# ON PAINLEVÉ AND DARBOUX-HALPHEN TYPE EQUATIONS

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Keywords: Darboux-Halphen, Chazy, monodromy, self-dual Yang-Mills

AMS classification numbers: 34A34, 53C07, 83C20

**Abstract**

It is now well known that a deep connection exists between soliton equations and ODE's of Painlevé type. As a consequence there has been a significant re-emergence of interest in the study of such ODE's and related issues. In this paper we demonstrate that a novel class of nonlinear ODE's, Darboux-Halphen (DH) type systems, can be obtained as reductions of the self-dual Yang-Mills (SDYM) equations. We show how to find by reduction from SDYM the associated linear pair for DH. This linear system is found to be monodromy evolving which is different from the linear systems associated with the Painlevé equations which are isomonodromy. The solution of the DH system can be obtained in terms of Schwarzian equations which are themselves linearizable. The DH system has solutions which are related to Painlevé equations but the solutions can have complicated analytic singularities such as natural boundaries and dense branching.

# 1 Introduction

This paper emanates from the lectures one of us (MJA) gave at a meeting in Cargese, Corsica organized during the summer of 1996 focusing on Painlevé equations and related issues. The meeting was held partly in commemoration of P. Painlevé's work one century ago. Painlevé and his school were well known at the end of the 19th century and early 20th century. But by the 1970's many of the results, although published in leading journals, were not known or appreciated by most mathematicians and physicists. The reader may wish to review Painlevé's collected works [17] for an in depth discussion of his contributions and point of view.

However this situation changed dramatically with the recognition that equations of "Painlevé type" were intimately connected with a class of integrable systems; i.e. soliton and related equations. Namely, it was demonstrated that ordinary differential equations (ODE's) obtained as reductions of the well known soliton equations yielded ODE's with the *Painlevé Property*. Namely the solutions of the resulting ODE's were free of movable branch points. Moreover, similarity reductions of the best known soliton equations often resulted in one of the classical second order Painlevé equations. Background information and a description of the research involving Painlevé equations and their relationship to integrable soliton systems can be found in [2, 4]. Ever since the discovery that such integrable systems and Painlevé equations were deeply connected there has been a major research effort which has involved many aspects of Painlevé equations. The scope of the work and results are far too numerous to discuss here.

In this paper we will:

*i.* describe, by example, some of the salient features of the connection of integrable systems and Painlevé equations. Notably we discuss the connection of the Inverse Scattering Transform (IST) to Painlevé equations and how one can obtain the linearization and solutions from this connection.

*ii.* discuss a generalized Darboux-Halphen (DH) system, which is a fifth order ODE and which reduces to the classical third order DH system first studied by Darboux in 1879 [11]. This equation was discussed in a recent letter [5]. This system, which is a reduction of the self-dual Yang-Mills equations (SDYM), has a compatible linear monodromy evolving system. The reason for this can be traced to the fact that the reduction from SDYM involves an infinite dimensional algebra: $\mathfrak{sdiff}(3)$.

*iii.* show how, for the first time, one can use the infinite dimensional algebra in SDYM to deduce the compatible linear monodromy system for DH.

*iv.* quote the main results of the monodromy analysis and demonstrate that although the generalized DH system is linearizable and hence integrable, nevertheless generically speaking the solution is densely branched. Only when a constant of the motion takes on a denumerably infinite set of values does the solution become single valued. In this case the solution is expressible in terms of automorphic functions. Indeed we note that in the special case of the classical third order DH system, the solution is always single valued. However in this case the solution contains natural boundaries in the complex plane.

## 2 Painlevé Equations and IST

In the Cargese lectures the importance of similarity reductions of PDE's was reviewed. Asymptotic analysis of Fourier integrals shows that the long time behavior of linear evolution equations with constant coefficients yields the fact that self-similar (i.e. similarity) solutions are leading order asymptotic states in certain regions of space (e.g. [4]). In the well known soliton systems a similar situation arises. For example associated with the modified Korteweg-de Vries (mKdV) equation,

$$u_t - 6u^2 u_x + u_{xxx} = 0, \tag{1}$$

is the self-similar reduction,

$$u(x,t) = w(x/(3t)^{1/3})/(3t)^{1/3} \tag{2}$$

which upon substitution into equation (1) yields, after an integration, the second ($P_{II}$) of the six classical Painlevé equations,

$$w'' - zw - 2w^3 = \alpha, \tag{3}$$

where $\alpha$ is an arbitrary constant.

Ablowitz and Segur (see e.g. [4]) showed that equation (3) with $\alpha = 0$ governed the dominant long time asymptotic state of the Cauchy problem associated with mKdV equation (1), in the region $|x/(3t)^{1/3}| = O(1)$. Not only can one associate the second Painlevé equation with mKdV, but the association yields an integral equation governing a one-parameter family of solutions to $P_{II}$ which is relevant to the Cauchy problem of mKdV.

