

Solvable models of relativistic charged spherically symmetric fluids

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Abstract

It is known that charged relativistic shear-free fluid spheres are described by the equation $y_{xx} = f(x)y^2 + g(x)y^3$, where f and g are arbitrary functions of x only and y is a function of x and an external parameter t . Necessary and sufficient conditions on f and g are obtained such that this equation possesses the Painlevé property. In this case the general solution y is given in terms of solutions of the first or second Painlevé equation (or their autonomous versions) and solutions of their linearizations. In the autonomous case we recover the solutions of Wyman, Chatterjee and Sussman and a large class of (apparently new) solutions involving elliptic integrals of the second kind. Solutions arising from the special Airy function solutions of the second Painlevé equation are also given. It is noted that, as in the neutral case, a three-parameter family of choices of f and g are described by solutions of an equation of Chazy type.

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1. Introduction

In [31] Shah and Vaidya studied the spherically symmetric Einstein–Maxwell equations for a shear-free non-rotating charged perfect fluid. In isotropic coordinates the metric takes the form

$$ds^2 = T^2(r, t) dt^2 - Y^2(r, t) \{dr^2 + r^2 d\Omega^2\}, \quad (1)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$, is the standard metric on the 2-sphere. With respect to the variables $x = r^2$ and $y(x, t) = 1/Y(r, t)$, the Einstein–Maxwell equations reduce to

$$\frac{\partial^2 y(x, t)}{\partial x^2} = f(x)y^2(x, t) + g(x)y^3(x, t), \quad (2)$$

and

$$T = h(t) \frac{\partial}{\partial t} \ln y(x, t), \quad (3)$$

where f , g and h are arbitrary functions of their arguments. All components of the electromagnetic tensor vanish except

$$F_{rt} = -F_{tr} = 2xg(x)h(t)\frac{\partial y(x,t)}{\partial t}.$$

The above results of Shah and Vaidya have been rediscovered several times: Faulkes [14], Mashhoon and Partovi [22], and Nduka [25, 26]. See Krasinski [21] for a review of the results in these and related papers. Sussman [34] notes that $h(t)$ cannot be absorbed in a new ‘ t ’ coordinate. Barnes [7] and White and Collins [37] have shown that (1) describes a spacetime of Petrov type D or O.

From equations (1) and (3) we see that y must be a non-constant differentiable function of t . This leads to the problem of finding a family of solutions to equation (2) (for fixed f and g) that depends on at least one arbitrary parameter. This parameter can then be replaced by an arbitrary function of t .

In this paper we will determine necessary and sufficient conditions on f and g such that the general solution of equation (2) is single-valued as a function of x about all points where f and g are analytic in the complex plane. This property (the Painlevé property) is closely related to integrability [2, 4, 5]. The property was first used by Kowalevskaya to find a new solvable case of the spinning top equations which she then solved in terms of ratios of hyperelliptic functions.

Painlevé, Fuchs and Gambier showed that all equations of the form

$$\frac{d^2 y}{dx^2} = F\left(x; y, \frac{dy}{dx}\right),$$

where F is rational in y and dy/dx and (locally) analytic in x , that possess the Painlevé property can be mapped to one of several canonical forms using a change of independent variable and a Möbius transformation in y (Painlevé [29, 30]—see also Ablowitz and Clarkson [2]). All of these equations were solved in terms of known functions or solutions of six new nonlinear equations now called the Painlevé equations (P_I – P_{VI}). These equations are considered to be integrable because of their associated linear (isomonodromy) problems. The first two Painlevé equations are

$$P_I: \quad \frac{d^2 u}{dz^2} = 6u^2 + z, \tag{4}$$

$$P_{II}: \quad \frac{d^2 u}{dz^2} = 2u^3 + zu + \kappa, \tag{5}$$

where κ is a constant.

Equation (2) falls into the class of equations studied by Painlevé and his results show that if equation (2) possesses the Painlevé property then it can be transformed into either equation (4) or (5) (depending on whether $g = 0$) or autonomous versions of these equations which are solvable in terms of elliptic functions. However, in Painlevé’s work the transformations from equation (2) to one of these canonical forms are only given implicitly as solutions of differential equations. Here we provide an analysis of these equations.

In section 2 we present the Painlevé (singularity) analysis of equation (2). Necessary and sufficient conditions on f and g such that equation (2) possesses the Painlevé property are given in theorems 1 and 2. In section 3 we solve the case in which Painlevé’s transformations map equation (2) to an autonomous equation which can be solved in terms of elliptic functions. We construct explicit parametrizations of all solutions of equation (2) under this constraint. We recover all solutions of Sussman [34] and Wyman [38, 39] as special cases. Chatterjee [8]

and Srivastava [32] also considered special cases of f and g such that equation (2) possesses the Painlevé property. These solutions fall into the class studied by Sussman.

The first Painlevé equation (4) is not known to have any solutions that can be written as closed-form expressions involving only classically known functions. However, it is known that if $\kappa = n + \frac{1}{2}$, where n is an integer, then the second Painlevé equation (5) admits a one-parameter family of solutions that are rational functions of the Airy functions and their derivatives [6, 16]. In section 4 this leads to a countable number of cases in which we find a family of solutions of equation (2) depending on one arbitrary function of t .

In [38], Wyman considered equation (2) with $g = 0$, which corresponds to neutral fluid spheres, and found necessary and sufficient conditions on f such that it possesses the Painlevé property and solved the equations. In [17], the author showed that a three-parameter family of choices of f for which this equation possesses the Painlevé property is given by the general solution of a special case of the generalized Chazy equation. In section 5 we show that in the charged case ($g \neq 0$) there is also a three-parameter family of choices for f and $g \neq 0$ for which equation (2) possesses the Painlevé property given by an equation of Chazy type. This equation is shown to be a reduction of the Darboux–Halphen system and the self-dual Yang–Mills equations.

