Midterm 1 practice

At the midterm, you will be asked to have your camera on and microphone off. If you need to ask a question, send me a private message through the Zoom chat. During the midterm, you are allowed to use two pages of notes that you wrote yourself. You are not allowed to use textbooks, the Internet, or any other resources. The exam will last 60 minutes. In the remaining 15 minutes of class, upload your solutions together with the two pages of notes to the Gradescope. If you encounter technical difficulties with the Gradescope or anything else, send me a message through the Zoom chat.

references

The midterm will be based on Lectures 1–7, Homework 1–3, and the problems from this practice midterm. It will cover the following topics:

1. Separable equations
2. First order linear equations
3. Existence and uniqueness for solutions
4. Autonomous equations
5. Autonomous systems and exact equations

In the lecture notes, you can find references to the relevant parts of the textbooks of Boyce–DiPrima and Braun.

The midterm problems will be very similar to the problems from this practice midterm and the following homework problems:

- 3, 4, 5.1, 6 from Homework 1,
- 1, 2, 6 from Homework 2,
- 3, 4 from Homework 3.

Problem 1. Solve the initial value problem

$$\begin{cases} y' = -t^{-1}y \\ y(1) = 2 \end{cases}$$

and determine the maximal interval of existence of the solution $y = y(t)$.

Similar to: Example 3 from Lecture 1, Example 2 from Lecture 2, Examples 1–3 from Boyce–DiPrima section 2.2.

Solution. This is a separable equation. We separate y and t and integrate with respect to t:

$$\int \frac{y'}{y} \mathrm{d}t = -\int \frac{1}{t} \mathrm{d}t = -\ln|t| + C,$$

for a constant C. We integrate the left-hand side is by substitution $y = y(t)$,

$$\int \frac{y'}{y} \mathrm{d}t = \int \frac{1}{y} \mathrm{d}y = \ln|y|.$$

Therefore,

$$\ln|y| = -\ln|t| + C.$$

Exponentiating both sides, we get

$$|y(t)| = e^C |t|^{-1},$$

where $A = e^C$ can be any positive constant. We can also drop the absolute value to get

$$y(t) = At^{-1}$$

where now A is allowed to have any sign. (We also allow $A = 0$ which we easily check to be a solution.) This is the general solution. To solve the initial value problem, we set $t = 1$:

$$2 = y(1) = A,$$

so the solution is

$$y(t) = 2t^{-1}.$$

We see that the maximal interval containing $t = 1$ for which $y(t)$ is well-defined is $(0, \infty)$, as $y(t) \to \infty$ as t approaches 0 from the right. This is the maximal interval of existence.

Problem 2. Find the general solution to the differential equation

$$y' + 2y = e^{-t}.$$

Hint: The function $\mu(t) = e^{2t}$ satisfies the differential equation $\mu' = 2\mu$.

Similar to: Example 2 from Lecture 3, Examples 2–5 from Boyce–DiPrima section 2.1.

Solution. This is a first order, non-homogenous linear equation. We look for the integrating factor $\mu = \mu(t)$. The integrating factor has to have the property that the equation multiplied by μ:

$$\mu y' + 2\mu y = \mu e^{-t}. \tag{0.1}$$

can be written as

$$(\mu y)' = \mu e^{-t}. \tag{0.2}$$

Since the derivative of the product is

$$(\mu y)' = \mu y' + \mu' y,$$

equations (0.1) and (0.2) are equivalent if μ satisfies the differential equation $\mu' = 2\mu$. The hint tells us that $\mu(t) = e^{2t}$ is a non-zero solution to this equation, so we will use this function. It remains to solve (0.2). Integrating both sides with respect to t, we get that

$$\mu(t)y(t) = \int \mu(t)e^{-t}\mathrm{d}t = \int e^{2t}e^{-t}\mathrm{d}t = \int e^{t}\mathrm{d}t = e^{t} + C,$$

for any constant C. Dividing by $\mu(t)$ we get a formula for the general solution

$$y(t) = e^{-2t}(e^{t} + C) = e^{-t} + Ce^{-2t}.$$

Problem 3. State the existence and uniqueness theorem for first order differential equations. Argue that the initial value problem

$$\begin{cases} y' = t^2 e^{t-y} \\ y(0) = 1 \end{cases} \tag{0.3}$$

has a unique solution $y = y(t)$ defined for t close to $t = 0$.

Similar to: Example 1 from Lecture 5, Problems 4–15 (without specifying the integral of existence) and 17 in Braun section 1.10.

