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Integrable relativistic models and the generalized Chazy equation

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Abstract. The equation $y'' = f(x)y^2$ arises in the study of a class of fluid models in relativity and possesses the Painlevé property (closely connected with integrability) if and only if f satisfies a certain sixth-order ODE which admits $GL(2, \mathbb{C})$ as its symmetry group. Using differential invariants of this non-solvable group, the general solution is obtained. A special case of the sixth-order equation is equivalent to the generalized Chazy equation with parameter $n = \frac{6}{7}$. All known explicit choices for f considered in the literature arise in a natural way in this framework. Generalizations of the techniques described here lead to a novel class of integrable equations.

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1. Introduction

Kustaanheimo and Qvist [19] showed that Einstein's field equations for an expanding spherically symmetric shear-free fluid lead to the equation

$$y'' = f(x)y^2, \tag{1}$$

where the primes denote differentiation with respect to x and f is an arbitrary function (which we take to be locally analytic). The algorithm for constructing the spacetime metric from a solution of equation (1) is outlined below. Several authors have addressed the problem of determining which functions f allow us to solve equation (1) explicitly (Kustaanheimo and Qvist [19], Chakravarty [6], Wyman [26, 27], Stephani [24], Srivastava [23], Maharaj *et al* [20], also see references in Krasiński [17]). Most of these approaches have involved analysing conditions on f to ensure that equation (1) admits two symmetries. Wyman [26] and later Maharaj *et al* [20] determined conditions on f that ensure equation (1) possesses the Painlevé property (that all solutions in the complex plane are single-valued about all movable singularities).

The Painlevé property is intimately connected with integrability [3–5, 18]. This property was first used by Kowalevskaya last century to find a new integrable case of the classical equations for the spinning top [14, 15]. Painlevé and his colleagues (Painlevé [21, 22], Fuchs [12], Gambier [13]) classified all equations of the form

$$y''(x) = \Phi(x, y(x), y'(x)), \tag{2}$$

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that possess the Painlevé property, where Φ is rational in y and y' with coefficients (locally) analytic in x. They showed that all of these equations can be transformed by a change of independent variable and a Möbius transformation of the dependent variable into one of approximately 50 canonical equations. These equations were solved in terms of known functions with the exception of six equations, now called the Painlevé equations: P_I–P_{VI}. The first Painlevé equation, P_I, has the form

$$y'' = 6y^2 + x, \tag{3}$$

and will appear again in the analysis below.

Although any case of equation (1) possessing the Painlevé property can in principle be mapped to one of the canonical equations discovered by Painlevé, it is important to note that these mappings themselves arise as solutions of differential equations and may be very complicated.

The following result follows from Wyman [26].

Theorem 1. *Equation (1) possesses the Painlevé property if and only if* $\lambda(x) := \{6/f(x)\}^{1/5}$ *satisfies the sixth-order ODE*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\lambda^2 \frac{\mathrm{d}}{\mathrm{d}x}(\lambda^3 F[\lambda])\right\} = 0 \tag{4}$$

where

$$F[\lambda] := 2(\lambda^4\lambda'')'' - \lambda^3(\lambda'')^2 \tag{5}$$

and the primes denote differentiation with respect to x.

In this paper we use symmetry methods to find the general solution of equation (4) (due to Wyman) and to systematically recover all known particular closed-form solutions in a natural way.

Equation (4) possesses a four-parameter non-solvable Lie group of symmetries (isomorphic to $GL(2;\mathbb{C})$). Following Clarkson and Olver [11] we identify two differential invariants of this Lie group and use them to reduce equation (4) to an equation of second order which is solvable in terms of either the Weierstrass elliptic function or solutions of the first Painlevé equation. A special case of equation (4) is shown to correspond to the generalized Chazy equation with parameter $n = \frac{6}{7}$. All of the known explicit choices of f that have appeared in the literature to date are shown to correspond to either the case where the above reduction fails (in which the would-be independent variable of the reduced equation is constant) or to the known explicit solutions of the generalized Chazy equation.

We conclude by discussing methods of deriving similar integrable equations to equation (4) that are associated with each of the Painlevé equations. The solutions of these equations have an interesting singularity structure in the complex domain and will be the subject of a further study.

2. Painlevé analysis of shear-free fluids

In this section we review a number of known results.

2.1. Spherically symmetric shear-free fluids

The metric for an expanding shear-free spherically symmetric spacetime can be written in the form

$$\mathrm{d}s^2 = \mathrm{e}^{2a(r,t)}(r^2\,\mathrm{d}\Omega^2 + \mathrm{d}r^2) - \mathrm{e}^{2b(r,t)}\,\mathrm{d}t^2,$$

where $d\Omega^2$ is the standard metric on the sphere (see, e.g., Kramer *et al* [16]). Einstein's equations for this metric admit two arbitrary functions of integration;

$$g(t) = \ln a_t - b, \tag{6}$$

$$h(x) = e^a(a_{rr} - a_r^2 - a_r/r), \tag{7}$$

(Wyman [25], Kustaanheimo and Qvist [19]). Given a, b, g and h satisfying equations (6) and (7), Einstein's equations allow us to determine the local energy and pressure densities (see, e.g., Kramer *et al* [16]).