Namely the inverse scattering transform (IST) shows that the mKdV equation can be solved (i.e. linearized) to the following integral equation

$$K(x,y;t) - F(x+y;t) - \int_{-\infty}^{x}\int_{-\infty}^{x} K(x,z;t)F(z+s;t)F(s+y;t)dzds = 0,$$

where

$$F(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} r(k)exp(ikx+8ik^3)tdk$$

and the solution to mKdV is given by

$$u(x,t) = 2K(x,x;t).$$

The long time asymptotic analysis of mKdV shows that to leading order in the region: $|x/(3t)^{1/3}| = O(1)$, the solution of mKdV (1) satisfies equation (3) with $\alpha = 0$ and the relevant solution of the $P_{II}$ equation is linearized via:

$$K_{\#}(x,y) - r_0 Ai(x+y) - (r_0)^2 \int_{-\infty}^{x}\int_{-\infty}^{x} K_{\#}(x,z)Ai(z+s)Ai(s+y)dzds = 0$$

where Ai(x) is the well known Airy function whose integral representation is given by

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ikx + ik^3/3)dk$$

and the solution of $P_{II}$ is given by,

$$w(x) = 2K_{\#}(x,x)$$

This solution satisfies the following "connection" formulae (c.f. [2, 4]):

As $x \to +\infty$,

$$w(x) \sim r_0\mathrm{Ai}(x) \sim \frac{r_0}{2\sqrt{\pi}x^{1/4}}\,\mathrm{e}^{-\frac{2}{3}x^{3/2}}$$

and as $x \to -\infty$,

$$w(x) \sim \frac{d_0}{(-x)^{1/4}} \sin\theta$$

where

$$\begin{aligned}
d_0 &= -(1/\pi)\log(1-(r_0)^2),\\
\theta &= (2/3)(-x)^{3/2} - (3/4)d_0^2\log(-x) + \theta_0,\\
\theta_0 &= \pi/4 - (3/2)d_0^2\log 2 - \arg[\Gamma(1 - i d_0^2/2)],
\end{aligned}$$

where $\Gamma(x)$ is the usual Gamma function.

Similarity reduction not only yields special solutions, but it is also the mechanism for one to be able to obtain the isomonodromy problems, which govern the general solution of the Painlevé equation. Indeed Flaschka and Newell [12] showed that the similarity reduction of the compatible linear system for mKdV also yields a compatible linear system monodromy problem governing $P_{II}$. Namely, equation (3) is the compatibility condition ($\Psi_{t\zeta} = \Psi_{\zeta t}$) for the following system:

$$\begin{aligned}
\Psi_\zeta &= \begin{pmatrix} -i(4\zeta^2 + z + 2w^2) & 4\zeta w + 2iw_z + \alpha\zeta^{-1} \\ 4\zeta w - 2iw_z + \alpha\zeta^{-1} & i(4\zeta^2 + z + 2w^2) \end{pmatrix}\Psi,\\
\Psi_z &= \begin{pmatrix} -i\zeta & w \\ w & i\zeta \end{pmatrix}\Psi.
\end{aligned}$$

In principle the compatible isomonodromy system yields the complete solution of the mKdV equation.

We shall not dwell on this aspect of the problem since we prefer to discuss a novel reduction of the self-dual Yang-Mills (SDYM) integrable system: the generalized Darboux-Halphen (gDH) system. SDYM reductions alone occupy a major aspect of the integrable systems literature. We only point out here that lower dimensional reductions of the four-dimensional SDYM system yield virtually all the well known soliton equations (see e.g. [2]) and all the classical Painlevé equations with their monodromy problems (see e.g. [15]). Just as the reduction (2) of the mKdV linear system gives rise to the isomonodromy problem for $P_{II}$, the SDYM linear system under this novel reduction yields a new class of *monodromy evolving* problems underlying the gDH systems. The evolution of this monodromy system is such that the temporal evolution of the scattering data can be tracked exactly and this allows us to find the exact solution of the gDH system.

The results associated with DH type systems also have an important connection in terms of Painlevé equations. Namely we will see that there is a parameter in their associated linear systems. When this parameter vanishes, the compatible linear system of DH reduces to an isomonodromy system associated with one of the classical Painlevé equations. This isomonodromic system is essential when solving the DH problem.

In this paper we present the solution of the following system of ODEs, which we refer to as the *generalized DH system*:

$$\begin{aligned}
\dot{\omega}_1 &= \omega_2\omega_3 - \omega_1(\omega_2+\omega_3) + \phi^2, \\
\dot{\omega}_2 &= \omega_3\omega_1 - \omega_2(\omega_3+\omega_1) + \theta^2, \\
\dot{\omega}_3 &= \omega_1\omega_2 - \omega_3(\omega_1+\omega_2) - \theta\phi, \\
\dot{\phi} &= \omega_1(\theta-\phi) - \omega_3(\theta+\phi), \\
\dot{\theta} &= \omega_2(\phi-\theta) - \omega_3(\theta+\phi),
\end{aligned} \tag{4}$$

where the dots denote differentiation with respect to $t$. They correspond to an off-diagonal Bianchi IX metric with self-dual Weyl curvature (see [5]). In

Chakravarty, Ablowitz, and Takhtajan [8], it was shown that this system arises as a reduction of the SDYM equations. From the reduction process we obtain a linear problem for the DH system from the linear problem for SDYM. As outlined earlier in Chakravarty and Ablowitz [5], this linear problem, which has non-constant monodromy, allows us to solve the initial value problem for the system (4) in terms of solutions of Schwarzian equations, which arise in the theory of conformal mappings.

We remark that this system with $\theta = \phi = 0$, i.e.

$$\begin{aligned} \dot{\omega}_1 &= \omega_2\omega_3 - \omega_1(\omega_2 + \omega_3), \\ \dot{\omega}_2 &= \omega_3\omega_1 - \omega_2(\omega_3 + \omega_1), \\ \dot{\omega}_3 &= \omega_1\omega_2 - \omega_3(\omega_1 + \omega_2), \end{aligned} \tag{5}$$

which we call the *classical DH system*, was originally analyzed by Darboux [11] in the context of certain triply orthogonal surfaces. Shortly thereafter the solution was found by Halphen [13]. The system is equivalent to the Einstein field equations for a (complex) self-dual Bianchi IX spacetime with diagonal metric. We also note that the symmetric combination of variables $y := -2(\omega_1 + \omega_2 + \omega_3)$ satisfies the Chazy equation (Chazy [9, 10]),

$$\frac{d^3y}{dt^3} = 2y\frac{d^2y}{dt^2} - 3\left(\frac{dy}{dt}\right)^2, \tag{6}$$

whose general solution possesses a natural boundary.

