of the DH system given by (15) are multi-valued functions of s with branch points at $s=0$, $s=1$, and $s=\infty$. The multi-valued behavior can be expressed in terms of the fundamental determinations:

$$\gamma_0: [I_1 \ I_2] \mapsto [I_1 \ I_2] M_0 e^{i\pi c}, \quad \gamma_1: [I_1 \ I_2] \mapsto [I_1 \ I_2] M_1 e^{i\pi(a+b-c)},$$

where M_0 and M_1 are the monodromy associated with a fundamental matrix solution of the hypergeometric equation (9) around the closed curves γ_0 and γ_1 , respectively.

Remark 1: The multi-valued behavior of the first integrals I_α may also be described in terms of the DH variables ω_i . It follows from Eq. (4) that the branch points $s=0$, $s=1$, and $s=\infty$ correspond to the complex diagonal hyperplanes $\omega_i = \omega_j$, $i \neq j$. The monodromy group generated by M_0 and M_1 determines a (complex) representation of the fundamental group $\pi_1(\mathcal{M}_3)$ on the complement $\mathcal{M}_3 = \mathcal{C}^3 \setminus \{\omega_i = \omega_j, i \neq j\}$ of the arrangement of the diagonal hyperplanes in \mathcal{C}^3 . Arnold,⁴ in his study of pure braid groups, discussed the cohomology of the complement \mathcal{M}_n of the diagonal hyperplane arrangement in \mathcal{C}^n . In particular, he proved that the integral cohomology ring $H^*(\mathcal{M}_n, \mathbb{Z})$ is isomorphic to the algebra generated by the closed differential one-forms: $\omega_{jk} = (1/2\pi i) d \ln(\omega_j - \omega_k)$, $j \neq k$ which satisfy $\omega_{kl} \wedge \omega_{lm} + \omega_{lm} \wedge \omega_{mk} + \omega_{mk} \wedge \omega_{kl} \equiv 0$. Note that for $n=3$, there is only one independent relation: $\omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{31} + \omega_{31} \wedge \omega_{12} \equiv 0$, which is indeed satisfied by the parametrization of the ω_i in Eq. (3).

Remark 2: The first integrals in Eq. (15) for the classical DH system ($\alpha_1 = \alpha_2 = \alpha_3 = 0$) are expressed in terms of the special hypergeometric Eq. (9) with $a = b = \frac{1}{2}$, $c = 1$. In this case, the monodromy matrices with respect to the basis $\chi_1 = F(\frac{1}{2}, \frac{1}{2}, 1; s)$ and $\chi_2 = iF(\frac{1}{2}, \frac{1}{2}, 1; 1-s)$, are given by

$$M_0 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

The corresponding monodromy group is the subgroup $\Gamma(2)$ (principal congruent subgroup of level 2) of the modular group $\text{SL}(2, \mathbb{Z})$, defined as $\Gamma(2) := \{g \in \text{SL}(2, \mathbb{Z}) \mid g \equiv \text{Id} \pmod{2}\}$. When $a = b$

$=\frac{1}{12}$, $c=\frac{1}{2}$ in Eq. (9), the associated monodromy group is the full modular group $SL(2, \mathbb{Z})$ which is isomorphic to the pure braid group B_3 of three colored strands. Similar representations of pure braid group B_n are given by the monodromy group associated with particular Picard–Fuchs equations with n regular singular points which arise in the theory of Frobenius manifolds.⁹ This is related to Arnold's work⁴ (see Remark 1) on the presentation of pure braid group as the fundamental group of the complement \mathcal{M}_n under the action of the Coxeter group A_n .