Kustaanheimo and Qvist [19] noted that under the change of variables

$$y(x,t) = e^{-a(r,t)}, \qquad x = r^2, \qquad f(x) = -4r^2 h(r),$$

equation (7) becomes (1). Although it is apparent from this transformation that y will, in general, be a function of t as well as x, we study equation (1) as an ODE. The t-dependence can be recovered by allowing any arbitrary constants that appear in solutions of equation (1) to be functions of t.

2.2. *Painlevé analysis*

Following Painlevé we introduce the transformation

$$z = \phi(x), \qquad y(x) = \lambda(x)Y(z) + \mu(x), \tag{8}$$

where $\lambda(x)$ and $\phi'(x)$ are nonzero. If we choose ϕ, λ and μ such that

$$\lambda\phi'' + 2\lambda'\phi' = 0, \tag{9}$$

$$\lambda'' - 2f\mu\lambda = 0, \tag{10}$$

$$f\lambda - 6(\phi')^2 = 0, \tag{11}$$

then equation (1) becomes

$$\frac{d^2Y(z)}{dz^2} = 6Y^2(z) - \nu(z), \tag{12}$$

where

$$\nu(z) = \frac{\mu'' - f\mu^2}{\lambda(\phi')^2}. \tag{13}$$

Painlevé also showed that equation (12) (and hence equation (1)) possesses the Painlevé property if and only if

$$\nu(z) = Az + B, \tag{14}$$

where A and B are constants (Painlevé [21, 22], see also Kruskal *et al* [18]). So from equation (13), we have

$$\mu'' - f\mu^2 = (A\phi + B)\lambda(\phi')^2. \tag{15}$$

We have shown that equation (1) possesses the Painlevé property provided we can solve the system of equations (9)–(11) and (15). By elimination we will reduce this system to a single equation (equation (4)).

Dividing equation (9) through by $\lambda\phi'$ and integrating with respect to x gives $\lambda^2\phi' = \kappa$, where κ is a constant. Since κ cannot be zero we take $\kappa = 1$ since the Painlevé property for equation (1) is invariant under a rescaling of t. Hence

$$\phi = \int \lambda^{-2}\,dx. \tag{16}$$

Equation (11) then gives

$$f = 6\lambda^{-5}, \tag{17}$$

and equation (10) gives

$$\mu = \tfrac{1}{12}\lambda^4\lambda''. \tag{18}$$

Substituting equations (16)–(18) into (15), we have

$$\lambda^3 F[\lambda] = 12\left(A\int \lambda^{-2}\,\mathrm{d}x + B\right), \tag{19}$$

where $F[\lambda]$ is defined by equation (5). Next we differentiate equation (19) with respect to x, multiply through by $2\lambda^2$, and differentiate once more with respect to x to obtain equation (4). This concludes our proof of theorem 1 (Wyman [26]).

Given any solution, $\lambda(x)$, of equation (4), then (1) with $f(x)$ given by (17) possesses the Painlevé property and is therefore integrable. If $A \neq 0$ then equation (12) is the first Painlevé equation (cf equation (3)) up to a translation in z and a rescaling of z and Y. If $A = 0$, equation (12) can be solved in terms of Weierstrass elliptic functions.

3. Symmetries

In this section we use symmetry methods to find solutions of equation (4). Three symmetries of equation (4) can be seen immediately: $t \mapsto t + \epsilon$, $t \mapsto \mathrm{e}^{\epsilon} t$, and $\lambda \mapsto \mathrm{e}^{\epsilon}\lambda$. A fourth symmetry follows from the observation of Kramer *et al* [16] that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - f(x)y^2 = \tilde{x}^3\left\{\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}\tilde{x}^2} - \tilde{f}(\tilde{x})\tilde{y}^2\right\},$$

where $\tilde{x} = 1/x$, $\tilde{y}(\tilde{x}) = x^{-1}y(x)$ and $\tilde{f}(\tilde{x}) = x^5 f(x)$. This transformation preserves the Painlevé property and therefore induces the (discrete) symmetry $(x, \lambda(x)) \mapsto (1/x, x\lambda(x^{-1}))$ of equation (4).