Solution. For the statement of the existence and uniqueness theorem, see Theorem 4 in Lecture 4; or Theorem 2.8.1 in Boyce–DiPrima section 2.8; or Theorem 2′ in Braun section 1.10. The textbooks state the theorem in a more detailed way. You don't need to memorize the exact formulation of the theorem, but you should understand the general idea. The theorem concerns initial value problems of the form

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0, \end{cases}$$

where f is a given function of two variables and t_0, y_0 are given numbers. The theorem says that if the function f is continuous around the point (t_0, y_0) and its partial derivative $\partial f / \partial y$ exists and is continuous around (t_0, y_0), then the above initial value problem has a solution $y = y(t)$ defined for t in some interval containing t_0, and that this solution is unique for t in that interval.

In order to apply the theorem to the specific initial value problem (0.4), we need to verify that the function

$$f(t, y) = t^2 e^{t-y}$$

and its partial derivative $\partial f / \partial y$ are continuous at all (t, y) close to $(t_0, y_0) = (0, 1)$. Recall from the Calculus course that polynomials, the exponential function, logarithm, trigonometric functions and their inverses are continuous, (whenever well-defined; for example, $\ln x$ is continuous on the interval $(0, \infty)$). Moreover, sums, products, and compositions of continuous functions are continuous. Since the functions

$$t \mapsto t^2, \quad (t, y) \mapsto t - y, \quad t \mapsto e^t$$

are continuous, so is f as it can be constructed by multiplying and composing the above functions. Similarly,

$$\frac{\partial f}{\partial y} = -t^2 e^{t-y}$$

is continuous for the same reason. We conclude that f and $\partial f / \partial y$ are continuous on the entire ty–plane, in particular they are continuous around the point $(t_0, y_0) = (0, 1)$ and the uniqueness and existence theorem applies in this case.

Problem 3 (Alternative version)**.** State the existence and uniqueness theorem for first order differential equations. Show that the initial value problem

$$\begin{cases} y' = y^{1/3}, \\ y(0) = 0, \end{cases} \tag{0.4}$$

has two different solutions $y = y(t)$ defined for $t \geq 0$. Explain why the existence and uniqueness theorem cannot be applied in this case.

Similar to: Example 2 from Lecture 5; Problems 18 and 19 from Braun section 1.10.

Solution. After stating the existence and uniqueness theorem as in the previous version of the problem, we need to find two solutions to this initial value problem. The first one is obvious: the constant function $y(t) = 0$. The second solution can be found using separation of variables and integrating (see the solution to Problem 1):

$$\int y'y^{-1/3}\mathrm{d}t = \int 1\mathrm{d}t = t + C.$$

The left integral can be computed by substitution $y = y(t)$:

$$\int y'y^{-1/3}\mathrm{d}t = \int y^{-1/3}\mathrm{d}y = \frac{3}{2}y^{2/3}.$$

Therefore,

$$\frac{3}{2}y(t)^{2/3} = t + C$$

for some constant C. Plugging $t = 0$ and $y(0) = 0$, we get $C = 0$, so the solution is

$$y(t) = \left(\frac{3}{2}t\right)^{3/2}.$$

We have found two different solutions to the given initial value problem, defined for $t \geq 0$. The existence and uniqueness theorem cannot be applied in this case because the function $f(t, y) = y^{1/3}$ does not have continuous partial derivative $\partial f/\partial y$ at $(t_0, y_0) = (0, 0)$. Indeed, for $y > 0$ we compute

$$\frac{\partial f}{\partial y} = \frac{1}{3}y^{-2/3}$$

and the right hand-side diverges to infinity as $y \to 0$.

Problem 4. Find equilibria of the autonomous differential equation

$$y' = y^2 - y.$$

Draw the direction field and integral curves. Determine whether the equilibria are asymptotically stable, asymptotically unstable, or neither of these.

Similar to: Examples 5, 6, 8, 9, 11 from Lecture 5; Example 3 and Problems 15–14 from Boyce–DiPrima section 11; Boyce–DiPrima section 2.5, in particular Problems 1–7.

Solution. We write the equation in the form

$$y' = f(y) = y(y-1).$$

The equlibria are the points where $f(y) = 0$, that is: $y = 0$ and $y = 1$. The draw the direction field, it is helpful to see where f is positive and where it is negative. We see, for example by drawing the graph of f, that

$$\begin{cases} f(y) > 0 & \text{for } y < 0, \\ f(y) < 0 & \text{for } 0 < y < 1, \\ f(y) > 0 & \text{for } 1 < y. \end{cases}$$

Remember that the direction field at a point with coordinates (t, y) on the plane is the line whose slope is $f(y)$.Based on the above information about f we can sketch the direction field as in class. The integral curves are curves which are tangent to the direction field. During office hours and review session, we will discuss again how to draw the direction field and integral curves.
From the picture we will see that:

- an integral curve starting at $y < 0$ goes to $y = 0$ as $t \to \infty$ and to $y \to -\infty$ as $t \to -\infty$,
- an integral curve starting at $0 < y < 1$ goes to $y = 0$ as $t \to \infty$ and to $y = 1$ as $t \to -\infty$,
- and integral curve starting at $y > 1$ goes to $y \to \infty$ as $t \to \infty$ and to $y = 1$ as $t \to -\infty$.

