THE GENERALIZED CHAZY EQUATION FROM THE SELF-DUALITY EQUATIONS [1]

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Abstract

It is shown that classically known generalizations of the Chazy equation and Darboux Halphen system are reductions of the self-dual Yang-Mills (SDYM) equations with an infinite-dimensional gauge algebra. The general ninth-order Darboux-Halphen system is reduced to a Schwarzian equation which governs conformal mappings of regions with piecewise circular sides. The generalized Chazy equation is shown to correspond to special mappings where either the triangles are equi-angular or two of the angles are $\pi/3$.

The self-dual Yang-Mills (SDYM) equations are known to be a rich source of integrable systems [1, 2, 3]. The complexified SDYM equations are a system of three partial differential equations for four Lie algebra-valued functions of $\mathbf{C}^4$. Most of the earlier work in the literature have considered the SDYM equations with finite-dimensional Lie algebras where the integrability of the equations is well understood [4, 5, 6, 7]. The study of the SDYM equations with infinite-dimensional Lie algebras, however, has led to several interesting and important symmetry reductions including the Kadomtsev-Petviashvili equation, the 2+1-dimensional N-wave equation, the self-dual Einstein equations, and the classical Darboux-Halphen system (which is equivalent to the classical Chazy equation) [8, 2, 4, 3].

In this paper we investigate a particular reduction of the SDYM equations with an infinite-dimensional Lie algebra to a ninth-order system of ODEs

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which we call DH–IX (equation 9, see also [9]). In the generic case, this system can be transformed into the following third-order system

$$\begin{aligned} \dot{\omega}_1 &= \omega_2\omega_3 - \omega_1(\omega_2 + \omega_3) + \tau^2, \\ \dot{\omega}_2 &= \omega_3\omega_1 - \omega_2(\omega_3 + \omega_1) + \tau^2, \\ \dot{\omega}_3 &= \omega_1\omega_2 - \omega_3(\omega_1 + \omega_2) + \tau^2, \end{aligned} \tag{1}$$

where

$$\tau^2 = \alpha^2(\omega_1 - \omega_2)(\omega_3 - \omega_1) + \beta^2(\omega_2 - \omega_3)(\omega_1 - \omega_2) + \gamma^2(\omega_3 - \omega_1)(\omega_2 - \omega_3), \tag{2}$$

and α, β, and γ are constants. Note that τ^2 is given by equation (2) only when the ω_i are distinct ($i.e.$ $\omega_i \neq \omega_j$, $i \neq j$). The case $\tau^2 = 0$ is the classical Darboux-Halphen system which appeared in Darboux's analysis of triply orthogonal surfaces [10] and was later solved by Halphen [11]. Halphen also studied and solved equations (1–2) [12] which are linearizable in terms of Fuchsian differential equations with three regular singular points. More recently, solutions of equations (1–2) for special choices of $\{\alpha, \beta, \gamma\}$ were determined in terms of automorphic forms. [13, 14]. We will show that equation (1–2) in the special case $\alpha = \beta = \gamma =: 2/n$ is equivalent to the equation

$$\frac{d^3y}{dt^3} - 2y\frac{d^2y}{dt^2} + 3\left(\frac{dy}{dt}\right)^2 = \frac{4}{36 - n^2}\left(6\frac{dy}{dt} - y^2\right)^2, \tag{3}$$

where $y = -2(\omega_1 + \omega_2 + \omega_3)$.

Equation (3) was studied by Chazy ([15, 16, 17]) and is usually referred to as the *generalized Chazy equation* (to contrast it with the special case $n = \infty$ which is called the Chazy equation).

We show that the general solution of the system (1–2) can be expressed in terms of the general solution of the Schwarzian equation that corresponds to the conformal mapping of circular triangles (and some degenerate cases which correspond to the mappings of crescent-shaped regions). These Schwarzian equations can be linearized via the hypergeometric equation.