# 3 Darboux-Halphen Systems and Their Linear Problems as Reductions of SDYM

As mentioned in the introduction, the self-dual Yang-Mills equations (SDYM) play a central role in the theory of integrable systems. In Chakravarty, Ablowitz, and Clarkson [6, 7] it was shown that the classical Darboux-Halphen system (5) is a reduction of SDYM with the gauge algebra $\mathfrak{sdiff}(SU(2))$. In this section we discuss the SDYM equations and their reduction to the generalized Darboux-Halphen system (4). We then go on to show that a linear problem for the Darboux-Halphen system can be obtained from that for SDYM. To our knowledge, all previously studied linear problems associated with integrable systems of ODEs have been *isomonodromy problems*. The linear problem obtained in this section, however, has nonconstant, i.e. evolving monodromy data. It is important to stress that we do not impose the evolving monodromy condition; rather it is a logical consequence of the SDYM reduction.

For a given Lie algebra $\mathfrak{g}$ corresponding to the Lie group G, let the *gauge potential 1-form* $\mathbf{A}$ be given by

$$\mathbf{A} = \sum_{\mu} A_{\mu}\, dx_{\mu},$$

where the $x_{\mu}$ are coordinates on $\mathbf{R}^4$, $A_{\mu} : \mathbf{R}^4 \to \mathfrak{g}$, $\mu = 0, \ldots, 3$. The *curvature 2-form* $\mathbf{F}$ is given by

$$\mathbf{F} = \frac{1}{2} \sum_{\mu,\nu} F_{\mu\nu} dx_{\mu} \wedge dx_{\nu},$$

where

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - [A_{\mu}, A_{\nu}]$$

and we define $\partial_\mu \equiv \frac{\partial}{\partial x_\mu}$.

With respect to the standard coordinates on $\mathbf{R}^4$, the *SDYM equations associated with the gauge algebra* $\mathfrak{g}$ are

$$F_{01} = F_{23}, \qquad F_{02} = F_{31}, \qquad F_{03} = F_{12}.$$

Alternatively, in terms of the so-called null coordinates

$$\alpha = x_0 + ix_1, \quad \bar{\alpha} = x_0 - ix_1, \quad \beta = x_3 + ix_2, \quad \bar{\beta} = x_3 - ix_2,$$

SDYM becomes

$$F_{\alpha\beta} = F_{\bar{\alpha}\bar{\beta}} = F_{\alpha\bar{\alpha}} + F_{\beta\bar{\beta}} = 0, \tag{7}$$

where

$$\begin{aligned} A_\alpha &= \frac{1}{2}(A_0 - iA_1) \qquad A_\beta = \frac{1}{2}(A_3 - iA_2), \\ A_{\bar{\alpha}} &= \frac{1}{2}(A_0 + iA_1) \qquad A_{\bar{\beta}} = \frac{1}{2}(A_3 + iA_2). \end{aligned}$$

It is especially easy to see that equations (7) are the compatibility of the system

$$(\partial_\alpha - \lambda\partial_{\bar{\beta}})\Psi = (A_\alpha - \lambda A_{\bar{\beta}})\Psi, \tag{8}$$

$$(\partial_\beta + \lambda\partial_{\bar{\alpha}})\Psi = (A_\beta + \lambda A_{\bar{\alpha}})\Psi, \tag{9}$$

since

$$0 = [\partial_\alpha - \lambda\partial_{\bar{\beta}}, \partial_\beta + \lambda\partial_{\bar{\alpha}}]\Psi = \{F_{\alpha\beta} + \lambda(F_{\alpha\bar{\alpha}} + F_{\beta\bar{\beta}}) + \lambda^2 F_{\bar{\alpha}\bar{\beta}}\}\Psi.$$

The SDYM equations are invariant under the *gauge transformation*

$$A_\mu \mapsto h^{-1}A_\mu h - h^{-1}\partial_\mu h,$$

for any G-valued function $h$ on $\mathbf{R}^4$ since this induces the map $\mathbf{F} \mapsto h^{-1}\mathbf{F}h$. This gauge invariance extends to the linear problem (8–9) by the transformation $\Psi \mapsto h^{-1}\Psi$.

Consider the one-dimensional reduction in which the $A_\mu$'s are functions of $x_0 =: t$ only. Choosing a gauge in which $A_0 = 0$, the SDYM equations reduce to

$$\begin{aligned} \partial_t A_1 + [A_2, A_3] &= 0, \\ \partial_t A_2 + [A_3, A_1] &= 0, \\ \partial_t A_3 + [A_1, A_2] &= 0. \end{aligned} \tag{10}$$

This is known as the *Nahm system* (see Nahm [16]).