Combining all of the above symmetries, we see that equation (4) is invariant under $GL(2;\mathbb{C})$ transformations of the form

$$x \mapsto \hat{x} := \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad \lambda(x) \mapsto \hat{\lambda}(\hat{x}) := (\gamma x + \delta)^{-1}\lambda(x),$$

where $\alpha\delta - \beta\gamma \neq 0$. The infinitesimal generators of this group are

$$X_0 = \partial_\lambda, \qquad X_1 = \partial_x, \qquad X_2 = x\partial_x, \qquad X_3 = x^2\partial_x + x\lambda\partial_\lambda.$$

It is important to note that since $\{X_\mu\}_{\mu=0,\ldots,3}$ span the non-solvable Lie algebra $\mathfrak{gl}(2;\mathbb{C})$, we cannot simply reduce equation (4) by these symmetries one at at time, in any order, without losing some of the symmetries. We begin by reducing equation (4) by the scaling symmetry generated by X_0. Writing

$$w(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\ln\lambda(x), \tag{20}$$

equation (4) is reduced to fifth order:

$$G_{xx} - 18wG_x + 8(w^2 - w')G = 0, \tag{21}$$

where

$$G[w] := 2w''' - 24ww'' - 13w'^2 + 98w^2w' - 49w^4.$$

Under this transformation, the remaining generators, X_1, X_2, X_3 have been mapped to

$$\tilde{X}_1 = \partial_x, \qquad \tilde{X}_2 = x\partial_x - w\partial_w, \qquad \tilde{X}_3 = x^2\partial_x - (1+2xw)\partial_w,$$

which generate the non-solvable $SL(2;\mathbb{C})$ group of symmetries

$$x \mapsto \hat{x} := \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad w(x) \mapsto \hat{w}(\hat{x}) := (\gamma x + \delta)^2 w(x) + \gamma(\gamma x + \delta),$$

where $\alpha\delta - \beta\gamma = 1$. Clarkson and Olver [11] note that this Lie group admits the differential invariants;

$$\eta = 2\lambda^4(w_x - w^2) = -2\lambda^3\lambda_{xx}, \tag{22}$$

$$\zeta = 2\lambda^6(w_{xx} - 6ww_x + 4w_x^2) = -2\lambda^4(\lambda\lambda_{xxx} + 3\lambda_x\lambda_{xx}) = \lambda^2\frac{\mathrm{d}\eta}{\mathrm{d}x}. \tag{23}$$

These invariants are central to our solution of equation (4).

3.1. The general case

In the general case $\eta_x \neq 0$, η can be used as a local coordinate. Define

$$\xi := \int \zeta^{-1}\,\mathrm{d}\eta = \int \lambda^{-2}\,\mathrm{d}x. \tag{24}$$

It follows from equations (23) and (24) that

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = -2\lambda^2(\lambda^2\lambda_{xx})_x, \tag{25}$$

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2} = 2\lambda^3(\lambda^3\lambda_{xx}^2 - (\lambda^4\lambda_{xx})_{xx}) = \lambda^3(\lambda^3\lambda_{xx}^2 - F[\lambda]), \tag{26}$$

where F is given by equation (5). Using equations (22), (24) and (26), equation (19) becomes

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2} = \frac{1}{4}\eta^2 - 24(A\xi + B). \tag{27}$$

A rescaling ξ and η and a translation in ξ shows that the general solution of equation (27) is given in terms of solutions of P_I, the first Painlevé equation (3), provided $A \neq 0$. If $A = 0$ the general solution of equation (27) can be written in terms of the Weierstrass elliptic function.

Given a solution $\eta(\xi)$ of equation (27), the general solution of (19) is given by

$$\lambda(x) = \frac{1}{\psi(\xi)}, \qquad x = \frac{1}{W(\psi,\psi_1)}\frac{\psi_1(\xi)}{\psi(\xi)}, \tag{28}$$

where ψ and ψ_1 are linearly independent solutions of

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} - \frac{1}{2}\eta(\xi)\psi = 0 \tag{29}$$

and $W(\psi,\psi_1) := \psi\partial_\xi\psi_1 - \psi_1\partial_\xi\psi$ is the corresponding Wronskian. The choices for f for which equation (1) possesses the Painlevé property are then given by equation (17). The solution of equation (1) is given by (12) and (14) where ϕ and μ are given by equations (16) and (18), respectively. Note that in the case where η is an elliptic function ($A = 0$), equation (29) is the Lamé equation.

3.2. The degenerate case

In the previous section we assumed $\eta_x \neq 0$. If η is a constant then

$$f(x) = (c_0 + c_1x + c_2x^2)^{-5/2},$$

which was known to Kustaanheimo and Qvist [19] and has been rediscovered many times since then (see Krasiński [17]).