We remark that for generic choices of α, β, γ, the general solution of equation (1) is densely branched about movable singularities and so possesses neither the Painlevé nor the poly-Painlevé properties which are often closely associated with integrability (see [18, 19, 20, 2, 21, 22]).

The Reduction of the SDYM Equations to DH–IX

With respect to the standard Cartesian coordinates $\{x_\mu\}_{\mu=0,1,2,3}$ on $\mathbf{R}^4$ the self-dual Yang-Mills equations are [3]

$$F_{01} = F_{23}, \qquad F_{02} = F_{31}, \qquad F_{03} = F_{12}, \tag{4}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu],$$

where $\partial_\mu = \partial\ /\partial x^\mu$ and the components A_μ of the Yang-Mills connection lie in the Lie algebra $\mathfrak{g}$. One of the simplest reductions of the SDYM equations is obtained by demanding that the A_μ are functions of $t =: x^0$ only. The SDYM equations then become the well known Nahm equations [23]

$$\partial_t A_i + \frac{1}{2}\sum_{j,k=1}^{3} \varepsilon_{ijk}[A_j, A_k] = 0, \tag{5}$$

where without loss of generality we have taken $A_0 \equiv 0$.

In order to obtain the generalized Darboux-Halphen system as a reduction of SDYM, we consider the infinite-dimensional Lie algebra of vector fields on S^3 and express the components of the connection as

$$A_i(t) = -\sum_{j,k=1}^{3} M_{ij}(t) O_{jk} X_k, \tag{6}$$

where the operators $\{X_k\}$ are the standard generators of $\mathfrak{sdiff}(S^3)$. The X_k are divergence-free vector fields which satisfy the $\mathfrak{su}(2)$ commutator relations

$$[X_i, X_j] = \sum_{k=1}^{3} \varepsilon_{ijk} X_k, \tag{7}$$

where ε_{ijk} is totally antisymmetric and $\varepsilon_{123} = 1$. The points of S^3 are represented by the $SO(3)$ matrix $[O_{ij}]$ (see, e.g. [24]) and the action of the vector fields X_i on O_{jk} is given by [25]

$$X_i O_{jk} = \sum_{l=1}^{3} \varepsilon_{ikl} O_{jl}. \tag{8}$$

Substituting equation (6) into equation (5) and using (7) and (8) together with the identities

$$\sum_{i,j,k=1}^{3} \varepsilon_{ijk} O_{ip} O_{jq} O_{kr} = \varepsilon_{pqr}, \qquad \sum_{i=1}^{3} \varepsilon_{ijk}\varepsilon_{imn} = \frac{1}{2}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}),$$

yields the following ordinary differential equation for the coefficient matrix $M(t)$

$$\dot{M} = (\mathrm{Adj}\, M)^T + M^T M - (\mathrm{Tr}\, M)M, \tag{9}$$

where $(\mathrm{Adj}\, M) := (\det M)M^{-1}$ is the adjoint of M and the dot denotes differentiation with respect to t (see also [26]). We refer to equation (9) as DH–IX (the ninth-order generalization of the classical Darboux-Halphen system). We have investigated special reductions of equation (9), e.g. the fifth-order reduction [27, 28] in previous work.

The Solution of DH–IX

The DH–IX system (equation 9) admits a simple factorization upon decomposing M into its symmetric (M_s) and antisymmetric (M_a) parts. The symmetric part, M_s, can be diagonalized by an orthogonal matrix P giving

$$M = M_s + M_a = P(d + a)P^{-1}, \tag{10}$$

where d is diagonal and $a := P^{-1}M_a P$ is antisymmetric. We begin by considering the case in which the eigenvalues of M_s are distinct. Substituting equation (10) into equation (9), we obtain

$$\begin{aligned} \dot{P} &= -Pa, && (11)\\ \dot{a} &= -ad - da, && (12)\\ \dot{d} &= 2\left\{d^2 - (\mathrm{Tr}\, d)d\right\} + \frac{1}{2}\left\{(\mathrm{Tr}\, d)^2 - \mathrm{Tr}(d^2) - \mathrm{Tr}\, a^2\right\} I, && (13) \end{aligned}$$