2. Painlevé analysis

In this section we find necessary and sufficient conditions on f and g such that equation (2) possesses the Painlevé property. Following Painlevé, we write $y(x, t)$ as a linear function of a new dependent variable u and we write equation (2) as an equation for u as a function of a new independent variable z (in our case we treat t as a parameter). That is, we set

$$y(x, t) = \frac{u(z, t) - v(z)}{w(z)}, \quad (6)$$

and

$$z = \phi(x), \quad (7)$$

where v , w and ϕ are yet to be determined. We choose to write the linear transformation between y and u in the form (6) as it leads to simple differential equations for v and w .

We substitute equations (6) and (7) into equation (2) and regard the resulting equation as an equation for $u(z, t)$. Demanding that this equation have no u_z term yields

$$\frac{\phi_{xx}}{\phi_x^2} = 2 \frac{w_z}{w}.$$

This equation is satisfied if we choose ϕ such that $\phi_x = w^2$. Hence

$$x = \int w^{-2}(z) dz. \quad (8)$$

Equation (2) now takes the form

$$u_{zz} = v_{zz} + w^{-1}w_{zz}(u - v) + fw^{-5}(u - v)^2 + gw^{-6}(u - v)^3. \quad (9)$$

We will choose v and w such that equation (9) takes a simple form as an equation in u . There are two cases depending on whether g is zero.

Case 1: $g \neq 0$ (charged fluids)

As an equation in u , equation (9) says that u_{zz} is cubic in u . We choose v and w such that the coefficients of u^3 and u^2 are 2 and 0, respectively. Hence $w(z) = (g(x)/2)^{1/6}$ and $v(z) = f(x)/[6w^5(z)]$. Equation (9) now takes the form

$$\frac{\partial^2 u(z, t)}{\partial z^2} = 2u^3(z, t) + Q(z)u(z, t) + R(z), \quad (10)$$

where

$$Q(z) = \frac{w_{zz}}{w} - 6v^2 \quad \text{and} \quad R(z) = v_{zz} - vQ(z) - 2v^3.$$

For $g \neq 0$, equation (2) possesses the Painlevé property if and only if equation (10) does. Painlevé showed that necessary and sufficient conditions for any equation of the form (10) to possess the Painlevé property are

$$Q(z) = az + b, \quad R(z) = c/2,$$

where a, b and c are arbitrary constants. We have proved the following.

Theorem 1. *Equation (2) with $g \neq 0$ possesses the Painlevé property if and only if*

$$f(x) = 6v(z)w^5(z), \quad g(x) = 2w^6(z), \quad (11)$$

where $w \neq 0$ and v are any solutions of

$$\frac{d^2 v}{dz^2} = 2v^3 + (az + b)v + c/2, \quad (12)$$

$$\frac{d^2 w}{dz^2} = (6v^2 + az + b)w, \quad (13)$$

and z is defined by (8). The general solution of equation (2) is then given by (6) where $u(z, t)$ is the general solution of equation (12) (t is treated as an independent parameter).

Theorem 1 is a special case of Painlevé's classification scheme [29, 30]. Note that equation (13) is the linearization of equation (12).

If $a = 0$ then the general solution of equation (12) is given in terms of (Jacobi) elliptic functions and equation (13) becomes a special case of the Lamé equation. If $a \neq 0$, a linear transformation in v maps equation (12) to the second Painlevé equation (5).

For generic solutions of equation (12) with $a = 0$, the general solution of equation (13) contains an incomplete elliptic integral of the second kind. Chatterjee [8] and Srivastava [32] have considered special cases for which equation (2) possesses the Painlevé property. We will see in the next section that the special case in which v is a constant corresponds to the class of solutions considered by Sussman [34] and Srivastava [32]. Chatterjee's solutions [8] form a subset of these.

Note that in general the solution of equation (13), and hence equation (2), will be branched at the poles of v . From equations (2) and (11), however, we see that these are fixed singularities of equation (2) and do not violate the Painlevé property.

Case 2: $g = 0$ (Neutral fluids)

This case has been considered previously by Wyman [38, 39] and Halburd [17]. Following a similar analysis to that given in case 1, we obtain the following.

Theorem 2. Equation (2) with $g = 0$ possesses the Painlevé property if and only if

$$f(x) = 6w^5(z), \quad (14)$$

where $w \neq 0$ and v are any solutions of

$$\frac{d^2v}{dz^2} = 6v^2 + az + b/2, \quad (15)$$

$$\frac{d^2w}{dz^2} = 12vw, \quad (16)$$

and z is defined by (8). The general solution of equation (2) is then given by (6) where $u(z, t)$ is the general solution of equation (15) (t is treated as an independent parameter).

As in case 1, equation (16) is the linearization of equation (15). If $a = 0$, the general solution of equation (15) is given in terms of (Weierstrass) elliptic functions and equation (16) becomes a special case of the Lamé equation. If $a \neq 0$, equation (15) can be mapped to the first Painlevé equation by a linear transformation in v .

If v is the general solution of equation (12) (respectively (15)) depending on two arbitrary independent parameters μ and ν , then two linearly independent solutions of equation (13) (respectively (16)) are $\partial v / \partial \mu$ and $\partial v / \partial \nu$. Note that the Wronskian $W(w_1, w_2) := w_1(w_2)_z - (w_1)_z w_2$ of any two independent solutions of equation (13) (respectively (16)) is constant. On writing $w = \gamma w_1 + \delta w_2$, the general solution of equation (2) is given parametrically as

$$\begin{aligned} y(x) &= \frac{u(z, t) - v(z)}{\gamma w_1(z) + \delta w_2(z)}, \\ x &= \int \frac{dz}{(\gamma w_1(z) + \delta w_2(z))^2} = \frac{\alpha w_1(z) + \beta w_2(z)}{\gamma w_1(z) + \delta w_2(z)}, \end{aligned} \quad (17)$$

where without loss of generality we have taken the integration constant to be zero and α, β, γ and δ are arbitrary constants satisfying

$$(\alpha\delta - \beta\gamma)W(w_1, w_2) + 1 = 0. \quad (18)$$

3. The case $a = 0$

When $a = 0$ the general solutions of equations (12) and (15) are both given in terms of elliptic functions (Jacobi and Weierstrass, respectively). In this section we provide explicit parametrizations of equation (2) and its general solution.