We conclude that $y = 0$ is an asymptotically stable equilibrium and $y = 1$ is an asymptotically unstable equilibrium.

Problem 4 (Alternative versions)**.** The same problem but for the equations

$$y' = y^2$$

and

$$y' = -y(y-1)(y-2).$$

Solution. In the first case, there is one equilibrium $y = 0$ and

1. an integral curve starting at $y > 0$ goes to $y \to \infty$ as $t \to \infty$ and to $y = 0$ as $t \to -\infty$,

2. an integral curve starting at $y < 0$ goes to $y = 0$ as $t \to \infty$ and to $y- \to -\infty$ as $t \to -\infty$.

We conclude that $y = 0$ is neither asymptotically stable nor asymptotically unstable: it is in the infinite future of some nearby solutions but in the infinite past of the others.

The second case is the example discussed in Lecture 5. The equilibria are $y = 0$ (asymptotically stable), $y = 1$ (asymptotically unstable), and $y = 2$ (asymptotically stable).

Problem 5. Explain:

1. what is an autonomous system of differential equations,
2. what it means for an autonomous system to be exact,
3. what it means for an autonomous system to be closed,
4. what is the relation between exact and closed systems.

(You can restrict yourself to two-dimensional systems, that is systems consisting of two differential equations for two unknown functions.)

Verify that the following autonomous system is closed and find an equation describing its integral curves:

$$\begin{cases} x' = 2xy, \\ y' = 3x^2 - y^2. \end{cases} \tag{0.5}$$

Similar to: Examples from Lecture 7; Examples 2, 3 and Problems 1–12 in Boyce–DiPrima section 2.6.

Solution. The definitions can be found in Lecture 7. A two-dimensional autonomous system is a system of differential equations of the form

$$\begin{cases} x' = f(x, y), \\ y' = g(x, y) \end{cases}$$

for two unknown functions $x = x(t)$ and $y = y(t)$. We say that such a system is exact if there is a function $H = H(x, y)$ such that

$$f(x, y) = -\partial H/\partial y \quad \text{and} \quad g(x, y) = \partial H/\partial x.$$

We say that an autonomous system is closed if

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0. \tag{0.6}$$

In Lecture 7 we proved a theorem which says that an autonomous system defined on a rectangle $[x_0, y_0] \times [x_1, y_1]$ in the xy–plane is exact if and only if it is closed.

To verify that the autonomous system (0.5) we compute that for $f(x, y) = 2xy$ and $g(x, y) = 3x^2 - y^2$ we have

$$\frac{\partial f}{\partial x} = 2y \quad \text{and} \quad \frac{\partial g}{\partial y} = -2y.$$

Therefore, (0.6) holds and the system is closed. To find the integral curves, we look for a function $H = H(x, y)$ such that

$$\frac{\partial H}{\partial y} = -f(x, y) = -2xy.$$

Integrating with respect to y we get that for some function $\varphi = \varphi(x)$,

$$H(x,y) = \varphi(x) - \int 2xy\mathrm{d}y = \varphi(x) - xy^2.$$

(We can incorporate the integration constant into the unknown function φ.) To find φ we use that:

$$\frac{\partial H}{\partial x} = g(x,y) = 3x^2 - y^2.$$

Therefore, computing the right-hand side from the formula for H:

$$\varphi'(x) - y^2 = 2x^2 - y^2 \implies \varphi'(x) = 3x^2.$$

Integrating the last equation with respect to x, we get

$$\varphi(x) = x^3 + C$$

for a constant C. The integral curves are described by $H(x,y) = 0$, that is:

$$x^3 - xy^2 + C = 0.$$

ORDINARY DIFFERENTIAL EQUATIONS (MATH 2030)

Midterm 1

11 February 2021

problems and solutions

Problem 1. Solve the initial value problem

$$\begin{cases} y' = 2ty^2 \\ y(0) = 1 \end{cases}$$

and determine the maximal interval of existence of the solution $y = y(t)$.

