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FINITE-ORDER MEROMORPHIC SOLUTIONS AND THE DISCRETE PAINLEVÉ EQUATIONS

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Abstract

Let $w(z)$ be an admissible finite-order meromorphic solution of the second-order difference equation

$$w(z+1) + w(z-1) = R(z, w(z)) \qquad (\dagger)$$

where $R(z, w(z))$ is rational in $w(z)$ with coefficients that are meromorphic in z. Then either $w(z)$ satisfies a difference linear or Riccati equation or else equation (†) can be transformed to one of a list of canonical difference equations. This list consists of all known difference Painlevé equations of the form (†), together with their autonomous versions. This suggests that the existence of finite-order meromorphic solutions is a good detector of integrable difference equations.

1. *Introduction*

A century ago Painlevé [**31**, **32**], Fuchs [**13**] and Gambier [**14**] classified a large class of second order differential equations in terms of a characteristic which is now known as the Painlevé property. An ordinary differential equation is said to possess the Painlevé property if all of its solutions are single-valued about all movable singularities (see, for example, [**1**].) Painlevé and his colleagues looked at the class

$$w'' = F(z, w, w'),$$

where F is rational in w and w', rejecting those equations which did not have the Painlevé property. They singled out a list of 50 equations, six of which could not be integrated in terms of known functions. These equations are now known as the Painlevé equations. During the twentieth century it was confirmed by different authors (and by different methods) that these equations indeed possess the Painlevé property [**31**, **28**, **27**, **24**, **17**].

The Painlevé property is a good detector of integrability. For instance, the six Painlevé equations are proven to be integrable by the inverse scattering techniques based on an associated isomonodromy problem, see, for instance, [**3**]. It is widely believed that all ordinary differential equations possessing the Painlevé property are integrable, although there are examples of equations which are solvable via an evolving monodromy problem but do not have the Painlevé property [**7**].

It is clear that when trying to distinguish discrete integrable equations from the non-integrable ones, a discrete analogue of the Painlevé property would be useful. Several candidates for the discrete Painlevé property have already been proposed. Ablowitz, Halburd and Herbst [**2**] considered discrete equations as delay equations in the complex plane which allowed them to analyze the equations with methods

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from complex analysis. The equations they consider to be of "Painlevé type" possess two properties: they are of finite order of growth in the sense of Nevanlinna theory, and they have no digamma functions in their series expansions. Ablowitz, Halburd and Herbst looked at, for instance, difference equations of the type

$$\overline{w} + \underline{w} = R(z, w), \tag{1.1}$$

where R is rational in both of its arguments, and the z-dependence is suppressed by writing $w \equiv w(z)$, $\overline{w} \equiv w(z+1)$ and $\underline{w} \equiv w(z-1)$. They showed that if equation (1.1) has at least one non-rational finite-order meromorphic solution, then the degree of $R(z, w)$ in w is less or equal to two. Indeed, a number of equations widely considered to be of Painlevé type lie within this class of equations. On the other hand, many equations within the class (1.1) with $\deg_w(R) \leq 2$ are generally considered to be non-integrable.

Costin and Kruskal [**10**] also applied complex analytic methods to detect integrability in discrete equations. Their idea of integrability is related to whether the sequence of iterates of solutions can be imbedded in the complex plane as an analyzable function.

Another method which has proved to be a good detector of integrability in discrete equations is the singularity confinement test by Grammaticos, Ramani and Papageorgiou [**16**]. The basic idea is to choose suitable initial conditions so that an iterate will become infinite at a certain point. The singularity is said to be confined if the iterates become finite after a certain finite number of steps and still contain information about the initial conditions. Singularity confinement has been a successful test. With it many important discrete equations, which are widely believed to be integrable, have been discovered [**35**].

However, implementation of the singularity confinement test is not without difficulty. In particular, how do we decide whether a given singularity sequence is truly confined and what exactly is the property for which we are testing? Also, an example of a numerically chaotic discrete equation possessing the singularity confinement property was found by Hietarinta and Viallet [**23**]. They suggest that singularity confinement needs to be augmented by a condition that a sequence of iterates possesses zero algebraic entropy. This is related to a number of approaches to the integrability of discrete equations or maps in which one considers the growth of the degree of the n^{th} iterate as a rational function of the initial conditions [**42**, **11**, **6**, **36**].

In this paper we consider the equation (1.1) where the coefficients of $R(z, w)$ have slow growth with respect to a meromorphic solution w in the sense of Nevanlinna theory. This type of solution is called *admissible*. For instance, all non-rational meromorphic solutions of an equation with rational coefficients are admissible. We show that if (1.1) has at least one admissible meromorphic solution of finite order, then either w satisfies a difference Riccati equation, or (1.1) can be transformed into a difference Painlevé or a linear equation. This indicates that the existence of a finite order meromorphic solution of a difference equation is a strong indicator of integrability of the equation.

An important subcase where equation (1.1) has rational coefficients will be analyzed in [**19**]. Choosing the subclass of rational coefficients as a starting point enables us to bypass a large number of technical details which may not be omitted in the analysis of the full case presented here. However, the field of rational functions do not contain all coefficients of the difference Painlevé equations in their full generality.

In what follows $\mathcal{S}(w)$ denotes the field of small functions with respect to w in terms of Nevanlinna theory. For example all rational functions are small with respect to any non-rational meromorphic function. (See (2.1) in Section 2 for the exact definitions of "admissible" and "small".)

THEOREM 1.1. *If the equation*

$$\overline{w} + \underline{w} = R(z, w), \tag{1.1}$$

where $R(z, w)$ is rational in w and meromorphic in z, has an admissible meromorphic solution of finite order, then either w satisfies a difference Riccati equation

$$\overline{w} = \frac{\overline{p}\, w + q}{w + p}, \tag{1.2}$$

where $p, q \in \mathcal{S}(w)$, or equation (1.1) *can be transformed by a linear change in w to one of the following equations:*

$$\overline{w} + w + \underline{w} = \frac{\pi_1 z + \pi_2}{w} + \kappa_1 \tag{1.3}$$

$$\overline{w} - w + \underline{w} = \frac{\pi_1 z + \pi_2}{w} + (-1)^z \kappa_1 \tag{1.4}$$

$$\overline{w} + \underline{w} = \frac{\pi_1 z + \pi_3}{w} + \pi_2 \tag{1.5}$$

$$\overline{w} + \underline{w} = \frac{\pi_1 z + \kappa_1}{w} + \frac{\pi_2}{w^2} \tag{1.6}$$

$$\overline{w} + \underline{w} = \frac{(\pi_1 z + \kappa_1) w + \pi_2}{(-1)^{-z} - w^2} \tag{1.7}$$

$$\overline{w} + \underline{w} = \frac{(\pi_1 z + \kappa_1) w + \pi_2}{1 - w^2} \tag{1.8}$$

$$\overline{w} w + w \underline{w} = p \tag{1.9}$$

$$\overline{w} + \underline{w} = p\, w + q \tag{1.10}$$

where $\pi_k, \kappa_k \in \mathcal{S}(w)$ are arbitrary finite-order periodic functions with period k.

Equations (1.3), (1.5) and (1.6) are known integrable discretizations of the Painlevé I equation, while equation (1.8) is often referred to as the difference Painlevé II. Equation (1.2) is a difference Riccati equation, and (1.10) a linear difference equation. These equations have been studied extensively in the literature and they are considered to be integrable [**15**, **12**, **33**, **34**]. They are of "Painlevé type" since in addition to being singled out by Theorem 1.1 they pass the singularity confinement test. Equation (1.9) is linear in $w\underline{w}$ and possesses finite-order meromorphic solutions for many choices of p. The list of equations (1.2) – (1.10) is complete in the sense that it contains all known integrable equations of the form (1.1) and apparently no non-integrable equations.

Equations (1.4) and (1.7) are slight variations of (1.3) and (1.8), respectively. Indeed, the transformation $w \to (-1)^z w$ maps (1.4) to (1.3) under a suitable redefinition of the periodic coefficients. However, such a transformation may affect the admissibility of the solution w if the order of w is at most one.

Although the notion of singularity confinement does not appear in the statement of Theorem 1.1, we have used ideas related to confinement in its proof. We use Nevanlinna theory to demonstrate that if generically we cannot associate a certain number (one or two, depending on the degree of R) of nearby poles of $\overline{w}+\underline{w}$ to each pole of w, then the order of the meromorphic solution w is infinite. Demanding that we can always associate enough poles of $\overline{w}+\underline{w}$ to each pole of w gives a number of possible constraints for the coefficients of (1.1). Generically these constraints lead to one of the difference Painlevé equations. The difference Riccati equation appears when the solution has a certain degenerate singularity structure. The singularity patterns of solutions of the chaotic difference equation studied in [**23**] are, although confined, not of the type allowed for a finite-order solution.

We are able to reduce the proof of Theorem 1.1 to a problem involving only the relative frequencies of special values of the solution by using a difference analogue [**18**] of the Lemma on the Logarithmic Derivative. In particular, we use corollaries which are natural difference analogues of results by Clunie [**9**] and A. A. Mohon'ko and V. D. Mohon'ko [**30**]. The results described in [**18**] have led to difference analogues of a number of other important theorems from Nevanlinna theory, including the Second Main Theorem [**20**]. Also, q-difference analogues of all these results were obtained in [**4**]. Chiang and Feng discovered another difference analogue of the Lemma on the Logarithmic Derivative in [**8**].

2. *Tools from Nevanlinna theory*

Nevanlinna theory is an efficient tool for studying the density of points in the complex plane at which a meromorphic function takes a prescribed value. It also provides a natural way to describe the growth of a meromorphic function. In this section we briefly recall some of the basic definitions and elementary results of Nevanlinna theory, and give some auxiliary results we need to prove Theorem 1.1. For a more comprehensive description of Nevanlinna theory we refer to [**21**].

2.1. *Basic definitions and notation*

The growth of a meromorphic function is described by the *Nevanlinna characteristic*

$$T(r,y) := N(r,y) + m(r,y),$$

where $m(r,y)$ is the *proximity function*

$$m(r,y) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ |y(re^{i\theta})|\,\mathrm{d}\theta, \qquad \log^+ x := \max(0,\log x),$$

and $N(r,y)$ is the *counting function*

$$N(r,y) := \int_0^r \frac{n(t,y)-n(0,y)}{t}\,\mathrm{d}t + n(0,y)\log r,$$

where $n(r,y)$ is the number of poles (counting multiplicities) of y in the disc $\{z : |z| \le r\}$. The *order of growth* of a meromorphic function is defined to be

$$\rho(y) := \limsup_{r\longrightarrow\infty} \frac{\log T(r,y)}{\log r}.$$

For entire functions $\rho(y)$ is equal to the classical growth order

$$\sigma(y) := \limsup_{r\longrightarrow\infty} \frac{\log\log M(r,y)}{\log r},$$

where $M(r,y)$ is the maximum modulus of y in the disc of radius r.

A quantity which is of the growth $o(T(r,y))$ as $r\to\infty$ outside of a set with finite logarithmic measure is denoted by $S(r,y)$. Then

$$\mathcal{S}(y) := \{w \text{ meromorphic} : T(r,w) = S(r,y)\} \tag{2.1}$$

is a field with respect to the usual addition and multiplication. In other words, a meromorphic function g is in $\mathcal{S}(y)$ if

$$\lim_{r\to\infty} \frac{T(r,g)}{T(r,y)} = 0$$

where r runs to infinity anywhere outside of a set E satisfying $\int_E \frac{dt}{t} < \infty$. The field $\mathcal{S}(y)$ is often referred to as the *field of small functions* with respect to y. A non-rational meromorphic solution y of a difference (or differential) equation is called *admissible* if all coefficients of the equation are in $\mathcal{S}(y)$. For example, if a difference equation has only rational coefficients then all non-rational meromorphic solutions are admissible. This is due to the fact that a meromorphic function y is rational if and only if $T(r,y) = O(\log r)$. We often omit the expression "with respect to y" when we talk about small functions with respect to an admissible solution of (1.1).

When applying Nevanlinna theory to differential and functional equations an identity due to Valiron [**41**] and Mohon'ko [**29**] has proved to be useful (see also [**26**]). It states that given a function $R(z,y)$ which is rational in y and meromorphic in z, we have

$$T(r,R(z,y)) = \deg_y(R)T(r,y) + S(r,y) \tag{2.2}$$

whenever all coefficients of $R(z,y)$ are small compared to y.

In what follows we often say that there are $S(r,y)$ points z_j with a certain property. By this we mean that the integrated counting function $N(r,\cdot\,)$ measuring the points with the property in question is at most of the growth $S(r,y)$. We also use the expressions like: "There are more than $S(r,y)$ points such that ..." This means in precise terms that

$$\limsup_{r\to\infty} \frac{N(r,\cdot\,)}{T(r,y)} = c > 0,$$

where $c \in \mathbb{R}^+ \cup \{+\infty\}$, and r runs to infinity in a set with infinite logarithmic measure. For instance, if a meromorphic function g has more than $S(r,y)$ poles, then $g \notin \mathcal{S}(y)$.

On several occasions we will encounter inequalities of the type

$$n(r,\cdot\,) \le \alpha\, n(r+k,y) + S'(r,y), \tag{2.3}$$

where by $S'(r,y)$ we mean a quantity which is at most of the growth $S(r,y)$ after a logarithmic integration. In exact terms, the quantity on the right side of (2.3) is $\alpha\, n(r+k,y) + \tilde{n}(r)$ where $\tilde{n}(r)$ is a piecewise continuous increasing function of r such that

$$\tilde{N}(r) := \int_0^r \frac{\tilde{n}(t)-\tilde{n}(0)}{t}\,\mathrm{d}t + \tilde{n}(0)\log r = S(r,y).$$

In other words, the (integrated) counting function $\tilde{N}(r)$ counting the number of exceptional points in (2.3) is small with respect to y.

While we only consider meromorphic solutions of difference equations in this paper, we sometimes end up in a situation where the coefficients of a considered equation may have some finite sheeted branching. The classical version of Nevanlinna theory we introduced earlier in this section deals only with meromorphic functions, and so it is unable to handle this kind of situation. However, there is a version of the theory introduced by Selberg [**37**, **38**, **39**], Ullrich [**40**] and Valiron [**41**] called the *algebroid Nevanlinna theory* which studies meromorphic functions on a finite sheeted Riemann surface. Such functions are called algebroid and they are allowed to have isolated branch points with finite branching. To make the proof of Theorem 1.1 watertight we have to assume that whenever the coefficients of a difference equation may have branching, $T(r,\cdot\,)$ denotes the Nevanlinna characteristic of a 2-sheeted algebroid function. Since all branched functions we consider are small with respect to the meromorphic solution of (1.1), the change in notation only effects the error term which needs to be redefined in terms of the algebroid characteristic. The "algebroid error term" will still be denoted by $S(r,\cdot\,)$ and it remains small with respect to the meromorphic solution of (1.1). An interested reader may refer, for instance, to [**25**] for more details on algebroid Nevanlinna theory. Also, in the special case when the coefficients of (1.1) are rational in the first place, algebroid Nevanlinna theory is not required [**19**].