3.1. The generic charged case (v non-constant)

Multiplying equation (12) with $a = 0$ by $v_z \neq 0$ and integrating, we obtain

$$v_z^2 = v^4 + bv^2 + cv + d, \quad (19)$$

where d is an arbitrary constant. Let $\eta(z; b, c, d)$ be any non-constant meromorphic solution of equation (19). Then

$$v(z) = \eta(z - z_0; b, c, d),$$

where z_0 is an arbitrary constant¹ is also a solution of equation (19). One solution of equation (13) is then given by

$$\tilde{w} = -\frac{\partial v}{\partial z_0} = \frac{\partial}{\partial z} \eta(z - z_0; b, c, d).$$

Variation of parameters gives a second independent solution of equation (13);

$$\tilde{w}(z) \int \frac{dz}{\tilde{w}^2(z)}.$$

Let ξ_1 and ξ_2 be two independent solutions of

$$\frac{d^2 \xi(z)}{dz^2} = \{6\eta^2(z; b, c, d) + b\} \xi(z). \quad (20)$$

Theorem 1 implies that the general solution of

$$\begin{aligned} \frac{\partial^2 y(x, t)}{\partial x^2} &= 6\eta(z - z_0; b, c, d) \{\gamma \xi_1(z - z_0) + \delta \xi_2(z - z_0)\}^5 y^2(z, t) \\ &\quad + \{\gamma \xi_1(z - z_0) + \delta \xi_2(z - z_0)\}^6 y^3(z, t), \end{aligned} \quad (21)$$

is given parametrically as

$$y(x) = \frac{\eta(z - \sigma(t); b, c, \tau(t)) - \eta(z - z_0; b, c, d)}{\gamma \xi_1(z - z_0) + \delta \xi_2(z - z_0)}, \quad (22)$$

$$x = \frac{\alpha \xi_1(z - z_0) + \beta \xi_2(z - z_0)}{\gamma \xi_1(z - z_0) + \delta \xi_2(z - z_0)}, \quad (23)$$

where the constants $\alpha, \beta, \gamma, \delta$ satisfy

$$(\alpha\delta - \beta\gamma)W(\xi_1, \xi_2) + 1 = 0 \quad (24)$$

and σ and τ are arbitrary functions of t (not both constant). Note that without loss of generality we can choose $z_0 = 0$ in the parametric solution (22) and (23).

Equation (19) can be factorized to yield

$$\eta_z^2 = (\eta - e_1)(\eta - e_2)(\eta - e_3)(\eta - e_4), \quad (25)$$

where e_1, e_2, e_3 and e_4 , are (possibly complex) constants such that

$$\sum_i e_i = 0, \quad \sum_{i < j} e_i e_j = b, \quad \sum_{i < j < k} e_i e_j e_k = -c, \quad e_1 e_2 e_3 e_4 = d.$$

In table 1 we write a non-constant solution of equation (21) for generic b, c, d listed according to the associated (distinct) roots e_1, e_2, e_3 and e_4 . Specifically, the solution of equation (21) can be reconstructed from table 1 as

$$\eta(z) = \frac{c_2 - c_3 F(z)}{1 - c_1 F(z)}. \quad (26)$$

We also list two independent solutions of equation (20). The particular form of ξ_1 and ξ_2 in table 1 were chosen for simplicity, rather than to ensure continuity in b, c and d . Recalling that $x = r^2$, intervals of z should be chosen so that $x \geq 0$.

The author believes these solutions to be new. Degenerate cases in which two or more of the roots $\{e_i\}_{i=1, \dots, 4}$ are equal will not be considered here. In the case where $\sigma'(t) \neq 0$ it is often convenient to define a new time coordinate $t^* := \sigma(t)$ in the metric 1.

¹ In the degenerate case $c = 0, b^2 = 4d$, we have $v_z = \pm(v^2 + b^2/4)$ which leads to two branches of solutions $\eta(\pm(z - z_0); b, 0, b^2/4) = \pm \frac{b}{2} \tan\left\{\frac{2}{b}(z - z_0)\right\}$ (for $b \neq 0$) and $\pm \frac{1}{z - z_0}$ for $b = 0$.

Table 1. Choices for η , ξ_1 and ξ_2 in the charged case.