Solution. This is a separable equation. We separate y and t and integrate with respect to t:

$$\int y'y^{-2}\mathrm{d}t = \int 2t\mathrm{d}t = t^2 + C.$$

for a constant C. We integrate the left-hand side is by substitution $y = y(t)$,

$$\int y'y^{-2}\mathrm{d}t = \int y^{-2}\mathrm{d}y = -y^{-1}.$$

Therefore,

$$y^{-1} = -t^2 - C$$

so

$$y(t) = \frac{1}{-C - t^2}.$$

This is the general solution. To solve the initial value problem, we set $t = 0$:

$$1 = y(0) = -C^{-1}$$

so $C = -1$ and the solution is

$$y(t) = \frac{1}{1 - t^2}.$$

We see that the maximal interval containing $t = 0$ for which $y(t)$ is well-defined is $(-1, 1)$ as $y(t) \to \infty$ when $t \to -1$ or $t \to 1$.

Problem 2. Find the general solution to the differential equation

$$y' + y = t.$$

Hint: The formula for integration by parts is

$$\int f'g = fg - \int fg'.$$

Solution. This is a first order, non-homogenous linear equation. We look for the integrating factor $\mu = \mu(t)$. The integrating factor has to have the property that the equation multiplied by μ:

$$\mu y' + \mu y = \mu t \tag{0.1}$$

can be written as

$$(\mu y)' = \mu t. \tag{0.2}$$

Since the derivative of the product is

$$(\mu y)' = \mu y' + \mu' y,$$

equations (0.1) and (0.2) are equivalent if μ satisfies the differential equation $\mu' = \mu$. We can take any function satisfying this equation, so we choose $\mu(t) = e^t$. It remains to solve (0.2). Integrating both sides with respect to t, and using integration by parts (with $f(t) = e^t$ and $g(t) = t$), we get

$$\mu(t)y(t) = \int \mu(t)t\mathrm{d}t = \int e^t t\mathrm{d}t = e^t t - \int e^t \mathrm{d}t = e^t t - e^t + C = e^t(t-1) + C$$

for any constant C. Dividing by $\mu(t)$ we get a formula for the general solution

$$y(t) = t - 1 + Ce^{-t}.$$

Problem 3. State the existence and uniqueness theorem for first order differential equations. Argue that the initial value problem

$$\begin{cases} y' = y^2 \sin(y-t) \\ y(0) = 0 \end{cases} \tag{0.3}$$

has a unique solution $y = y(t)$ defined for t close to $t = 0$.

Bonus question: (3 points) Can you guess this solution?

Solution. For the statement of the existence and uniqueness theorem, see Theorem 4 in Lecture 4; or Theorem 2.8.1 in Boyce–DiPrima section 2.8; or Theorem 2′ in Braun section 1.10. The textbooks state the theorem in a more detailed way. You don't need to memorize the exact formulation of the theorem, but you should understand the general idea. The theorem concerns initial value problems of the form

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0, \end{cases}$$

where f is a given function of two variables and t_0, y_0 are given numbers. The theorem says that if the function f is continuous around the point (t_0, y_0) and its partial derivative $\partial f/\partial y$ exists and is continuous around (t_0, y_0), then the above initial value problem has a solution $y = y(t)$ defined for t in some interval containing t_0, and that this solution is unique for t in that interval.

In order to apply the theorem to the specific initial value problem (0.3), we need to verify that the function

$$f(t,y) = y^2 \sin(y-t)$$

and its partial derivative $\partial f/\partial y$ are continuous at all (t,y) close to $(t_0, y_0) = (0,0)$. Recall from the Calculus course that polynomials, the exponential function, logarithm, trigonometric functions and their inverses are continuous, (whenever well-defined; for example, $\ln x$ is continuous on the interval $(0, \infty)$). Moreover, sums, products, and compositions of continuous functions are continuous. Since the functions

$$y \mapsto y^2, \quad (y,t) \mapsto y - t, \quad t \mapsto \sin t$$

are continuous, so is f as it can be constructed by multiplying and composing the above functions. Similarly,

$$\frac{\partial f}{\partial y} = 2y \sin(y-t) + y^2 \cos(y-t)$$

is continuous for the same reasons. We conclude that f and $\partial f/\partial y$ are continuous on the entire ty–plane, in particular they are continuous around the point $(t_0, y_0) = (0,0)$ and the uniqueness and existence theorem applies in this case.

We easily verify that the constant function $y(t) = 0$ satisfies the given initial value problem. The general theorem guarantees that this solution is unique.

Problem 4. Find equilibria of the autonomous differential equation

$$y' = (1+y)(2-y)$$

Draw the direction field and integral curves. Determine whether the equilibria are asymptotically stable, asymptotically unstable, or neither of these.