2.2. *Nevanlinna theory and difference equations*

Assume that w is an admissible meromorphic solution of (1.1). In other words, the coefficients of (1.1) are all small with respect to w, and in particular they are in the field $\mathcal{S}(w)$. Following the reasoning used by Yanagihara [**43**] and by Ablowitz, Halburd and Herbst [**2**], which combines the Valiron-Mohon'ko identity (2.2) and the fact that

$$T(r,w(z\pm 1)) \leq (1+\varepsilon)T(r+1,w)+O(1)$$

holds for $\varepsilon>0$ when r is sufficiently large, see [**43**], we have

$$T(r,w) \leq \frac{2(1+\varepsilon)}{\deg_w(R)}T(r+1,w)+S(r,w).$$

If the degree of $R(z,w)$ with respect to w is at least three, there is an $\alpha<1$ such that

$$T(r,w) \leq \alpha T(r+1,w) \tag{2.4}$$

outside of a possible set E of r-values with finite logarithmic measure. Intuitively the iteration of (2.4) gives $T(r+j,w)\geq (1/\alpha)^j T(r,w)$ which seems to imply that w is of infinite order by letting $j\to\infty$ similarly as in [**2**]. However, we need to be very careful here due to the exceptional set E. Namely, if $r+j\in E$ for any $j\in\mathbb{N}$ then the iteration process is terminated after a finite number of steps, and no conclusion about the order of w can be made by this argument.

Nevertheless it turns out that it is sufficient that (2.4) holds in a set $\mathbb{R}^+\setminus E$ with infinite logarithmic measure, as the following lemma shows. In its proof we show that (2.4) implies that w is of infinite order by a careful choice of a sequence (r_n) in $\mathbb{R}^+\setminus E$. We conclude that if (1.1) has a meromorphic solution of finite order then $\deg_w(R)\leq 2$. The rest of the proof of Theorem 1.1 can be found in Section 3.

Lemma 2.1. *Let f be a non-constant meromorphic function, $s > 0$, $\alpha < 1$, and let $F \subset \mathbb{R}^+$ be the set of all r such that*

$$T(r, f) \leq \alpha T(r + s, f). \tag{2.5}$$

If the logarithmic measure of F is infinite, that is, $\int_F \frac{dt}{t} = \infty$, then f is of infinite order of growth.

Proof. Suppose that $\int_F \frac{dt}{t} = \infty$. By the definition (2.5) the set F is closed and so it has a smallest element. We define a sequence $(r_n) \subset F$ inductively as follows:

(i) Let r_0 be the smallest element of F.

(ii) For all $n \in N$ let $r_n = \min(F \cap [r_{n-1} + s, \infty))$.

Then (r_n) satisfies $r_{n+1} - r_n \geq s$ for all $n \in \mathbb{N}$. Moreover

$$F \subset \bigcup_{n=0}^{\infty} [r_n, r_n + s]$$

and

$$T(r_n, f) \leq \alpha T(r_{n+1}, f) \tag{2.6}$$

for all $n \in \mathbb{N}$.

We show next that (r_n) has a subsequence (r_{n_k}) such that $r_{n_k} \leq n_k^2$ for all $k \in \mathbb{N}$. To this end, assume conversely that $r_n \geq n^2$ for all $r_n \geq m$, where m is a sufficiently large constant. This implies that

$$\begin{aligned}
\int_F \frac{dt}{t} &\leq \sum_{n=0}^{\infty} \int_{r_n}^{r_n+s} \frac{dt}{t} \\
&\leq \int_1^m \frac{dt}{t} + \sum_{n=1}^{\infty} \int_{n^2}^{n^2+s} \frac{dt}{t} \\
&= \log m + \log \prod_{n=1}^{\infty} \left(1 + \frac{s}{n^2}\right) \\
&= \log m + \log(\sinh(\sqrt{s}\pi)) - \frac{1}{2}\log s - \log \pi < \infty,
\end{aligned}$$

which is a contradiction since F was assumed to be of infinite logarithmic measure. Therefore there is (r_{n_k}) such that $r_{n_k} \leq n_k^2$ for all $k \in \mathbb{N}$. By iterating (2.6) we then have

$$T(r_n, f) \geq \frac{1}{\alpha^n} T(r_0, f)$$

for all $n \in \mathbb{N}$. In particular,

$$T(r_{n_k}, f) \geq \frac{1}{\alpha^{n_k}} T(r_0, f)$$

for all $k \in N$, and so

$$\begin{aligned}
\rho(y) &\geq \limsup_{k\to\infty} \frac{\log T(r_{n_k}, f)}{\log r_{n_k}} \\
&\geq \limsup_{k\to\infty} \frac{n_k \log(1/\alpha) + \log T(r_0, f)}{\log r_{n_k}} \\
&\geq \limsup_{k\to\infty} \frac{n_k \log(1/\alpha) + \log T(r_0, f)}{2 \log n_k} = \infty
\end{aligned}$$

since $r_{n_k} \leq n_k^2$ for all $k \in \mathbb{N}$. □

In the remainder of this subsection we state a number of recent results on difference equations and Nevanlinna theory [**18**]. They are concerned with functions which are polynomials in $f(z+c_j)$, where $c_j \in \mathbb{C}$, with coefficients in the field $\mathcal{S}(f)$. Such functions are called *difference polynomials in* $f(z)$. We also denote

$$|c| := \max\{|c_j|\}.$$

The first result is an analogue of the Lemma on the Logarithmic Derivative.

THEOREM 2.2. *Let f be a non-constant meromorphic function of finite order, $c \in \mathbb{C}$ and $\mu < 1$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r+|c|, f)}{r^\mu}\right)$$

for all r outside of a possible exceptional set with finite logarithmic measure.

The second result is a difference analogue of the Clunie Lemma [**9**].

THEOREM 2.3. *Let $f(z)$ be a non-constant finite-order meromorphic solution of*

$$f(z)^n P(z,f) = Q(z,f),$$

where $P(z,f)$ and $Q(z,f)$ are difference polynomials in $f(z)$, and let $\mu < 1$. If the degree of $Q(z,f)$ as a polynomial in $f(z)$ and its shifts is at most n, then

$$m\big(r, P(z,f)\big) = o\left(\frac{T(r+|c|, f)}{r^\mu}\right) + o(T(r,f)) \tag{2.7}$$

for all r outside of a possible exceptional set with finite logarithmic measure.

The third result is a difference analogue of a result due to A. A. Mohon'ko and V. D. Mohon'ko on algebraic differential equations [**30**].

THEOREM 2.4. *Let $f(z)$ be a non-constant finite-order meromorphic solution of*

$$P(z,f) = 0$$

where $P(z,f)$ is difference polynomial in $f(z)$, and let $\mu < 1$. If $P(z,a) \not\equiv 0$ for a meromorphic function $a \in \mathcal{S}(f)$, then

$$m\left(r, \frac{1}{f-a}\right) = o\left(\frac{T(r+|c|, f)}{r^\mu}\right) + o(T(r,f))$$

for all r outside of a possible exceptional set with finite logarithmic measure.

2.3. *The counting function and the order of solutions*

We conclude this section with a theorem on the order of meromorphic solutions of certain difference equations within the class (1.1). It will be frequently applied in the proof of Theorem 1.1 in Section 3.

THEOREM 2.5. *Let w be an admissible meromorphic solution of one of the equations*

$$\overline{w} + \sigma\underline{w} = \frac{c_2 w^2 + c_1 w + c_0}{w^2 + aw + b} \tag{2.8}$$

$$\overline{w} + \underline{w} - c_2 w = \frac{c_1 w + c_0}{w} \tag{2.9}$$

where the right sides are irreducible, $\sigma := \pm 1$, and all coefficients c_j, a and b are in $\mathcal{S}(w)$. If w satisfies (2.8) *and there exist $k \geq 1$ and $\alpha < 2$ such that*

$$n(r, \overline{w} + \sigma\underline{w}) \leq \alpha\, n(r+k, w) + S'(r, w), \tag{2.10}$$

then w is of infinite order of growth. Similarly, if w satisfies (2.9) *and there exist $k \geq 1$ and $\alpha < 1$ such that*

$$n(r, \overline{w} + \underline{w} - c_2 w) \leq \alpha\, n(r+k, w) + S'(r, w), \tag{2.11}$$

then w is of infinite order of growth.

Proof. Assume first that w is a solution of (2.8). By integrating (2.10) we obtain

$$\begin{aligned} N(r, \overline{w} + \sigma\underline{w}) &\leq \alpha \int_{r_0}^{r+k} \frac{t}{t-k} \frac{n(t, w)}{t}\, \mathrm{d}t + S(r, w) \\ &\leq \alpha(1+\varepsilon)\, N(r+k, w) + S(r, w), \end{aligned} \tag{2.12}$$

where $\varepsilon > 0$ is chosen so that $\tilde{\alpha} := \alpha(1+\varepsilon) < 2$ and r_0 is a sufficiently large constant. By the Valiron-Mo'honko identity (2.2), we have

$$\begin{aligned} 2T(r, w) &= T(r, \overline{w} + \sigma\underline{w}) + S(r, w) \\ &= N(r, \overline{w} + \sigma\underline{w}) + m(r, \overline{w} + \sigma\underline{w}) + S(r, w). \end{aligned}$$

Therefore, by (2.12) and by Theorem 2.3 where we have chosen $P(z, w) = \overline{w} + \sigma\underline{w}$ and $Q(z, w) = c_2 w^2 + c_1 w + c_0 - (aw + b)(\overline{w} + \sigma\underline{w})$, we obtain

$$2T(r, w) \leq \tilde{\alpha} N(r+k, w) + o\left(\frac{T(r+1, w)}{r^{\mu}}\right) + S(r, w),$$

outside a set E of finite logarithmic measure, where $\mu < 1$. Hence

$$T(r, w) \leq \left(\frac{\tilde{\alpha}}{2} + \varepsilon\right) T(r+k, w)$$

holds for any $\varepsilon > 0$ in a set with infinite logarithmic measure. The assertion follows by choosing ε such that $\frac{\tilde{\alpha}}{2} + \varepsilon < 1$ and applying Lemma 2.1.

Now suppose that w satisfies (2.9). By integrating (2.11) we obtain

$$N(r, \overline{w} + \underline{w} - c_2 w) \leq \alpha(1+\varepsilon)\, N(r+k, w) + S(r, w), \tag{2.13}$$

where $\varepsilon > 0$ is chosen so that $\tilde{\alpha} := \alpha(1+\varepsilon) < 1$. By (2.2), we obtain

$$T(r, w) = N(r, \overline{w} + \underline{w} - c_2 w) + m(r, \overline{w} + \underline{w} - c_2 w) + S(r, w).$$

Therefore, by (2.13) and applying Theorem 2.3 with $P(z, w) = \overline{w} + \underline{w} - c_2 w$ and $Q(z, w) = c_1 w + c_0$, we have

$$T(r, w) \leq (\tilde{\alpha} + \varepsilon)\, T(r+k, w)$$

where $\varepsilon > 0$ and r is in a set with infinite logarithmic measure. The assertion follows by choosing ε such that $\tilde{\alpha} + \varepsilon < 1$ and applying Lemma 2.1. □

3. *Analysis of solutions near singularities*

In Section 2.2 we showed that if equation (1.1) has an admissible meromorphic solution of finite order, then $\deg_w(R) \leq 2$. In this section we complete the proof of Theorem 1.1 by a careful consideration of a number of subcases depending on the exact form of $R(z, w)$.

Since we allow the coefficients of (1.1) to be non-rational meromorphic functions, they may in general have infinitely many zeros and poles. Therefore, even though we demand that all coefficients of (1.1) are small compared to a meromorphic solution w with a large number of poles, *counting* multiplicities, it may happen that w has in fact fewer poles than some of the coefficients if we *ignore* multiplicities. In particular, this means there might not be a point z_0 such that $w(z_0) = \infty$ and no coefficient of (1.1) has a pole or zero at z_0. However, at most points the multiplicity of a pole of w is much greater than the multiplicities of poles and zeros of the coefficients, which is enough for our purposes.

Notation: In what follows we use the notation $D(z_0, \tau)$ to denote an open disc of radius τ centered at $z_0 \in \mathbb{C}$. Also, $\infty^{\mathbf{k}}$ denotes a pole of w with multiplicity k. Similarly, $0^{\mathbf{k}}$ and $a + 0^{\mathbf{k}}$ denote a zero and an a-point of w, respectively, with multiplicity k. For instance, $w(z_0) = a + 0^{\mathbf{k}}$ is a short notation for

$$w(z) = a + c_0(z - z_0)^k + O\left((z - z_0)^{k+1}\right)$$

for all $z \in D(z_0, \tau_0)$, where $c_0 \neq 0$ and τ_0 is a sufficiently small constant.

Lemma 3.1. *Let w be an admissible meromorphic solution of* (1.1) *with more than $S(r, w)$ poles (counting multiplicities) and let z_j denote the zeros and poles of the coefficients a_i of $R(z, w)$. Let*

$$m_j := \max_{i=1,\ldots,n} \{l_i \in \mathbb{N} : a_i(z_j) = 0^{\boldsymbol{l_i}} \text{ or } a_i(z_j) = \infty^{\boldsymbol{l_i}}\}$$

be the maximal order of zeros and poles of the functions a_i at z_j. Then for any $\epsilon > 0$ there are at most $S(r, w)$ points z_j such that

$$w(z_j) = \infty^{\boldsymbol{k_j}} \tag{3.1}$$

where $m_j \geq \epsilon k_j$.