It is important to note that the first integrals I_α , J_α remain multi-valued independent of the choice of parameters, even in the particular cases where the general solution is single-valued in its domain of existence. For instance, the classical DH system [Eq. (2) with $\tau=0$] can be solved in terms of the elliptic modular function and the general solution is analytic inside a circle D in the complex t -plane (see, e.g., Refs. 7 and 1). It was shown in Ref. 19 that the classical DH system does not possess a meromorphic first integral. This is consistent with our results that first integrals do indeed exist, but they are nonalgebraic and multi-valued functions of the ω_i . Thus there is no natural connection between the analyticity properties of the solution and the first integrals for the DH system. To establish such connection for nonlinear differential equations is a very delicate issue. For specific cases of Hamiltonian dynamical systems, it was proved under certain assumptions that if the system admits solutions that are branched, then the system can not possess analytic first integrals independent of the Hamiltonian.¹⁸ Furthermore, Ziglin's work^{27,28} reveals that branching of solutions and the absence of single-valued first integrals in certain Hamiltonian systems are *both* consequences of the same complex singularity structure of the solutions (although one does not necessarily imply the other). However, it should be noted that these results do not rule out the possibility that multi-valued first integrals may exist. Indeed this is the case for the DH system which serves as an important example of equations that are integrable in the sense that the general solutions can be expressed in terms of linear equations, yet the constants of integrations are not single-valued functions of the dependent variables.

IV. POISSON STRUCTURES

The DH equations (2) may be viewed as a complex dynamical system on a manifold \mathcal{M} of (complex) dimension 3 where the DH variables ω^i , $i=1,2,3$, are local holomorphic coordinates on \mathcal{M} . (Note: In this section the standard notation for coordinate functions ω^i is used instead of ω_i to denote the DH variables.) Solutions of Eq. (2) determine a flow given by the integral curves of a holomorphic vector field $X \in \mathcal{TM}$ expressed in local coordinates ω^i as $X = X^i \partial_i$, $X^i := \omega^j \omega^k - \omega^i(\omega^j + \omega^k) + \tau^2$, $i \neq j \neq k$, and cyclic. Here $\partial_i := \partial/\partial \omega^i$, and summation over repeated indices is implied. Denote by $\Lambda^p(\mathcal{M})$ and $\Lambda_q(\mathcal{M})$ the respective spaces of (holomorphic) p -forms and q -vectors (contravariant, skew-symmetric q -tensor fields) on \mathcal{M} . Let $\nu \in \Lambda^3(\mathcal{M})$ be a nondegenerate three-form given in terms of local coordinates by

$$\nu = \frac{1}{\Delta(\omega^1, \omega^2, \omega^3)} d\omega^1 \wedge d\omega^2 \wedge d\omega^3, \quad (16)$$

for some function $\Delta \in C^\infty(\mathcal{M})$, $\Delta \neq 0$, which is to be determined later. Using the three-form ν we define the dual map $\Phi: \Lambda_q(\mathcal{M}) \rightarrow \Lambda^{3-q}(\mathcal{M})$ and its inverse $\Phi^{-1}: \Lambda^p(\mathcal{M}) \rightarrow \Lambda_{3-p}(\mathcal{M})$ by the inner products

$$\Phi(A) := i_A \nu, \quad \Phi^{-1}(\beta) := i_{\tilde{\nu}} \beta,$$

where $A \in \Lambda_q(\mathcal{M})$, $\beta \in \Lambda^p(\mathcal{M})$, and $\tilde{\nu} := \Delta \partial_1 \wedge \partial_2 \wedge \partial_3 \in \Lambda_3(\mathcal{M})$ is the inverse of the three-form ν . In particular, note that for $\beta_1, \beta_2 \in \Lambda^1(\mathcal{M})$, the vector $v = \Phi^{-1}(\beta_1 \wedge \beta_2)$ satisfies $i_v \beta_1 = i_v \beta_2 = 0$.

