THE COALESCENCE LIMIT OF THE SECOND PAINLEVÉ EQUATION

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Abstract

In this paper, we study a well known asymptotic limit in which the second Painlevé equation (P_{II}) becomes the first Painlevé equation (P_I). The limit preserves the Painlevé property (i.e. that all movable singularities of all solutions are poles). Indeed it has been commonly accepted that the movable simple poles of opposite residue of the generic solution of P_{II} must coalesce in the limit to become movable double poles of the solutions of P_I, even though the limit naively carried out on the Laurent expansion of any solution of P_{II} makes no sense. Here we show rigorously that a coalescence of poles occurs. Moreover we show that locally all analytic solutions of P_I arise as limits of solutions of P_{II}.

1 Introduction

An ordinary differential equation is said to be of Painlevé type (or to possess the Painlevé property) if the only movable singularities of its solutions are poles. The property is strongly related to integrable systems (systems which can be solved via related linear problems) [1, 2, 9, 13]. Knowledge of equations with the Painlevé property, including various methods of classification, is therefore valuable in the search for integrable systems. Asymptotic limits of differential equations that preserve the Painlevé property provide another mechanism for such searches.

In a series of papers published around the turn of the century, Painlevé [11], Gambier [5], and Fuchs [4] conducted an exhaustive search for all equations of

Painlevé type of the form

$$u'' = \Phi(x; u, u'),$$

where Φ is analytic in x and rational in u and u'. They discovered six equations of Painlevé type which, up to a transformation, are the only equations of this type whose general solutions are new transcendental functions. These equations are known as the Painlevé equations P_I–P_{VI}. The first two are

$$\begin{array}{llcl} P_I: & u''(x) & = & 6u^2 + x; \\ P_{II}: & u''(x) & = & 2u^3 + xu + \alpha; \end{array}$$

where α is a complex constant.

Painlevé [12] noted that under the transformation

$$\begin{aligned} x &= \epsilon^2 z - 6\epsilon^{-10}; \\ u &= \epsilon y + \epsilon^{-5}; \\ \alpha &= 4\epsilon^{-15}, \end{aligned} \tag{1}$$

P_{II} becomes

$$y''(z) = 6y^2 + z + \epsilon^6 \left\{ 2y^3 + zy \right\}. \tag{2}$$

If ϵ vanishes, equation (2), which we will refer to as $P_{II}(\epsilon)$, becomes P_I with x replaced by z and u replaced by y. We will say that under (1), P_{II} degenerates to P_I and write $P_{II} \to P_I$. Painlevé gave a series of such degeneracies which is summarized in Figure 1.

FIGURE 1 NEAR HERE.

Using a method based on maximal dominant balances, Joshi and Kruskal [7] have found a new degeneracy of P_{IV} to another equation of Painlevé type (equation XXXIV on p.340 of Ince [6]). Their paper raises the possibility of using asymptotic limits between differential equations which preserve the Painlevé property as new tools in the search for, and classification of, integrable systems.

The central concern of the present paper is an exploration of the convergence of solutions of $P_{II}(\epsilon)$ to solutions of P_I as ϵ vanishes. In particular we are concerned with the way in which simple poles of oppositely signed residues in solutions of $P_{II}(\epsilon)$ coalesce to form the double poles of P_I. Unfortunately, a purely local analysis of this coalescence is problematic as the radius of convergence of any Laurent expansion centred on any pole necessarily decreases to zero if the poles coalesce. Rather than attempt to find an accurate upper bound on the radius of convergence we use techniques based on steepest ascent curves similar to those first expounded by Joshi and Kruskal in their direct proof that P_I to P_{VI} possess the Painlevé property [8]. Before embarking on this problem we analyse a model problem given by a similar degeneracy between the autonomous versions of P_I and P_{II} whose general solutions are expressible in terms of elliptic functions. We show that the poles here coalesce by estimating the distance between them. Such estimates are obtained in section 2.