Proof. Assume on the contrary that there are more than $S(r, w)$ points z_j such that (3.1) holds and $m_j \geq \epsilon k_j$. Let $N_{z_j}(r, w)$ denote the counting function for those poles of w which are in the set $\{z_j\}$, and let $N_\Sigma(r, a_i)$ be the counting function for the poles and zeros of all a_i. Then by assumption

$$\limsup_{r\to\infty} \frac{N_\Sigma(r, a_i)}{T(r, w)} \geq \limsup_{r\to\infty} \frac{\epsilon N_{z_j}(r, w)}{T(r, w)} > 0,$$

where r runs to infinity in a set with infinite logarithmic measure. This implies that at least one of the functions a_i has more than $S(r, w)$ poles or zeros, which contradicts the fact that all a_i are small with respect to w. We conclude that $N_{z_j}(r, w) = S(r, w)$. □

The gamma function Γ has a simple pole at each of the points $\{-n+1 : n \in \mathbb{N}\}$, and the order of growth of Γ is one. We may construct a meromorphic function G

which has a pole of order n^2 at the points $\{-n^2 : n \in \mathbb{N}\}$ [**21**]. Then the order of G is at least $3/2$, and so Γ is small with respect to G. Also, G has far fewer poles than Γ when we ignore multiplicities, and in particular there are no points where G has a pole and Γ does not. However, G has many more poles than Γ when we take the multiplicities into account.

In an attempt to make the somewhat awkward notation involved in dealing with this issue more readable, we introduce the following notation. Whenever a small quantity arises from reasoning related to Lemma 3.1, we use the notation $\epsilon > 0$, rather than the usual $\varepsilon > 0$. Note that if the multiplicities of the poles of a solution w of (1.1) have a uniform upper bound, or if the coefficients have only finitely many zeros and poles, then each ϵ in the below reasoning may be replaced by zero. In the treatment of (1.1) with rational coefficients [**19**] the technicalities with ϵ are avoided.

3.1. *The Difference Painlevé II Equation*

Assume that the denominator of $R(z,w)$ has exactly two distinct roots, which implies that the degree of $R(z,w)$ is also two. Then equation (1.1) takes the form

$$\overline{w} + \underline{w} = \frac{u_2 w^2 + u_1 w + u_0}{w^2 + aw + b}, \tag{3.2}$$

where the coefficients of the right side belong to $\mathcal{S}(w)$, and $a^2 \not\equiv 4b$. The transformation $w \to w - a/2$ takes (3.2) into the form

$$\overline{w} + \underline{w} = \frac{c_2 w^2 + c_1 w + c_0}{w^2 - p^2} =: \frac{P(z,w)}{Q(z,w)}, \tag{3.3}$$

where the coefficients c_j are in $\mathcal{S}(w)$ and $p^2 = a^2/4 - b \not\equiv 0$. Theorem 2.5 states that if

$$n(r, \overline{w} + \underline{w}) \le \alpha\, n(r+k, w) + S'(r,w) \tag{3.4}$$

for $\alpha < 2$, $k \ge 1$ and for all r sufficiently large, then w is of infinite order. (Recall the exact definition of $S'(r,w)$ from Section 2.1.) Roughly speaking this means that if w has finite order then for "most" poles of w there are two points (counting multiplicities) where $w(z) = \pm p(z)$.

Assume first that a meromorphic solution of equation (3.3) has at most $S(r,w)$ poles. Then Valiron-Mohon'ko identity (2.2), Theorem 2.3 and equation (3.3) yield

$$\begin{aligned} 2T(r,w) &= T(r, \overline{w} + \underline{w}) + S(r,w) \\ &\le 2N(r+1,w) + m(r, \overline{w} + \underline{w}) + S(r,w) \\ &= o\left(\frac{T(r+1,w)}{r^{\mu}}\right) + S(r+1,w), \end{aligned} \tag{3.5}$$

where $\mu < 1$ and r lies outside of an exceptional set E with finite logarithmic measure. Therefore

$$T(r,w) \le \varepsilon T(r+1,w),$$

where $0 < \varepsilon < 1$, holds in a set with infinite logarithmic measure. Hence w is of infinite order by Lemma 2.1, which is a contradiction. We conclude that w has more than $S(r,w)$ poles. Therefore also $\overline{w} + \underline{w}$ has more than $S(r,w)$ poles (which are the $\pm p(z)$ -points of w) since otherwise (2.10) holds with any $\alpha > 0$ and w would be of infinite order by Theorem 2.5.

We have shown that both w and $\overline{w}+\underline{w}$ have more than $S(r,w)$ poles. In addition, the number of points z' where $Q(z',w(z')) = P(z',w(z')) = 0$ is at most $S(r,w)$, since otherwise it would follow that $c_2p^2 \pm c_1p + c_0 = 0$, which is impossible due to the irreducibility of the right side of (3.3). Also, since the coefficients of $R(z,w)$ are in $\mathcal{S}(w)$, they have altogether at most $S(r,w)$ poles. Hence there are more than $S(r,w)$ points z_j such that $Q(z_j-1,w(z_j-1)) = 0$ and w has a pole at either z_j or z_j-2. We assume, without loss of generality, that $w(z_j) = \infty$. Moreover, denoting the multiplicity of $Q(z_j-1,w(z_j-1)) = 0$ by k_j, Lemma 3.1 implies that there are more than $S(r,w)$ points such that the multiplicity of $w(z_j) = \infty$ is at least $(1-\epsilon)k_j$ for an arbitrarily small $\epsilon \geq 0$. If for all but $S(r,w)$ many such z_j we have $Q(z_j+1,w(z_j+1)) = 0$ with the multiplicity less than $\frac{1}{3}k_j$ (this includes the case $Q(z_j+1,w(z_j+1)) \neq 0$) then there are more than $S(r,w)$ poles of $\overline{w}+\underline{w}$ at $z_j \pm 1$ with multiplicities k_j and $< \frac{1}{3}k_j$, respectively, which can be associated with the pole of w at z_j with multiplicity at least $(1-\epsilon)k_j$, and only $S(r,w)$ poles of $\overline{w}+\underline{w}$ which cannot be associated to a pole of w in this way. Therefore inequality (3.4) is satisfied with $\alpha = \frac{4}{3} + 2\epsilon/(1-\epsilon)$, and so Theorem 2.5 implies that w is of infinite order.

Recall that $D(z_0,\tau)$ denotes the open disc of radius τ centered at $z_0 \in \mathbb{C}$. Since we assumed that w is of finite order, there must be more than $S(r,w)$ points z_j such that $w(z_j) = \infty$ with multiplicity k_j and

$$Q(z \pm 1, w(z \pm 1)) = O\left((z-z_j)^{\frac{1}{3}k_j}\right) \tag{3.6}$$

for both choices of the $\pm$ sign and for all $z \in D(z_j,\tau_j)$ with sufficiently small constants τ_j. Then by equation (3.3) and Lemma 3.1 there are more than $S(r,w)$ points z_j for an arbitrarily small $\epsilon \geq 0$ such that

$$w(z+1) + w(z-1) = c_2(z) + O\left((z-z_j)^{(1-\epsilon)k_j}\right)$$

for all $z \in D(z_j,\tau_j)$ where τ_j is small enough. Hence, by taking (3.6) into account, we obtain

$$(c_2(z) - w(z-1))^2 = p(z+1)^2 + O\left((z-z_j)^{\frac{1}{3}k_j}\right) \tag{3.7}$$

for all $z \in D(z_j,\tau_j)$. Since also

$$w(z-1)^2 = p(z-1)^2 + O\left((z-z_j)^{\frac{1}{3}k_j}\right), \tag{3.8}$$

we obtain

$$2c_2(z)w(z-1) + p(z+1)^2 - p(z-1)^2 - c_2(z)^2 = O\left((z-z_j)^{\frac{1}{3}k_j}\right) \tag{3.9}$$

for all $z \in D(z_j,\tau_j)$ at more than $S(r,w)$ points z_j.

We now consider two cases depending on whether or not c_2 is identically zero. If $c_2 \equiv 0$ then equation (3.9) yields

$$p(z-1)^2 - p(z+1)^2 = h(z), \tag{3.10}$$

where h is a small meromorphic function with respect to w such that

$$h(z) = O\left((z-z_j)^{\frac{1}{3}k_j}\right)$$

for all $z \in D(z_j,\tau_j)$ at more than $S(r,w)$ points z_j. Hence h has more than $S(r,w)$ zeros counting multiplicities, which implies that $h \equiv 0$ since $T(r,h) = S(r,w)$. We conclude that

$$p(z-1)^2 - p(z+1)^2 = 0. \tag{3.11}$$

In other words $p(z)^2 \not\equiv 0$ is an arbitrary finite order periodic function with period two. Therefore, by making the transformation $w \to p\,w$, equation (3.3) takes the form

$$\overline{w} + \sigma \underline{w} = \frac{a_1 w + a_0}{1 - w^2}, \tag{3.12}$$

where $\sigma := \pm 1$, and a_0, a_1 are small functions compared to w depending on p and on the coefficients c_j. After this transformation the admissible solution w of (3.3) corresponds to an admissible solution of (3.12). One should keep in mind that the coefficients a_j may, at least in principle, have some square root type branching. An explanation of how to deal with branched coefficients was given in Section 2.1.

The dependence between the multiplicity of the pole of w at the points z_j and the multiplicity of zeros of h in (3.10) is important. If no information about the multiplicities of the zeros of h would be available we could not rule out the possibility that $N(r, \frac{1}{h}) = \bar{N}(r, \frac{1}{h}) = S(r, w)$ in the case when w has more than $S(r, w)$ poles at points z_j counting multiplicities, but only $S(r, w)$ poles ignoring multiplicities.

We continue with a closer analysis of the singularity structure of meromorphic solutions of (3.12). (Recall the notation used below in (3.14) from the beginning of Section 3.)

Lemma 3.2. *Let w be an admissible meromorphic solution of equation* (3.12). *Then either*

$$n(r, \overline{w} + \sigma \underline{w}) \le \left(\frac{8}{5} + \epsilon\right) n(r+1, w) + S'(r, w) \tag{3.13}$$

for any $\epsilon > 0$, or there are more than $S(r, w)$ points z_j such that

$$\begin{aligned} &w(z_j - 2) = \infty^{l_j}, \quad w(z_j - 1) = \delta + 0^{k_j}, \quad w(z_j) = \infty^{k_j}, \\ &w(z_j + 1) = -\sigma\delta + 0^{k_j}, \quad w(z_j + 2) = \infty^{m_j}, \end{aligned} \tag{3.14}$$

where $\delta = \pm 1$, and l_j and m_j are strictly less than $\frac{3}{4}k_j$.

Note that there is nothing to stop $w(z_j \pm 2)$ being finite, and indeed for most z_j this seems to be the case, but for our purposes the assertion of the lemma is sufficient.

Proof. By Lemma 3.1, given $\epsilon > 0$, there are at most $S(r, w)$ points z_j where $w(z_j)^2 = 1$ with the multiplicity k_j, but where $\overline{w} + \sigma \underline{w}$ has a pole with order higher than $(1 + \epsilon)k_j$. We include all such points in the error term, and in what follows consider the rest of the δ-points of w.

We will next associate each δ-point of w with a certain number of nearby poles of w. To this end, we look at sequences of iterates

$$\big(w(z_j + n)\big)_{n=l}^{m} \qquad l, m \in I \subseteq \mathbb{Z} \cup \{\pm\infty\}$$

of (3.12) consisting of poles and δ-points of w such that all iterates within a sequence have the same constant multiplicity. If the multiplicity of $w(z_j + n)$ is different than the multiplicity of $w(z_j + n + 1)$ we say that the iterates $w(z_j + n)$ and $w(z_j + n + 1)$ are in different sequences. For example, the iterates $w(z_j - 1)$, $w(z_j)$ and $w(z_j + 1)$

in (3.14) are in the same sequence, but $w(z_j+1)$ and $w(z_j+2)$ are not. We will systematically go through all different possible types of sequences containing δ-points of w.

Consider first a sequence with only one iterate, say $w(z_j)$. By assumption $w(z_j) = \delta + 0^{\mathbf{k_j}}$ for some $k_j \in \mathbb{N}$. From (3.12) it follows that either $w(z_j-1) = \infty$ or $w(z_j+1) = \infty$. Since the sequence contains only one iterate $w(z_j)$, the multiplicity of the neighboring pole is not equal to the multiplicity of the δ-point at z_j. The only way this is possible without contradicting (3.12) is when $w(z_j-1) = \infty^{\mathbf{l_j}}$ and $w(z_j+1) = \infty^{\mathbf{l_j}}$ where $l_j > k_j$. Then by (3.12) we have $w(z_j \pm 2) = -\sigma\delta + 0^{\mathbf{k_j}}$ and $w(z_j \pm 3) = \infty^{\mathbf{l_j}}$. Hence any δ-point of w in a length one sequence can be associated with at most one pole of w, with the possible exception of at most $S(r,w)$ points where certain coefficients of (3.12) have zeros or poles.

By iteration of (3.12) we see that poles and δ-points of w alternate in each sequence which consists of two or more iterates. Therefore, in a sequence with n δ-points of w there are $n-1$, n or $n+1$ points where w has a pole. If a sequence consists of an even number of iterates, then exactly half of them are poles of w and the other half are δ-points of w. Therefore those δ-points of w which are part of a sequence with even or infinite length can be associated with exactly one pole of w.

For a sequence with an odd number of iterates, say j, the number of δ-points of w is at most $(j+1)/2$, and the number of poles of w at least $(j-1)/2$. Therefore the number of δ-points of w divided by the number of poles w within the sequence is at most

$$\frac{j+1}{j-1} \leq \frac{3}{2}$$

when j is at least five. If $j=3$ there are two possibilities: the sequence can have either one or two δ-points of w. In the former case the ratio is $1/2$, and the latter case is (3.14).

It remains to be shown that either there are more than $S(r,w)$ points z_j such that (3.14) holds with the multiplicities l_j and m_j strictly less than $\frac{3}{4}k_j$, or (3.13) is true. Assume that $m_j \geq \frac{3}{4}k_j$. Within the five points in (3.14) there is one complete sequence consisting of the points $w(z_j), w(z_j \pm 1)$, and two starting points $w(z_j \pm 2)$ of other sequences. The sequence starting from $w(z_j+2) = \infty^{\mathbf{m_j}}$ ends (at least from one end) at a pole. The number of δ-points of w divided by the number of poles of w within such sequence is at most one. We "remove" one third (worth $m_j/3$) of the iterate $w(z_j+2) = \infty^{\mathbf{m_j}}$ from its original sequence and associate it with the three central points of (3.14) instead. The pole and δ-point ratio in the remaining part of the sequence containing $w(z_j+2)$ is at most $3/2$ even if the removal has to be done from both of its ends. Since we assumed $m_j \geq \frac{3}{4}k_j$ and the sequence consisting of the three central points in (3.14) contains exactly k_j poles of w and $2k_j$ δ-points of w, the combined pole and δ-point ratio for the middle sequence in (3.14) and the extra third of a point (with the multiplicity at least $\frac{1}{4}k_j$) is at most $8/5$. If also $l_j \geq \frac{3}{4}k_j$ we may similarly attach a third of a point from the other end into (3.14). In this case the combined pole and δ-point ratio is at most $4/3$. We illustrate the situation in Table 1.