from the off-diagonal symmetric, anti-symmetric, and diagonal parts respectively. Note we have used the characteristic polynomial equation for M,

$$M^3 - (\mathrm{Tr}\, M)M^2 + \frac{1}{2}\left[(\mathrm{Tr}\, M)^2 - (\mathrm{Tr}\, M^2)\right] M - (\det M)I = 0,$$

in deriving equations (11–13). Since P does not appear in equations (12–13), these equations form an independent subsystem. After equations (12–13) have been solved, P can be obtained by solving the linear equation (11). If two or more eigenvalues of M_s are the same, equation (11) is no longer valid but a suitable choice of P can still be determined (*e.g.* it is convenient to take P to be the 3×3 identity matrix when $\omega_1 = \omega_2 = \omega_3$). Equations (12–13) remain unchanged in these degenerate cases.

We introduce the parameterization $d = \mathrm{diag}(\omega_1, \omega_2, \omega_3)$ and $a_{ij} = \sum_k \varepsilon_{ijk}\tau_k$. Equations (12–13) generate the three equations (1) together with the three equations

$$\dot{\tau}_1 = -\tau_1(\omega_2 + \omega_3), \qquad \dot{\tau}_2 = -\tau_2(\omega_3 + \omega_1), \qquad \dot{\tau}_3 = -\tau_3(\omega_1 + \omega_2), \tag{14}$$

where

$$\tau^2 = \tau_1^2 + \tau_2^2 + \tau_3^2. \tag{15}$$

Hence the solution of equations (1) and (14) together with the determination of P provide a complete solution of the DH–IX system (9). Using equations (1) and (14), it can be directly verified that the quantities

$$\begin{aligned}
\alpha^2 &:= \frac{\tau_1^2}{(\omega_1 - \omega_2)(\omega_3 - \omega_1)}, \\
\beta^2 &:= \frac{\tau_2^2}{(\omega_2 - \omega_3)(\omega_1 - \omega_2)}, \\
\gamma^2 &:= \frac{\tau_3^2}{(\omega_3 - \omega_1)(\omega_2 - \omega_3)},
\end{aligned} \tag{16}$$

are constants, for $\omega_i \neq \omega_j$, $i \neq j$. Without loss of generality we choose α, β, and γ to have non-negative real parts. Solving equation (16) for τ_i and substituting into equation (15) gives equation (2) and so the generalized Darboux-Halphen system can be written as the third-order system (1–2).

We remark that the reduction of the SDYM equations to DH–IX induces a corresponding reduction from the associated linear problem of SDYM [7] to a linear problem for DH–IX [27, 28]. This linear problem is monodromy-evolving in contrast to the isomonodromy problems associated with the Painlevé equations. The variable

$$s := \frac{\omega_1 - \omega_3}{\omega_2 - \omega_3} \tag{17}$$

plays a special role in linearizing the generalized Darboux-Halphen system. Repeatedly differentiating equation (17) and using equation (1) gives

$$\dot{s} = 2s(\omega_1 - \omega_2), \tag{18}$$

$$\ddot{s} = 2(\dot{s} - 2\omega_3 s)(\omega_1 - \omega_2), \tag{19}$$

where the dots denote differentiation with respect to t. From equations (17–19) we see that the ω's are parameterized as

$$
\begin{aligned}
\omega_1 &= -\frac{1}{2}\frac{d}{dt}\ln\frac{\dot{s}}{s(s-1)}, \\
\omega_2 &= -\frac{1}{2}\frac{d}{dt}\ln\frac{\dot{s}}{s-1}, \\
\omega_3 &= -\frac{1}{2}\frac{d}{dt}\ln\frac{\dot{s}}{s}.
\end{aligned}
\tag{20}
$$

