The Darboux-Halphen System and the Singularity Structure of its Solutions

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Abstract

It is shown that a ninth-order generalization of the classical Darboux-Halphen system arises as a reduction of the self-dual Yang-Mills equations (SDYM). Using a convenient factorization we solve this system exactly and obtain its general solution in terms of a (linearizable) three-parameter Schwarzian equation. The general solution of the system is found to have a complicated singularity structure including movable natural barriers and dense branching which implies that the system does not possess the Painlevé property, although it is exactly solvable.

1 Introduction

The Painlevé property for ODEs (that all solutions are single-valued about all movable singularities) has been used as an effective detector of integrability for over a century now (see [5, 22, 23]). The Painlevé property was first used in this context by Kowalevskaya to find a new integrable case of the equations of motion for a spinning top [20, 21]. Since the late 1970's there has been a resurgence of interest in equations of Painlevé type after it appeared that all ODE reductions of equations solvable via the inverse scattering method possess the Painlevé property, possibly after a transformation of variables [9, 7, 8]. This has led to the discovery of a great many integrable equations.

In this paper we discuss the ninth-order system

$$\dot{M} = (\det M)\left(M^{-1}\right)^T + M^T M - (\operatorname{Tr} M)M, \tag{1}$$

where M is a 3×3 matrix-valued function of t, and we show that it can be obtained as a reduction of the self-dual Yang-Mills (SDYM) equations associated with the infinite-dimensional Lie algebra $\mathfrak{sdiff}(S^3)$. This equation is integrable in the sense that its general solution can be written explicitly in terms of solutions of the (linearizable) Schwarzian equation. Equation (1), which we refer to as DH–IX, is a generalization of the classical Darboux-Halphen system

$$\dot{\omega}_1 = \omega_2\omega_3 - \omega_1(\omega_2 + \omega_3), \quad \dot{\omega}_2 = \omega_3\omega_1 - \omega_2(\omega_3 + \omega_1), \quad \dot{\omega}_3 = \omega_1\omega_2 - \omega_3(\omega_1 + \omega_2), \tag{2}$$

which is obtained by setting $M = \operatorname{diag}(\omega_1, \omega_2, \omega_3)$ in equation (1). The system (2) originally arose in Darboux's study of triply orthogonal surfaces [16, 17] and was later solved by Halphen [19]. Later it was shown that this system is also equivalent to the Einstein field equations for a diagonal self-dual Bianchi–IX metric with Euclidean signature [18]. On

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setting $y = -2(\omega_1 + \omega_2 + \omega_3)$ it can be shown that the Darboux-Halphen system (2) is equivalent to the Chazy equation

$$\frac{d^3y}{dt^3} = 2y\frac{d^2y}{dt^2} - 3\left(\frac{dy}{dt}\right)^2 \tag{3}$$

(Chazy [12, 13, 14], see also Ablowitz and Clarkson [5]). The general solution of equations (2) and (3) contain a movable natural boundary (in this case, a circle or infinite line across which the solution cannot be analytically extended). However, every solution of (2) is single-valued and therefore possesses the Painlevé property.

Solutions of (1), however, are generically densely branched about movable singularities and so the system is not of Painlevé type. A fifth-order case of equation (1) was analyzed in [11, 4, 1, 2].

2 The ninth-order Darboux-Halphen system

With respect to standard coordinates $\{x_\mu\}$ on $\mathbf{R}^4$ the self-dual Yang-Mills equations are

$$F_{01} = F_{23}, \qquad F_{02} = F_{31}, \qquad F_{03} = F_{12}, \tag{4}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu],$$

with $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and where the A_μ are functions from $\mathbf{R}^4$ to a Lie algebra (the *gauge algebra*) $\mathfrak{g}$. These equations are integrable for a certain class of gauge algebras [10, 15]. Many well known integrable equations are reductions of the SDYM equations [26, 5].