We have shown that the only type of sequence where each δ-points of w cannot be associated with at most $8/5$ poles of w is of the type (3.14). Therefore, if the number of δ-points of w which are part of a (3.14) is at most $S(r,w)$, we have the inequality (3.13) by Lemma 3.1. □

TABLE 1. *The multiplicities l_j and m_j in (3.14). The values of w which are to be grouped together are marked by "$*$". The notation "$\dagger$" means that only a third of the multiplicity of the point is associated with the other points in the group.*

$l_j, m_j < \frac{3}{4}k_j$	$\infty^{\mathbf{l_j}}$	$\delta + 0^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}}$	$-\sigma\delta + 0^{\mathbf{k_j}}$	$\infty^{\mathbf{m_j}}$	(3.14)
$l_j < \frac{3}{4}k_j,\ m_j \geq \frac{3}{4}k_j$	$\infty^{\mathbf{l_j}}$	$\delta + 0^{\mathbf{k_j}}*$	$\infty^{\mathbf{k_j}}*$	$-\sigma\delta + 0^{\mathbf{k_j}}*$	$\infty^{\mathbf{m_j}}\dagger$	*ratio* $\leq 8/5$
$l_j \geq \frac{3}{4}k_j,\ m_j < \frac{3}{4}k_j$	$\infty^{\mathbf{l_j}}\dagger$	$\delta + 0^{\mathbf{k_j}}*$	$\infty^{\mathbf{k_j}}*$	$-\sigma\delta + 0^{\mathbf{k_j}}*$	$\infty^{\mathbf{m_j}}$	*ratio* $\leq 8/5$
$l_j, m_j \geq \frac{3}{4}k_j$	$\infty^{\mathbf{l_j}}\dagger$	$\delta + 0^{\mathbf{k_j}}*$	$\infty^{\mathbf{k_j}}*$	$-\sigma\delta + 0^{\mathbf{k_j}}*$	$\infty^{\mathbf{m_j}}\dagger$	*ratio* $\leq 4/3$

We now return back to the proof of Theorem 1.1. By manipulating equation (3.12), we obtain

$$\begin{aligned}(1-\overline{w}^2)(\overline{\overline{w}} - \underline{\underline{w}}) &= \overline{a}_0 + \overline{a}_1\overline{w} - \sigma(\underline{a}_0 + \underline{a}_1\underline{w}) \\ &\quad - (w + \sigma\underline{\underline{w}})\left[\frac{2\underline{w}(a_0+a_1w)}{1-w^2} - \sigma\left(\frac{a_0+a_1w}{1-w^2}\right)^2\right].\end{aligned} \tag{3.15}$$

If inequality (3.13) holds the meromorphic solution w of (3.12) is of infinite order by Theorem 2.5. On the other hand, if (3.14) is true for more than $S(r,w)$ points z_j, we have by (3.15)

$$\sigma a_1(z_j+1) - 2a_1(z_j) + \sigma a_1(z_j-1) - \delta\left[a_0(z_j+1) - \sigma a_0(z_j-1)\right] = 0, \tag{3.16}$$

where $\delta = \pm 1$. Also, if the multiplicity of the pole of w at z_j is k_j, then (3.16) holds with the multiplicity at least $\frac{1}{4}k_j$. Hence also (3.16) holds at more than $S(r,w)$ points. Since $T(r,a_j) = S(r,w)$ for $j = 0,1$ by assumption, it follows that (3.16) holds for all z, so that

$$\sigma a_1(z+1) - 2a_1(z) + \sigma a_1(z-1) - \delta\left[a_0(z+1) - \sigma a_0(z-1)\right] = 0. \tag{3.17}$$

Equation (3.12) *with* $\sigma = -1$

Denote by $n_{fin}(r,w)$ the counting function for those poles of w which are one of the three middle iterates of a sequence of the type (3.14), and by $n_{inf}(r,w)$ the counting function for the rest of the poles of w. From the proof of Lemma 3.2 it can be seen that

$$n_{inf}(r, \overline{w} + \sigma\underline{w}) \leq \left(\frac{8}{5} + \frac{\varepsilon}{2}\right) n_{inf}(r+1,w) + S'(r,w)$$

for any $\varepsilon > 0$, and so by integrating, we obtain

$$N_{inf}(r, \overline{w} + \sigma\underline{w}) \leq \left(\frac{8}{5} + \varepsilon\right) N_{inf}(r+1,w) + S(r,w). \tag{3.18}$$

Assuming that $\sigma = -1$, sequence (3.14) becomes

$$(\infty^{\mathbf{l_j}}, \delta + 0^{\mathbf{k_j}}, \infty^{\mathbf{k_j}}, \delta + 0^{\mathbf{k_j}}, \infty^{\mathbf{m_j}}) \tag{3.19}$$

where $l_j, m_j < \frac{3}{4}k_j$. Suppose that there are more than $S(r,w)$ δ-points of w which are not part of a sequence (3.19). Then by (3.18) there is a constant $c > 0$ and a set F with infinite logarithmic measure such that

$$\frac{N_{inf}(r+1,w)}{T(r,w)} \geq c \tag{3.20}$$

for all $r \in F$. By Theorem 2.3, the Valiron-Mohon'ko identity (2.2), and inequalities

(3.18) and (3.20), we have

$$\begin{aligned}2T(r,w) &= T(r,\overline{w}-\underline{w})+S(r,w)\\ &= m(r,\overline{w}-\underline{w})+N(r,\overline{w}-\underline{w})+S(r,w)\\ &\le 2N_{fin}(r+1,w)+\left(\frac{8}{5}+\varepsilon\right)N_{inf}(r+1,w)+S(r+1,w)\\ &\le 2N(r+1,w)-\left(\frac{2}{5}-\varepsilon\right)N_{inf}(r+1,w)+S(r+1,w)\\ &\le 2T(r+1,w)-c\left(\frac{2}{5}-\varepsilon\right)T(r,w)+S(r+1,w)\end{aligned}$$

for all $r\in F$. Hence,

$$T(r,w)\le\left(\frac{2}{2+c\left(\frac{2}{5}-\varepsilon\right)}+\varepsilon\right)T(r+1,w)$$

holds in a set with infinite logarithmic measure. Therefore w is of infinite order by Lemma 2.1. So if w is of finite order then the number of δ-points which are not part of (3.19) is at most $S(r,w)$.

Since $a_0\not\equiv\pm a_1$ due to the irreducibility of the right side of (3.12), Theorem 2.4 yields

$$N\left(r,\frac{1}{w-1}\right)=T(r,w)+S(r+1,w) \tag{3.21}$$

and

$$N\left(r,\frac{1}{w+1}\right)=T(r,w)+S(r+1,w). \tag{3.22}$$

If $N(r,\frac{1}{w\pm1})=S(r,w)$ it follows by (3.21) or (3.22) that $T(r,w)=S(r+1,w)$. In this case w is of infinite order by Lemma 2.1. Therefore w has more than $S(r,w)$ δ-points for both choices of $\delta=\pm1$. Since all except possibly at most $S(r,w)$ δ-points of w are in a sequence of the type (3.19) we conclude that (3.19) holds for more than $S(r,w)$ points for $\delta=1$ and $\delta=-1$. Thus also equation (3.17) holds with both choices of $\delta=\pm1$. Hence,

$$a_1(z+1)+2a_1(z)+a_1(z-1)=0$$

and

$$a_0(z+1)+a_0(z-1)=0.$$

By solving these equations we obtain $a_1(z)=(\lambda z+\mu)(-1)^z$ and $a_0(z)=\nu\, i^z+\gamma(-i)^z$ where $\lambda,\mu,\nu,\gamma\in\mathcal{S}(w)$ are arbitrary finite order periodic functions with period one. Therefore, (3.12) becomes

$$\overline{w}-\underline{w}=\frac{(\lambda z+\mu)(-1)^z w+\nu\, i^z+\gamma(-i)^z}{1-w^2}. \tag{3.23}$$

By the transformation $w\to i^z w$ equation (3.23) takes the form (1.7).

Equation (3.12) *with* $\sigma=1$

We will now look at the case $\sigma=1$. If (3.14) holds for both choices of $\delta=\pm1$ for more than $S(r,w)$ points, then so does (3.17). In this case (3.17) can be solved to obtain $a_1(z)=\pi_1 z+\kappa_1$ and $a_0(z)=\pi_2$, where π_k and κ_k are arbitrary small periodic functions of finite order with period k. Therefore equation (3.12) reduces to the difference Painlevé II equation (1.8). In Section 3.2 we will show that the case where (3.14) holds for only one choice of ± 1, with the possible exception of $S(r,w)$ points, leads to a difference Riccati equation.

Equation (3.3) *with* $c_2 \not\equiv 0$

Let us now return back to relation (3.9). If $c_2 \not\equiv 0$ then by (3.8) and (3.9), we have

$$c_2(z)^4 - 2\big(p(z+1)^2 + p(z-1)^2\big)c_2(z)^2 + \big(p(z+1)^2 - p(z-1)^2\big)^2 = 0,$$

which can be solved for c_2 to obtain

$$c_2(z) = \sigma_1 p(z+1) + \sigma_2 p(z-1), \tag{3.24}$$

where $\sigma_j^2 = 1$ and $p(z)$ is meromorphic on a suitable Riemann surface (recall from (3.3) that $p = \sqrt{a^2/4 - b}$ where a and b are small meromorphic functions.) Since c_2 is meromorphic by assumption, the right side of (3.24) cannot have any branching, although p can. We conclude that equation (3.3) takes the form

$$\overline{w} + \underline{w} = \frac{(\sigma_1 \overline{p} + \sigma_2 \underline{p})w^2 + c_1 w + c_0}{w^2 - p^2}, \tag{3.25}$$

where $\sigma_j^2 = 1$ for $j = 1,2$. This equation will also lead to a difference Riccati equation, and it will be dealt with in Section 3.2.

3.2. *Difference Riccati Equation*

In the previous section we considered equation (1.1) in the case when the denominator of $R(z,w)$ has two distinct roots. In this section we finish this case by showing that in all remaining subcases which were not dealt with in Section 3.1 the meromorphic solution w of (3.3) satisfies a difference Riccati equation.

Equation (3.12)

In Section 3.1 we have shown that if w has more than $S(r,w)$ singularities of the two types $(\infty^{\mathbf{l_j}}, 1+0^{\mathbf{k_j}}, \infty^{\mathbf{k_j}}, -1+0^{\mathbf{k_j}}, \infty^{\mathbf{m_j}})$ and $(\infty^{\mathbf{l_j}}, -1+0^{\mathbf{k_j}}, \infty^{\mathbf{k_j}}, 1+0^{\mathbf{k_j}}, \infty^{\mathbf{m_j}})$, the equation (3.12) reduces into the difference Painlevé II equation (1.8). If there are only $S(r,w)$ singularities of the types (3.14) altogether, then w is of infinite order by Lemma 3.2. For equation (3.12), the only remaining case to be considered is the one where w has more than $S(r,w)$ singularities of only one type, say $(\infty^{\mathbf{l_j}}, 1+0^{\mathbf{k_j}}, \infty^{\mathbf{k_j}}, -1+0^{\mathbf{k_j}}, \infty^{\mathbf{m_j}})$, and at most $S(r,w)$ of the other type. (The reasoning in the case where there are more than $S(r,w)$ singularities of the other type is almost identical to the following one, and will not be repeated.) By making a substitution

$$U = (w-1)(\overline{w}+1), \tag{3.26}$$

we have

$$(\underline{U} + U + a_1)w = \underline{U} - U - a_0. \tag{3.27}$$

Note that for the three middle points in the sequence $(\infty^{\mathbf{l_j}}, 1+0^{\mathbf{k_j}}, \infty^{\mathbf{k_j}}, -1+0^{\mathbf{k_j}}, \infty^{\mathbf{m_j}})$ the function U is finite and non-zero. The possible poles and zeros of U arise from the $S(r,w)$ singularities of the other type $(\infty^{\mathbf{l_j}}, -1+0^{\mathbf{k_j}}, \infty^{\mathbf{k_j}}, 1+0^{\mathbf{k_j}}, \infty^{\mathbf{m_j}})$, and from singularities in certain other type of sequences.

If $\underline{U} + U + a_1 \equiv 0$, then also $\underline{U} - U - a_0 \equiv 0$. Therefore,

$$U = -\frac{1}{2}(a_0 + a_1), \tag{3.28}$$

and so U is a small function with respect to w. Hence, it follows from (3.26) and (3.28) that w satisfies the difference Riccati equation (1.2) with $p = -1$ and $q = (2 - a_0 - a_1)/2$. Note that the coefficients satisfy

$$a_1(z+1) - a_1(z) = a_0(z+1) + a_0(z).$$

Suppose on the contrary that $\underline{U} + U + a_1 \not\equiv 0$, and assume first that

$$N_{inf}(r, w) = o(T(r, w)) \tag{3.29}$$

in a set E of r-values with infinite logarithmic measure (recall the definitions of $N_{inf}(r, w)$ and $N_{fin}(r, w)$ from Section 3.1.) By substituting (3.27) into (3.26), we have

$$\begin{aligned}
&U^3\left(\frac{\underline{U}}{U} + \frac{\underline{U}\overline{U}}{U^2} + \frac{\overline{U}}{U} + 1\right) + U^2\left(a_1(z+1)\frac{\underline{U}}{U} + a_1(z)\frac{\overline{U}}{U} + a_1(z+1) + a_1(z) + 4\right)\\
&+U\left(a_1(z)a_1(z+1) + 2a_1(z+1) - 2a_0(z+1) + 2a_1(z) + 2a_0(z)\right)\\
&+\left(a_0(z) + a_1(z)\right)\left(a_1(z+1) - a_0(z+1)\right) = 0
\end{aligned}$$

which we denote by

$$A_3(z)U^3 + A_2(z)U^2 + A_1(z)U + A_0 = 0 \tag{3.30}$$

for brevity. By the construction of U and relation (3.29), we obtain

$$\begin{aligned}
N(r, A_j) &= O\left(N\left(r+1, \frac{1}{U}\right) + N(r+1, U)\right) + S(r+1, w)\\
&= O\left(N_{inf}(r+2, w)\right) + S(r+1, w)\\
&= o(T(r+2, w))
\end{aligned} \tag{3.31}$$

in a set with infinite logarithmic measure for all $j = 0, \ldots, 3$. Also, given $\mu < 1$, Theorem 2.2 and equation (3.26) imply that

$$\begin{aligned}
m(r, A_j) &= o\left(\frac{T(r+1, U)}{r^\mu}\right) + S(r+1, w)\\
&= o\left(\frac{T(r+2, w)}{r^\mu}\right) + S(r+1, w)\\
&= S(r+2, w)
\end{aligned} \tag{3.32}$$

for all $j = 0, \ldots, 3$. By combining equations (3.27) and (3.30)–(3.32), we have

$$\begin{aligned}
T(r, w) &\le 5T(r+1, U) + S(r, w)\\
&= O\left(\sum_{j=1}^{3} T(r+1, A_j)\right) + S(r, w)\\
&= o(T(r+2, w))
\end{aligned} \tag{3.33}$$

in a set with infinite logarithmic measure unless all $A_j(z)$, $j = 0, \ldots, 3$, vanish identically. If (3.33) holds we obtain

$$T(r, w) \le \varepsilon T(r+3, w),$$

where $0 < \varepsilon < 1$ and r is in a set with infinite logarithmic measure. Therefore w

is of infinite order by Lemma 2.1. In the case where all coefficients of (3.30) vanish identically we have

$$A_0(z) = (a_0(z) + a_1(z))\,(a_1(z+1) - a_0(z+1)) = 0,$$

which implies that $a_0 = \pm a_1$ contradicting the irreducibility of (3.12).