Since the first integrals I_1 and I_2 of Eq. (2) are constant along the integral curves of X , it follows that $\dot{I}_\alpha = i_X(dI_\alpha) = 0$, $\alpha=1,2$. The one-forms dI_1 and dI_2 span a two-dimensional, integrable (in the Frobenius sense) co-distribution of $T^*\mathcal{M}$, dual to the vector field X . Hence the vector field can be expressed as $X = G \Phi^{-1}(dI_1 \wedge dI_2) = G i_{\tilde{\nu}}(dI_1 \wedge dI_2)$ for some function G

$\in C^\infty(\mathcal{M})$. Without any loss of generality, we can set $G=1$ and thus determine the function Δ in Eq. (16). A straightforward calculation using the explicit forms of the I_α in Eq. (12) yields

$$\Delta(\omega^1, \omega^2, \omega^3) = 4(\omega^2 - \omega^3)(\omega^3 - \omega^1)(\omega^1 - \omega^2). \quad (17)$$

Therefore we have the following characterization of the DH vector field X .

Proposition 4: The DH system (2) defines a flow in a three-dimensional, complex manifold \mathcal{M} equipped with a nondegenerate three-form ν given in terms of local coordinates by Eqs. (16) and (17). The flow is an integral submanifold of \mathcal{M} generated by the vector field $X \in T\mathcal{M}$ which is dual to the integrable codistribution spanned by the one-forms dI_1 and dI_2 . That is,

$$X = \Phi^{-1}(dI_1 \wedge dI_2) = \tilde{\nu}(\cdot, dI_1, dI_2). \quad (18)$$

Let H denote the union of the complex hyperplanes given by $\omega^i = \omega^j$, $i \neq j$. It is evident from Eqs. (17) and (12) that the three-form ν and the one-forms dI_1 , dI_2 are singular on H . Hence the manifold \mathcal{M} is prescribed by $\mathcal{M} = \mathbb{C}^3 \setminus H$ on which Eq. (18) is valid and defines the holomorphic vector field X . The flow defined by Eq. (18) on \mathcal{M} corresponds to the functions $\omega^i(t)$ which remain distinct for all t in the domain of analyticity of the DH solutions. It should be noted, however, that the DH flow itself [given by Eq. (2)] is *not* singular on H , but the corresponding vector field can no longer be defined via Eq. (18). In fact, the complex planes $\omega^i = \omega^j$, $i \neq j$, are invariant manifolds of the DH flow. The flow restricted to these planes corresponds to the special cases of Eq. (2) which are solved either by quadratures or in terms of Bessel's equation.³

It follows from Proposition 4 that the intersection of the two-dimensional level sets of the first integrals I_1 and I_2 defines (locally) a unique solution curve for Eq. (2) on \mathcal{M} . We will next show that \mathcal{M} is a Poisson manifold with a pair of Poisson structures defined in a natural way via the first integrals I_α . Furthermore, the DH vector field X is locally Hamiltonian with respect to both Poisson structures.

A Poisson structure on \mathcal{M} is specified by a bi-vector $B \in \Lambda_2(\mathcal{M})$ whose Nijenhuis–Schouten bracket with itself, defined by the three-vector $[B, B]_S = 0$. In terms of the coordinates ω^i ,

$$B = B^{ij} \partial_i \wedge \partial_j, \quad [B, B]_S^{ijk} := \partial_l (B^{ij}) B^{lk} + \partial_l (B^{jk}) B^{li} + \partial_l (B^{ki}) B^{lj} = 0.$$

The Poisson bracket of functions $f, g \in C^\infty(\mathcal{M})$ is the pairing defined by

$$\{f, g\} := B(df, dg),$$

which is skew-symmetric and satisfies the Leibniz rule $\{fg, h\} = f\{g, h\} + g\{f, h\}$ and the Jacobi identity $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = [B, B]_S(df, dg, dh) = 0$, for all $f, g, h \in C^\infty(\mathcal{M})$. A Hamiltonian vector field X_H with respect to a Poisson structure B is defined as $X_H := B(\cdot, dH)$ where $H(\omega^i)$ is the Hamiltonian function on \mathcal{M} . The Hamiltonian flow given by the integral curves of X_H corresponds to the solution of the system

$$\dot{\omega}^i = X_H(\omega^i) = \{\omega^i, H\}, \quad i = 1, 2, 3.$$

In three dimensions it is convenient to introduce the Poisson one-form $\theta \in \Lambda^1(\mathcal{M})$ (see, e.g., Ref. 12) by $\theta = \Phi(B) = i_B \nu$, which is the dual of the Poisson bi-vector. The Jacobi identity can be reformulated as the Frobenius integrability condition for the Poisson one-form θ . Specifically, we have the following.