(Distinct) Roots	Choices for η , ξ_1 , and ξ_2
All real	$F(z) = \text{sn}^2(\Omega z, k)$
$e_1 < e_2 < e_3 < e_4$	$\xi_1(z) = \frac{\text{sn cn dn}}{(1 - c_1 \text{sn}^2)^2}$
$e_1 + e_2 + e_3 + e_4 = 0$	$\xi_2(z) = \xi_1(z) \left\{ \theta_1 \Omega z + \theta_2 E(\Omega z) + \theta_3 \frac{\text{sn cn}}{\text{dn}} + \theta_4 \frac{\text{cn dn}}{\text{sn}} + \theta_5 \frac{\text{sn dn}}{\text{cn}} \right\}$ $k^2 = \frac{(e_3 - e_2)(e_4 - e_1)}{(e_4 - e_2)(e_3 - e_1)}, \quad \Omega = \frac{\sqrt{(e_4 - e_2)(e_3 - e_1)}}{2}$ $\theta_1 = k'^4(k^2 c_1^4 + 2c_1^4 - 4k^2 c_1^3 + k^4) + k^4 k'^2 (c_1 - 1)^4,$ $\theta_2 = -\{k'^4 c_1^4 + k^4 k'^4 + k^4 (c_1 - 1)^4 + (c_1 - k^2)^4\},$ $\theta_3 = k^2 (c_1 - k^2)^4, \theta_4 = -k^4 k'^4, \theta_5 = k^4 (c_1 - 1)^4$
$\eta \leq e_1$ or $\eta \geq e_4$	$c_1 = \frac{e_4 - e_1}{e_4 - e_2}, \quad c_2 = e_1, \quad c_3 = e_2 c_1$
$e_2 \leq \eta \leq e_3$	$c_1 = \frac{e_3 - e_2}{e_3 - e_1}, \quad c_2 = e_2, \quad c_3 = e_1 c_1$
2 real and 2 complex	$F(z) = \text{cn}(\Omega z, k)$
$e_1 < e_2$, real	$\xi_1(z) = \frac{\text{sn dn}}{(1 - c_1 \text{cn})^2}$
$e_3 = \mu + i\nu = \bar{e}_4$	$\xi_2(z) = \xi_1(z) \left\{ \theta_1 \Omega z + \theta_2 \frac{\text{dn}}{\text{sn}} + \theta_3 \frac{\text{sn}}{\text{dn}} + \theta_4 E(\Omega z) + \theta_5 \frac{\text{sn cn}}{\text{dn}} + \theta_6 \frac{\text{cn dn}}{\text{sn}} \right\}$
$e_1 + e_2 + 2\mu = 0$	$M^2 = (e_1 - \mu)^2 + \nu^2, \quad N^2 = (e_2 - \mu)^2 + \nu^2$
$\nu \neq 0$	$k^2 = \frac{(M + N)^2 - (e_1 - e_2)^2}{4MN}, \quad \Omega = \sqrt{MN}$ $\theta_1 = k^2 k'^2 (c_1^4 + 6c_1^2 + 1) - k'^2 c_1^4, \quad \theta_2 = 4k^2 k'^2 c_1 (1 + c_1^2)$ $\theta_3 = 4k^2 k'^2 c_1 (c_1 k'^2 - k^2), \quad \theta_4 = k^4 - 6k^2 k'^2 c_1^2 + k'^4 c_1^4 - k^2 k'^2 (c_1^4 + 6c_1^2 + 1),$ $\theta_5 = -k^2 (k^4 - 6k^2 k'^2 c_1^2 + k'^4 c_1^4), \quad \theta_6 = -k^2 k'^2 (c_1^4 + 6c_1^2 + 1)$
$\eta \leq e_1$ or $\eta \geq e_2$	$c_1 = \frac{N + M}{N - M}, \quad c_2 = \frac{e_1 N - e_2 M}{N - M}, \quad c_3 = \frac{e_1 N + e_2 M}{N - M}$
all complex	$F(z) = \text{tn}(\Omega z, k)$
$e_1 = \mu + i\nu_1 = \bar{e}_3,$	$\xi_1(z) = \frac{\text{dn}}{(\text{cn} - c_1 \text{sn})^2}$
$e_2 = -\mu + i\nu_2 = \bar{e}_4$	$\xi_2(z) = \xi_1(z) \left\{ \theta_1 \Omega z + \theta_2 E(\Omega z) + \theta_3 \frac{\text{sn cn}}{\text{dn}} + \theta_4 \text{dn} + \theta_5 \frac{1}{\text{dn}} \right\}$
$\mu \geq 0$	$\theta_1 = k^4 c_1^4 - k'^4 (c_1^4 - 6c_1^2 + 1) - c_1^4 + 6k'^2 c_1^2 - k'^4,$
$\nu_1, \nu_2 > 0$	$\theta_2 = k'^2 (c_1^4 - 6c_1^2 + 1) + c_1^4 - 6k'^2 c_1^2 + k'^4,$
	$\theta_3 = -k^2 (c_1^4 - 6k'^2 c_1^2 + k'^4), \quad \theta_4 = 4k'^2 c_1 (1 - c_1^2), \quad \theta_5 = 4k'^2 c_1 (k'^2 - c_1^2)$
$-\infty < \eta < \infty$	$M^2 = 4\mu^2 + (\nu_1 + \nu_2)^2, N^2 = 4\mu^2 + (\nu_1 - \nu_2)^2, k^2 = \frac{4MN}{(M + N)^2}$ $\Omega = \frac{M + N}{2}, c_1 = \sqrt{\frac{4\nu_1^2 - (M - N)^2}{(M + N)^2 - 4\nu_1^2}}, c_2 = \mu + \nu_1 c_1, c_3 = -\nu_1 + \mu c_1$
$\eta(z) = \frac{c_2 - c_3 F(z)}{1 - c_1 F(z)},$	$\text{sn} \equiv \text{sn}(\Omega z, k), \text{cn} \equiv \text{cn}(\Omega z, k), \text{dn} \equiv \text{dn}(\Omega z, k), \text{tn} \equiv \text{tn}(\Omega z, k)$
$k^2 + k'^2 = 1$	$E(u) = \int_0^u \text{dn}^2 t \, dt \equiv E(\text{am } u, k) = \int_0^{\text{am } u} \sqrt{1 - k^2 \sin^2 \vartheta} \, d\vartheta$ is Legendre's incomplete elliptic integral of the second kind.