If we choose the Lie algebra $\mathfrak{g}$ to be $\mathfrak{su}(2)$ and take

$$A_i(t) = -\omega_i(t) X_i,$$

where the $\{X_i\}_{i=1,2,3}$ is a basis for $\mathfrak{su}(2)$ which satisfies $[X_i, X_j] = \sum_k \varepsilon_{ijk} X_k$, then we obtain the system

$$\dot\omega_1 = \omega_2\omega_3, \qquad \dot\omega_2 = \omega_3\omega_1, \qquad \dot\omega_3 = \omega_1\omega_2, \tag{11}$$

where the dots denote differentiation with respect to $t$. This system, which dates back to Lagrange, admits a general solution in terms of elliptic functions. Equation (11) can be viewed (very loosely speaking) as an "unperturbed" version of the system (5).

Actually, since we are interested in the "off diagonal" system (4), we first consider the "unperturbed" off diagonal system associated with it which is obtained by assuming

$$-A_1(t) \equiv T_1(t) = \omega_1(t)X_1 + \theta(t)X_2, \qquad -A_2(t) \equiv T_2(t) = \phi(t)X_1 + \omega_2(t)X_2,$$

$$-A_3(t) \equiv T_3(t) = \omega_3(t) X_3, \tag{12}$$

whereupon (10) implies

$$\begin{aligned}
\dot{\omega}_1 &= \omega_2\omega_3, \\
\dot{\omega}_2 &= \omega_3\omega_1, \\
\dot{\omega}_3 &= \omega_1\omega_2 - \theta\phi, \\
\dot{\theta} &= -\phi\omega_3, \\
\dot{\phi} &= -\theta\omega_3.
\end{aligned} \tag{13}$$

As with equations (11), equations (13) are solvable in terms of elliptic functions. Indeed, we note that

$$\theta^2 - \phi^2 = C^2, \qquad \omega_1^2 - \omega_2^2 = E^2,$$

where $C$ and $E$ are constants. The parameterizations

$$\theta = C\cosh\psi(t), \quad \phi = C\sinh\psi(t), \quad \omega_1 = E\cosh\mu(t), \quad \omega_2 = E\sinh\mu(t),$$

imply

$$\psi + \mu = k, \quad \text{and} \quad \ddot{\psi} = \alpha\sinh(2\psi - \beta),$$

where $k$ is a constant, $\alpha^2 = (C^4 + E^4 + 2C^2E^2\cosh 2k)/4$, and $\tanh\beta = E^2\sinh 2k/(C^2 + E^2\cosh 2k)$. This equation can be solved via elliptic functions. In our study of equation (4) we will deform (12) in equation (10) appropriately.

In fact the Darboux-Halphen system arises from the Nahm system (equation 10) with a different choice of gauge algebra [6, 8]. We also again mention that unlike equations (11) or (13), the solution of equation (4) or (5) is expressible in terms of Schwarzian functions; hence the structure of the solution is more complicated.

We begin by briefly reviewing the double covering of $SO(3)$ by $SU(2)$. Let

$$\mathbf{x} = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} = \sum_{i=1}^{3} x_i \sigma_i,$$

be any trace-free Hermitian matrix, where $x_i \in \mathbf{R}$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{14}$$

are the Pauli matrices. For any $g \in SU(2)$ (i.e. $g$ is a unit determinant $2 \times 2$ matrix over $\mathbf{C}$ satisfying $gg^\dagger = I$),

$$\mathbf{y} = \begin{pmatrix} y_3 & y_1 - iy_2 \\ y_1 + iy_2 & -y_3 \end{pmatrix} := g\mathbf{x}g^{-1}, \tag{15}$$

is also a trace-free Hermitian matrix. Furthermore, $y_1^2 + y_2^2 + y_3^2 = -\det \mathbf{y} = -\det \mathbf{x} = x_1^2 + x_2^2 + x_3^2$. In other words, the mapping $\mathbf{O} : (x_1, x_2, x_3) \mapsto (y_1, y_2, y_3)$ is in $SO(3)$. Using the standard representation of $\mathbf{O}$ as a $3 \times 3$ matrix, the components, $O_{ij}$, are given by

$$g^{-1}\sigma_i g = \sum_j O_{ij}(g)\sigma_j \tag{16}$$

for all $g \in SU(2)$. The matrices $g$ and $O_{ij}$ can be parameterized by the Euler angles $\theta$, $\phi$, $\psi$ (see, for example, Vilenkin [19]):

$$g = \begin{pmatrix} \mathrm{e}^{-i(\phi+\psi)/2}\cos(\theta/2) & -i\mathrm{e}^{i(\psi-\phi)/2}\sin(\theta/2) \\ -i\mathrm{e}^{i(\phi-\psi)/2}\sin(\theta/2) & \mathrm{e}^{i(\phi+\psi)/2}\cos(\theta/2) \end{pmatrix}$$

and

$$\mathbf{O} = \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta & -\cos\phi\sin\psi - \sin\phi\cos\psi\cos\theta & \sin\phi\sin\theta \\ \sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta & -\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta & -\cos\phi\sin\theta \\ \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta \end{pmatrix},$$

where $0 < \theta < \pi$, $0 \leq \phi < 2\pi$, and $-2\pi \leq \psi < 2\pi$. Note that for this range of the Euler angles $SU(2)$ (parameterized by $g$) is covered once and $SO(3)$ (parameterized by $\mathbf{O}$) is covered twice.