In section 3 we consider general equations of the form

$$\frac{dy_i}{dz} = f_i(z, y_1, \ldots, y_n; \epsilon), \qquad 1 \leq i \leq n \tag{3}$$

where the f_i are entire functions of $(z, y_1, \ldots, y_n; \epsilon)$. We will say that equations (3) degenerate to the equations

$$\frac{dy_i}{dz} = f_i(z, y_1, \ldots, y_n; 0), \qquad 1 \leq i \leq n \tag{4}$$

in the limit as ϵ approaches zero. We show that, locally, any analytic solution of the target equations (4) can be obtained in the limit as $\epsilon \to 0$ of a solution to equations (3). A corollary of the theorem states that if y_I is a solution of P_I then given any compact subset K on which y_I is analytic then there is a solution, y of $P_{II}(\epsilon)$ such that $y \to y_I$ on K with respect to the sup norm as $\epsilon \to 0$. Hence, by considering the maximal analytic extension of y we see that $y \to y_I$ everywhere. In section 4 we examine the rate of coalescence of poles. We obtain estimates of the distances between coalescing poles and show that these are of order ϵ^3.

2 Two Autonomous Painlevé Equations

Consider the following autonomous versions of P_I and P_{II},

$$E_I \qquad u'' = 6u^2 + \lambda$$

$$E_{II} \qquad u'' = 2u^3 + \mu u + \alpha$$

where $\lambda, \mu \in \mathbf{C}$ are constants and the primes denote differentiation with respect to x. The solutions of E_I and E_{II} are either constants or may be expressed in terms of elliptic integrals.

Following the analogy of the $P_{II} \to P_I$ coalescence we transform the variables in E_{II} as follows:

$$x = \epsilon^2 z, \quad \mu = \lambda\epsilon^2 - 6\epsilon^{-10}, \quad u = \epsilon y + \epsilon^{-5}, \qquad \alpha = 4\epsilon^{-15}.$$

Under this transformation E_{II} becomes

$$\ddot{y} = 6y^2 + \lambda + \epsilon^6 \left(2y^3 + \lambda y\right), \tag{5}$$

where a dot denotes differentiation with respect to z, giving us the degeneracy $E_{II} \to E_I$. In order to examine the nonconstant solutions of (5) we multiply the equation through by $\dot{y}$ and integrate. In this way we obtain

$$\dot{y}^2 = \epsilon^6 P_\epsilon(y) := h + 2\lambda y + \epsilon^6\lambda y^2 + 4y^3 + \epsilon^6 y^4 \tag{6}$$

where $h \in \mathbf{C}$ is a constant of integration. Take h given and fixed in the following analysis. The nonconstant solutions of equation (6) satisfy

$$\epsilon^3 \frac{dz}{dy} = Q_\epsilon(y) := \frac{1}{\sqrt{P_\epsilon(y)}}.$$

Now, for $\epsilon \neq 0$,

$$P_\epsilon(y) =: (y - a_0)(y - a_1)(y - a_2)(y - a_3),$$

where

$$\begin{aligned} a_0 = a_0(\epsilon) &= -\frac{4}{\epsilon^6} + \frac{\lambda}{8}\epsilon^6 + O(\epsilon^{12}), \qquad (7)\\ a_i = a_i(\epsilon) &= \eta_i + O(\epsilon^6), \qquad i = 1, 2, 3 \end{aligned}$$

are the zeros of $P_\epsilon(y)$ and the η_i are zeros of $P_0(\eta)$.

We briefly recall some of the standard results from the theory of elliptic integrals, beginning with a description of a Riemann surface for $Q_\epsilon(y)$ (see, for example Siegel [14]). We will assume that h is such that for small ϵ, $P_\epsilon(y)$ has distinct zeros (note that this is the generic case). Cut two nonintersecting slits in the Riemann sphere, say one from a_0 to a_1 and the other from a_2 to a_3. Make two copies of the resulting manifold and label them $\mathfrak{M}_1$ and $\mathfrak{M}_2$; these two slit spheres correspond to the two branches of the square root operation in the definition of $Q_\epsilon(y)$. Now take each side of both slits on $\mathfrak{M}_1$ and identify them with the opposite sides of the corresponding slits of $\mathfrak{M}_2$. The resulting Riemann surface, $\mathfrak{R}$, is homeomorphic to the 2-torus T^2. $Q_\epsilon(y)$ is meromorphic throughout $\mathfrak{R}$ and the elliptic integral

$$I(\gamma) := \int_\gamma Q_\epsilon(y)dy$$

is well defined for any piecewise smooth curve γ in $\mathfrak{R}$ where $\tilde{y}$ varies over the natural projection of γ to the Riemann sphere $\mathbf{CP}^1$.