3.2. Sussman's solutions (constant v)

In section 3.1 we assumed that v was non-constant when we multiplied equation (12) by the integrating factor v_z to obtain equation (19). If $v = v_0$, a constant, then equations (12) and (13) become

$$4v_0^3 + 2bv_0 + c = 0, \quad \frac{d^2w(z)}{dz^2} = \frac{\Delta}{4}w(z),$$

where $\Delta = 4(6v_0^2 + b)$. If we define $U(z, t) = u(z, t) - v_0$ we see that the solution of equation (2) is $y(x, t) = U(z, t)/w(z)$, where U and w satisfy

$$U_z^2 = U^4 + 4v_0U^3 + \frac{\Delta}{4}U^2 + L(t), \quad (27)$$

$$ww_{xx} - 2w_x^2 = \frac{\Delta}{4}w^6, \quad (28)$$

respectively, where $L(t)$ is an arbitrary differentiable function of t , and we have used equation (8) to derive equation (28). Note that we have used the fact that U is not constant which follows from the fact that $y_t \neq 0$.

The general solution of equation (28) is

$$w = \frac{\pm 1}{\sqrt{Ax^2 + Bx + C}}, \quad (29)$$

where A , B and C are constants satisfying $\Delta = B^2 - 4AC$. Without loss of generality we choose the positive sign in equation (29). From equation (27) and the definition of η , we see that

$$y(x, t) = \sqrt{Ax^2 + Bx + C} \eta(z - \sigma(t); 4v_0, \Delta/4, L(t)),$$

$$z - z_0 = \int w^2 dx = \begin{cases} \frac{1}{\sqrt{\Delta}} \ln \left| \frac{2Ax + B - \sqrt{\Delta}}{2Ax + B + \sqrt{\Delta}} \right|; & A \neq 0, \quad \Delta > 0, \\ -\frac{2}{2Ax + B}; & A \neq 0, \quad \Delta = 0, \\ \frac{2}{\sqrt{-\Delta}} \tan^{-1} \left(\frac{2Ax + B}{\sqrt{-\Delta}} \right); & A \neq 0, \quad \Delta < 0, \\ \frac{1}{B} \ln |Bx + C|; & A = 0, \quad B \neq 0, \\ \frac{x}{C}; & A = B = 0, \quad C \neq 0. \end{cases} \quad (30)$$

The solutions presented in this subsection correspond precisely to those of Sussman [34]².

3.3. Wyman's solutions (the neutral case)

In this section we repeat the analysis of subsections 3.1 and 3.2 for theorem 2 with $a = 0$. The solutions described in this subsection were first found by Wyman [38, 39].

If $v_z \neq 0$ and $a = 0$ then equation (15) integrates to give

$$v_z^2 = 4v^3 + bv + c = 4(v - e_1)(v - e_2)(v - e_3), \quad (31)$$

² Note that B and Δ in the above are, respectively, twice and four times the corresponding parameters used by Sussman.

Table 2. Choices for η , ξ_1 and ξ_2 in the uncharged case.

(Distinct) Roots	Choices for η , ξ_1 and ξ_2
All real	$F(z) = \text{sn}^2(\Omega z, k)$
$e_1 < e_2 < e_3$	ξ_1, ξ_2 are as in table 1 ('all real' case) with
$e_1 + e_2 + e_3 = 0$	$k^2 = \frac{e_2 - e_1}{e_3 - e_1}, \quad \Omega = \sqrt{e_3 - e_1},$
$e_1 \leq \eta \leq e_2$	$c_1 = 0, \quad c_2 = e_1, \quad c_3 = e_1 - e_2$
$e_3 \leq \eta$	$c_1 = 1, \quad c_2 = e_3, \quad c_3 = e_2$
One real and two complex	$F(z) = \text{cn}(\Omega z, k)$
e_1 real	ξ_1, ξ_2 are as in table 1 ('two real and two complex' case) with
$e_2 = \mu + iv = \bar{e}_3$	$M^2 = (\mu - e_1)^2 + v^2, \quad k^2 = \frac{M + \mu - e_1}{2M}, \quad \Omega = 2\sqrt{M}$
$v \neq 0, e_1 + 2\mu = 0$	
$e_1 \leq \eta$	$c_1 = 1, \quad c_2 = e_1 + M, \quad c_3 = e_1 - M$

where c is an integration constant and at least one of the roots, say e_1 , is real. Note that $e_1 + e_2 + e_3 = 0$. Table 2 provides a list of solutions $\eta(z; b, c)$ of equation (31) for the case of distinct roots e_1, e_2, e_3 together with solutions ξ_1, ξ_2 of equation (16). The solution η again has the form (26). Theorem 2 says that the differential equation

$$\frac{\partial^2 y(x, t)}{\partial x^2} = 6 \{ \gamma \xi_1(z - z_0) + \delta \xi_2(z - z_0) \}^5 y^2(x, t),$$

with x given by (23) and (24), has the general solution

$$y(x, t) = \frac{\eta(z - \sigma(t); b, \tau(t)) - \eta(z - z_0; b, c)}{\gamma \xi_1(z - z_0) + \delta \xi_2(z - z_0)},$$

where σ and τ are arbitrary differentiable functions of t (not both constant).

If $v = v_0$ is a constant then $b = -12v_0^2$ and w is given by (29) where $\Delta := B^2 - 4AC = 3v_0$. Theorem 2 then says that the general solution of

$$\frac{\partial^2 y(x, t)}{\partial x^2} = 6(Ax^2 + Bx + C)^{-5/2} y^2(x, t)$$

is given parametrically as

$$y(x, t) = \sqrt{Ax^2 + Bx + C} \left\{ \eta(z - \sigma(t); -12v_0^2, \tau(t)) - \frac{1}{3}\Delta \right\},$$

where z is given by equation (30) and η is given in table 2. In the above we have assumed that $y_t \neq 0$ and hence u is not a constant.