Lemma 1. $B \in \Lambda_2(\mathcal{M})$ is a Poisson bi-vector if and only if the dual one-form $\Phi(B) := \theta \in \Lambda^1(\mathcal{M})$ satisfies $\theta \wedge d\theta = 0$.

Proof: If $B \in \Lambda_2(\mathcal{M})$ and $\nu \in \Lambda^3(\mathcal{M})$, then we have the contraction formula (see, e.g., Ref. 20)

$$\nu([B, B]_S) = 2i_B di_B \nu - i_B i_B d\nu.$$

Since ν is a top-degree holomorphic form, $d\nu=0$. Furthermore, we have $B=\Phi^{-1}(\theta)=\tilde{\nu}(\theta)$. Hence

$$\nu([B, B]_s)=2i_B d\theta=2\tilde{\nu}(\theta\wedge d\theta)$$

and the result follows.

In terms of the functions I_1 and I_2 , define the bi-vectors

$$B_\alpha:=\Phi^{-1}(dI_\alpha)=\tilde{\nu}(\cdot, \cdot, dI_\alpha), \quad \alpha=1, 2, \quad (19)$$

on \mathcal{M} . The corresponding dual one-forms $\Phi(B_\alpha)=dI_\alpha$ are exact. Therefore it follows immediately from Lemma 1 that the B_α are Poisson bi-vectors. The DH vector field X in Eq. (18) can be expressed as

$$X=-B_1(\cdot, dI_2)=B_2(\cdot, dI_1), \quad (20)$$

which is a Hamiltonian vector field with respect to both Poisson structures B_α . As a result, the DH equations (2) satisfy the Poisson bracket formulations

$$\dot{\omega}^i=X(\omega^i)=\{\omega^i, I_1\}_2=\{\omega^i, -I_2\}_1,$$

where $\{g, h\}_\alpha=B_\alpha(dg, dh)$, $\alpha=1, 2$. Moreover, B_1 and B_2 are compatible Poisson structures, namely, there exist functions λ_1, λ_2 such that the linear combination $B=\lambda_1 B_1+\lambda_2 B_2$ is also a Poisson bi-vector. It is easy to verify that the corresponding dual one-form $\theta=\Phi(B)=\lambda_1 dI_1+\lambda_2 dI_2$ satisfies Lemma 1 when λ_1, λ_2 are arbitrary differentiable functions of I_1 and I_2 . For a given Poisson structure B , it is also possible to find a corresponding Hamiltonian function $H(I_1, I_2)$ such that $X=B(\cdot, dH)=\mu^{-1}(dH\wedge\theta)$ gives the DH vector field as in Eq. (18). This is equivalent to the first-order, linear partial differential equation $\lambda_2(\partial H/\partial I_1)-\lambda_1(\partial H/\partial I_2)=1$, which can be solved by the method of characteristics. Thus X does not have a unique representation as a Hamiltonian vector field; the simplest forms are the ones given in Eq. (20). A Hamiltonian system with compatible Poisson structures is called a bi-Hamiltonian system. The DH vector field X in Eq. (20) is therefore a bi-Hamiltonian vector field with respect to the pair of compatible Hamiltonian structures $\{(B_1, -I_2), (B_2, I_1)\}$.

Remark 3: Since \mathcal{M} is odd-dimensional [$\dim(\mathcal{M})=3$], the B_α are degenerate (rank 2) bi-vector fields on \mathcal{M} . It follows from Eq. (19) that $B_1(\cdot, dI_1)=B_2(\cdot, dI_2)=0$. Therefore, I_1 and I_2 are the Casimir functions for the Poisson structures B_1 and B_2 respectively, and satisfy $\{g, I_\alpha\}_\alpha=0$, $\alpha=1, 2$, for any $g\in C^\infty(\mathcal{M})$. Furthermore, since $B_\alpha(dI_1, dI_2)=\{I_1, I_2\}_\alpha=0$, the first integrals I_1 and I_2 are in involution.