Suppose that y has poles with residues of opposite sign at z_+ and z_-, then

$$z_+ - z_- = \epsilon^{-3} \int_\gamma Q_\epsilon(y)dy, \tag{8}$$

where γ is a path connecting ∞_1 and ∞_2 — the subscripts distinguish the points at infinity on the two slit spheres $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respectively. Such a path must pass through one of the open slits connecting the two spheres. Its projection onto the Riemann sphere must loop around the points a_k, $k = 0, \ldots, 3$ an odd number of times (note that if it encloses an even number of the points a_k, the resultant integral is just a period of the elliptic function y). For small ϵ the point a_0 is closest to infinity so we consider a path which begins at ∞_1 and remains in $\mathfrak{M}_1$ until it reaches the point a_0, loops around it, and then retraces the corresponding path in $\mathfrak{M}_2$, terminating at ∞_2. Since an arbitrarily small loop around a_0 contributes nothing to (8), the distance between the two poles is simply

$$|z_+ - z_-| = 2\left|\epsilon^{-3}\int_\infty^{a_0} Q_\epsilon(y)dy\right|. \tag{9}$$

Next we refine our choice of the path of integration for the right hand side of equation (9). At any point where y is analytic and neither y nor y' vanishes, there is a unique direction of fastest increase in $|y|$. Hence we can define a steepest ascent curve through any such point. A simple calculation using the Cauchy-Riemann equations shows that on such a curve, $d|y| = |dy|$ (on a path of steepest descent, $d|y| = -|dy|$).

Let R_+ be the connected component of the region

$$\Omega := \{z : \ |y(z)| > |a_0|\}$$

containing z_+ in its closure. Note that the only pole in the closure of R_+ is z_+ (since there is a unique level curve of $|y|$ passing through every point in R_+). Expanding y about a point z_1 in the boundary of R_+ such that $y(z_1) = a_0$ gives

$$y(z) = a_0 + \frac{1}{2}y''(z_1)(z - z_1)^2 + O\left((z - z_1)^3\right),$$

where $a_0 \approx -4\epsilon^{-6}$ and, from equation (5), $y''(z_1) \approx -32\epsilon^{-6}$. We see that z_1 is a (complex) saddle point. This implies that z_1 is the initial point for two steepest ascent curves (and two steepest descent curves). One of the steepest ascent curves must enter R_+ and terminate at z_+. This is the path, Γ, over which we integrate in equation (9).

Choose $r > 0$ so small that for all ϵ such that $|\epsilon| < r$,

$$|a_0| \geq 2 \max_{1 \leq i \leq 3} \{|a_i|\}.$$

Note that this can be achieved because the expansion (7) shows that a_0 is large for small ϵ. Then, since $|y(z)| > |a_0|$ on Γ, for $|\epsilon| < r$ and $1 \leq i \leq 3$, we get

$$|y| \leq |y - a_i| + |a_i| \leq |y - a_i| + |y|/2 \quad \Rightarrow \quad |y|/2 \leq |y - a_i|.$$

So

$$|Q_\epsilon(y)| \leq \frac{2\sqrt{2}}{\sqrt{|y|^3\left(|y| - |a_0|\right)}}. \tag{10}$$

In particular, notice that the integral in (9) is convergent at infinity.

Using (10), we find from equation (9) that

$$\begin{aligned} |z_+ - z_-| &\leq -2|\epsilon|^{-3}\int_\infty^{|a_0|} \frac{2\sqrt{2}\ d|y|}{\sqrt{|y|^3\left(|y| - |a_0|\right)}} \\ &= -\frac{8\sqrt{2}}{|a_0|\,|\epsilon|^3}\sqrt{1 - \frac{|a_0|}{|y|}}\Bigg|_{y=\infty}^{y=a_0} = \frac{8\sqrt{2}}{|a_0(\epsilon)|\,|\epsilon|^3} = O(\epsilon^3). \end{aligned}$$

Therefore the two oppositely signed poles coalesce as ϵ vanishes.

In the above analysis we have only considered the generic case in which $P_\epsilon(y)$ has four distinct zeros. In the nongeneric case the Riemann surface, $\mathfrak{R}$, of y is no longer a torus. Our analysis, however, does not depend critically on the global topology of $\mathfrak{R}$ and the same estimates apply.