3.3. The generalized Chazy equation

In this section we consider the special case $\nu(z) = Az + B = 0$ for all z. The substitution (20) in equation (19) (with $A = B = 0$) gives

$$G[w]/2 = w_{xxx} - 12ww_{xx} - \tfrac{13}{2}w_x^2 + 49w^2w_x - \tfrac{49}{2}w^4 = 0. \tag{30}$$

If we rescale w such that $q(x) = 6w(x)$, then equation (30) becomes

$$q_{xxx} = 2qq_{xx} - 3q_x^2 + \frac{4}{36 - n^2}(6q_x - q^2)^2, \tag{31}$$

for the special choice of parameter $n = \frac{6}{7}$. Equation (31) was first written down and solved by Chazy [8–10] and is known today as the *generalized Chazy equation*. We note that the classical Chazy equation, which is equivalent to the Darboux–Halphen system, corresponds to the limit $n \to \infty$ of equation (31).

Chazy [9, 10] showed that the general solution of equation (31) is

$$q(x) = -6\frac{\mathrm{d}}{\mathrm{d}x}\ln r(t), \qquad x = \frac{r_1(t)}{r(t)},$$

where r and r_1 are independent solutions of the hypergeometric equation

$$t(1-t)\frac{\mathrm{d}^2r}{\mathrm{d}t^2} + \left(\frac{1}{2} - \frac{7t}{6}\right)\frac{\mathrm{d}r}{\mathrm{d}t} - \frac{1}{4}\left(\frac{1}{36} - \frac{1}{n^2}\right)r = 0. \tag{32}$$

We note that equation (32) can be mapped to (29) in this case (see Clarkson and Olver [11]).

Chazy also showed that equation (31) admits the particular solutions

$$q(x) = \frac{6}{x - x_0}, \tag{33}$$

and

$$q(x) = \frac{n-6}{2(x - x_1)} - \frac{n+6}{2(x - x_2)}. \tag{34}$$

The solution (33) corresponds to

$$f(x) = c(x - x_0)^{-5},$$

where c and x_0 are constant. The solution (34) in the cases $n = \frac{6}{7}$ (corresponding to equation (30)) we have $w(x) = -4/(7(x + a_1)) - 3/(7(x + a_2))$, which corresponds to

$$f(x) = (c_1x + c_2)^{-20/7}(c_3x + c_4)^{-15/7}, \tag{35}$$

where the a_i and the c_μ are constants (only three of the constants $c_1, \ldots, c_4$ are essential). This solution was discovered by Srivastava [23].

4. Discussion and generalizations

We have used symmetry methods to find all choices of the function f for which equation (1) possesses the Painlevé property and we proceeded to give its general solution in terms of solutions of the first Painlevé equation (or its autonomous form which is solvable in terms of Weierstrass elliptic functions).

For the special case $A = B = 0$ the choice of f for which equation (1) possesses the Painlevé property is related to the generalized Chazy equation (31) for the special choice of parameter $n = \frac{6}{7}$. Chazy noted that the general solution of equation (31) has a rich singularity structure in the complex domain including movable natural barriers (curves across which a particular solution cannot be analytically continued). He showed that equation (31) possesses the Painlevé property (i.e. it is single-valued about movable singularities) if n is an integer greater than 1 and does not possess the Painlevé property for non-integer n, despite the fact that it is solvable via the hypergeometric equation (32). The case considered above clearly gives rise to a general solution that is branched about movable singularities. However, the movable singularities of equation (4) become fixed singularities of (1) and so the branching in f does not violate the fact that equation (1) possesses the Painlevé property.

We note that equation (31) is a reduction of the self-dual Yang–Mills equations with an infinite-dimensional gauge algebra (Ablowitz *et al* [1, 2]). The classical Chazy equation ($n = \infty$ in equation (31)) was shown to be a reduction of the self-dual Yang–Mills equations by Chakravarty *et al* [7].

It is easy to generalize equation (19) (or, equivalently, equation (4)) to a large class of equations that are integrable through similar techniques. Equation (27) can be rescaled to have the form

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2} = \alpha\eta^2 + 2\beta\xi + 2\gamma, \tag{36}$$

where α, β, and γ are constants. Using equations (22), (24) and (26), equation (36) becomes

$$\lambda^3\left[(\lambda^4\lambda_{xx})_{xx} + (2\alpha - 1)\lambda^3\lambda_{xx}^2\right] + \beta\int\lambda^{-2}\,\mathrm{d}x + \gamma = 0. \tag{37}$$

The general solution of equation (37) is given by (28) where ψ and ψ_1 are independent solutions of equation (29) and $\eta(\xi)$ solves (36). In the special case $\beta = \gamma = 0$, $q := -6\lambda_x/\lambda$ solves the generalized Chazy equation (31) where

$$\alpha = -\frac{12n^2}{(n-6)(n+6)}.$$

Note that rescaling x and λ in equation (37) will not remove α from the equation despite the fact that a rescaling of η can remove α from equation (36). We can replace equation (36) with any integrable ODE and repeat the above procedure.

The behaviour of solutions of equations such as (37) will be the subject of future study.

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