Solution.
Let $f(y) = (1+y)(2-y)$. The equlibria are the points where $f(y) = 0$, that is: $y = -1$ and $y = 2$. The draw the direction field, it is helpful to see where f is positive and where it is negative. We see, for example by drawing the graph of f, that

$$\begin{cases} f(y) < 0 & \text{for } y < -1, \\ f(y) > 0 & \text{for } -1 < y < 2, \\ f(y) < 0 & \text{for } 2 < y. \end{cases}$$

Remember that the direction field at a point with coordinates (t, y) on the plane is the line whose slope is $f(y)$. Based on the above information about f we can sketch the direction field as in class. The integral curves are curves which are tangent to the direction field. During office hours and review session, we will discuss again how to draw the direction field and integral curves. From the picture we will see that:

- an integral curve starting at $y < -1$ goes to $y = -\infty$ as $t \to \infty$ and to $y \to -1$ as $t \to -\infty$,
- an integral curve starting at $-1 < y < 2$ goes to $y = 2$ as $t \to \infty$ and to $y = -1$ as $t \to -\infty$,
- an ntegral curve starting at $y > 2$ goes to $y = 2$ as $t \to \infty$ and to $y \to \infty$ as $t \to -\infty$.

We conclude that $y = -1$ is an asymptotically unstable equilibrium and $y = 2$ is an asymptotically stable equilibrium.

Problem 5. Explain:

1. what is an autonomous system of differential equations,
2. what it means for an autonomous system to be exact,
3. what it means for an autonomous system to be closed,
4. what is the relation between exact and closed systems.

(You can restrict yourself to two-dimensional systems, that is systems consisting of two differential equations for two unknown functions.)

Verify that the following autonomous system is closed and find an equation describing its integral curves:

$$\begin{cases} x' = -3xy^2, \\ y' = y^3 + 4x^3. \end{cases} \tag{0.4}$$

Solution. The definitions can be found in Lectures 7 and 8. A two-dimensional autonomous system is a system of differential equations of the form

$$\begin{cases} x' = f(x,y), \\ y' = g(x,y) \end{cases}$$

for two unknown functions $x = x(t)$ and $y = y(t)$. We say that such a system is exact if there is a function $H = H(x,y)$ such that

$$f(x,y) = -\partial H/\partial y \quad \text{and} \quad g(x,y) = \partial H/\partial x.$$

We say that an autonomous system is closed if

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0. \tag{0.5}$$

In Lecture 7 we proved a theorem which says that an autonomous system defined on a rectangle $[x_0, y_0] \times [x_1, y_1]$ in the xy–plane is exact if and only if it is closed.

To verify that the autonomous system (0.4) we compute that for $f(x,y) = -3xy^2$ and $g(x,y) = y^3 + 4x^3$ we have

$$\frac{\partial f}{\partial x} = -3y^2 \quad \text{and} \quad \frac{\partial g}{\partial y} = 3y^2.$$

Therefore, (0.5) holds and the system is closed. To find the integral curves, we look for a function $H = H(x,y)$ such that

$$\frac{\partial H}{\partial y} = -f(x,y) = 3xy^2.$$

Integrating with respect to y we get that for some function $\varphi = \varphi(x)$,

$$H(x,y) = \varphi(x) + \int 3xy^2 \mathrm{d}y = \varphi(x) + xy^3.$$

(We can incorporate the integration constant into the unknown function φ.) To find φ we use that:

$$\frac{\partial H}{\partial x} = g(x, y) = y^3 + 4x^3.$$

Therefore, computing the right-hand side from the formula for H:

$$\varphi'(x) = y^3 = y^3 + 4x^3 \implies \varphi'(x) = 4x^3.$$

Integrating the last equation with respect to x, we get

$$\varphi(x) = x^4 + C$$

for a constant C. The integral curves are described by $H(x, y) = 0$, that is:

$$x^4 + xy^3 + C = 0.$$

ORDINARY DIFFERENTIAL EQUATIONS (MATH 2030)

Midterm 1: second approach

26 February 2021

PROBLEMS AND SOLUTIONS

Problem 1. Solve the initial value problem

$$\begin{cases} y' = (1+y)^2 \\ y(0) = 0 \end{cases}$$

and determine the maximal interval of existence of the solution $y = y(t)$.

Solution. This is a separable equation. We separate y and t and integrate with respect to t:

$$\int y'(1+y)^{-2} \mathrm{d}t = \int 1 \mathrm{d}t = t + C.$$

for a constant C. We integrate the left-hand side is by substitution $y = y(t)$,

$$\int y'(1+y)^{-2} \mathrm{d}t = \int (1+y)^{-2} \mathrm{d}y = -(1+y)^{-2}$$

Therefore,

$$-(1+y)^{-1} = t + C$$

so solving for y we find

$$y(t) = -1 - \frac{1}{t+C}.$$

This is the general solution. To solve the initial value problem, we set $t = 0$:

$$0 = y(0) = -1 - C^{-1}$$

so $C = -1$ and the solution is

$$y(t) = -1 + \frac{1}{1-t}.$$

We see that the maximal interval containing $t = 0$ for which $y(t)$ is well-defined is $(-\infty, 1)$ as $y(t) \to \infty$ when t approaches 1 from the left.