Consider the reduction of SDYM in which the A_μ are functions of $x_0 =: t$ only and take values in the infinite-dimensional Lie algebra $\mathfrak{sdiff}(S^3)$. Then the SDYM equations become

$$\dot{A}_i + \frac{1}{2}\sum_{j,k=1}^{3} \varepsilon_{ijk}[A_j, A_k] = 0, \tag{5}$$

where, without loss of generality, we have taken $A_0 \equiv 0$. Equation (5) is known as the Nahm system [24]. We define the A_i's as

$$A_i(t) = -\sum_{j,k=1}^{3} M_{ij}(t)O_{jk}X_k,$$

where the operators X_i are the standard generators of $\mathfrak{sdiff}(S^3)$. These are divergence-free vector fields on S^3 satisfying the SU(2) Lie bracket $[X_i, X_j] = \sum_k \varepsilon_{ijk}X_k$. The points on S^3 are represented by the $SO(3)$ matrix $[O_{ij}]$ and the generators X_i act on the O_{ij} as

$$X_i(O_{jk}) = \sum_{l=1}^{3} \varepsilon_{ikl}O_{jl}.$$

The Nahm system (5) now reduces to the DH–IX system (1) for the matrix M_{ij}.

3 A convenient factorization

DH–IX admits a natural factorization upon decomposing M into symmetric and antisymmetric parts. The symmetric part, M_s, can be diagonalized by an orthogonal matrix P giving

$$M = M_s + M_a = P(D + a)P^{-1}, \tag{6}$$

where $a := P^{-1}M_aP$ is antisymmetric. If the components of D are distinct then under this factorization equation (1) splits into two linear equations for P and a while D satisfies an equation similar to system (2). These equations are

$$\dot{P} = -Pa, \tag{7}$$

$$\dot{a} = -aD - Da, \tag{8}$$

$$\dot{D} = 2\left\{D^2 - (\operatorname{Tr} D)D\right\} + \frac{1}{2}\left\{(\operatorname{Tr} D)^2 - \operatorname{Tr}(D^2) - \operatorname{Tr} a^2\right\} I. \tag{9}$$

In deriving these equations we have used the characteristic polynomial equation for M.

Note that once a solution for the sixth-order system (8–9) is known, the orthogonal matrix P is given by equation (7) which is linear. The solution of equation (1) is then given by (6).

Equations (7–9) can be written as the compatibility condition for a linear problem which arises as a nontrivial reduction of the isospectral problem for the SDYM equations. It is interesting to note that unlike most known auxiliary linear problems related to integrable systems, the resulting linear problem is monodromy evolving. Details of these facts can be found in [3].

Using the parameterization $D = \operatorname{diag}(\omega_1, \omega_2, \omega_3)$ and $a_{ij} = \sum_k \varepsilon_{ijk}\tau_k$, equations (8–9) become

$$\begin{aligned}
\dot{\omega}_1 &= \omega_2\omega_3 - \omega_1(\omega_2 + \omega_3) + \tau^2, & \dot{\tau}_1 &= -\tau_1(\omega_2 + \omega_3),\\
\dot{\omega}_2 &= \omega_3\omega_1 - \omega_2(\omega_3 + \omega_1) + \tau^2, & \dot{\tau}_2 &= -\tau_2(\omega_3 + \omega_1),\\
\dot{\omega}_3 &= \omega_1\omega_2 - \omega_3(\omega_1 + \omega_2) + \tau^2, & \dot{\tau}_3 &= -\tau_3(\omega_1 + \omega_2),
\end{aligned}$$

where $\tau^2 := \tau_1^2 + \tau_2^2 + \tau_3^2$.