From equation (1) we see that s is the general solution of the Schwarzian equation

$$\{s,t\} + \frac{\dot{s}^2}{2}V(s) = 0 \tag{21}$$

where

$$\{s,t\} := \frac{d}{dt}\left(\frac{\ddot{s}}{\dot{s}}\right) - \frac{1}{2}\left(\frac{\ddot{s}}{\dot{s}}\right)^2$$

is the Schwarzian derivative and V is given by

$$V(s) = \frac{1-\beta^2}{s^2} + \frac{1-\gamma^2}{(s-1)^2} + \frac{\beta^2+\gamma^2-\alpha^2-1}{s(s-1)}. \tag{22}$$

The general solution of the Schwarzian equation (21) is given implicitly by

$$t(s) = \frac{\chi_2(s)}{\chi_1(s)}, \tag{23}$$

where $\chi_1(s)$ and $\chi_2(s)$ are two independent solutions of the hypergeometric equation

$$s(1-s)\frac{d^2\chi}{ds^2} + [c-(a+b+1)s]\frac{d\chi}{ds} - ab\chi = 0, \tag{24}$$

where $a = (1+\alpha-\beta-\gamma)/2$, $b = (1-\alpha-\beta-\gamma)/2$, and $c = 1-\beta$ (see, e.g. [29, 30]). Thus the general solution of the DH–IX system (equation (1)) is given in terms of solutions of the *linear* differential equation (24). The solution procedure may be summarized as follows: The Schwarz function $s(t)$ obtained by inverting $t(s)$ in equation (23) provides the explicit solution for the ω_i's in equation (20) and hence of the system (1–2). Then the τ_i's obtained from equations (14) determine P via the solution of equation (11). Finally, the DH–IX matrix M is reconstructed from equation (10).

The Generalized Chazy Equation
We will now derive a third-order differential equation for

$$y := -2(\omega_1 + \omega_2 + \omega_3) = -2\mathrm{Tr}\, M, \tag{25}$$

which we write as

$$\frac{d^3y}{dt^3} = P\left(\frac{d^2y}{dt^2}, \frac{dy}{dt}, y; t\right). \tag{26}$$

We will consider the general case in which the ω's are distinct. From equations (20) and (25), we find that y can be written in terms of the Schwarzian s-function as

$$y(t) = \frac{1}{2}\frac{d}{dt} \ln \frac{\dot{s}^6}{s^4(s-1)^4}. \tag{27}$$

The Schwarzian equation (21) is invariant under the $\mathrm{SL}(2, \mathbf{C})$-transformation

$$s(t) \mapsto s(\mu(t)), \qquad \text{where} \qquad \mu(t) := \frac{at+b}{ct+d}, \qquad ad - bc = 1. \tag{28}$$

Under this transformation the derivative of s transforms as

$$\dot{s}(t) \mapsto (ct+d)^{-2}\dot{s}(\mu(t)).$$

This induces, via equation (27), corresponding transformations for y and its derivatives that leaves equation (26) invariant. In particular,

$$y(t) \mapsto (ct+d)^{-2}y(\mu(t)) - 6c(ct+d)^{-1}.$$

$\mathrm{SL}(2, \mathbf{C})$-invariant equations of the form $\Delta(t; y, \dot{y}, \ddot{y}, \ldots) = 0$ (c.f. equation 26) can be generated by the forms F_j which are polynomial in y and its derivatives and transform as

$$F_j(t) \mapsto (ct+d)^{-(2j+2)}F_j(\mu(t)). \tag{29}$$

The first three are given by

$$\begin{aligned} F_1 &:= 6\frac{dy}{dt} - y^2, \\ F_2 &:= 9\frac{d^2y}{dt^2} - 9y\frac{dy}{dt} + y^3, \\ F_3 &:= \frac{d^3y}{dt^3} - 2y\frac{d^2y}{dt^2} + 3\left(\frac{dy}{dt}\right)^2. \end{aligned} \tag{30}$$