Choose the gauge algebra to be $\mathfrak{sdiff}(SU(2))$ with now

$$A_i(t) = -\widetilde{T}_i, \quad \text{where} \quad \widetilde{T}_i := \sum_j O_{ij} T_j \tag{17}$$

and the $T_i$ are given by (12), where $\{X_i\}$ are the standard left-invariant vector fields generating $\mathfrak{su}(2)$ which satisfy

$$X_j(g) = \frac{1}{2i} g\sigma_j. \tag{18}$$

In this notation the $X_j$ are given explicitly as vector fields in the Euler angles $(\theta, \phi, \psi)$ as

$$\begin{aligned} X_1 &= \cos\psi \frac{\partial}{\partial\theta} + \sin\psi \operatorname{cosec}\theta \frac{\partial}{\partial\phi} - \cot\theta \, \sin\psi \frac{\partial}{\partial\psi}, \\ X_2 &= -\sin\psi \frac{\partial}{\partial\theta} + \cos\psi \operatorname{cosec}\theta \frac{\partial}{\partial\phi} - \cot\theta \, \cos\psi \frac{\partial}{\partial\psi}, \\ X_3 &= \frac{\partial}{\partial\psi}. \end{aligned}$$

Using the properties (18) (or equivalently $X_i(O_{jk}) = \sum_l \varepsilon_{ikl} O_{jl}$) it can be shown that under the present reduction, the SDYM equations become the gDH system (4) (see also [6, 8]). We note that if $O_{ij}$ is replaced by $\delta_{ij}$ in equation (17) we recover the system (13). It is in this sense that equations (4) are a perturbed form of equations (13).

Next we will find a linear problem associated with equation (4) from the linear problem for SDYM. In (8–9) we take

$$-2A_\alpha = i\widetilde{T}_3 = 2A_{\bar{\alpha}}, \qquad 2iA_\beta = \widetilde{T}_1 - i\widetilde{T}_2 \equiv \widetilde{T}_-, \qquad -2iA_{\bar{\beta}} = \widetilde{T}_1 + i\widetilde{T}_2 \equiv \widetilde{T}_+.$$

Noting $\partial_\alpha \Psi = \partial_{\bar\alpha}\Psi = \partial_t \Psi/2$, $\partial_\beta \Psi = \partial_{\bar\beta}\Psi = 0$ and defining $\widetilde{T}_0 \equiv \partial\ /\partial t$, we then obtain

$$\left(\widetilde{T}_0 + i\widetilde{T}_3 + i\lambda\widetilde{T}_+\right)\Psi = 0, \tag{19}$$

$$\left(\lambda\widetilde{T}_0 + i\widetilde{T}_- - i\lambda\widetilde{T}_3\right)\Psi = 0. \tag{20}$$

We will find it convenient to parameterize $\lambda$ by writing it as a "projective coordinate" $\lambda = \pi^1/\pi^0$. Assuming $\Psi$ to be a function of $t$ and $\lambda$ only, we obtain the following linear pair from (19–20):

$$l_1\Psi = l_2\Psi = 0,$$

where

$$\begin{aligned} (\, l_1 \quad l_2 \,) &= (\, \pi^0(\widetilde{T}_0 + i\widetilde{T}_3) + i\pi^1\widetilde{T}_+ \quad i\pi^0\widetilde{T}_- + \pi^1(\widetilde{T}_0 - i\widetilde{T}_3) \,) \\ &= (\, \pi^0 \quad \pi^1 \,) \begin{pmatrix} \widetilde{T}_0 + i\widetilde{T}_3 & i\widetilde{T}_- \\ i\widetilde{T}_+ & \widetilde{T}_0 - i\widetilde{T}_3 \end{pmatrix} = (\, \pi^0 \quad \pi^1 \,)\, \widetilde{\mathbf{T}} \end{aligned}$$

$\widetilde{T}_0 = \partial_t$ and $\widetilde{T}_\pm = \widetilde{T}_1 \pm i\widetilde{T}_2$.

This linear problem is unfortunately too complicated to work with concretely because $\mathbf{T} \in \mathfrak{sdiff}(SU(2))$. In order to simplify it, we first note from equation (15) that equation (17) can be written as an $SU(2)$ action:

$$\widetilde{\mathbf{T}} = g\mathbf{T}g^{-1}, \tag{21}$$

where

$$\mathbf{T} = \begin{pmatrix} T_0 + iT_3 & iT_- \\ iT_+ & T_0 - iT_3 \end{pmatrix},$$

$T_0 = \partial_t$, $T_\pm := T_1 \pm iT_2$. Equation (21) says $\widetilde{\mathbf{T}}(\phi) = g\mathbf{T}(\phi)g^{-1}$ for all SU(2)-valued functions $\phi$.

Next we take two independent linear combinations of $l_1$ and $l_2$, only one of which, M, contains a derivative with respect to $t$.

$$
\begin{aligned}
L &:= \begin{pmatrix} l_1 & l_2 \end{pmatrix} \begin{pmatrix} -\pi^1 \\ \pi^0 \end{pmatrix} = \begin{pmatrix} \pi^0 & \pi^1 \end{pmatrix} g\mathbf{T}g^{-1} \begin{pmatrix} -\pi^1 \\ \pi^0 \end{pmatrix}, \qquad (22)\\
M &:= \begin{pmatrix} l_1 & l_2 \end{pmatrix} \begin{pmatrix} -v^1 \\ v^0 \end{pmatrix} = \begin{pmatrix} \pi^0 & \pi^1 \end{pmatrix} g\mathbf{T}g^{-1} \begin{pmatrix} -v^1 \\ v^0 \end{pmatrix}, \qquad (23)
\end{aligned}
$$

where $\pi^0 v^1 - \pi^1 v^0 = 1$ and we have used equation (21).