3 Local Analytic Solutions

The aim of this section is to prove the following theorem:-

Theorem 1 *Let $(\eta_1 \ldots, \eta_n)$ be a given solution of the system of ODEs in (4) which is analytic in some pathwise connected region $\Omega \subseteq \mathbf{C}$ and choose $z_0 \in \Omega$. Given any simply connected compact subspace $K \subset \Omega$ containing z_0, there exists a solution $(y_1, \ldots, y_n)$ of equations (3) and a number $r_K > 0$ such that,*

1. *the y_i are analytic in (z, ϵ) for $z \in K$, $|\epsilon| < r_K$;*
2. *$y_i(z, 0) = \eta_i(z)$ $\forall z \in K$;*
3. *$y_i(z_0, \epsilon) = \eta_i(z_0)$ $\forall \epsilon$ such that $|\epsilon| < r_K$.*

Note that, regardless of the choice of K, the y_i satisfy the same initial value problem at z_0. This theorem shows us that, locally, solutions of equations (3) converge onto solutions of equations (4). It shows that the singularities of this family of solutions of equations (3) lie arbitrarily close to those of equations (4) (or go to infinity), for small ϵ. In the proof of this theorem given below we will make use of the following lemma which can be proved using elementary arguments involving majorant series (see, for example, Cartan [3]).

Lemma 2 *Consider the system of ODEs*

$$\frac{dy_i}{dz} = f_i(z, y_1, \ldots, y_n; \epsilon), \qquad 1 \le i \le n \tag{11}$$

together with the initial conditions

$$y_i(z_0, \epsilon) = \phi_i(\epsilon), \qquad 1 \le i \le n$$

where the ϕ_i are analytic for $|\epsilon| \le r$ and the f_i are analytic on

$$S := \{(z, y_1, \ldots, y_n; \epsilon) : |z - z_0| \le \rho,\ |y_i - \phi_i| \le R,\ |\epsilon| \le r,\ 1 \le i \le n\}.$$

Then there is a unique solution $\mathbf{y} := (y_1, \ldots, y_n)$ of (11) which is analytic in (z, ϵ) whenever $|\epsilon| < r$ and

$$|z - z_0| < Z^M_{\rho,r,R}(\epsilon) := \rho \left(1 - \exp\left[-\frac{(1 - |\epsilon|/r)R}{(n+1)\rho M}\right]\right),$$

where

$$M \ge \sup_S |f_i|, \qquad 1 \le i \le n.$$

Proof of Theorem 1: Since the f_i are entire, we may expand them as power series,

$$f_i(z, y_1, \ldots, y_n; \epsilon) = \sum a^i_{jk_1\cdots k_n l} z^j y_1^{k_1} \cdots y_n^{k_n} \epsilon^l,$$

which converge everywhere.

Fix $\rho, R, r_0 > 0$. Let Γ be any finite length curve connecting z_0 to ∂K. We will first prove existence in a thin neighbourhood of Γ. Define

$$\begin{aligned} B &:= \sup\{|z| : |z - \tilde{z}| = \rho,\ \tilde{z} \in \Gamma\}, \\ L &:= \sup_{\substack{z\in\Gamma \\ 1\le i\le n}} |\eta_i(z)|, \end{aligned}$$

and $M := \max_{1\le i\le n} M_i$ where

$$M_i \ := \ 2\sum |a^i_{jk_1\cdots k_n l}| B^j (R + L)^{k_1\cdots k_n} r_0^l. \tag{12}$$

This last series converges because the f_i are entire.

Let $S_0 = \{(z, y_1, \ldots, y_n; \epsilon) : |z - z_0| \le \rho,\ |\epsilon| \le r_0,\ |y_i - \eta_i(z_0)| \le R,\ 1 \le i \le n\}$. Then for $(z, y_1, \ldots, y_n; \epsilon) \in S_0$ we have

$$\begin{aligned} & |f_i(z, y_1, \ldots y_n; \epsilon)| \\ \le\ & \sum |a^i_{jk_1,\cdots k_n l}||z|^j (|y_1 - \eta_1(z_0)| + |\eta_1(z_0)|)^{k_1} \cdots (|y_n - \eta_n(z_0)| + |\eta_n(z_0)|)^{k_n} |\epsilon|^l \\ \le\ & \sum |a^i_{jk_1,\ldots k_n l}| B^j (R + L)^{k_1\cdots k_n} r_0^l \\ =\ & \frac{1}{2} M_i. \end{aligned}$$

Therefore $\sup_{z\in S_0}|f_i| \le M$ and so we deduce from Lemma 2 that there is a solution $\mathbf{y}^{(0)} := (y_1^{(0)},\ldots,y_n^{(0)})$ of equations (3) satisfying the initial condition

$$\mathbf{y}^{(0)}(z_0,\epsilon) = \eta(z_0), \qquad |\epsilon| < r_0.$$

Furthermore, $\mathbf{y}^{(0)}$ is analytic in (z,ϵ) provided $|z-z_0| < Z^M_{\rho,r_0,R}(\epsilon)$ (see Lemma 2).