Now assume that $\underline{U} + U + a_1 \not\equiv 0$, and that (3.29) holds only for a set of finite logarithmic measure. In this case

$$N_{inf}(r,w) \geq cT(r,w) \tag{3.34}$$

outside of a set with finite logarithmic measure and for an absolute constant $c > 0$. Combining (3.12), (3.18) and (3.34) with Theorem 2.3, we obtain

$$\begin{aligned} 2T(r,w) &= T(r,\overline{w}+\underline{w}) + S(r,w) \\ &= m(r,\overline{w}+\underline{w}) + N(r,\overline{w}+\underline{w}) + S(r,w) \\ &\leq K\frac{T(r+1,w)}{r^{\mu}} + 2N(r+1,w) \\ &\quad - \left(\frac{2}{5} - \varepsilon\right) N_{inf}(r+1,w) + S(r,w) \\ &\leq \left(\frac{K}{r^{\mu}} + 2 - c\left(\frac{2}{5} - \varepsilon\right)\right) T(r+1,w) + S(r,w), \end{aligned} \tag{3.35}$$

where $\varepsilon > 0$, $K > 0$, $\mu < 1$ and $c > 0$. Hence there exists an absolute constant $\varepsilon' > 0$ such that

$$T(r,w) \leq (1-\varepsilon')T(r+1,w)$$

outside of a set with finite logarithmic measure. Thus w is of infinite order by Lemma 2.1.

Equation (3.25) *with* $\sigma_1\sigma_2 = -1$

We conclude this part of the proof by looking at the equation (3.25). First, assume that w is a solution of

$$\overline{w} + \underline{w} = \frac{\sigma(\overline{p} - \underline{p})w^2 + c_1 w + c_0}{w^2 - p^2}, \tag{3.36}$$

where $\sigma = \pm 1$ and $\sigma(\overline{p} - \underline{p}) \not\equiv 0$. We redefine the counting function $N_{fin}(r,w)$ to count the singularities of a meromorphic solution of equation (3.36) appearing as part of a sequence

$$(\infty^{\mathbf{l_j}}, \delta p + 0^{\mathbf{k_j}}, \infty^{\mathbf{k_j}}, \pm\delta p + 0^{\mathbf{k_j}}, \infty^{\mathbf{m_j}}), \tag{3.37}$$

where $\delta = \pm 1$, and l_j and m_j are strictly less than $\frac{3}{4}k_j$. $N_{inf}(r,w)$ is the counting function for the rest of the singularities.

Similarly as for (3.21) and (3.22), we obtain by Theorem 2.4 that a finite-order meromorphic solution w of (3.36) has more than $S(r,w)$ p and $-p$ points. Therefore, by Lemma 3.1 we may choose more than $S(r,w)$ points z_j for an arbitrarily small $\epsilon \geq 0$ such that $w(z-1) = -\sigma p(z-1) + O((z-z_j)^{k_j})$ for all $z \in D(z_j, \tau_j)$ and w has a pole of order at least $(1-\epsilon)k_j$ at either z_j or $z_j - 2$. Say, $w(z_j) = \infty$. Then $w(z+1) = \sigma p(z+1) + O((z-z_j)^{(1-\epsilon')k_j})$ again by using Lemma 3.1. On the other hand if $w(z-1) = \sigma p(z-1) + O((z-z_j)^{k_j})$ and $w(z_j) = \infty$ with the multiplicity at least $(1-\epsilon)k_j$, then $w(z+1) = \sigma(p(z+1) - 2p(z-1)) + O((z-z_j)^{(1-\epsilon')k_j})$.

Therefore there can be only $S(r,w)$ points z_j such that

$$\begin{aligned} w(z-1) &= \sigma p(z-1) + O\left((z-z_j)^{k_j}\right), \\ w(z) &= \beta(z-z_j)^{-k_j} + O\left((z-z_j)^{1-k_j}\right), \quad \beta \neq 0, \\ w(z+1) &= \pm\sigma p(z+1) + O\left((z-z_j)^{k_j}\right), \end{aligned} \tag{3.38}$$

since otherwise $\sigma(p(z+1) - 2p(z-1)) = \pm\sigma p(z+1) + O((z-z_j)^{(1-\epsilon')k_j})$ in neighborhoods of more than $S(r,w)$ points, and p would either be identically zero or a periodic function with period two. In either case $\sigma(\overline{p} - \underline{p})$ would vanish which is a contradiction. Similarly there can be only $S(r,w)$ points such that

$$\begin{aligned} w(z-1) &= -\sigma p(z-1) + O\left((z-z_j)^{k_j}\right), \\ w(z) &= \beta(z-z_j)^{-k_j} + O\left((z-z_j)^{1-k_j}\right), \quad \beta \neq 0, \\ w(z+1) &= -\sigma p(z+1) + O\left((z-z_j)^{k_j}\right). \end{aligned} \tag{3.39}$$

Therefore, if w has more than $S(r,w)$ singularities in sequences of the type (3.37), then there are more than $S(r,w)$ sequences of the type

$$(\infty^{\mathbf{l_j}}, -\sigma p + 0^{\mathbf{k_j}}, \infty^{\mathbf{k_j}}, \sigma p + 0^{\mathbf{k_j}}, \infty^{\mathbf{m_j}}), \tag{3.40}$$

and only $S(r,w)$ of the types (3.38) and (3.39).

Since all except possibly $S(r,w)$ sequences of the type (3.37) are in fact of the form (3.40) we make a change of variable

$$U = (w + \sigma p)(\overline{w} - \sigma\overline{p}) \tag{3.41}$$

which takes equation (3.36) to the form

$$(U + \overline{U} - c_1)w = \sigma p(U - \underline{U}) + c_0 + \sigma(\overline{p} - \underline{p})p^2. \tag{3.42}$$

If the left and the right side of (3.42) both vanish, we obtain

$$U = \frac{1}{2}\left(c_1 - \frac{\sigma c_0}{p} - (\overline{p} - \underline{p})p\right), \tag{3.43}$$

and so (3.41) is a difference Riccati equation (1.2). If not, then by combining (3.41) and (3.42), we have

$$B_3(z)U^3 + B_2(z)U^2 + B_1(z)U + B_0(z) = 0,$$

where

$$\begin{aligned} B_0(z) &= \left(\sigma(\overline{p} - \underline{p})p^2 + c_0 - \sigma p\, c_1\right)\left(\sigma(\overline{\overline{p}} - p)\overline{p}^2 + \overline{c}_0 + \sigma\overline{p}\,\overline{c}_1\right), \\ B_1(z) &= 2\sigma p\left(\sigma(\overline{p} - \underline{p})p^2 + c_0 - \sigma p\, c_1\right) - 2\sigma\overline{p}\left(\sigma(\overline{\overline{p}} - p)\overline{p}^2 + \overline{c}_0 + \sigma\overline{p}\,\overline{c}_1\right), \\ B_2(z) &= 4p\overline{p} - c_1\left(1 + \frac{\overline{U}}{U}\right) - \overline{c}_1\left(1 + \frac{\underline{U}}{U}\right), \\ B_3(z) &= \frac{\underline{U}}{U} + \frac{\overline{U}\underline{U}}{U^2} + \frac{\overline{U}}{U} + 1. \end{aligned}$$

If $N_{inf}(r,w) = S(r,w)$ then, by a similar reasoning as for the equation (3.12), either w is of infinite order, or all coefficients $B_j(z)$ must vanish identically. In particular, since $B_0(z) \equiv B_1(z) \equiv 0$, we have

$$\sigma(\overline{p} - \underline{p})p^2 + c_0 - \sigma p\, c_1 = 0$$

and

$$\sigma(\overline{p}-\underline{p})p^2+c_0+\sigma p\, c_1=0.$$

But this means that there is a drop of two in the degree of the right side of (3.36), which contradicts the irreducibility of the equation. We therefore conclude that w satisfies the Riccati equation

$$\overline{w}=\frac{\sigma\overline{p}w+p\overline{p}+U}{w+\sigma p},$$

where U is as in (3.43). Conversely, any meromorphic solution of

$$\overline{w}=\frac{\sigma\overline{p}w+c}{w+\sigma p}$$

is also a solution of

$$\overline{w}+\underline{w}=\frac{\sigma(\overline{p}-\underline{p})w^2+(c+\underline{c}-p(\overline{p}+\underline{p}))w+\sigma p(\underline{c}-c)}{w^2-p^2}. \tag{3.44}$$

Note that when p is a non-zero periodic function, equation (3.44) reduces into (3.12).

If $N_{inf}(r,w)\neq S(r,w)$ then w is of infinite order by a similar calculation to that of (3.35).

Equation (3.25) *with* $\sigma_1\sigma_2=1$

Suppose now that w is a solution of

$$\overline{w}+\underline{w}=\frac{\sigma(\overline{p}+\underline{p})w^2+c_1w+c_0}{w^2-p^2}. \tag{3.45}$$

The functions $\pm\sigma p$ (which may have square root type branching) cannot satisfy $\sigma(\overline{p}+\underline{p})p^2\pm\sigma c_1 p+c_0\equiv 0$, since otherwise the right side of (3.45) would not be irreducible. Therefore, by Theorem 2.4 we have

$$m\left(r,\frac{1}{w\pm\sigma p}\right)=S(r+1,w) \tag{3.46}$$

for both choices of $\pm\sigma p$. Hence w has more than $S(r,w)$ σp points and $-\sigma p$ points (in other words $w\pm\sigma p$ has more than $S(r,w)$ zeros for both choices of the sign), since otherwise by (3.46) we would have $T(r,w)=S(r+1,w)$ which implies that w is of infinite order by Lemma 2.1.

Assuming that $w(z_j)=-\sigma p(z_j)$, either $w(z_j+1)=\infty$ or $w(z_j-1)=\infty$, and we have by Lemma 3.1, provided that $w(z_j+2)\neq\pm\sigma p(z_j+2)$,

$$\begin{aligned}
w(z)&=-\sigma p(z)+O\left((z-z_j)^{k_j}\right)\\
w(z+1)&=\beta_1(z-z_j)^{-(1-\epsilon_1)k_j}+O\left((z-z_j)^{1-(1-\epsilon_1)k_j}\right)\\
w(z+2)&=2\sigma p(z)+\sigma p(z+2)+O\left((z-z_j)^{(1-\epsilon_2)k_j}\right)\\
w(z+3)&=\beta_3(z-z_j)^{-(1-\epsilon_3)k_j}+O\left((z-z_j)^{1-(1-\epsilon_3)k_j}\right)\\
w(z+4)&=-2\sigma p(z)+\sigma p(z+4)+O\left((z-z_j)^{(1-\epsilon_4)k_j}\right),
\end{aligned} \tag{3.47}$$

where $\beta_1\beta_3\neq 0$ and $\epsilon_4\geq\epsilon_3\geq\epsilon_2\geq\epsilon_1\geq 0$ are arbitrarily small constants such that by construction $(1-\epsilon_i)k_j\in\mathbb{N}$ for all $i=1,\ldots,4$. If there are more than $S(r,w)$ points counting multiplicities such that $w(z_j+2)=\pm\sigma p(z_j+2)$, then either $p\equiv 0$

or p satisfies the equation $p(z+2) = -p(z)$. But in both cases $\overline{p}+\underline{p} \equiv 0$, which is a contradiction. Therefore, with the possible exception of $S(r,w)$ points, for each k_j points z_j such that $w(z_j) = -\sigma p(z_j)$ there are at least $(1-\epsilon)k_j$ poles of w which may be uniquely associated to the point $w(z_j) = -\sigma p(z_j)$. Similarly, assuming that $w(z_j) = \sigma p(z_j)$, we obtain

$$\begin{aligned}
w(z) &= \sigma p(z) + O\left((z-z_j)^{k_j}\right)\\
w(z+1) &= \beta_1 (z-z_j)^{-(1-\epsilon_1)k_j} + O\left((z-z_j)^{1-(1-\epsilon_1)k_j}\right), \quad \beta_1 \neq 0\\
w(z+2) &= \sigma p(z+2) + O\left((z-z_j)^{(1-\epsilon_2)k_j}\right)\\
w(z+3) &= \beta_3 (z-z_j)^{-(1-\epsilon_3)k_j} + O\left((z-z_j)^{1-(1-\epsilon_3)k_j}\right)\\
w(z+4) &= \sigma p(z+4) + O\left((z-z_j)^{(1-\epsilon_4)k_j}\right).
\end{aligned}$$

If there are more than $S(r,w)$ points such that $w(z_j+2) = -\sigma p(z_j+2)$, then $\overline{p}+\underline{p} \equiv 0$ which is a contradiction. Hence, excluding at most $S(r,w)$ points, for each k_j points z_j such that $w(z_j) = \sigma p(z_j)$ there are at least $(1/2-\epsilon)k_j$ poles of w which may be uniquely associated to the point $w(z_j) = \sigma p(z_j)$. We conclude that

$$N(r+1,w) \geq (1-\epsilon)\left(N\left(r, \frac{1}{w+\sigma p}\right) + \frac{1}{2}N\left(r, \frac{1}{w-\sigma p}\right)\right) + S(r,w) \tag{3.48}$$

for any $\epsilon > 0$. Since by (3.46)

$$N\left(r, \frac{1}{w \pm \sigma p}\right) = T(r,w) + S(r+1,w)$$

for both choices of $\pm\sigma p$, (3.48) yields

$$\begin{aligned}
\frac{3}{2}T(r,w) &= N\left(r, \frac{1}{w+\sigma p}\right) + \frac{1}{2}N\left(r, \frac{1}{w-\sigma p}\right) + S(r+1,w)\\
&\leq \frac{1}{1-\epsilon}T(r+1,w) + S(r+1,w).
\end{aligned}$$

Hence

$$T(r,w) \leq \left(\frac{2}{3} + \epsilon'\right) T(r+1,w)$$

for any $\epsilon' > 0$ and for all r outside of a set of finite logarithmic measure. Thus w is of infinite order by Lemma 2.1.