For distinct ω_i's, the general solution of the system (8–9) is

$$\begin{aligned}
\omega_1 &= -\frac{1}{2}\frac{d}{dt}\ln\frac{\dot{s}}{s(s-1)}, & \tau_1 &= \frac{\kappa_1\dot{s}}{\sqrt{s(s-1)}},\\
\omega_2 &= -\frac{1}{2}\frac{d}{dt}\ln\frac{\dot{s}}{s-1}, & \tau_2 &= \frac{\kappa_2\,\dot{s}}{s\sqrt{s-1}},\\
\omega_3 &= -\frac{1}{2}\frac{d}{dt}\ln\frac{\dot{s}}{s}, & \tau_3 &= \frac{\kappa_3\dot{s}}{\sqrt{s}(s-1)},
\end{aligned}$$

where the κ_i are constants and s solves the Schwarzian equation

$$\{s, t\} + \frac{\dot{s}^2}{2}V(s) = 0 \tag{10}$$

where

$$\{s, t\} := \frac{d}{dt}\left(\frac{\ddot{s}}{\dot{s}}\right) - \frac{1}{2}\left(\frac{\ddot{s}}{\dot{s}}\right)^2$$

is the Schwarzian derivative and V is given by

$$V(s) = \frac{1-\beta^2}{s^2} + \frac{1-\gamma^2}{(s-1)^2} + \frac{\beta^2 + \gamma^2 - \alpha^2 - 1}{s(s-1)},$$

with $\alpha^2 = -4\kappa_1^2$, $\beta^2 = 4\kappa_2^2$, and $\gamma^2 = -4\kappa_3^2$. Equation (10) is of great importance in the theory of conformal mappings (see, for example, Nehari [25] or Ablowitz and Fokas [6]). It's general solution can be written in terms of two independent solutions of a linear ODE (the hypergeometric equation). When α, β, and γ are non-negative real numbers satisfying $\alpha+\beta+\gamma < 1$, the function $s(t)$ maps a given circular triangle with internal angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$ to the upper half plane.

Equation (10) is singular when $s = 0$, 1, or ∞. The expansions for s at these points are summarized below

TABLE 1

Expansions of the solution of the Schwarzian equation

Singular point	Expansion for s
$s = 0$	$s = (t-t_0)^{1/\beta} U_0$
$s = 1$	$s = 1 + (t-t_0)^{1/\gamma} U_1$
$s = \infty$	$s = (t-t_0)^{-1/\alpha} U_\infty$

where t_0 is an arbitrary constant and the U_i's are analytic in a neighborhood of $t = t_0$. It follows that if $\alpha = 1/p$, $\beta = 1/q$, and $\gamma = 1/r$, for integers p, q, and r, then s is single-valued. For general α, β, γ, however, the solutions are densely branched.

The general solution described above assumes the ω_i's are distinct. When exactly two of the ω_i's are equal the general solution is again in terms of solutions of an equation involving the Schwarzian derivative. The corresponding s-variable in this case maps crescent regions rather than circular triangles [3]. When $\omega_1 = \omega_2 = \omega_3 =: \omega$, the general solution is

$$\omega = \frac{t-t_0}{(t-t_0)^2+\kappa^2}, \qquad \tau_i = \frac{\kappa_i}{(t-t_0)^2+\kappa^2},$$

where t_0 and the κ_i are constants and $\kappa^2 = \kappa_1^2 + \kappa_2^2 + \kappa_3^2$.

References

[1] M. J. Ablowitz, S. Chakravarty, and R. Halburd, *On Painlevé and Darboux-Halphen type equations*, in the Painlevé property, one century later, CRM series in mathematical physics, Springer, Berlin, ed. R. Conte, 1998.

[2] ———, *Darboux-Halphen type equations and evolving monodromy problems*, (to appear).

[3] ———, *The general Darboux-Halphen system, Schwarzian functions, and the sixth Painlevé equation*, (to appear).

[4] M. J. Ablowitz, S. Chakravarty, and B. M. Herbst, *Integrability, computation and applications*, Acta Appl. Math., 39 (1995), pp. 5–37.

[5] M. J. Ablowitz and P. A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, Lond. Math. Soc. Lecture Note Series, **149**, CUP, Cambridge, 1991.