4. The Airy function solutions of P_{II}

In this section we consider theorem 1 with $a \neq 0$. After transforming variables in equations (12) and (13) to $\zeta = \mu(z + b/a)$, $\tilde{v}(\zeta) = \mu^{-1}v(z)$, $\tilde{w}(\zeta) = w(z)$, and $\tilde{u}(\zeta, t) = \mu^{-1}u(z, t)$, where $\mu^2 = -a/2$, we see that \tilde{u} and \tilde{v} satisfy

$$\tilde{v}_{\zeta\zeta} = 2\tilde{v}^3 - 2\zeta\tilde{v} - \tilde{\kappa}, \quad (32)$$

where the sign of μ is chosen so that $\tilde{\kappa} := -c/(2\mu^3) \geq 0$. Equation (13) becomes

$$\tilde{w}_{\zeta\zeta} = 2(3\tilde{v}^2 - \zeta)\tilde{w}, \quad (33)$$

which is still the linearization of equation (32).

In 1909, Gambier [15] noted that if $\tilde{\kappa} = 1$ then a one-parameter family of solutions of equation (32) is given by the general solution of the Riccati equation

$$\frac{d\tilde{v}}{d\zeta} = \tilde{v}^2 - \zeta,$$

which is

$$\tilde{v} = -\frac{d}{d\zeta} \ln \phi(\zeta),$$

where ϕ solves the Airy equation

$$\phi_{\zeta\zeta} = \zeta \phi. \quad (34)$$

That is, ϕ is a linear combination of the Airy functions $\text{Ai}(\zeta)$ and $\text{Bi}(\zeta)$. Let ψ be a second solution of the Airy equation (34) such that $W(\phi, \psi) = 1$. A one-parameter family of solutions of equation (32) with $\tilde{\kappa} = 1$ is then given by

$$h(\zeta, \epsilon) = -\frac{\partial}{\partial \zeta} \ln(\phi(\zeta) + \epsilon \psi(\zeta)),$$

which satisfies $h(\zeta, 0) = \tilde{v}(\zeta)$. Therefore,

$$\tilde{w}_1(\zeta) = -\frac{\partial}{\partial \epsilon} h(\zeta, \epsilon) \Big|_{\epsilon=0} = \phi^{-2}(\zeta)$$

is a solution of equation (33). Variation of parameters yields a second solution,

$$\tilde{w}_2 = \phi^{-2}(\zeta) \int \phi^4(\zeta) d\zeta.$$

We can also find a family of solutions for \tilde{u} (which also satisfies equation (32))

$$\tilde{u}(\zeta, t) = -\frac{\partial}{\partial \zeta} \ln(\psi(\zeta) + \sigma(t)\phi(\zeta)),$$

containing one arbitrary function $\sigma(t)$. This gives $y(x, t) = \{\tilde{u}(\zeta, t) - \tilde{v}(\zeta)\} / \{\gamma \tilde{w}_1(\zeta) + \delta w_2(\zeta)\}$, where $x = \{\alpha w_1(\zeta) + \beta w_2(\zeta)\} / \{\gamma w_1(\zeta) + \delta w_2(\zeta)\}$ and $\alpha, \beta, \gamma, \delta$ satisfy equation (18).

When $\tilde{\kappa} = 2n + 1$, where n is an integer, then equation (32) has a one-parameter family of solutions that are rational functions of the Airy functions and their derivatives [6, 16]. Okamoto [28] showed that these solutions can be expressed simply in terms of determinants as follows. Define the functional

$$G_n[\phi] = \begin{pmatrix} \phi & \phi' & \dots & \phi^{(n-1)} \\ \phi' & \phi'' & \dots & \phi^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi^{(n-1)} & \phi^{(n)} & \dots & \phi^{(2n-2)} \end{pmatrix}$$

and let $\Phi_n = G_n[\phi]$ and $\Psi = G_n[\psi]$ where ϕ and ψ are solutions of the Airy equation (34) satisfying $W(\phi, \psi) = 1$. Okamoto showed that

$$\tilde{v}(\zeta) = -\frac{d}{d\zeta} \ln \left(\frac{\det \Phi_{n+1}(\zeta)}{\det \Phi_n(\zeta)} \right) \quad (35)$$

solves equation (32) with $\tilde{\kappa} = 2n + 1$. Repeating the above argument given for Gambier's solution and using Cramer's rule, we see that one solution of equation (33) with \tilde{v} given by (32) is

$$\tilde{w}_1(\zeta) = \frac{d}{d\zeta} \left\{ \text{Tr}(\Phi_{n+1}^{-1} \Psi_{n+1}) - \text{Tr}(\Phi_n^{-1} \Psi_n) \right\}$$

where Tr is the trace. Variation of parameters then gives a second solution $\tilde{w}_2 = \tilde{w}_1 \int d\zeta / \tilde{w}_1^2$. A one-parameter family of choices for \tilde{u} is

$$\tilde{u}(\zeta, t) = -\frac{\partial}{\partial \zeta} \ln \left(\frac{\det G_{n+1}[\psi(\zeta) + \sigma(t)\phi(\zeta)]}{\det G_n[\psi(\zeta) + \sigma(t)\phi(\zeta)]} \right).$$

5. Equations of Chazy type

In this section we show that when $a = b = c = 0$, the equations that define v and w in theorems 1 and 2 are related to equations of Chazy type. These equations are shown to be reductions of the generalized Darboux–Halphen system and the self-dual Yang–Mills equations and solvable via hypergeometric equations, yet they do not possess the Painlevé property.

5.1. The neutral case

In this subsection we consider equations (15) and (16) with $a = b = 0$. In [17] the author showed that if we let

$$q(x) = 6 \frac{d}{dx} \ln w, \quad (36)$$

then q satisfies

$$q_{xxx} = 2q q_{xx} - 3q_x^2 + \frac{4}{36 - n_0^2} (6q_x - q^2)^2, \quad (37)$$

where $n_0 = \frac{6}{7}$. Equation (37) was studied by Chazy [9–11] and is now known as the generalized Chazy equation (in order to distinguish it from the case $n_0 = \infty$ which is the classical Chazy equation). Chazy showed that the general solution of equation (37) is branched unless n_0 is an integer.