3.3. *The Difference Painlevé I Equation*

In subsections 3.1 and 3.2 we dealt with the case where the denominator of $R(z,w)$ in (1.1) has two distinct roots. All remaining cases are of the form

$$\overline{w} + \underline{w} = \frac{u_2 w^2 + u_1 w + u_0}{(w+a)^q}, \tag{3.49}$$

where the coefficients of the right side are small meromorphic functions, and $q \in \{0,1,2\}$. The transformation $w \to w - a$ takes (3.49) into the form

$$\overline{w} + \underline{w} = \frac{a_2 w^2 + a_1 w + a_0}{w^q}, \tag{3.50}$$

where the coefficients a_j are meromorphic and small with respect to w. In [**2**] and [**22**] it was proven that if the coefficients a_j are rational functions, then all meromorphic solutions of (3.50) where $q = 0$ are of infinite order, provided that $a_2 \not\equiv 0$. On the other hand if $q = 0$ and $a_2 \equiv 0$ equation (3.50) is the linear equation (1.10). In the remainder of this paper we consider the three cases $q \in \{0, 1, 2\}$ separately.

Equation (3.50) *with* $q = 0$

Suppose that $a_2 \not\equiv 0$ and $q = 0$ in equation (3.50). Assume first that $N(r, w) = S(r, w)$. Since by Theorem 2.3 we have $m(r, w) = S(r + 1, w)$ it follows that

$$T(r, w) \leq \varepsilon T(r + 1, w)$$

with any $\varepsilon > 0$ in a set with infinite logarithmic measure. Therefore w is of infinite order in this case by Lemma 2.1.

Now assume that w has more than $S(r, w)$ poles. By Lemma 3.1 there are more than $S(r, w)$ points z_j such that the multiplicity of $w(z_j) = \infty$ is greater than K times the multiplicity of $a_2(z_j) = 0$ for any $K > 1$. Suppose that w has such a pole at z_j, say of multiplicity k_j. Then either $w(z_j + 1) = \infty$ or $w(z_j - 1) = \infty$, at least with the multiplicity $(2 - 1/K)k_j$. Without loss of generality we assume that $w(z_j + 1) = \infty^{\mathbf{(2\text{-}1/K)k_j}}$. Then either $w(z_j + 2) = \infty$ with the multiplicity at least $(4 - 3/K)k_j$, or a_2 has a zero with multiplicity greater than k_j/K at $z_j + 1$. In the former case $w(z_j + 3) = \infty^{\mathbf{(8\text{-}7/K)k_j}}$ (at least), which implies that either $w(z_j + 4) = \infty^{\mathbf{(16\text{-}15/K)k_j}}$ (at least), or there is a zero of a_2 at $z_j + 2$ with the multiplicity greater than k_j/K. And so on. Not all sequences of iterates of this type can have a zero of a_2 with the multiplicity greater than k_j/K in them, since otherwise a_2 would have more than $S(r, w)$ zeros (counting multiplicities) which implies $a_2 \equiv 0$ contradicting the assumption. Hence there is at least one infinite sequence, say $(z_0 + n)$, $n \in \mathbb{N}$, such that the multiplicities of $a_2(z_0 + n) = 0$ are all less than k_0/K for all $n \in \mathbb{N}$, and so

$$n(r, w) \geq \left(1 - \frac{1}{K}\right) 2^{r - r_0}$$

for some $r_0 \geq 0$ and for any $K > 1$. Therefore w is of infinite order.

We conclude that if w is of finite order then $a_2 \equiv 0$, and (3.50) with $q = 0$ reduces to the linear difference equation (1.10).

Equation (3.50) *with* $q = 2$

This subcase is very similar to the derivation of the difference Painlevé II in the beginning of Section 3.1, and so the details are kept to a minimum.

Similarly as in (3.5) we conclude that w has more than $S(r, w)$ poles. Also, Theorem 2.4 implies that all admissible finite-order solutions of (3.50) with $q = 2$ also have more than $S(r, w)$ zeros, provided that $a_0(z) \not\equiv 0$. On the other hand if $a_0(z) \equiv 0$ the degree of the right side of (3.50) drops contradicting irreducibility of the rational expression.

Choose a point z_j such that $w(z_j - 1) = 0$ with multiplicity k_j. Then by (3.50) and Lemma 3.1 w has a pole of multiplicity $(1 - \epsilon)k_j$, with an arbitrarily small $\epsilon \geq 0$, at either z_j or $z_j - 2$. We assume, without loss of generality, that $w(z_j) = \infty$. If $w(z_j + 1) = 0$ with multiplicity less than $\frac{1}{3}k_j$ for all but $S(r, w)$ points z_j, then inequality (3.4) is satisfied with $\alpha = \frac{4}{3} + 2\epsilon/(1 - \epsilon)$. In this case Theorem 2.5 implies that w is of infinite order. Hence there must be more than $S(r, w)$ points z_j such that $w(z_j) = \infty$ with multiplicity k_j and $w(z_j \pm 1) = 0$ with multiplicities at least $\frac{1}{3}k_j$ for both choices of the sign. For such points equation (3.50) shows that

$$w(z+1) + w(z-1) = a_2(z) + O((z - z_j)^{\frac{1}{3}k_j})$$

in a small neighborhood $D(z_j, \tau_j)$ of z_j. Since $w(z_j + 1) = w(z_j - 1) = 0$, at least with multiplicity $\frac{1}{3}k_j$, and a_2 is small with respect to w, we have $a_2(z) \equiv 0$ and so (3.50) (with $q = 2$) reduces into

$$\overline{w} + \underline{w} = \frac{a_1 w + a_0}{w^2}. \tag{3.51}$$

To reduce the equation further we need the following analogue of Lemma 3.2.

LEMMA 3.3. *Let w be an admissible meromorphic solution of equation* (3.51). *Then either,*

$$n(r, \overline{w} + \underline{w}) \leq \left(\frac{16}{9} + \epsilon\right) n(r+1, w) + S'(r, w) \tag{3.52}$$

for any $\epsilon > 0$, or there are more than $S(r, w)$ points z_j such that

$$\begin{aligned} &w(z_j - 2) = \infty^{l_j}, \quad w(z_j - 1) = 0^{k_j}, \quad w(z_j) = \infty^{2k_j}, \\ &w(z_j + 1) = 0^{k_j}, \quad w(z_j + 2) = \infty^{m_j}, \end{aligned} \tag{3.53}$$

where l_j and m_j are strictly less than $\frac{3}{4}k_j$.

The proof of Lemma 3.3 is almost identical to that of Lemma 3.2, and hence will not be repeated. The essential difference between the proofs of these lemmas can be seen by comparing Table 2 with Table 1.

TABLE 2. *The multiplicity l_j and m_j in* (3.53). *The poles and zeros of w which are to be grouped together are marked by "$*$". The notation "†" means that only a third of the multiplicity of the point is associated with the other points in the group.*

$l_j, m_j < \frac{3}{4}k_j$	∞^{l_j}	0^{k_j}	∞^{2k_j}	0^{k_j}	∞^{m_j}	(3.53)
$l_j < \frac{3}{4}k_j$, $m_j \geq \frac{3}{4}k_j$	∞^{l_j}	$0^{k_j}*$	$\infty^{2k_j}*$	$0^{k_j}*$	$\infty^{m_j}†$	*ratio* $\leq 16/9$
$l_j \geq \frac{3}{4}k_j$, $m_j < \frac{3}{4}k_j$	$\infty^{l_j}†$	$0^{k_j}*$	$\infty^{2k_j}*$	$0^{k_j}*$	∞^{m_j}	*ratio* $\leq 16/9$
$l_j, m_j \geq \frac{3}{4}k_j$	$\infty^{l_j}†$	$0^{k_j}*$	$\infty^{2k_j}*$	$0^{k_j}*$	$\infty^{m_j}†$	*ratio* $\leq 8/5$

Now, by manipulating equation (3.51), we obtain

$$\begin{aligned} \overline{w}^2(\overline{\overline{w}} - \underline{\underline{w}}) &= \overline{a}_0 + \overline{a}_1\overline{w} - \underline{a}_0 - \underline{a}_1\underline{w} \\ &\quad + (w + \underline{\underline{w}}) \left[\frac{2\underline{w}(a_0 + a_1 w)}{w^2} - \left(\frac{a_0 + a_1 w}{w^2}\right)^2 \right]. \end{aligned} \tag{3.54}$$

If inequality (3.52) holds the solution w of (3.51) is of infinite order by Theorem 2.5. On the other hand, if (3.53) is true for more than $S(r, w)$ points z_j, we have by (3.54)

$$a_0(z+1) - a_0(z-1) = 0. \tag{3.55}$$

Equation (3.54) may then be written as

$$\overline{w}(\overline{\overline{w}} - \underline{\underline{w}}) = \overline{a}_1 + \underline{a}_1 \left(1 - \frac{a_1 w + a_0}{\overline{w} w^2}\right) + (w + \underline{\underline{w}}) \left[\frac{2(a_0 + a_1 w)}{w^2} \left(-1 + \frac{a_1 w + a_0}{\overline{w} w^2}\right) - \frac{1}{\overline{w}} \left(\frac{a_0 + a_1 w}{w^2}\right)^2\right],$$

and so

$$a_1(z+1) - 2a_1(z) + a_1(z-1) = 0. \tag{3.56}$$

By solving equations (3.55) and (3.56) we obtain $a_1(z) = \pi_1 z + \kappa_1$ and $a_0(z) = \pi_2$, where π_k and κ_k are arbitrary finite order periodic functions with period k, and small compared to w. Therefore equation (3.51) reduces to the difference Painlevé I equation (1.6).

3.4. *The first-degree case, and related equations*

So far we have considered equations of the form (1.1) where the degree of R as a function of w has been exactly two. We have shown that unless for each pole of w there are at most two poles of the left side of the equation, the solution is of infinite order. By a singularity analysis of solutions, the only equations of the form (1.1) satisfying this condition are (1.6), (1.7), (1.8) and (1.2).

In this section we will look at the case $\deg_w R(z, w) = 1$, and some equations with $\deg_w R(z, w) = 2$ which behave similarly in the sense of Nevanlinna theory. Now the solution may be of finite order if for each pole of w there is at most one nearby pole of the left side, instead of two. This means that we will have to be more careful in the singularity analysis to get to the difference Painlevé equations.

Equation (3.50) *with* $q = 1$

We write (3.50) with $q = 1$ in the form

$$\overline{w} + \underline{w} - a_2 w = \frac{a_1 w + a_0}{w}. \tag{3.57}$$

There can be at most $S(r, w)$ points z_j such that

$$w(z_j) = a_1(z_0) w(z_j) + a_0(z_j) = 0 \tag{3.58}$$

since otherwise the right side of (3.57) (and so also of (3.50)) would be reducible. Like before, we include all such points in the error term $S(r, w)$, as well as all points where a coefficient of (3.50) has a high multiplicity zero in the sense of Lemma 3.1.

Also, all finite order solutions of (3.57) have more than $S(r, w)$ poles and zeros. This can be seen by using Lemma 2.1 together with Theorems 2.3 and 2.4.

Choose a point z_j such that $w(z_j - 1) = 0$ with multiplicity k_j. Then by Lemma 3.1 and (3.57) w has a pole of order at least $(1 - \epsilon)k_j$ at either z_j or $z_j - 2$ for an arbitrarily small constant $\epsilon \geq 0$. We assume, without loss of generality, that $w(z_j)=\infty$. Then

$$\begin{aligned} w(z-1) &= \alpha(z-z_j)^{k_j} + O\left((z-z_j)^{k_j+1}\right), \quad \alpha\neq 0 \\ w(z) &= \beta(z-z_j)^{-(1-\epsilon_1)k_j} + O\left((z-z_j)^{1-(1-\epsilon_1)k_j}\right), \quad \beta\neq 0 \\ w(z+1) &= a_2(z)\beta(z-z_j)^{-(1-\epsilon_1)k_j} + O\left((z-z_j)^{1-(1-\epsilon_2)k_j}\right) \\ w(z+2) &= \beta(a_2(z+1)a_2(z)-1)(z-z_j)^{-(1-\epsilon_1)k_j} + O\left((z-z_j)^{1-(1-\epsilon_3)k_j}\right) \end{aligned} \tag{3.59}$$

for all $z\in D(z_j,\tau_j)$, where $\epsilon_3\geq\epsilon_2\geq\epsilon_1\geq 0$ are arbitrarily small constants satisfying, by construction, $(1-\epsilon_i)k_j\in\mathbb{N}$ for $i=1,2,3$.

Since w has more than $S(r,w)$ zeros, it also has more than $S(r,w)$ iteration sequences of the type (3.59). Assume that within these sequences only $S(r,w)$ points z_j satisfy

$$a_2(z_j)=0 \tag{3.60}$$

or

$$a_2(z_j+1)a_2(z_j)=1. \tag{3.61}$$

If $w(z_j+3)=0$ with the multiplicity less or equal to k_j we may associate the zero at z_j+3 with the other iterates in (3.59) and so for these iterates the inequality (2.11) holds with $\alpha=2/3+\epsilon$, $\epsilon\geq 0$. If the multiplicity of $w(z_j+3)=0$ is strictly greater than k_j then the inequality (2.11) holds for the iterates in (3.59) with $\alpha=1/3+\epsilon$, and z_j+3 is a starting point for another sequence of the type (3.59). Therefore we have (2.11) with $\alpha=2/3+\epsilon$ and so w is of infinite order by Theorem 2.5.

Thus either (3.60) or (3.61) holds at more than $S(r,w)$ points z_j and since a_2 is of finite order we have $a_2\equiv 0$ or $a_2\equiv\pm 1$. We consider the equations

$$\overline{w}+\underline{w}=\frac{a_1w+a_0}{w} \tag{3.62}$$

and

$$\overline{w}+\underline{w}+\sigma w=\frac{a_1w+a_0}{w},\quad \sigma=\pm 1, \tag{3.63}$$

separately.