Remark 4: The flow associated with the vector field X preserves the three-form ν on \mathcal{M} . Indeed we have

$$\mathcal{L}_X \nu = d\Phi(X) = d[\Phi \circ \Phi^{-1}(dI_1 \wedge dI_2)] = d(dI_1 \wedge dI_2) = 0.$$

Note that on a three-dimensional *real* phase space, ν would be phase volume element that is invariant along the flow of X . Thus the condition $\mathcal{L}_X \nu=0$ on the DH phase space \mathcal{M} can be regarded as the holomorphic extension of the Liouville theorem on an odd-dimensional (complex) phase space.

We summarize the results discussed above.

Proposition 5: The DH system (2) represents a bi-Hamiltonian flow on \mathcal{M} corresponding to the Poisson structures $B_1=\Phi^{-1}(dI_1)$, $B_2=\Phi^{-1}(dI_2)$; and Hamiltonians $-I_2, I_1$ respectively. The DH vector field X is Hamiltonian with respect to both Poisson structures as given by Eq. (20). Furthermore, the first integrals I_1 and I_2 are in involution with respect to both Poisson structures.

The local expressions for the Poisson structures B_k are considerably simple in terms of the “action-angle” variables $\{I_\alpha, \phi_\alpha, \alpha=1, 2\}$ introduced via Eqs. (13) and (14) in Sec. II. Any three of the four variables can be taken to form a natural set of local coordinates on \mathcal{M} while the

remaining variable is solved algebraically using the constraint equation (14). For example, if we take $\{\phi_1, I_1, I_2\}$ as new local coordinates on \mathcal{M} and use the relations between the ω_i and $\{I_\alpha, \phi_\alpha\}$ from Eq. (12), then in the new coordinates the three-vector $\tilde{\nu}$ [inverse of ν in Eq. (16)] takes the form

$$\tilde{\nu} = I_1 \frac{\partial}{\partial \phi_1} \wedge \frac{\partial}{\partial I_1} \wedge \frac{\partial}{\partial I_2}.$$

Furthermore, from Eqs. (19) and (20) we have the following expressions for the Poisson bi-vectors and the DH vector field:

$$B_1 = -I_1 \frac{\partial}{\partial \phi_1} \wedge \frac{\partial}{\partial I_2}, \quad B_2 = I_1 \frac{\partial}{\partial \phi_1} \wedge \frac{\partial}{\partial I_1}, \quad X = I_1 \frac{\partial}{\partial \phi_1}.$$

Both Hamiltonian structures $(B_1, -I_2)$ or (B_2, I_1) yield the same dynamical equations: $\dot{\phi}_1 = I_1$, $\dot{I}_1 = \dot{I}_2 = 0$ which together with the algebraic constraint [Eq. (14)] are then equivalent to the DH dynamics given by Eqs. (13).

Note that the two sets of fundamental Poisson brackets,

$$\begin{aligned} \{\phi_1, I_1\}_1 &= 0, & \{I_2, \phi_1\}_1 &= I_1, & \{I_1, I_2\}_1 &= 0, \\ \{\phi_1, I_1\}_2 &= I_1, & \{I_2, \phi_1\}_2 &= 0, & \{I_1, I_2\}_2 &= 0, \end{aligned} \quad (21)$$