Define $\tilde{\pi}^A$, $\tilde{v}^A$, which we interpret as spinors (see e.g. [18]), as

$$
\begin{pmatrix} \tilde{\pi}^0 & \tilde{\pi}^1 \\ \tilde{v}^0 & \tilde{v}^1 \end{pmatrix} = \begin{pmatrix} \pi^0 & \pi^1 \\ v^0 & v^1 \end{pmatrix} g,
$$

and clearly $\tilde{\pi}^0\tilde{v}^1 - \tilde{\pi}^1\tilde{v}^0 = 1$. In this way the $g$-action is absorbed into the spinors $\tilde{\pi}^A$, $\tilde{v}^A$. From (22–23) we see that these operators have an elegant representation in terms of the spinors $\pi^A$, $v^A$, $\tilde{\pi}^A$, and $\tilde{v}^A$:

$$
\begin{aligned}
L &= \sum_{A,B} \pi^A \widetilde{\mathbf{T}}_A^{\ B} \pi_B = \sum_{A,B} \tilde{\pi}^A \mathbf{T}_A^{\ B} \tilde{\pi}_B,\\
M &= \sum_{A,B} \pi^A \widetilde{\mathbf{T}}_A^{\ B} v_B = \sum_{A,B} \tilde{\pi}^A \mathbf{T}_A^{\ B} \tilde{v}_B,
\end{aligned}
$$

and where the dual spinors $\pi_A$, $v_A$, etc., are given by

$$
\pi_B = \sum_A \pi^A \varepsilon_{AB}, \qquad \varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
$$

We also note that by multiplication equations (22–23) become

$$
\begin{aligned}
-L &= i\left\{(\tilde{\pi}^1)^2 T_+ + 2\tilde{\pi}^0\tilde{\pi}^1 T_3 - (\tilde{\pi}^0)^2 T_-\right\}, \qquad (24)\\
-M &= \partial_t + i\left\{\tilde{\pi}^1\tilde{v}^1 T_+ + (\tilde{\pi}^0\tilde{v}^1 + \tilde{\pi}^1\tilde{v}^0) T_3 - \tilde{\pi}^0\tilde{v}^0 T_-\right\}. \qquad (25)
\end{aligned}
$$

Notice that in the operators L and M we have moved the explicit g-dependence from the operators $\widetilde{\mathbf{T}}^A{}_B$ to the spinors $\tilde{\pi}^A$, $\tilde{v}^A$. This will help us simplify the linear problem.

We need to take into account the action of $\mathbf{T}$ on the spinors $\tilde{\pi}^A$ and $\tilde{v}^A$, which depend on $g$. Without this action of $\mathbf{T}$ on $\tilde{\pi}^A, \tilde{v}^A$ — i.e. replacing $\tilde{\pi}^A \mapsto \pi^A$, $\tilde{v}^A \mapsto v^A$, we actually reduce to the unperturbed system (13). Specifically, using equation (18), we obtain a convenient *linear* representation of the $X_j =: \frac{1}{2i}\widehat{X_j}$ on the space of $\tilde{\pi}^A$ and $\tilde{v}^A$:

$$\widehat{X_j} = \sum_{A,B} \left( \tilde{\pi}^A \sigma_{jA}{}^B \frac{\partial}{\partial \tilde{\pi}^B} + \tilde{v}^A \sigma_{jA}{}^B \frac{\partial}{\partial \tilde{v}^B} \right), \tag{26}$$

where $\sigma_{jA}{}^B$ is the $(A, B)$–entry of the Pauli matrix $\sigma_j$ (c.f. 14). Explicitly,

$$\begin{aligned}
\widehat{X_1} &= \tilde{\pi}^1 \frac{\partial}{\partial \tilde{\pi}^0} + \tilde{\pi}^0 \frac{\partial}{\partial \tilde{\pi}^1} + \tilde{v}^1 \frac{\partial}{\partial \tilde{v}^0} + \tilde{v}^0 \frac{\partial}{\partial \tilde{v}^1}, \\
\widehat{X_2} &= i \left( \tilde{\pi}^1 \frac{\partial}{\partial \tilde{\pi}^0} - \tilde{\pi}^0 \frac{\partial}{\partial \tilde{\pi}^1} + \tilde{v}^1 \frac{\partial}{\partial \tilde{v}^0} - \tilde{v}^0 \frac{\partial}{\partial \tilde{v}^1} \right), \\
\widehat{X_3} &= \tilde{\pi}^0 \frac{\partial}{\partial \tilde{\pi}^0} - \tilde{\pi}^1 \frac{\partial}{\partial \tilde{\pi}^1} + \tilde{v}^0 \frac{\partial}{\partial \tilde{v}^0} - \tilde{v}^1 \frac{\partial}{\partial \tilde{v}^1}.
\end{aligned}$$

Using equations (24–25), the operators $L$ and $M$ can be written as

$$L = -\left(\tilde{\pi}^1\right)^2 \mathcal{L}, \qquad M = -\left(\partial_t + \mathcal{M}\right) - \tilde{\pi}^1 \tilde{v}^1 \mathcal{L},$$

where

$$\mathcal{L} = i\left\{T_+ - 2i\tilde{\lambda}T_3 + \tilde{\lambda}^2 T_-\right\}, \qquad \mathcal{M} = \tilde{\lambda}T_- - iT_3, \quad \text{and} \quad \tilde{\lambda} = i\tilde{\pi}^0/\tilde{\pi}^1.$$