Equation (3.62)

Equation (3.62) with $a_1\equiv 0$ is just (1.9), and so assume from now on that $a_1\not\equiv 0$.

We will show that each pole of $\overline{w}+\underline{w}$ in (3.62) (i.e. the zero of w) may be grouped together with a finite number of nearby poles of w in such a manner that the number of poles of $\overline{w}+\underline{w}$ divided by the number of poles of w (both counting multiplicities) is less than $4/5+\varepsilon$, unless (3.62) is the equation (1.5).

By equation (3.62), Theorem 2.3 and Lemma 3.1 there are more than $S(r,w)$ points z_j such that w has a pole of order at least k_j at z_j+1 or z_j-1 whenever w has a zero of multiplicity $(1+\epsilon)k_j\in\mathbb{N}$ at z_j, where $\epsilon\geq 0$ is an arbitrarily small constant, and there are only $S(r,w)$ other points where w has a pole.

We begin by considering the case in which both $w(z_j+1)$ and $w(z_j-1)$ are poles of the same order k_j. Denote $\delta=\pm 1$. Since w is meromorphic there is a disc

$D(z_j, \tau_j)$ centered at z_j with a suitably small radius τ_j such that

$$\begin{aligned} w(z) &= \alpha\,(z-z_j)^{(1+\epsilon)k_j} + O\left((z-z_j)^{(1+\epsilon)k_j+1}\right) \\ w(z+\delta) &= \beta_\delta\,(z-z_j)^{-k_j} + O\left((z-z_j)^{1-k_j}\right) \end{aligned} \tag{3.64}$$

for all $z \in D(z_j, \tau_j)$ where α and $\beta_{\pm 1}$ are non-zero. By iteration of equation (3.62), we obtain

$$\begin{aligned} w(z+2\delta) &= a_1(z+\delta) + O\left((z-z_j)^{(1-\epsilon_1)k_j}\right) \\ w(z+3\delta) &= -\beta_\delta\,(z-z_j)^{-k_j} + O\left((z-z_j)^{1-k_j}\right) \\ w(z+4\delta) &= a_1(z+3\delta) - a_1(z+\delta) + O\left((z-z_j)^{(1-\epsilon_2)k_j}\right) \\ w(z+5\delta) &= \beta_\delta(z-z_j)^{-k_j} + \frac{a_0(z+4\delta)}{a_1(z+3\delta) - a_1(z+\delta) + O\left((z-z_0)^{(1-\epsilon_2)k_j}\right)} \\ &\quad + O\left((z-z_0)^{1-k_j}\right) \end{aligned} \tag{3.65}$$

for all $z \in D(z_j, \tau_j)$, where $\epsilon_2 \geq \epsilon_1 \geq 0$ are arbitrarily small constants. Some of this information is summarised in the second row of Table 3.

Since a_1 has at most $S(r, w)$ poles we may include all cases where $a_1(z)$ has a pole with multiplicity greater than ϵk_j at $z_j + 3\delta$ or at $z_j + \delta$ into the error term of (2.10). Otherwise $w(z_j + 4\delta)$ is finite or has a pole with multiplicity at most ϵk_j. If $w(z_j + 4\delta)$ is non-zero, or a zero with the multiplicity $l_j < k_j$, then $w(z_j + 5\delta)$ has a pole of order k_j. If $w(z_j + 4\delta)$ has a zero with the multiplicity $M_j > k_j$ then $w(z_j + 5\delta)$ has a pole of order M_j. If $w(z_j + 4\delta)$ has a zero of order k_j then $w(z_j + 5\delta)$ is either regular at $z = z_j$ or it has a pole of order at most k_j. This information is summarised in Table 3.

TABLE 3. *Iteration of equation* (3.62)*. Here L_j and M_j are used to denote any integer greater than k_j, while l_j and m_j are integers less than k_j. The symbol "–" denotes either a pole of order strictly less than k_j or a regular point (i.e. a finite value, including zero.) The quantity $n\delta$ on the first row is a short notation for $w(z_j + n\delta)$, and f denotes a finite non-zero value or a pole or zero of a_1, not necessarily the same one each time the symbol is repeated.*

-5δ	-4δ	-3δ	-2δ	$-\delta$	0	δ	2δ	3δ	4δ	5δ
$\infty^{\mathbf{k_j}}$	$0^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}*$	$0^{\mathbf{(1+\epsilon)k_j}}*$	$\infty^{\mathbf{k_j}}*$	f	$\infty^{\mathbf{k_j}}$	$0^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}}$
$\infty^{\mathbf{k_j}}$	$0^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}*$	$0^{\mathbf{(1+\epsilon)k_j}}*$	$\infty^{\mathbf{k_j}}*$	f	$\infty^{\mathbf{k_j}}*$	$0^{\mathbf{k_j}}*$	–
$\infty^{\mathbf{k_j}}$	$0^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}*$	$0^{\mathbf{(1+\epsilon)k_j}}*$	$\infty^{\mathbf{k_j}}*$	f	$\infty^{\mathbf{k_j}}$	$0^{\mathbf{M_j}}$	$\infty^{\mathbf{M_j}}$
$\infty^{\mathbf{k_j}}$	$0^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}*$	$0^{\mathbf{(1+\epsilon)k_j}}*$	$\infty^{\mathbf{k_j}}*$	f	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}$
–	$0^{\mathbf{k_j}}*$	$\infty^{\mathbf{k_j}}*$	f	$\infty^{\mathbf{k_j}}*$	$0^{\mathbf{(1+\epsilon)k_j}}*$	$\infty^{\mathbf{k_j}}*$	f	$\infty^{\mathbf{k_j}}*$	$0^{\mathbf{k_j}}*$	–
–	$0^{\mathbf{k_j}}*$	$\infty^{\mathbf{k_j}}*$	f	$\infty^{\mathbf{k_j}}*$	$0^{\mathbf{(1+\epsilon)k_j}}*$	$\infty^{\mathbf{k_j}}*$	f	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}$
$\infty^{\mathbf{M_j}}$	$0^{\mathbf{M_j}}$	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}*$	$0^{\mathbf{(1+\epsilon)k_j}}*$	$\infty^{\mathbf{k_j}}*$	f	$\infty^{\mathbf{k_j}}*$	$0^{\mathbf{k_j}}*$	–
$\infty^{\mathbf{L_j}}$	$0^{\mathbf{L_j}}$	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}*$	$0^{\mathbf{(1+\epsilon)k_j}}*$	$\infty^{\mathbf{k_j}}*$	f	$\infty^{\mathbf{k_j}}$	$0^{\mathbf{M_j}}$	$\infty^{\mathbf{M_j}}$
$\infty^{\mathbf{M_j}}$	$0^{\mathbf{M_j}}$	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}*$	$0^{\mathbf{(1+\epsilon)k_j}}*$	$\infty^{\mathbf{k_j}}*$	f	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}$
$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}*$	$0^{\mathbf{(1+\epsilon)k_j}}*$	$\infty^{\mathbf{k_j}}*$	f	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}$
$\infty^{\mathbf{k_j}}$	$0^{\mathbf{l_j}}$	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}*$	$0^{\mathbf{(1+\epsilon)k_j}}*$	$\infty^{\mathbf{k_j}}*$	f	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}$
$\infty^{\mathbf{k_j}}$	$0^{\mathbf{l_j}}$	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}*$	$0^{\mathbf{(1+\epsilon)k_j}}*$	$\infty^{\mathbf{k_j}}*$	f	$\infty^{\mathbf{k_j}}$	$0^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}}$
$\infty^{\mathbf{k_j}}$	$0^{\mathbf{l_j}}$	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}*$	$0^{\mathbf{(1+\epsilon)k_j}}*$	$\infty^{\mathbf{k_j}}*$	f	$\infty^{\mathbf{k_j}}$	$0^{\mathbf{M_j}}$	$\infty^{\mathbf{M_j}}$
$\infty^{\mathbf{k_j}}$	$0^{\mathbf{l_j}}$	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}*$	$0^{\mathbf{(1+\epsilon)k_j}}*$	$\infty^{\mathbf{k_j}}*$	f	$\infty^{\mathbf{k_j}}*$	$0^{\mathbf{k_j}}*$	–
$\infty^{\mathbf{k_j}}$	$0^{\mathbf{l_j}}$	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}*$	$0^{\mathbf{(1+\epsilon)k_j}}*$	$\infty^{\mathbf{k_j}}*$	f	$\infty^{\mathbf{k_j}}$	$0^{\mathbf{m_j}}$	$\infty^{\mathbf{k_j}}$
		$\infty^{\mathbf{M_j}}$	f	$\infty^{\mathbf{M_j}}*$	$0^{\mathbf{(1+\epsilon)k_j}}*$	$\infty^{\mathbf{M_j}}*$	f	$\infty^{\mathbf{M_j}}$		

In the last row of Table 3 we have included part of the iteration sequence in the case where $w(z_j-\delta)$ has a pole of order greater than k_j. In each row of Table 3 we have indicated with "$*$" the zeros and poles of w that are to be grouped together. Note that in each grouping, the number of zeros divided by the number of poles is less than $3/4+\epsilon$ (counting multiplicities) for any $\epsilon>0$.

We still need to examine the case in which w has a zero of order $(1+\epsilon)k_j$ at z_j but does not have a pole of order k_j or higher at $z_j-\delta$ (it could have a zero, another finite value, or a pole of order less than k_j.) In this case

$$\begin{aligned} w(z) &= \alpha(z-z_j)^{(1+\epsilon)k_j} + O\left((z-z_j)^{(1+\epsilon)k_j+1}\right), \quad \alpha\neq 0 \\ w(z+\delta) &= \frac{a_0(z)}{\alpha}(z-z_j)^{-(1+\epsilon)k_j} + O\left((z-z_j)^{1-(1+\epsilon)k_j}\right) \\ &= \beta(z-z_j)^{-k_j} + O\left((z-z_j)^{1-k_j}\right) \end{aligned} \tag{3.66}$$

and iteration of equation (3.62) yields

$$\begin{aligned} w(z+2\delta) &= a_1(z+\delta) + \frac{(a_0(z+\delta)-a_0(z))\,\alpha}{a_0(z)}(z-z_j)^{(1+\epsilon)k_j} + O\left((z-z_j)^{(1+\epsilon)k_j+1}\right) \\ w(z+3\delta) &= -\frac{a_0(z)}{\alpha}(z-z_j)^{-(1+\epsilon)k_j} + O\left((z-z_j)^{1-(1+\epsilon)k_j}\right) \\ w(z+4\delta) &= a_1(z+3\delta)-a_1(z+\delta) \\ &\quad - \alpha\frac{(a_0(z+3\delta)+a_0(z+\delta)-a_0(z))}{a_0(z)}(z-z_j)^{(1+\epsilon)k_j} + O\left((z-z_j)^{(1+\epsilon)k_j+1}\right) \\ w(z+5\delta) &= \frac{a_0(z)}{\alpha}(z-z_j)^{-(1+\epsilon)k_j} + O\left((z-z_j)^{1-(1+\epsilon)k_j}\right) \\ &\quad + \frac{a_0(z+4\delta)}{a_1(z+3\delta)-a_1(z+\delta)-\alpha\dfrac{a_0(z+3\delta)+a_0(z+\delta)-a_0(z)}{a_0(z)}(z-z_j)^{(1+\epsilon)k_j}+\cdots} \end{aligned} \tag{3.67}$$

for all z in a suitably small neighborhood of z_j.

Note that $w(z_j+4\delta)$ is finite unless $a_1(z)$ has a pole at $z_j+3\delta$ or at $z_j+\delta$ in which case w has a pole of order at most ϵk_j at $z_j+4\delta$. If $w(z_j+4\delta)$ is non-zero, or a zero with the multiplicity $l_j<k_j$, then $w(z_j+5\delta)$ has a pole of order k_j. If $w(z_j+4\delta)$ has a zero with the multiplicity $M_j>k_j$ then $w(z_j+5\delta)$ has a pole of order M_j. If $w(z_j+4\delta)$ has a zero of order k_j then $w(z_j+5\delta)$ has a pole of order k_j unless

$$a_1(z_j+3\delta)=a_1(z_j+\delta) \tag{3.68}$$

with the multiplicity at least k_j, and $w(z_j+5\delta)$ has a pole of order at least $\frac{2}{3}k_j$ unless

$$a_0(z_j)-a_0(z_j+\delta)-a_0(z_j+3\delta)+a_0(z_j+4\delta)=0 \tag{3.69}$$

with the multiplicity at least $\frac{1}{3}k_j$. Assuming that equations (3.68) and (3.69) do not both hold then we can construct the grouping of the zeros of w described in Table 4.

The only case in Tables 3 and 4 where there may be some overlap when a pole of w is associated to a zero of w is with the rows (†) and (‡) in Table 4. Combining them together we obtain

$-$	$0^{(1+\epsilon_1)l_j}*$	$\infty^{l_j}*$	f	$\infty^{l_j}*$	$0^{k_j}*$	$\infty^{k_j}*$	f	$\infty^{k_j}*$	$0^{(1+\epsilon_2)k_j}*$	$-$

TABLE 4. *The rest of the iteration of equation* (3.62). *Here l_j is such that $k_j > l_j \geq \frac{2}{3}k_j$, and otherwise the notation is as in Table 3.*

$-\delta$	0	δ	2δ	3δ	4δ	5δ	Notes:
–	$0^{(1+\epsilon)k_j}*$	$\infty^{k_j}*$	f	$\infty^{k_j}*$	f	∞^{k_j}	
–	$0^{(1+\epsilon)k_j}*$	$\infty^{k_j}*$	f	$\infty^{k_j}*$	0^{M_j}	∞^{M_j}	(†)
–	$0^{(1+\epsilon)k_j}*$	$\infty^{k_j}*$	f	$\infty^{k_j}*$	$0^{m_j}*$	$\infty^{k_j}*$	Compare with the last row of Table 3
–	$0^{(1+\epsilon)k_j}*$	$\infty^{k_j}*$	f	$\infty^{k_j}*$	$0^{k_j}*$	$\infty^{l_j}*$	(‡)
–	$0^{(1+\epsilon)k_j}$	∞^{k_j}	f	∞^{k_j}	0^{k_j}	∞^{k_j}	Apply rules from Table 3 to the zero of $w(z_j+4\delta)$

where $l_j \geq \frac{2}{3}k_j$, and so the number of zeros divided by the number of poles is less or equal to $4/5+\epsilon$ for any $\epsilon > 0$.