On repeatedly differentiating equation (27), using equation (21) with (22), and substituting the resulting expressions for y and its derivatives in terms of s, $\dot{s}$ and $\ddot{s}$ into the definitions of F_1, F_2, F_3 in equation (30) we find that all the F_i are of the form $(\dot{s})^{i+1}$ multiplied by a function of s. Hence equation (26) can be expressed as

$$F_3 = G(s)F_1^2, \tag{31}$$

where $G(s)$, which depends on the SL$(2, \mathbf{C})$-invariant s only, is given by

$$G(s) = \frac{\left(\alpha^2 - \dfrac{\beta^2}{s} + \dfrac{\gamma^2}{s-1}\right)}{\left[(9\alpha^2 - 1) - \dfrac{(9\beta^2 - 1)}{s} + \dfrac{(9\gamma^2 - 1)}{s-1}\right]}$$
$$+ \frac{\dfrac{\alpha^2 - \gamma^2}{s} - \dfrac{\alpha^2 - \beta^2}{s-1}}{\left[(9\alpha^2 - 1) - \dfrac{(9\beta^2 - 1)}{s} + \dfrac{(9\gamma^2 - 1)}{s-1}\right]^2}. \tag{32}$$

Note that equation (31) is consistent with the transformation property (29). It also follows from the transformation property (29) that $I := F_2/F_1^{3/2}$ is another SL$(2, \mathbf{C})$-invariant and hence a function of s only. So in equation (31) s, and hence $G(s)$, can be expressed in terms of y, $\dot{y}$, and $\ddot{y}$ via the invariant I. Therefore, in general, the form of P in equation (26) will be algebraic. In order for equation (26) to be of polynomial type in y and its derivatives, G must be constant (*i.e.* independent of s). This is possible only for certain values of the parameters α, β, and γ.

Note that $G(s)$ is regular at $s = 0, 1, \infty$ provided $\beta^2 \neq 1/9$, $\gamma^2 \neq 1/9$, and $\alpha^2 \neq 1/9$ respectively. Hence we evaluate $G(s)$ in equation (32) at the special points $s = 0, 1, \infty$ and set them equal to obtain restrictions on the parameters α, β, and γ. There are two cases.

Case I: If none of α, β, and γ is equal to 1/3, then we have

$$G(0) = \frac{\beta^2}{9\beta^2 - 1}, \qquad G(1) = \frac{\gamma^2}{9\gamma^2 - 1}, \qquad G(\infty) = \frac{\alpha^2}{9\alpha^2 - 1},$$

and so, from $G(0) = G(1) = G(\infty)$ we have $\alpha^2 = \beta^2 = \gamma^2$.

Case II: One or more of the parameters are 1/3. We first consider the case when exactly one of the parameters, say β, is 1/3. From $G(1) = G(\infty)$, we still have $\gamma^2 = \alpha^2$, hence

$$G(s) = \frac{1}{9\alpha^2 - 1}\left(\alpha^2 - \frac{2}{9}\frac{s-1}{s^2}\right).$$

So $G(s)$ is not constant if exactly one of the parameters is $1/3$. However, if exactly two of the parameters are $1/3$ then $G(s)$ will be a constant, regardless of the value of the other parameter. In particular, if $\beta^2 = \gamma^2 = 1/9$, then

$$G(s) = \frac{\alpha^2}{9\alpha^2 - 1}.$$

Both cases I and II lead to the same third-order equation for $y(t)$ with an arbitrary parameter. Namely, from equation (31) and the constant $G(s)$ from case I or II, we find that y satisfies

$$\frac{d^3y}{dt^3} - 2y\frac{d^2y}{dt^2} + 3\left(\frac{dy}{dt}\right)^2 = \frac{A^2}{9A^2 - 1}\left(6\frac{dy}{dt} - y^2\right)^2. \tag{33}$$

In case I, $\alpha^2 = \beta^2 = \gamma^2 = A^2$ whereas in case II, $\alpha^2 = A^2$, $\beta^2 = \gamma^2 = 1/9$ et cyc. Note, however, that the Schwarzian $s(t)$ underlying equation (33) is different for the two cases since $s(t)$ satisfies equation (21) with

$$V(s) = (1 - A^2)\left(\frac{1}{s^2} + \frac{1}{(s-1)^2} - \frac{1}{s(s-1)}\right)$$

for case I and

$$V(s) = \frac{8}{9}\left(\frac{1}{s^2} + \frac{1}{(s-1)^2}\right) - \frac{7 + 9A^2}{9s(s-1)}$$

for case II.