Notice that $Z^M_{\rho,r_1,R}(\epsilon)$ has the maximal value

$$d := Z^M_{\rho,r_1,R}(0) = \rho\left(1-\exp\left\{-\frac{R}{(n+1)\rho M}\right\}\right).$$

Let z_1 be the first point on Γ such that $|z_1 - z_0| = d/2$ (if no such point exists then we have finished).

Next we show that by restricting the range of ϵ we can ensure that the initial value problem at z_1 gives us a solution whose radius of convergence in z is again bounded below by $d/2$. At $z = z_1$, $\mathbf{y}^{(0)}$ is analytic in ϵ for $|\epsilon| < \tilde r$ for some $\tilde r < r_0$. Let $S_1(\tilde r) := \{(z,y_1,\ldots,y_n;\epsilon) : |z-z_0| \le \rho,\ |\epsilon| \le \tilde r,\ |y_i - \eta_i(z_0)| \le R,\ 1 \le i \le n\}$. Then

$$\begin{aligned}\sup_{S_1(\tilde r)}|f_i| \;\le\; & \sum |a^i_{jk_1\ldots k_n l}| B^j \left(R + \sup_{|\epsilon|<\tilde r}\left|y_1^{(0)}(z_1,\epsilon)\right|\right)^{k_1}\cdots \\ & \cdots\left(R + \sup_{|\epsilon|<\tilde r}\left|y_n^{(0)}(z_1,\epsilon)\right|\right)^{k_n} r_0^l. \qquad (13)\end{aligned}$$

Now as $\tilde r \to 0$, $\sup_{|\epsilon|<\tilde r}|y_i^{(0)}(z_1,\epsilon)| \to |\eta_i(z_1)| \le L$, and so (13) approaches $\frac{1}{2}M$. Therefore there exists r_1 such that $0 < r_1 \le \tilde r < r_0$ and

$$\sup_{S_1(r_1)}|f_i| \le M.$$

Invoking Lemma 2 again we see that there is a solution, $\mathbf{y}^{(1)}$, of equations (3) satisfying

$$\mathbf{y}^{(1)}(z_1,\epsilon) = \mathbf{y}^{(0)}(z_1,\epsilon)$$

for all $|\epsilon| < r_1$, which is analytic in (z,ϵ) provided $|z - z_1| < Z^M_{\rho,r_1,R}(\epsilon)$. We then look for the next point, z_2, on Γ such that $|z_2 - z_1| = d/2$ (if such a point exists) and repeat the above argument for a finite number of points $z_2, z_3, \ldots, z_N$ in order to cover the curve. $\mathbf{y}^{(i+1)}$ analytically continues $\mathbf{y}^{(i)}$. $\mathbf{y}(z) := (y_1(z),\ldots,y_n(z))$ is then defined to be $\mathbf{y}^{(k)}$ whenever z lies in the domain of analyticity of $\mathbf{y}^{(k)}$. Since we proceed in steps of $d/2$ in z, the radius of convergence of $\mathbf{y}(z)$ about any point of Γ is bounded below. The compactness and pathwise connectedness of K then ensure that we can analytically extend $\mathbf{y}(z)$ to all of K by using a finite number of curves Γ_j from z_0. The existence of the number r_K then follows because we require only a finite number of reductions of r in the above analytic continuation of $\mathbf{y}(z)$.

□

The condition that K be simply connected is essential for the single-valuedness of $\mathbf{y}(z)$. For example, consider the equation

$$y'' = 6y^2 + \epsilon z^2.$$

The solutions of this equation for $\epsilon = 0$ are elliptic functions and therefore meromorphic. However, Painlevé analysis (see [9]) reveals that generic solutions to this equation for $\epsilon \neq 0$ possess logarithmic singularities. So locally analytic solutions of the equation with $\epsilon = 0$ whose domain of analyticity (Ω in Theorem 1) is not simply connected do not necessarily arise from analytic solutions of the general equation, but rather from multivalued ones.