The operators L, M still have the group action contained, but the representations are greatly simplified. Observe that the compatibility of the system $L\Psi = M\Psi = 0$ is, after recombination and some calculation,

$$[L, M] = (\tilde{\pi}^1)^2 \left(-\mathcal{L}_t + [\mathcal{L}, \mathcal{M}] + A\mathcal{L}\right) = 0, \tag{27}$$

where $A = \tilde{\pi}^1 \mathcal{L} \tilde{v}^1 - \tilde{v}^1 \mathcal{L} \tilde{\pi}^1 - (2/\tilde{\pi}^1)\mathcal{M}\tilde{\pi}^1$ which depends *only* on $\tilde{\lambda}$ and $t$. It is important to note that since the $T_i$ (through the $\widehat{X_j}$) act on the $\tilde{\pi}^A$ and

$\tilde{v}^A$, we cannot replace the equation $L\Psi = 0$ with $\mathcal{L}\Psi = 0$ without changing the compatibility conditions. Thus equation (27) reduces the compatibility of L, M to a compatibility equation involving $\mathcal{L}$, $\mathcal{M}$ which depend on the "unperturbed" operators $T_i$ and $\tilde{\lambda}$.

Using the representation (26), we see that

$$A = -\frac{1}{2}\left\{[(\omega_1 + \omega_2 + 2\omega_3) - i(\theta - \phi)] - \tilde{\lambda}^2[(\omega_1 - \omega_2) - i(\theta + \phi)]\right\},$$

and

$$\widehat{X_1}(\tilde{\lambda}) = i(\tilde{\lambda}^2 + 1), \qquad \widehat{X_2}(\tilde{\lambda}) = \tilde{\lambda}^2 - 1, \qquad \widehat{X_3}(\tilde{\lambda}) = 2\tilde{\lambda}.$$

We replace the operators $\widehat{X_i}$ by the operators $\widehat{Y_i} := \sigma_i + \widehat{X_i}$ in order to calculate the compatibility condition (27). In this extended algebra the $\widehat{Y_i}$ form a suitable basis that can be multiplied by $\tilde{\lambda}$-dependent functions (the original $\widehat{X_i}$ acting on $\Psi(\tilde{\lambda}, t)$ are not linearly independent when multiplied by such functions). Explicitly, the $\widehat{Y_i}$ have the form

$$\begin{aligned}
\widehat{Y_1} &= \sigma_1 + i(\tilde{\lambda}^2 + 1)I\partial_{\tilde{\lambda}}, \\
\widehat{Y_2} &= \sigma_2 + (\tilde{\lambda}^2 - 1)I\partial_{\tilde{\lambda}}, \\
\widehat{Y_3} &= \sigma_3 + 2\tilde{\lambda}I\partial_{\tilde{\lambda}},
\end{aligned}$$

where the $\sigma_i$ are the Pauli matrices (14). Thus equation (27) can now be written as the compatibility condition for a system in $t$ and $\tilde{\lambda}$ only:

$$\mathcal{L}\Psi = \frac{i}{2}\mu\Psi, \tag{28}$$

$$\partial_t\Psi = -(\mathcal{M} + \nu I)\Psi, \tag{29}$$

subject to the condition

$$\mathcal{L}_d(\nu) = A\mu, \tag{30}$$

where

$$
\begin{aligned}
\mathcal{L} &= \frac{i}{2}\left\{P(\tilde{\lambda})\frac{\partial}{\partial\tilde{\lambda}} + l\right\} = \frac{i}{2}(\mathcal{L}_d + l), \\
\mathcal{M} &= \frac{1}{2}\left\{(\alpha_-\lambda^3 + \beta_+\lambda)\frac{\partial}{\partial\tilde{\lambda}} + l_1\right\}, \\
\alpha_\pm &= (\omega_1 - \omega_2) \pm i(\theta + \phi), \quad \beta_\pm = \omega \pm i(\theta - \phi), \\
\omega &= \omega_1 + \omega_2 - 2\omega_3, \quad P(\tilde{\lambda}) = \alpha_+ + \tilde{\lambda}^2(\beta_+ + \beta_-) + \tilde{\lambda}^4\alpha_-,
\end{aligned}
$$

and the matrices $l_1$ and $l$ are given by

$$
l_1 = \tilde{\lambda}X + Z, \qquad l = \tilde{\lambda}^2 X + 2\tilde{\lambda}Z + Y,
$$

$$
\begin{aligned}
X &= -(\phi + i\omega_1)\sigma_1 - (\omega_2 + i\theta)\sigma_2, \\
Y &= (\phi - i\omega_1)\sigma_1 + (\omega_2 - i\theta)\sigma_2, \\
Z &= -\omega_3\sigma_3.
\end{aligned}
$$

# 4 The Monodromy Evolving System and the Solution of the Generalized DH System

In this section we recapitulate the main result and we shall summarize the results of the monodromy analysis. Full details of the monodromy analysis will be published separately. From (28–30) the linear monodromy evolving system is given by

$$
\Psi_\lambda = (1/P)(\mu I - l)\Psi, \tag{31}
$$

$$
\Psi_t = -(\nu I + \frac{1}{2}l_1 + \frac{1}{2}f_1\partial_\lambda)\Psi, \tag{32}
$$

where

$$f_1 = \alpha_- \lambda^3 + \beta_+ \lambda,$$

and we have dropped the tildes on the $\lambda$'s. From (30) the parameter $\nu$ satisfies

$$\frac{\partial \nu}{\partial \lambda} = \frac{A}{P} \mu. \tag{33}$$

Detailed analysis of the linear system (31–32) shows that (33) governs the evolution of the monodromy associated with $\Psi(\lambda)$ [5].