Equation (21) is of central importance in the theory of conformal mappings. When the parameters α, β, γ are non-negative real numbers satisfying $\alpha+\beta+\gamma < 1$, the function $s(t)$ maps a circular triangle (a region bounded by arcs of three circles) with angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$ to the upper or lower half-plane. The corners of the triangles are mapped to the singular points of equation (21–2), namely $s = 0$, $s = 1$ and $s = \infty$.

The solutions of equation (21) and hence equations (1–2) and (33) can be analytically extended across each of the sides of the circular triangle by the Schwarz Reflection Principle (see e.g. [29, 30]). The Schwarz function

$s(\alpha, \beta, \gamma; t)$ has the following expansions in the neighborhoods of the singular points of the Schwarzian equation (21):

Singular point	Leading Order Behavior for s as $t \to t_0$
$s = 0$	$s \sim \kappa_0 (t - t_0)^{1/\beta}$
$s = 1$	$s \sim 1 + \kappa_1 (t - t_0)^{1/\gamma}$
$s = \infty$	$s \sim \kappa_\infty (t - t_0)^{-1/\alpha}$

where κ_0, κ_1, and κ_∞ are constants. If the parameters are of the form $\alpha = 1/p$, $\beta = 1/q$, and $\gamma = 1/r$, where p, q, r are integers, then the solution is single-valued but possesses a movable natural barrier (a circle or line across which the solution cannot be extended). In general, however, the solution of equations (1–2) and (33) will be densely branched about movable singularities.

Equation (33) becomes the generalized Chazy equation (3) if we set $A = 2/n$. Chazy [15, 16, 17] analyzed equation (3) and showed that its solution is related to the Schwarz function J which solves equation (21) with (22) and $\alpha = 1/n$, $\beta = 1/3$, $\gamma = 1/2$. The general solution of equation (3) is given by

$$y = \frac{1}{2}\frac{d}{dt} \ln \frac{\dot{J}^6}{J^4 (J-1)^3}. \tag{34}$$

The function J, and hence y, is single-valued if n is an integer greater than 1. The choice $n = \infty$ ($A = 0$) corresponds to the classical Chazy equation. We note that when n is odd the Schwarz functions $s(2/n, 2/n, 2/n; t)$ (in case I) and $s(2/n, 1/3, 1/3; t)$ (in case II) appearing in equation (27) are branched yet y given by (25) is single-valued. In the case of $s(2/n, 2/n, 2/n; t)$

$$J = \frac{4}{27} \frac{(s^2 - s + 1)^3}{s^2 (s-1)^2},$$

and similarly, for the function $s(2/n, 1/3, 1/3; t)$ we have

$$J = -4s(s-1),$$

(see [31]). It follows from the above identities that

$$\frac{\dot{J}^6}{J^4(J-1)^3} = k_s \frac{\dot{s}^6}{s^4(s-1)^4}, \tag{35}$$

for *both* Schwarzian functions $s(2/n, 2/n, 2/n; t)$ (corresponding to $k_s = 432$) and $s(2/n, 1/3, 1/3; t)$ (corresponding to $k_s = -16$). Consequently the two expressions for y given in equations (27) and (34) are equivalent. Moreover, $y(t)$ given by equation (34) is single-valued when $n > 1$ is an integer (because J is single-valued in this case).

In the special case $\alpha = \beta = \gamma = A = 1/3$, differentiating equation (27) and using equation (21) with (22) we obtain

$$F_1 = 6\dot{y} - y^2 = 0,$$

giving

$$y(t) = -\frac{6}{t - t_0},$$

where t_0 is a constant. This is the case $n = 6$ in the generalized Chazy equation (3). Note, however, that in general the ω_i depend on t in a nontrivial fashion thru (20–22) despite the fact that their sum $(-y/2)$ only has a simple pole.