In the case of P_I and P_{II}, however, the solutions are meromorphic [11, 8]. So, on recalling the form of transformation (1), we see that all solutions of $P_{II}(\epsilon)$ are meromorphic and therefore single valued. Analytically extending any solution of $P_{II}(\epsilon)$ along any path connecting z_0 to any other point will give a result which is independent of the particular path chosen. Hence, when we apply Theorem 1 to $P_{II}(\epsilon)$ we can weaken the requirement that K be simply connected, instead demanding only that it be pathwise connected. The theorem then has the following corollary.

Corollary 3 *Choose $z_0, \alpha, \beta \in \mathbf{C}$. Let y_I and y be maximally extended solutions of P_I and $P_{II}(\epsilon)$ respectively, both satisfying the initial value problem given by*

$$y(z_0) = \alpha, \qquad y'(z_0) = \beta.$$

Let $\Omega \subset \mathbf{C}$ be the domain of analyticity of y_I. Given any compact $K \subset \Omega$, $\exists r_K > 0$ such that y is analytic in (z, ϵ) for $z \in K$, $|\epsilon| < r_K$ and $y \to y_I$ with respect to the sup norm as $\epsilon \to 0$.

Proof: Apply Theorem 1 using some compact pathwise connected subspace $\widetilde{K} \subset \Omega$ such that $\{z_0\} \cup K \subseteq \widetilde{K}$.

□

4 Coalescence of Poles and the Second Painlevé Equation

We now return to the problem of estimating the rate of coalescence of poles in a family of solutions to $P_{II}(\epsilon)$

$$y'' = 2\epsilon^6 y^3 + 6y^2 + \epsilon^6 zy + z,$$

as $\epsilon \to 0$. Choose $z_0 \in \mathbf{C}$. We will consider a family of solutions to $P_{II}(\epsilon)$ given by $y(z_0) = \alpha$, $y'(z_0) = \beta$. Multiplying $P_{II}(\epsilon)$ through by y' and integrating along some path γ from z_0 to z gives

$$[y'(z)]^2 = \epsilon^6 y^4 + 4y^3 + 2zy + \epsilon^6 zy^2 - \int_\gamma^z \left\{2y + \epsilon^6 y^2\right\} dz + k_\epsilon =: \epsilon^6 F_\epsilon\{z, y\} \quad (14)$$

where

$$k_\epsilon = \beta^2 - \left\{\epsilon^6 \alpha^4 + 4\alpha^3 + 2z_0\alpha + \epsilon^6 z_0 \alpha^2\right\}.$$

From the corollary to Theorem 1 in section 3 we see that as $\epsilon \to 0$ the solution to $P_{II}(\epsilon)$ given by $y(z_0) = \alpha$, $y'(z_0) = \beta$ converges to the solution y_I of P_I satisfying the same initial conditions, on any compact subset K of the domain of analyticity of y_I.

Suppose y_I has a double pole at $\hat{z}$. Let D be the closed disc of radius ρ centred at 0 containing both z_0 and $\hat{z}$ in its interior. Let K be D after we have deleted open discs of small radius δ centred at each of the poles of y_I which lie in D. From Corollary 3 of Section 3 we see that for sufficiently small ϵ, any simple pole of y which lies in D must be within δ of a double pole of y_I (since it cannot lie in K).

Let $z_+, z_- \in B_\delta(\hat{z})$ be the positions of two poles of y of oppositely signed residues. The distance between these poles is given by

$$|z_+ - z_-| = \left| \int_{z_-}^{z_+} dz \right| = \left| \epsilon^{-3} \int_\Gamma \frac{dy}{\sqrt{F_\epsilon\{z, y\}}} \right| \tag{15}$$

for some path Γ between points whose natural projection to $\mathbf{CP}^1$ is $y = \infty$. The final integral in (15) makes sense if we use the fact that *locally* on Γ, z may be given as a function of y. Also, since all solutions of $P_{II}(\epsilon)$ are nonconstant meromorphic functions, the points on Γ at which $F_\epsilon\{z, y\}$ vanishes do not accumulate.

As was the case in the coalescence of poles induced by the $E_{II} \to E_I$ degeneracy, the opposite signs of the residues of the poles of y at z_- and z_+ indicates that Γ must loop around a zero, z_1, of y'. The close proximity of the poles for ϵ small indicates that y must be large at this stationary point.