If equations (3.68) and (3.69) hold for more than $S(r,w)$ points z_j then $a_0(z) = \pi_1 z + \pi_3$ and $a_1(z) = \pi_2$ where $\pi_k \in \mathcal{S}(w)$ are arbitrary periodic functions with period k, of finite order. Therefore (3.62) reduces to the difference Painlevé I equation (1.5). If on the contrary either of equations (3.68) and (3.69) hold for only $S(r,w)$ points z_j, we have been able to associate more than $S(r,w)$ zeros of w at z_j (which are poles of $\overline{w}+\underline{w}$) with an appropriate number of zeros and poles of "nearby" iterates $w(z_j+n\delta)$ such that within each grouping the number of zeros divided by the number of poles is less than $4/5+\epsilon$, and there are at most $S(r,w)$ exceptional zeros which cannot be grouped in this way. Therefore in this case

$$n(r,\overline{w}+\underline{w}) \leq \left(\frac{4}{5}+\epsilon\right) n(r+1,w) + S'(r,w), \tag{3.70}$$

where $\epsilon > 0$ is arbitrary, and so by Theorem 2.5 w is of infinite order.

Equation (3.63)

We will now complete the proof of Theorem 1.1 by looking at the equation (3.63). This final subcase is very similar to the derivation of (1.5) in the previous section. We try to avoid any unnecessary repetition.

As before, $\overline{w}+\underline{w}$ has more than $S(r,w)$ poles. If both $w(z_j+1)$ and $w(z_j-1)$ are poles of order k_j, then, as in (3.64) and (3.65), we have

$$\begin{aligned}
w(z) &= \alpha\,(z-z_j)^{(1+\epsilon)k_j} + O\left((z-z_j)^{(1+\epsilon)k_j+1}\right)\\
w(z+\delta) &= \sum_{i=-k_j}^{k_j-1} \beta_{i,\delta}\,(z-z_j)^i + O\left((z-z_j)^{k_j}\right)\\
w(z+2\delta) &= -\sigma \sum_{i=-k_j}^{k_j-1} \beta_{i,\delta}\,(z-z_j)^i + a_1(z+\delta) + O\left((z-z_j)^{k_j}\right)\\
w(z+3\delta) &= (\sigma^2-1) \sum_{i=-k_j}^{k_j-1} \beta_{i,\delta}\,(z-z_j)^i + a_1(z+2\delta) - \sigma a_1(z+\delta) + O\left((z-z_j)^{k_j}\right)\\
&= a_1(z+2) - \sigma a_1(z+1) + O\left((z-z_j)^{k_j}\right)\\
w(z+4\delta) &= +\sigma\beta_{-k_j,\delta}(z-z_j)^{-k_j} + \frac{a_0(z+3\delta)}{a_1(z+2\delta)-\sigma a_1(z+\delta)+\cdots} + O\left((z-z_j)^{1-k_j}\right)
\end{aligned}$$

for all z in a small enough neighborhood of z_j, where α and $\beta_{-k_j,\delta}$ are non-zero. The information obtained from the iteration above is summarised in Table 5, which is analogous to Table 3.

Table 5. *Iteration of equation (3.63). For the explanation of the notation see the caption of Table 3.*

-4δ	-3δ	-2δ	$-\delta$	0	δ	2δ	3δ	4δ
$\infty^{\mathbf{k_j}}$	$0^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}*}$	$0^{\mathbf{(1+\epsilon)k_j}*}$	$\infty^{\mathbf{k_j}*}$	$\infty^{\mathbf{k_j}}$	$0^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}}$
$\infty^{\mathbf{k_j}}$	$0^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}*}$	$0^{\mathbf{(1+\epsilon)k_j}*}$	$\infty^{\mathbf{k_j}*}$	$\infty^{\mathbf{k_j}*}$	$0^{\mathbf{k_j}*}$	–
$\infty^{\mathbf{k_j}}$	$0^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}*}$	$0^{\mathbf{(1+\epsilon)k_j}*}$	$\infty^{\mathbf{k_j}*}$	$\infty^{\mathbf{k_j}}$	$0^{\mathbf{M_j}}$	$\infty^{\mathbf{M_j}}$
$\infty^{\mathbf{k_j}}$	$0^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}*}$	$0^{\mathbf{(1+\epsilon)k_j}*}$	$\infty^{\mathbf{k_j}*}$	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}$
–	$0^{\mathbf{k_j}*}$	$\infty^{\mathbf{k_j}*}$	$\infty^{\mathbf{k_j}*}$	$0^{\mathbf{(1+\epsilon)k_j}*}$	$\infty^{\mathbf{k_j}*}$	$\infty^{\mathbf{k_j}*}$	$0^{\mathbf{k_j}*}$	–
–	$0^{\mathbf{k_j}*}$	$\infty^{\mathbf{k_j}*}$	$\infty^{\mathbf{k_j}*}$	$0^{\mathbf{(1+\epsilon)k_j}*}$	$\infty^{\mathbf{k_j}*}$	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}$
$\infty^{\mathbf{M_j}}$	$0^{\mathbf{M_j}}$	$\infty^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}*}$	$0^{\mathbf{(1+\epsilon)k_j}*}$	$\infty^{\mathbf{k_j}*}$	$\infty^{\mathbf{k_j}*}$	$0^{\mathbf{k_j}*}$	–
$\infty^{\mathbf{L_j}}$	$0^{\mathbf{L_j}}$	$\infty^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}*}$	$0^{\mathbf{(1+\epsilon)k_j}*}$	$\infty^{\mathbf{k_j}*}$	$\infty^{\mathbf{k_j}}$	$0^{\mathbf{M_j}}$	$\infty^{\mathbf{M_j}}$
$\infty^{\mathbf{M_j}}$	$0^{\mathbf{M_j}}$	$\infty^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}*}$	$0^{\mathbf{(1+\epsilon)k_j}*}$	$\infty^{\mathbf{k_j}*}$	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}$
$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}*}$	$0^{\mathbf{(1+\epsilon)k_j}*}$	$\infty^{\mathbf{k_j}*}$	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}$
$\infty^{\mathbf{k_j}}$	$0^{\mathbf{l_j}}$	$\infty^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}*}$	$0^{\mathbf{(1+\epsilon)k_j}*}$	$\infty^{\mathbf{k_j}*}$	$\infty^{\mathbf{k_j}}$	f	$\infty^{\mathbf{k_j}}$
$\infty^{\mathbf{k_j}}$	$0^{\mathbf{l_j}}$	$\infty^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}*}$	$0^{\mathbf{(1+\epsilon)k_j}*}$	$\infty^{\mathbf{k_j}*}$	$\infty^{\mathbf{k_j}}$	$0^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}}$
$\infty^{\mathbf{k_j}}$	$0^{\mathbf{l_j}}$	$\infty^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}*}$	$0^{\mathbf{(1+\epsilon)k_j}*}$	$\infty^{\mathbf{k_j}*}$	$\infty^{\mathbf{k_j}}$	$0^{\mathbf{M_j}}$	$\infty^{\mathbf{M_j}}$
$\infty^{\mathbf{k_j}}$	$0^{\mathbf{l_j}}$	$\infty^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}*}$	$0^{\mathbf{(1+\epsilon)k_j}*}$	$\infty^{\mathbf{k_j}*}$	$\infty^{\mathbf{k_j}*}$	$0^{\mathbf{k_j}*}$	–
$\infty^{\mathbf{k_j}}$	$0^{\mathbf{l_j}}$	$\infty^{\mathbf{k_j}}$	$\infty^{\mathbf{k_j}*}$	$0^{\mathbf{(1+\epsilon)k_j}*}$	$\infty^{\mathbf{k_j}*}$	$\infty^{\mathbf{k_j}}$	$0^{\mathbf{m_j}}$	$\infty^{\mathbf{k_j}}$
		$\infty^{\mathbf{M_j}}$	$\infty^{\mathbf{M_j}*}$	$0^{\mathbf{(1+\epsilon)k_j}*}$	$\infty^{\mathbf{M_j}*}$	$\infty^{\mathbf{M_j}}$		

In the case where w has a zero of order $(1+\epsilon)k_j$ at z_j but does not have a pole of order k_j or higher at $z_j-\delta$, we use the fact that (3.63) may be written in the form

$$w(z+\delta) = \sigma w(z-2\delta) + a_1(z) - \sigma a_1(z-\delta) + \frac{a_0(z)}{w(z)} - \frac{\sigma a_0(z-\delta)}{w(z-\delta)}$$

to obtain the following iteration sequence, analogous to (3.66) and (3.67):

$$\begin{aligned}
w(z) &= \alpha(z-z_j)^{(1+\epsilon)k_j} + O\left((z-z_j)^{(1+\epsilon)k_j+1}\right), \quad \alpha \neq 0 \\
w(z+\delta) &= \frac{a_0(z)}{\alpha}(z-z_j)^{-(1+\epsilon)k_j} + O\left((z-z_j)^{1-(1+\epsilon)k_j}\right) \\
w(z+2\delta) &= -\frac{\sigma a_0(z)}{\alpha}(z-z_j)^{-(1+\epsilon)k_j} + O\left((z-z_j)^{1-(1+\epsilon)k_j}\right) \\
w(z+3\delta) &= a_1(z+2\delta) - \sigma a_1(z+\delta) \\
&\quad +\alpha\left(\sigma - \frac{\sigma a_0(z+\delta)}{a_0(z)} - \frac{a_0(z+2\delta)}{\sigma a_0(z)}\right)(z-z_j)^{(1+\epsilon)k_j} + O\left((z-z_j)^{(1+\epsilon)k_j+1}\right) \\
w(z+4\delta) &= \frac{\sigma a_0(z)}{\alpha}(z-z_j)^{-(1+\epsilon)k_j} + O\left((z-z_j)^{1-(1+\epsilon)k_j}\right)
\end{aligned}$$

$$+\frac{a_0(z+3\delta)}{a_1(z+2\delta) - \sigma a_1(z+\delta) + \alpha\left(\sigma - \frac{\sigma a_0(z+\delta)}{a_0(z)} - \frac{a_0(z+2\delta)}{\sigma a_0(z)}\right)(z-z_j)^{(1+\epsilon)k_j} + \cdots}$$

Therefore, similarly as in Table 4, assuming that

$$a_1(z_j+2\delta) = \sigma a_1(z_j+\delta) \tag{3.71}$$

and

$$a_0(z_j) - a_0(z_j + \delta) - a_0(z_j + 2\delta) + a_0(z_j + 3\delta) = 0 \tag{3.72}$$

do not both hold we can construct the grouping of the zeros of w described in Table 6.

TABLE 6. *The rest of the iteration of equation* (3.63). *Here* l_j *is such that* $k_j > l_j \geq \frac{2}{3}k_j$, *and otherwise the notation is as in Table 3.*

$-\delta$	0	δ	2δ	3δ	4δ	Notes:
$-$	$0^{(1+\epsilon)k_j}*$	$\infty^{k_j}*$	$\infty^{k_j}*$	f	∞^{k_j}	
$-$	$0^{(1+\epsilon)k_j}*$	$\infty^{k_j}*$	$\infty^{k_j}*$	0^{M_j}	∞^{M_j}	(†)
$-$	$0^{(1+\epsilon)k_j}*$	$\infty^{k_j}*$	$\infty^{k_j}*$	$0^{m_j}*$	$\infty^{k_j}*$	Compare with the last row of Table 5
$-$	$0^{(1+\epsilon)k_j}*$	$\infty^{k_j}*$	$\infty^{k_j}*$	$0^{k_j}*$	$\infty^{l_j}*$	(‡)
$-$	$0^{(1+\epsilon)k_j}$	∞^{k_j}	∞^{k_j}	0^{k_j}	∞^{k_j}	Apply rules from Table 5 to the zero of $w(z_j + 3\delta)$

The only possible overlap in Tables 5 and 6 is with the rows (†) and (‡) in Table 6. Combining them together we obtain

$-$	$0^{(1+\epsilon_1)l_j}*$	$\infty^{l_j}*$	$\infty^{l_j}*$	$0^{k_j}*$	$\infty^{k_j}*$	$\infty^{k_j}*$	$0^{(1+\epsilon_2)k_j}*$	$-$

where $l_j \geq \frac{2}{3}k_j$, and so, the number of zeros divided by the number of poles is less or equal to $4/5 + \epsilon$ for any $\epsilon > 0$.

Now if equations (3.71) and (3.72) hold for more than $S(r, w)$ points then $a_0(z) = \pi_1 z + \pi_2$ for periodic functions $\pi_k \in \mathcal{S}(w)$ with period k. Moreover, if $\sigma = 1$ we have that $a_1(z)$ is an arbitrary periodic function with period 1, and if $\sigma = -1$ it follows that $a_1(z) = (-1)^z \kappa_1$ where κ_1 is periodic with period 1. Therefore (3.63) reduces to the difference Painlevé I equation (1.3) if $\sigma = 1$ and to equation (1.4) if $\sigma = -1$. If on the other hand either of equations (3.71) and (3.72) hold for only $S(r, w)$ points, then (3.70) holds and so by Theorem 2.5 w is of infinite order. This completes the proof of Theorem 1.1. □

4. *Discussion*

We have shown that the existence of at least one admissible finite-order meromorphic solution w is sufficient to reduce a large class of difference equations into one of the difference Painlevé equations or to a linear difference equation, provided that w does not satisfy a difference Riccati equation.

The existence of finite-order meromorphic solutions of the equations (1.2) – (1.10) is guaranteed in the autonomous case. In this case each of the difference Painlevé equations (1.3) – (1.8) are solved by a two periodic function family of elliptic functions, see, e.g., [**5**]. The autonomous form of (1.9) has finite-order meromorphic solutions expressed in terms of certain periodic functions with period two. The difference Riccati and the linear difference equations have large classes of meromorphic solutions also in the non-autonomous case, but so far the growth order of these solutions is unknown when the coefficients of the equation depend on the independent variable. In the autonomous case these solutions are also of finite order.

We have shown that methods related to the singularity confinement test can be used to estimate the relative frequency of points at which a solution of a difference equation takes special values. In the class of examples considered here, if sufficiently many singularities are not confined then the solution was shown to be of infinite order. If most of the singularities are confined, then we arrive at an estimate which is consistent with the solution being of finite order. This is not true for more general classes of equations. The confinement of singularities in a non-rational meromorphic solution of the equation of Hietarinta and Viallet [**23**] again leads to an estimate for the order of the solution. However, in this case the estimate implies that the order is infinite. For more details see [**19**].

We conclude that the existence of a finite-order meromorphic solution of a difference equation is a strong indicator of integrability. Indeed, it is enough to single out all known integrable equations out of a large class of difference equations, although the question of existence of finite-order meromorphic solutions still needs to addressed in the non-autonomous case.

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