[6] M. J. Ablowitz and A. S. Fokas, *Complex variables: introduction and applications*, CUP, Cambridge, 1997.

[7] M. J. Ablowitz, A. Ramani, and H. Segur, *Nonlinear evolution equations and ordinary differential equations of Painlevé type*, Lett. Nouvo Cim., 23 (1978), pp. 333–338.

[8] ——, *A connection between nonlinear evolution equations and ordinary differential equations of P-type. I and II*, J. Math. Phys., 21 (1980), pp. 715–721 and 1006–1015.
[9] M. J. Ablowitz and H. Segur, *Exact linearization of a Painlevé transcendent*, Phys. Rev. Lett., 38 (1977), pp. 1103–1106.
[10] M. F. Atiyah and R. S. Ward, *Instantons and algebraic geometry*, Commun. Math. Phys., 55 (1977), pp. 117–124.
[11] *Integrability, monodromy evolving deformations, and self-dual Bianchi IX systems*, Phys. Rev. Lett., 76 (1996), pp. 857–860.
[12] J. Chazy, *Sur les équations différentielles dont l'intégrale générale est uniforme et admet des singularities essentielles mobiles*, C.R. Acad. Sc. Paris, 149 (1909), pp. 563–565.
[13] ——, *Sur les équations différentielles dont l'intégrale générale possède une coupure essentielle mobile*, C.R. Acad. Sc. Paris, 150 (1910), pp. 456–458.
[14] ——, *Sur les équations différentielles du troisième et d'ordre supérieur dont l'intégrale générale à ses points critiques fixés*, Acta Math., 34 (1911), pp. 317–385.
[15] E. F. Corrigan, D. B. Fairlie, R. G. Yates, and P. Goddard, *Construaction of self-dual solutions to $SU(2)$ gauge theory*, Commun. Math. Phys., 58 (1978), pp. 223–240.
[16] G. Darboux, *Sur la théorie des coordonnées curvilignes et les systémes orthogonaux*, Ann. Ec. Normale Supér., 7 (1878), pp. 101–150.
[17] ——, *Leçons sur la théorie générale des surfaces et les applications géométrique du calcul infinitésimal*, Vol 1–4, Gauthier-Villars, Paris, 1887–1896.
[18] G. W. Gibbons and C. N. Pope, *The positive action conjecture and asymptotically Euclidean metrics in quantum gravity*, Commun. Math. Phys., 66 (1979), pp. 267–290.
[19] G. Halphen, *Sur un système d'équations différentielles*, C. R. Acad. Sci. Paris, 92 (1881), pp. 1101–1103.
[20] S. Kowalevski, *Sur le Problème de la Rotation d'un Corps Solid Autour d'un Point Fixè*, Acta Math., 12 (1889), pp. 177–232.
[21] ——, *Sur une Propriété d'un Système d'Équations Différentielles qui Definit la Rotation d'un Corps Solide Autour d'un Point Fixè*, Acta Math., 14 (1889), pp. 81–93.
[22] M. D. Kruskal and P. A. Clarkson, *The Painlevé-Kowalevski and poly-Painlevé tests for integrability*, Stud. Appl. Math., 86 (1992), pp. 87–165.
[23] M. D. Kruskal, N. Joshi, and R. Halburd, *Analytic and asymptotic methods for nonlinear singularity analysis: a review and extensions of tests for the Painlevé property* in Proceedings of the CIMPA Summer School on Nonlinear Systems, Pondicherry, India, eds B. Grammaticos and K. Tamizhmani, 1997.
[24] W. Nahm, *The algebraic geometry of multimonopoles* in Group Theoretical Methods in Physics, Lect. Notes Phys., 180, eds M. Serdaroglu and E. Inonu (1982).
[25] Z. Nehari, *Conformal mapping*, McGraw-Hill, New York, 1952.
[26] R. S. Ward, *Integrable and solvable systems, and relations among them*, Phil. Trans. R. Soc. Lond. A, 315 (1985), pp. 451–457.