We take Γ to loop around a zero, z_1, of $F_\epsilon\{z, y\}$, to be specified below. Let $A := |y(z_1)|$. Define the region

$$\Omega := \{z \in D : \ |y(z)| > A\}\,.$$

Then Ω is a union of regions surrounding poles of y. We take z_1 so that the connected component, R_+, of Ω whose closure contains z_+ contains no other stationary points of y; *i.e.* $y'(z) \neq 0$ for all $z \in R_+$. An analogous argument to that outlined in Section 2 shows that R_+ contains no pole other than z_+ and that z_1 is the initial point for two curves of steepest ascent and two curves of steepest descent (each separated by one of the four level curves of $|y|$ which pass through z_1). This follows from the fact that $y(z_1) \neq 0$, $y'(z_1) = 0$, and $y''(z_1) \neq 0$. One of these steepest ascent curves, Γ_+ say, lies in R_+ and so connects z_1 to z_+. The other steepest ascent curve, Γ_-, lies in $\Omega \setminus R_+$ and is of finite length (necessarily terminating at a pole, z_- say) since, for large A, Ω is a union of small disjoint regions containing the poles of y_I. We take the path of integration, Γ, in equation (15) to be the union of these two paths.

Since z_1 is in the boundary of Ω and is the initial point for a curve of steepest descent, there is a curve connecting z_0 to z_1 contained in $D \setminus \Omega$. The (initial) path of integration, γ, in equation (14) connecting z_0 to $z \in \Gamma$ is taken to be this descent curve followed by one of the steepest ascent curves, Γ_1 or Γ_2, from z_1 to the point z.

We now estimate $F_\epsilon\{z, y\}$ for large y on such a curve. For $z \in \Gamma$, we have

$$\sup_{\zeta \in \gamma} |y(\zeta)| = |y(z)|.$$

Since $D \setminus \Omega$ only contains a finite number of small holes, the length of the path γ from z_0 to any point on Γ can be bounded by some $d > 0$ which is independent of ϵ for ϵ small. Also, for small ϵ, k_ϵ can be bounded above by c^2, say, where $c > 0$ is independent of ϵ.

From equation (14) we see that $A := |y(z_1)|$ is asymptotically close to $4|\epsilon|^{-6}$. Let $r > 0$ be an upper bound on ϵ^6 which is so small that

$$r < \max\left\{d^{-1}, \rho^{-1}\right\}, \quad \text{and} \quad A > \max\{c, d, \rho\}.$$

We now see that on Γ, for $|\epsilon|^6 < r$,

$$\epsilon^6 F_\epsilon\{z, y\} = \epsilon^6 y^4 + 4y^3 + \phi(z, y),$$

where

$$\begin{aligned} |\phi(z,y)| &\leq |2zy| + \left|\epsilon^6 z y^2\right| + \left|\int_\gamma^z 2y dz\right| + \left|\int_\gamma^z \epsilon^6 y^2 dz\right| + |k_\epsilon| \\ &\leq 2\rho|y| + r\rho|y|^2 + 2d|y| + rd|y|^2 + c^2 \\ &\leq \kappa|y|^2, \end{aligned}$$

where $\kappa = 5 + r(d + \rho)$.

So on Γ, $\phi(z, y) = \psi(z, y)y^2$ where $|\psi(z, y)| < \kappa$, giving

$$\epsilon^6 F_\epsilon\{z, y\} = \epsilon^6 y^4 + 4y^3 + \psi(z, y)y^2. \tag{16}$$

Now $\arg(y)$ is a constant along any path of steepest ascent for y (since $d|y| = |dy|$ there). Hence Γ_+ can be parameterized by $t \in (1, \infty)$, where $y = ty_1$, $y_1 := y(z_1)$. Since $F_\epsilon\{z_1, y_1\} = 0$, we see from (16) that

$$\left|\frac{\epsilon^6 y_1}{4} + 1\right| \leq \frac{\kappa}{A} \to 0$$

as $\epsilon \to 0$, giving

$$\lim_{\epsilon \to 0} \frac{\epsilon^6 y_1}{4} = -1. \tag{17}$$

So if we hold t fixed as $\epsilon \to 0$ we have

$$\lim_{\epsilon \to 0} \frac{\epsilon^6 y}{4} = -t. \tag{18}$$

Consider the following ratio

$$R_\epsilon := \frac{F_\epsilon\{z, y\}}{y^3(y - y_1)} = \frac{\left(\frac{\epsilon^6 y}{4}\right)^2 + \left(\frac{\epsilon^6 y}{4}\right) + \frac{\epsilon^6 \psi}{16}}{\left(\frac{\epsilon^6 y}{4}\right)^2 - \left(\frac{\epsilon^6 y_1}{4}\right)\left(\frac{\epsilon^6 y}{4}\right)}.$$