Problem 2. Find the general solution to the differential equation

$$y' - 3y = e^t.$$

Solution. This is a first order, non-homogenous linear equation. We look for the integrating factor $\mu = \mu(t)$. The integrating factor has to have the property that the equation multiplied by μ:

$$\mu y' - 3\mu y = \mu e^t \tag{0.1}$$

can be written as

$$(\mu y)' = \mu e^t. \tag{0.2}$$

Since the derivative of the product is

$$(\mu y)' = \mu y' + \mu' y,$$

equations (0.1) and (0.2) are equivalent if μ satisfies the differential equation $\mu' = -3\mu$. We can take any function satisfying this equation, so we choose $\mu(t) = e^{-3t}$. (If this solution is not obvious to you, you can always find it by solving $\mu' = -3\mu$, since this is a separable equation.) It remains to solve (0.2). Integrating both sides with respect to t, we get

$$\mu(t)y(t) = \int \mu(t)e^t \mathrm{d}t = \int e^{-3t}e^t \mathrm{d}t = \int e^{-2t}\mathrm{d}t = -\frac{1}{2}e^{-2t} + C$$

for any constant C. Dividing by $\mu(t) = e^{-3t}$ we get a formula for the general solution

$$y(t) = -\frac{1}{2}e^t + Ce^{3t}.$$

Problem 3. State the existence and uniqueness theorem for first order differential equations. Consider two initial value problems for the same differential equation but with different initial values:

$$\begin{cases} y' = y^{1/2}(y+t) \\ y(1) = 0. \end{cases} \tag{0.3}$$

and

$$\begin{cases} y' = y^{1/2}(y+t) \\ y(1) = 5. \end{cases} \tag{0.4}$$

In which of these two cases can we apply the existence and uniqueness theorem to conclude that there exists a solution $y(t)$ defined for t in some interval containing the initial time $t = 1$, and that the solution is unique in that interval? In each case, explain we can or cannot apply the theorem.

Solution. For the statement of the existence and uniqueness theorem, see Theorem 4 in Lecture 4; or Theorem 2.8.1 in Boyce–DiPrima section 2.8; or Theorem 2′ in Braun section 1.10. The textbooks state the theorem in a more detailed way. You don't need to memorize the exact formulation of the theorem, but you should understand the general idea. The theorem concerns initial value problems of the form

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0, \end{cases}$$

where f is a given function of two variables and t_0, y_0 are given numbers. The theorem says that if the function f is continuous around the point (t_0, y_0) and its partial derivative $\partial f/\partial y$ exists and is continuous around (t_0, y_0), then the above initial value problem has a solution $y = y(t)$ defined for t in some interval containing t_0, and that this solution is unique for t in that interval.

In order to apply the theorem to the specific initial value problem (0.3), we need to check whether the function

$$f(t,y) = y^{1/2}(y+t)$$

and its partial derivative $\partial f/\partial y$ are continuous at all (t,y) close to (t_0, y_0). The function

$$(y,t) \mapsto y+t$$

is a polynomial so it is continues for all y and t. Moreover, its partial derivatives are also continuous. However, the function

$$y \mapsto y^{1/2}$$

is continuous for $y \in [0,\infty)$ but differentiable only for $y \in (0,\infty)$. Indeed, its derivative with respect to y is $(1/2)y^{-1/2}$ which is not defined at $y = 0$. Since the product of continuous functions is continuous, we see that $f(t,y)$ is continuous at those (t,y) with $y \geq 0$. Moreover, it is differentiable at those (t,y) with $y > 0$ (and similar argument shows that its partial derivative $\partial f/\partial y$ is continuous in that region). We conclude that the existence and uniqueness theorem applies to the initial value problem (0.4) since f and $\partial f/\partial y$ are continuous around $(t_0, y_0) = (1,5)$. However $\partial f/\partial y$ does not exist at $(t_0, y_0) = (1,0)$, so we cannot apply the existence and uniqueness theorem to the initial value problem (0.3).

Problem 4. Find equilibria of the autonomous differential equation

$$y' = y(y-2).$$

Draw the direction field and integral curves. Determine whether the equilibria are asymptotically stable, asymptotically unstable, or neither of these.