If v is non-constant then from equation (15) with $a = b = 0$, we have

$$v_z^2 = 4(v^3 - k), \quad (38)$$

where k is a constant. Equation (16) is a special case of the Lamé equation which can be written in algebraic form by changing the independent variable from z to v :

$$w_{vv} + \frac{3v^2}{2(v^3 - k)} w_v - \frac{3v}{v^3 - k} w = 0. \quad (39)$$

For $k \neq 0$ we set $s = v^3/k$ and equation (39) becomes the hypergeometric equation

$$s(1-s) w_{ss} + [\hat{c} - (\hat{a} + \hat{b} + 1)s] w_s - \hat{a}\hat{b} w = 0 \quad (40)$$

with

$$\hat{a} = \frac{2}{3}, \quad \hat{b} = -\frac{1}{2}, \quad \hat{c} = \frac{2}{3}. \quad (41)$$

Using equations (8) and (38) we find $w = 6^{-1/2}k^{-1/12}s^{-1/3}(s-1)^{-1/4}s_x^{1/2}$. Hence, from equation (36),

$$q(x) = \frac{1}{2} \frac{d}{dx} \ln \frac{s_x^6}{s^4(s-1)^3}. \quad (42)$$

We have recovered the two standard representations of the solution for the generalized Chazy equation (37). In the first,

$$x = w_1(s)/w(s), \quad (43)$$

where w and w_1 are solutions of the hypergeometric equation with \hat{a} , \hat{b} and \hat{c} given by (41), and $q(x)$ is given by (36). In the second approach, q is given by (42) where s is the general solution of the Schwarzian equation

$$\{s, x\} + \frac{1}{2} \left\{ \frac{1 - \hat{\beta}^2}{s^2} + \frac{1 - \hat{\gamma}^2}{(s-1)^2} + \frac{\hat{\beta}^2 + \hat{\gamma}^2 - \hat{\alpha}^2 - 1}{s(s-1)} \right\} s_x^2 = 0, \quad (44)$$

where

$$\{s, x\} := \frac{d}{dx} \left(\frac{s_{xx}}{s_x} \right) - \frac{1}{2} \left(\frac{s_{xx}}{s_x} \right)^2$$

is the Schwarzian derivative and $\hat{a} = (1 + \hat{\alpha} - \hat{\beta} - \hat{\gamma})/2$, $\hat{b} = (1 - \hat{\alpha} - \hat{\beta} - \hat{\gamma})/2$ and $\hat{c} = 1 - \hat{\beta}$ (see, e.g., [3, 27]). In our case

$$\hat{\alpha} = \frac{7}{6}, \quad \hat{\beta} = \frac{1}{3}, \quad \hat{\gamma} = \frac{1}{2}. \quad (45)$$

5.2. The charged case

In this subsection we consider equations (12) and (13) with $a = b = c = 0$. Proceeding as in subsection 5.1, we note that the transformations (8) and (36) yield

$$(6q_x - q^2)(q_{xxx} - 2qq_{xx} + 3q_x^2) = \frac{1}{27}(9q_{xx} - 9qq_x + q^3)^2 + \frac{37}{324}(6q_x - q^2)^3. \quad (46)$$

We call equation (46) an equation of Chazy type because it is a polynomial equation in the $SL(2; \mathbb{C})$ forms $6q_x - q^2$, $9q_{xx} - 9qq_x + q^3$ and $q_{xxx} - 2qq_{xx} + 3q_x^2$ (see [1]) and it can be solved in terms of solutions of the Schwarzian equation (44) as will be described below.

Non-constant solutions of equation (12) satisfy

$$v_z^2 = v^4 - k. \quad (47)$$

For $k \neq 0$ we change independent variables in equation (13) from z to $s := v^4/k$ and again obtain a special case of the hypergeometric equation (40), this time with parameters $\hat{b} = -\frac{1}{2}$, and $\hat{a} = \hat{c} = \frac{3}{4}$. Using $s = v^4/k$, and equations (8) and (47), we find that equation (36) becomes

$$q(x) = \frac{3}{4} \frac{d}{dx} \ln \frac{s_x^4}{s^3(s-1)^2}, \quad (48)$$

where s is the general solution of the Schwarzian equation (44) with parameters given by

$$\hat{\alpha} = \frac{5}{4}, \quad \hat{\beta} = \frac{1}{4}, \quad \hat{\gamma} = \frac{1}{2}. \quad (49)$$

5.3. Connection to a Darboux–Halphen system and the SDYM equations

In [1], the system

$$\begin{aligned}(\omega_1)_x &= \omega_2\omega_3 - \omega_1(\omega_2 + \omega_3) + \rho, \\(\omega_2)_x &= \omega_3\omega_1 - \omega_2(\omega_3 + \omega_1) + \rho, \\(\omega_3)_x &= \omega_1\omega_2 - \omega_3(\omega_1 + \omega_2) + \rho,\end{aligned}\tag{50}$$

where

$$\rho = \hat{\alpha}^2(\omega_1 - \omega_2)(\omega_3 - \omega_1) + \hat{\beta}^2(\omega_2 - \omega_3)(\omega_1 - \omega_2) + \hat{\gamma}^2(\omega_3 - \omega_1)(\omega_2 - \omega_3),\tag{51}$$

was obtained as a reduction of the self-dual Yang–Mills equations with an infinite-dimensional Lie algebra. The system (50) was first studied and solved by Halphen [18]. When $\hat{\alpha} = \hat{\beta} = \hat{\gamma} = 0$, the system (50) reduces to the classical Darboux–Halphen system which first appeared in the study of triply orthogonal surfaces [13]. The connection between this system and modular forms has also been studied by Harnad and McKay [19].