Using the limits in (17) and (18), we see that

$$\lim_{\epsilon \to 0} R_\epsilon = 1.$$

The definition of limit then shows for some given $0 < \nu < 1$, $\exists r > 0$ sufficiently small such that for all ϵ with $\epsilon^6 < r$,

$$|F_\epsilon\{z, y\}| \geq \nu^2 \left| y^3(y - y_1) \right|.$$

Since the same argument holds on Γ_-, we have from equation (15),

$$|z_+ - z_-| \leq \frac{2}{\nu |\epsilon|^3} \left| \int_\infty^{y_1} \frac{dy}{\sqrt{y^3(y - y_1)}} \right|,$$

where the integration is along a path of steepest descent (from a pole of y). So, recalling that along such a path, $|dy| = -d|y|$, we have

$$|z_+ - z_-| \leq -\frac{2}{\nu |\epsilon|^3} \int_\infty^A \frac{d|y|}{\sqrt{|y|^3(|y| - A)}} = \frac{4}{\nu A |\epsilon|^3}.$$

Since $A \approx 4|\epsilon|^{-6}$, this shows us that the distance between the poles of solutions of $P_{II}(\epsilon)$ is of order ϵ^3.

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References

[1] Ablowitz, M. J., Ramani, A., and Segur, H. (1978) *Nonlinear Evolution Equations and Ordinary Differential Equations of Painlevé Type*, Lett. Nouvo Cim., **23**, 333-338

[2] Ablowitz, M. J., Ramani, A., and Segur, H. (1980) *A Connection between Nonlinear Evolution Equations and Ordinary Differential Equations of P-type. I and II*, J. Math. Phys., **21**, 715-721 and 1006-1015

[3] Cartan, H. (1963) "Elementary Theory of Analytic Functions of One or Several Variables," Editions Scientifiques Hermann, Paris - Addison-Wesley Pub. Co., Reading, Massachusetts (Chapter VII)

[4] Fuchs, R. (1907) *Über lineare homogene differentialgleichungen zweiter ordnung mit drei im endlichen gelegene wesentlich singulären stellen*, Math. Ann. **63**, 301-321

[5] Gambier, B. (1909) *Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes*, Acta Math. **33**, 1-55

[6] Ince, E.L. (1956) "Ordinary Differential Equations," Dover, New York

[7] Joshi, N. and Kruskal, M. (1993) *A New Coalescence of Movable Singularities in the Fourth Painlevé Equation*, University of New South Wales Applied Mathematics Preprint AM93/11

[8] Joshi, N. and Kruskal, M. (1993) *A Direct Proof that Solutions of the Six Painlevé Equations Have No Movable Singularities Except Poles*, Stud. Appl. Math. (to appear)

[9] Kruskal, M.D. and Clarkson, P.A. (1992) *The Painlevé-Kowalevski and Poly-Painlevé Tests for Integrability*, Stud. Appl. Math. **86**, 87-165

[10] Okamoto, K. (1986) *Isomonodromic Deformation and Painlevé Equations and the Garnier System*, J. Fac. Sci. Univ. Tokyo Sec. IA, Math. **33**, 575-618

[11] Painlevé, P. (1902) *Sur les équations différentielles du second ordre et d'ordre supeérieur dont l'intégrale générale est uniforme*, Acta Math. **25**, 1-85

[12] Painlevé, P. (1906) *Sur les équations différentielles du second ordre à points critiques fixes*, C. R. Acad. Sc. Paris **143**, 1111-1117

[13] Ramani, A., Grammaticos, B., and Bountis, T. (1989) *The Painlevé Property and Singularity Analysis of Integrable and non-Integrable Systems*, Physics Reports **180**, (3) 159-245

[14] Siegel, C.L., (1969) "Topics in Complex Function Theory" Vol. 1; *Elliptic Functions and Uniformization Theory.* Translated from the original German by A. Shenitzer and D. Solitar. Wiley, New York.

Figure Captions

Figure 1: Degeneracies among the Painlevé equations.

$$
\begin{tikzcd}
 & & P_{III} \arrow[dr] & & \\
P_{VI} \arrow[r] & P_V \arrow[ur] \arrow[dr] & & P_{II} \arrow[r] & P_I \\
 & & P_{IV} \arrow[ur] & &
\end{tikzcd}
$$