Solution.
Let $f(y) = y(y-2)$. The equlibria are the points where $f(y) = 0$, that is: $y = 0$ and $y = 2$. The draw the direction field, it is helpful to see where f is positive and where it is negative. We see, for example by drawing the graph of f, that

$$\begin{cases} f(y) > 0 & \text{for } y < 0, \\ f(y) < 0 & \text{for } 0 < y < 2, \\ f(y) > 0 & \text{for } 2 < y. \end{cases}$$

Remember that the direction field at a point with coordinates (t, y) on the plane is the line whose slope is $f(y)$. Based on the above information about f we can sketch the direction field as in class. The integral curves are curves which are tangent to the direction field. During office hours and review session, we will discuss again how to draw the direction field and integral curves. From the picture we will see that:

- an integral curve starting at $y < 0$ goes to $y = 0$ as $t \to \infty$ and to $y \to -\infty$ as $t \to -\infty$,
- an integral curve starting at $0 < y < 2$ goes to $y = 0$ as $t \to \infty$ and to $y = 2$ as $t \to -\infty$,
- an integral curve starting at $y > 2$ goes to $y \to \infty$ as $t \to \infty$ and to $y \to 2$ as $t \to -\infty$.

We conclude that $y = 0$ is an asymptotically stable equilibrium and $y = 2$ is an asymptotically unstable equilibrium.

Problem 5. Explain:

1. what is an autonomous system of differential equations,
2. what it means for an autonomous system to be exact,
3. what it means for an autonomous system to be closed,
4. what is the relation between exact and closed systems.

(You can restrict yourself to two-dimensional systems, that is systems consisting of two differential equations for two unknown functions.)

Verify that the following autonomous system is closed and find an equation describing its integral curves:

$$\begin{cases} x' = -xe^y, \\ y' = e^y + 3x^2. \end{cases} \tag{0.5}$$

Solution. The definitions can be found in Lectures 7 and 8. A two-dimensional autonomous system is a system of differential equations of the form

$$\begin{cases} x' = f(x,y), \\ y' = g(x,y) \end{cases}$$

for two unknown functions $x = x(t)$ and $y = y(t)$. We say that such a system is exact if there is a function $H = H(x,y)$ such that

$$f(x,y) = -\partial H/\partial y \quad \text{and} \quad g(x,y) = \partial H/\partial x.$$

We say that an autonomous system is closed if

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0. \tag{0.6}$$

In Lecture 7 we proved a theorem which says that an autonomous system defined on a rectangle $[x_0, y_0] \times [x_1, y_1]$ in the xy–plane is exact if and only if it is closed.

To verify that the autonomous system (0.5) we compute that for $f(x,y) = -xe^y$ and $g(x,y) = e^y + 3x^2$ we have

$$\frac{\partial f}{\partial x} = -e^y \quad \text{and} \quad \frac{\partial g}{\partial y} = e^y.$$

Therefore, (0.6) holds and the system is closed. To find the integral curves, we look for a function $H = H(x,y)$ such that

$$\frac{\partial H}{\partial y} = -f(x,y) = xe^y.$$

Integrating with respect to y we get that for some function $\varphi = \varphi(x)$,

$$H(x,y) = \varphi(x) + \int xe^y \mathrm{d}y = \varphi(x) + xe^y.$$

(We can incorporate the integration constant into the unknown function φ.) To find φ we use that:

$$\frac{\partial H}{\partial x} = g(x, y) = e^y + 3x^2.$$

Therefore, computing the right-hand side from the formula for H:

$$\varphi'(x) + e^y = e^y + 3x^2 \quad \Longrightarrow \quad \varphi'(x) = 3x^2.$$

Integrating the last equation with respect to x, we get

$$\varphi(x) = x^3 + C$$

for a constant C. Therefore,

$$H(x, y) = x^3 + xe^y + C.$$

The integral curves are described by $H(x, y) = 0$, that is:

$$x^3 + xe^y + C = 0.$$

Midterm 2 practice

Midterm 2 scheduled at 10:10–11:25 Thursday 18 March

At the midterm, you will be asked to have your camera on and microphone off. If you need to ask a question, send me a private message through the Zoom chat. During the midterm, you are allowed to use two pages of notes that you wrote yourself. You are not allowed to use textbooks, the Internet, or any other resources. The exam will last 60 minutes. In the remaining 15 minutes of class, upload your solutions together with the two pages of notes to the Gradescope. If you encounter technical difficulties with the Gradescope or anything else, send me a message through the Zoom chat.

REFERENCES

The midterm will be based on Lectures 9–14, Homework 4–6, and the problems from this practice midterm. It will cover the following topics:

1. Second order equations as systems of first order equations
2. Existence and uniqueness theorem for second order equations
3. Second order linear equations; linear combinations of solutions; systems of linear algebraic equations; Wronskian; Abel's formula
4. Homogenous equations with constant coefficients
5. Complex numbers
6. Non-homogenous equations with constant coefficients

In the lecture notes, you can find references to the relevant parts of the textbooks of Boyce–DiPrima and Braun.

The midterm problems will be very similar to the problems from this practice midterm and Homework 4–6 (you can ignore bonus problems).