In the case of distinct ω s we set

$$s = \frac{\omega_1 - \omega_3}{\omega_2 - \omega_3}$$

which, as shown in [1], gives

$$\omega_1 = -\frac{1}{2} \frac{d}{dx} \ln \frac{s_x}{s(s-1)}, \quad \omega_2 = -\frac{1}{2} \frac{d}{dx} \ln \frac{s_x}{s-1}, \quad \omega_3 = -\frac{1}{2} \frac{d}{dx} \ln \frac{s_x}{s},\tag{52}$$

where s is the general solution of the Schwarzian equation (44). From equations (36) and (52) we see that $q = -(\omega_1 + 2\omega_2 + 3\omega_3)$ solves the generalized Chazy equation (37) when $\hat{\alpha} = 1/n_0$, $\hat{\beta} = \frac{1}{3}$ and $\hat{\gamma} = \frac{1}{2}$. In [1] it was also shown that given a solution $\omega_1, \omega_2, \omega_3$ of equation (50), then $q := -2(\omega_1 + \omega_2 + \omega_3)$ solves equation (37), provided $\hat{\alpha} = \hat{\beta} = \hat{\gamma} = 2/n_0$ or two of the parameters $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ are $\frac{1}{3}$ and the other is $2/n_0$.

We note that equation (46) is also a reduction of the system (49)–(51) in which we choose

$$q = -\frac{3}{2}(\omega_1 + \omega_2 + 2\omega_3).\tag{53}$$

Equation (46) is therefore a reduction of the generalized Darboux–Halphen system, and hence of the self-dual Yang–Mills equations. The other (non-negative) choices of parameters $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ for which q given by (53) satisfies equation (46) where ω_1, ω_2 and ω_3 satisfy the generalized Darboux–Halphen system (50) and (51) are $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}), (\frac{1}{4}, \frac{1}{4}, \frac{5}{2}), (\frac{1}{4}, \frac{5}{4}, \frac{1}{2}), (\frac{5}{4}, \frac{5}{4}, \frac{1}{2})$ and $(\frac{5}{4}, \frac{5}{4}, \frac{5}{2})$.

6. Properties of the fluids

Using $x = r^2$ and equation (2), the remaining Einstein equations reduce to the following expressions for the density ρ and pressure p of the fluid:

$$8\pi\rho = 3k(t) - 12xy_x^2 + 12yy_x + 8xfy^3 + 6xgy^4,\tag{54}$$

$$8\pi p = 4y(y - 2xy_x) \frac{y_{xt}}{y_t} + 12xy_x^2 - 8yy_x + 2xgy^4 + k_t(t) \frac{y}{y_t} - 3k,\tag{55}$$

where $k(t) = h^{-2}(t)$. Mashhoon and Partovi [23, 24], Srivastava and Prasad [33], and Collins and Wainwright [12] have addressed the problem of determining which charged shear-free spherically symmetric solutions admit a barotropic equation of state $p \equiv p(\rho)$. Following

the argument in Mashhoon and Partovi [23] (together with the exceptional case discussed in [24]) it can be shown that requiring the fluids described by the solutions given in sections 2 and 3 to be barotropic implies that v is a constant. This means that all such solutions fall into the class discussed by Sussman and the generic solutions described in the present paper are non-barotropic. Krasiński [20] has argued that a ‘... barotropic equation of state is a strong and rather artificial constraint on cosmological models’.

A thorough discussion of the spacetime singularities of the solutions described in sections 2–4 and the problem of matching these solutions to external electrovacuum solutions is beyond the scope of the present paper. Since all solutions of equations (12) and (15) are meromorphic and all singular points of equations (13) and (16) in the finite complex plane are regular, it should be possible to conduct a detailed local analysis of the types of spacetime singularities that can occur in solutions described by theorems 1 and 2. For an analysis of the (spacetime) singularities that arise in the constant v case and a discussion of the general case in which f and g are arbitrary, see Sussman [35, 36].

The problem of matching the (interior) metric given by (1) across a boundary Σ to an external Reissner–Nordström metric

$$ds^2 = \gamma d\hat{t}^2 - \gamma^{-1} d\hat{r}^2 - \hat{r}^2 d\Omega^2,$$

where $d\Omega^2$ is the standard metric on the 2-sphere and

$$\gamma = 1 - \frac{2m}{\hat{r}} + \frac{4\pi e^2}{\hat{r}^2},$$

and m and e are constants, is as follows. Choose the time coordinate t in (1) such that the boundary Σ is given by $r = r_0$ where r_0 is a constant (i.e. t is the proper time for an observer on the boundary Σ). The Darboux matching conditions demand that $\hat{r}(r = r_0, t) = r_0/y(r_0^2, t)$ and $p(r = r_0, t) = 0$. Expressed in terms of $\hat{t}(r = r_0, t)$, the coordinate $\hat{r}(r = r_0, t)$ reflects the radius of the boundary Σ as seen by a distant observer. Setting $p(r = r_0, t) = 0$ in equation (55) gives a linear equation for k .

7. Summary

We have determined necessary and sufficient conditions such that equation (2) possesses the Painlevé property (that all solutions are single-valued about all movable singularities) and we have described the general solution in theorems 1 and 2 in terms of elliptic functions or Painlevé transcendents, and solutions of associated linear equations. We have derived a large class of solutions for $g \neq 0$ in both the elliptic function and Painlevé transcendent cases and showed connections with some generalizations of the Chazy equation. The solutions in the elliptic function case generalize those of Chatterjee, Sussman and Strivastava.

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