We remark upon the important point that when $\mu = 0$ the system (31–32) is isomonodromic. Indeed equation (31) has four singular points corresponding to the zeros of P:

$$\begin{aligned} \lambda_1 &= \sqrt{(-r+v)/\delta}, \qquad & \lambda_2 &= -\sqrt{(-r-v)/\delta}, \\ \lambda_3 &= -\sqrt{(-r+v)/\delta}, \qquad & \lambda_4 &= \sqrt{(-r-v)/\delta}, \end{aligned} \tag{34}$$

where we have defined $r = \omega/\sqrt{\alpha_+ \alpha_-}$, $\delta^2 = \alpha_-/\alpha_+$ and $v^2 = r^2 - 1$. The fact that equation (31) has four distinct singular points indicates that it is related to the isomonodromy problem for the sixth Painlevé equation, $P_{VI}$ [14]. The canonical form of this isomonodromy problem has singularities at 0, 1, $\infty$, and $s$ — the independent variable of $P_{VI}$. To find $s$ we map $\lambda$ to a new spectral parameter, $\hat{\lambda}$, given by

$$\hat{\lambda}(\lambda) = \frac{(\lambda_2 - \lambda_3)(\lambda - \lambda_1)}{(\lambda_2 - \lambda_1)(\lambda - \lambda_3)}.$$

So $\hat{\lambda}(\lambda_1) = 0$, $\hat{\lambda}(\lambda_2) = 1$, $\hat{\lambda}(\lambda_3) = \infty$, and we define the isomonodromy variable $s$ by

$$s := \hat{\lambda}(\lambda_4) = \frac{r+1}{r-1}. \tag{35}$$

This variable plays a central role in the solution of the gDH system. Although will not go through the analysis in this paper we note that the Lax pair (31–32) can be used to express $\omega_i$, $\theta$, and $\phi$ in terms of $s$ and its derivatives. The field equations (i.e. the system 4) are then used to show that $s$ must satisfy a third-order Schwarzian equation. Namely, the analysis of (31–32) establishes the following:

$$\frac{(\phi-\theta)^2}{\omega^2-\alpha_+\alpha_-} = C_0^2, \tag{36}$$

where $C_0$ is a constant. Also, it can be shown that

$$\begin{aligned} \omega &= \frac{1}{2}\left(\frac{s+1}{s-1}\right)\frac{\dot{s}}{s}, && (37)\\ \omega_3 &= -\frac{1}{2}\left(\frac{\ddot{s}}{\dot{s}}-\frac{\dot{s}}{s}\right), && (38)\\ \theta-\phi &= \frac{C_0\dot{s}}{\sqrt{s}(s-1)}, && (39)\\ \alpha_\pm &= \kappa_\pm\frac{\dot{s}}{s}\mathrm{e}^{\pm iu(t)}, && (40) \end{aligned}$$

where $\kappa_\pm$ are constants satisfying $\kappa_+\kappa_- = 1/4$ and

$$u(t) = C_0 \ln\left(\frac{\sqrt{s}-1}{\sqrt{s}+1}\right).$$

Equations (37–40) show that $(\omega_1+\omega_2)$, $\omega_3$, $(\phi-\theta)^2$, and $(\omega_1-\omega_2)^2+(\phi+\theta)^2$ are rational functions of $s$, $\dot{s}$, and $\ddot{s}$. We see that we can solve for $\Omega = (\omega_1, \omega_2, \omega_3, \theta, \phi)$ in terms of $s$ and its first and second derivatives. A direct calculation shows that $\Omega$ is a solution of the gDH system if and only if the following Schwarzian equation is satisfied:

$$\{s,t\} + \frac{\dot{s}^2}{2}V(s) = 0, \tag{41}$$

where

$$\{s,t\} \equiv \frac{d}{dt}\left(\frac{\ddot{s}}{\dot{s}}\right) - \frac{1}{2}\left(\frac{\ddot{s}}{\dot{s}}\right)^2$$

is the Schwarzian derivative and

$$V(s) = \frac{1}{s^2} - \frac{1 + C_0^2}{s(s-1)} + \frac{1 + C_0^2}{(s-1)^2}.$$

This solution can be verified by direct substitution. It can be shown (see, e.g., Ablowitz and Fokas [3]) that, although this equation is linearizable, in general the solutions are densely branched! The general solution is single valued only when $C_0 = 0$ or $C_0 = i/n$ for some integer $n$.

# 5 Discussion

In this paper we have demonstrated how reductions of the SDYM equations with an infinite dimensional gauge algebra lead to DH type systems. These equations are linearized by monodromy evolving systems. Thus the gDH system (4) is a reduction of the SDYM equations and is solvable via an associated linear problem. The gDH system has been concretely linearized and therefore must be considered to be integrable in terms of real variables. It does not, however, share one of the other properties normally associated with integrable systems obtained by reduction from soliton equations. In particular, solutions of the gDH system do not, in general, have "nice" singularity structure in the complex plane. In the well-known special case of the classical DH system ($\theta = \phi = 0$) (5) admits solutions with a movable natural boundary — a circle on the complex sphere across which the solution cannot be analytically continued and whose center and radius depend on initial conditions. Although these solutions possess movable singularities other than poles, nevertheless they are single-valued in their domain of existence, hence the classical system can still be considered to possess the Painlevé property.

The general solution of the system (4), however, is densely branched in the complex plane and is definitely not of Painlevé type. This example shows that integrability as solvability via an associated linear problem does not imply integrability in the complex plane (see also [1]). Nevertheless the gDH system is deeply connected to the equations of Painlevé type as is demonstrated here.

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