with respect to the respective Poisson structures B_1 and B_2 , are *linear* in the coordinate I_1 . Each set corresponds to a Lie–Poisson bracket on \mathcal{M} induced by certain three-dimensional Lie algebra \mathfrak{g} . The Lie–Poisson structure can be defined by identifying \mathcal{M} with the dual \mathfrak{g}^* of \mathfrak{g} , and the linear coordinate functions $\{y_k, k=1,2,3\}$ on \mathfrak{g}^* with the coordinates $\{\phi_1, I_1, I_2\}$. Then the fundamental Lie–Poisson brackets induced by \mathfrak{g} on \mathcal{M} are defined as $\{y_i, y_j\} := c_{ij}^k y_k$, where c_{ij}^k are the structure constants associated with the Lie algebra bracket $[e_i, e_j] = c_{ij}^k e_k$ with respect to a basis $\{e_i, i=1,2,3\}$ of \mathfrak{g} . Let \mathfrak{g}_1 and \mathfrak{g}_2 denote the Lie algebras corresponding to the first and second set of fundamental Poisson brackets, respectively. Then it is evident from Eq. (21) that both \mathfrak{g}_1 and \mathfrak{g}_2 are solvable Lie algebras with one-dimensional centers corresponding to the respective Casimir functions I_1 and I_2 . However, \mathfrak{g}_1 is nilpotent of degree 2, whereas \mathfrak{g}_2 contains a one-dimensional ideal generated by the element corresponding to I_1 whose normalizer is \mathfrak{g}_2 itself. In fact, it is easy to verify that choosing *any* three of the four “action-angle” variables as local coordinates on \mathcal{M} yields two distinct, canonical Lie–Poisson structures which correspond to solvable Lie algebras, moreover, one of the Lie algebras is nilpotent.

The volume form ν together with the Hamiltonians I_1 and $-I_2$ induce a Nambu–Poisson structure on the manifold \mathcal{M} . Nambu²¹ proposed a generalization of the Poisson bracket to study the dynamics of a “canonical triplet” of variables in a three-dimensional real phase space. In its simplest form, the canonical Nambu bracket of functions $g_i \in C^\infty(R^3)$, $i=1,2,3$, is given by the Jacobian

$$\{g_1, g_2, g_3\} = \frac{\partial(g_1, g_2, g_3)}{\partial(x^1, x^2, x^3)} = \tilde{\epsilon}(dg_1, dg_2, dg_3),$$

where x^i , $i=1,2,3$, are local coordinates and $\tilde{\epsilon}$ is the inverse of the standard volume form $\epsilon = dx^1 \wedge dx^2 \wedge dx^3$ on R^3 . The Nambu dynamics is prescribed as $\dot{x}^i = \{x^i, H_1, H_2\}$ in terms of two “Hamiltonian” functions H_1 and H_2 . Takhtajan²⁶ extended the Nambu formalism to higher dimensions and introduced the analog of the Jacobi identity for Nambu brackets—the so-called “fundamental identity.” An example of a Nambu–Poisson structure (of order n) on an n -dimensional manifold \mathcal{N} with a volume form $\nu_{\mathcal{N}} \in \Lambda^n(\mathcal{N})$ is the n -linear map $\{\cdot, \dots, \cdot\}: \underbrace{C^\infty(\mathcal{N}) \otimes \dots \otimes C^\infty(\mathcal{N})}_{n \text{ factors}} \rightarrow C^\infty(\mathcal{N})$ defined as

$$dg_1 \wedge dg_2 \wedge \cdots \wedge dg_n := \{g_1, g_2, \dots, g_n\} \nu_{\mathcal{N}},$$

for functions $g_j \in C^\infty(\mathcal{M})$, $j=1,2,\dots,n$. It can be shown that the bracket defined above is a Nambu–Poisson bracket,¹⁰ namely, it is skew-symmetric, a derivation, and satisfies the “fundamental identity”

$$\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} = \sum_{i=1}^n \{g_1, \dots, g_{i-1}, \{f_1, \dots, f_{n-1}, g_i\}, g_{i+1}, \dots, g_n\}.$$

The Nambu formulation of the DH system arises as a special case ($n=3$) of the above example with a Nambu–Poisson structure on \mathcal{M} prescribed by

$$\{g_1, g_2, g_3\} := \Phi^{-1}(dg_1 \wedge dg_2 \wedge dg_3) = \tilde{\nu}(dg_1, dg_2, dg_3). \quad (22)$$

Then from Eq. (18), the vector field X is the generator of a Nambu–Hamilton flow on the DH phase space \mathcal{M} given by the action $\dot{g} = X(g) = \{g, I_1, I_2\}$ on functions $g \in C^\infty(\mathcal{M})$. Therefore, we have the following.