The Degenerate Cases

In this section we consider the case in which the eigenvalues of M_s are not distinct. Motivated by the classical Darboux-Halphen system, in the case $\omega_3 = \omega_2 \neq \omega_1$ we define

$$s := \mathrm{e}^{2\int(\omega_1 - \omega_2)dt}$$

which gives

$$\omega_1 = -\frac{1}{2}\frac{d}{dt}\ln\frac{\dot{s}}{s^2}, \qquad \omega_2 = \omega_3 = -\frac{1}{2}\frac{d}{dt}\ln\frac{\dot{s}}{s},$$
$$\tau_1 = A\frac{\dot{s}}{s}, \qquad \tau_2 = B\frac{\dot{s}}{s^{3/2}}, \qquad \tau_3 = C\frac{\dot{s}}{s^{3/2}},$$

where A, B, and C are constants and s satisfies equation (21) with

$$V(s) = \frac{\mu^2}{s^3} + \frac{1-\nu^2}{s^2}, \tag{36}$$

where $\mu^2 = 4(B^2 + C^2)$ and $\nu^2 = -4A^2$.

If $\mu \neq 0$, the general solution of the Schwarzian equation (21) with (36) is given implicitly by

$$t(s) = \frac{\Theta_2(x)}{\Theta_1(x)}, \qquad s = (\mu/x)^2,$$

where Θ_1 and Θ_2 are independent solutions of Bessel's equation

$$x^2 \frac{d^2\Theta}{dx^2} + x\frac{d\Theta}{dx} + (x^2 - \nu^2)\Theta = 0. \tag{37}$$

Note that Bessel's equation has an irregular singular point at infinity in contrast with the hypergeometric equation (24) which has three regular singular points.

If $\mu = 0$, then the equation (21) with (36) becomes another well known case of the Schwarzian equation whose solutions map crescent-shaped regions to the upper or lower half-plane. The "corners" of the crescent-shaped regions both form angles $\nu\pi$ and are mapped to the singular points $s = 0$ and $s = \infty$. Note that $s = 1$ is not a singular point of equation (21) with (36). For y defined by (25) we find that y solves

$$\left(9\frac{d^2y}{dt^2} - 9y\frac{dy}{dt} + y^3\right)^2 = \frac{1}{9\nu^2 - 1}\left(6\frac{dy}{dt} - y^2\right)^3,$$

which has the general solution

$$y = \begin{cases} \frac{1-3\nu}{\nu}(t-t_1)^{-1} - \frac{1+3\nu}{\nu}(t-t_2)^{-1}, & \text{for } \nu \neq 0, \\ -6(t-t_0)^{-1} + k(t-t_0)^{-2}, & \text{for } \nu = 0, \end{cases}$$

where t_0, t_1, t_2, and k are arbitrary constants.

Finally we consider the totally degenerate case $\omega_1 = \omega_2 = \omega_3 =: \omega$. The system (1,14) reduces to

$$\dot{\omega} + \omega^2 = \tau^2, \qquad \dot{\tau}_j + 2\omega\tau_j = 0,$$

which has the general solution

$$\omega = -\frac{1}{2}\frac{d}{dt}\ln s, \qquad \tau_j = \kappa_j s, \qquad \tau^2 = \tau_1^2 + \tau_2^2 + \tau_3^2,$$

where

$$s = \frac{1}{At^2 + Bt + C}, \qquad 4(\kappa_1^2 + \kappa_2^2 + \kappa_3^2) = 4AC - B^2$$

and, if $A \neq 0$ then, without loss of generality, we take $A = 1$. In this case ω satisfies a simple second-order polynomial differential equation:

$$\ddot{\omega} + 6\omega\dot{\omega} + 4\omega^3 = 0.$$

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