Proposition 6: The DH system (2) is equivalent to the Nambu–Hamilton equation of motions $\dot{\omega}^i = \{\omega^i, I_1, I_2\}$, $i=1,2,3$, with respect to the Nambu–Poisson bracket defined by Eq. (22) together with the “Hamiltonians” I_1 and I_2 . The vector field X in Eq. (18) is a Nambu–Hamiltonian vector field.

Remark 5: The essential difference between the DH bracket and the canonical Nambu bracket is the “discriminant” function $\Delta(\omega_1, \omega_2, \omega_3)$. In the DH case, Δ is given by Eq. (17), whereas $\Delta \equiv 1$ for the canonical Nambu bracket.

Remark 6: It is possible to construct an infinite family of Poisson brackets characterized by functions $I \in C^\infty(\mathcal{M})$ as $\{f, g\}_I = \{f, g, I\}$, from the Nambu–Poisson bracket in Eq. (22). The brackets defined by the Poisson bi-vectors B_α in Eq. (19) are in fact induced in this way from Eq. (22) with $I = I_\alpha$, $\alpha=1,2$. In general, a Nambu bracket of order $n>2$ on a manifold of dimension $k \geq n$ can induce infinite families of lower order Nambu structures, including families of Poisson brackets.²⁶

Remark 7: The “fundamental identity” for the bracket defined by Eq. (22) is equivalent to the statement that any Nambu–Hamiltonian vector field is a derivation of the Nambu bracket. Indeed, consider the vector field $Y = \tilde{\nu}(\cdot, df_1, df_2)$ where $f_1, f_2 \in C^\infty(\mathcal{M})$ are the “Hamiltonians.” Clearly from Eq. (22), $Y(g) = \{g, f_1, f_2\}$ for all $g \in C^\infty(\mathcal{M})$. Y also preserves the volume form (and its inverse $\tilde{\nu}$), since $\mathcal{L}_Y \nu = di_Y \nu = d(df_1 \wedge df_2) = 0$. Now taking the Lie derivative of Eq. (22) with respect to Y and using the Leibniz rule to expand the right-hand side gives the “fundamental identity” for the bracket in Eq. (22).

V. CONCLUSION

In this article, we studied the general solution and first integrals of the generalized DH system (2). We showed that the integral curves of the solution are locally defined by the intersection of the level sets of the first integrals in a three-dimensional phase space \mathcal{M} which is a Nambu–Poisson manifold. In order to study the global dynamics, it is necessary to consider the phase flow on the covering manifolds associated with the multi-valued first integrals. The covering manifolds are generally densely branched for the DH system, although it is possible to obtain finite or denumerable infinite sheeted covering of \mathcal{M} corresponding to particular choices of the DH parameters. In these latter cases, there may be several interesting avenues of investigation including the topological properties of the DH phase space as well as the conformal class of $SU(2)$ -invariant hypercomplex manifolds which correspond to these special DH solutions.

It is also worth mentioning that the DH system can be regarded as a gradient flow: $X = \eta(\cdot, dV)$ for some flat, indefinite metric η^{-1} . The potential function V is a homogeneous polynomial of degree 3 in the ω_i , invariant under cyclic permutation of $(\omega_1, \omega_2, \omega_3)$. It is conceivable

that further insights into the complex dynamics of the DH system may be gained by considering it as a gradient flow with a polynomial potential rather than a Nambu–Poisson flow with multi-valued Hamiltonians.

ACKNOWLEDGMENTS

The authors acknowledge useful discussions with Mark Ablowitz and Alexander Veselov. S.C. would especially like to thank Leon Takhtajan for his valuable comments and encouragement during early stages of this work. This work has been partially supported by Grant No. NAL/00344/G from the Nuffield Foundation and a Grigsby-Sensenbaugh foundation grant from the University of Colorado.

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