Problem 1 (Existence and uniqueness)**.** State the existence and uniqueness theorem for second order differential equations. Which of the following initial value problems satisfy the conditions of the theorem and therefore admit a unique solution for t close to the initial time? In each case, justify your answer.

1. $y'' - (y')^2 + y = 0$ with initial condition $y(2) = 1$;
2. $y'' - (y')^2 + y = 0$ with initial conditions $y(2) = 1$, $y'(2) = 0$;
3. $y'' - (y')^{1/4} + y = 0$ with initial conditions $y(2) = 1$, $y'(2) = 0$.

Solution. The theorem is explained in Lecture 9.

Theorem (Existence and uniqueness for second order equations)**.** *Given numbers t_0, y_0, y_0' and a function f of three variables, consider the initial value problem*

$$\begin{cases} y'' = f(t, y, y') \\ y(t_0) = y_0, \\ y'(t_0) = y_0'. \end{cases}$$

If the functions f, $\partial f/\partial y$ and $\partial f/\partial y'$ (by this we mean the partial derivatives of f with respect to the second and third variable) are continuous in a neighborhood of the point (t_0, y_0, y_0'), then there exists a solution $y(t)$ to the initial value problem defined for t from some interval $(t_0 - \epsilon, t_0 + \epsilon)$ containing t_0. Moreover, the solution is unique in that interval.

In the first case, we do not specify the value of y' at $t_0 = 2$, so the theorem does not apply: in an initial value problem for second order equations we have prescribe both $y(t_0)$ and $y'(t_0)$ to get uniqueness.

In the second case, the initial value problem prescribes both $y(t_0)$ and $y'(t_0)$. Moreover, the function

$$f(t, y, y') = (y')^2 + y$$

is continuous because it is a polynomial in y' and y. Similarly, its partial derivatives are also polynomials in y' and y, so are continuous. We conclude that the theorem applies in this case.

Finally, in the third case, the initial value problem prescribes both $y(t_0)$ and $y'(t_0)$. The function

$$f(t, y, y') = (y')^{1/4} + y$$

is continuous for $y \in (-\infty, \infty)$ and $y' \in [0, \infty)$. However, its partial derivative with respect to y' is

$$\partial f/\partial y' = \frac{1}{4}(y')^{-3/4}$$

not defined at $y' = 0$. The existence and uniqueness theorem assumes that $\partial f/\partial y'$ is continuous at (t_0, y_0, y_0'), in this case $(2, 1, 0)$. However, in this case this is not true. Therefore, the theorem does not apply in this case.

Problem 2 (Wronskian)**.** Verify that the functions

$$y_1(t) = t^2 \quad \text{and} \quad y_2(t) = t^{-1}$$

are both solutions of the linear differential equation

$$t^2 y'' - 2y = 0.$$

Compute their Wronskian. What is the general solution of this differential equation? Justify your answer.

Solution. Let us verify that these functions solve the equation. We have

$$y_1'(t) = 2t \quad \text{and} \quad y_1''(t) = 2.$$

We compute

$$t^2 y_1'' - 2y_1 = 2t^2 - 2t^2 = 0,$$

so y_1 is a solution. Similarly,

$$y_2'(t) = -t^{-2} \quad \text{and} \quad y_2''(t) = 2t^{-3}.$$

We compute

$$t^2 y_2 - 2y_2 = 2t^{-1} - 2t^{-1} = 0,$$

so y_2 is a solution. The Wronskian is

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = t^2(-t^{-2}) - 2tt^{-1} = -3.$$

Since the Wronskian is non-zero, the general theorem about solutions to homogenous second order equations discussed in class tells us that any other solution is a linear combination of y_1 and y_2. That is, the general solution is

$$y(t) = C_1 t^2 + C_2 t^{-1}.$$

Problem 3 (Complex numbers)**.** Write the following complex numbers in the form $a + bi$, with a and b real:

1. $(2+i)^{-1}$,
2. $(5+i)(1-i)$,
3. $e^{\ln 2 + \pi i/4}$,
4. $\overline{1+3i}$,
5. i^5.

Solution

$$(2+i)^{-1} = \frac{1}{2+i} = \frac{2-i}{(2+i)(2-i)} = \frac{2-i}{4-i^2} = \frac{2}{5} - \frac{i}{5}.$$

$$(5+i)(1-i) = 5 + i - 5i - i^2 = 6 - 4i.$$

$$e^{\ln 2 + \pi i/4} = e^{\ln 2} e^{\pi i/4} = 2(\cos \pi/4 + i \sin \pi/4) = \frac{2}{\sqrt{2}} + i\frac{2}{\sqrt{2}}.$$

$$\overline{1+3i} = 1 - 3i.$$

$$i^5 = i \cdot i^2 \cdot i^2 = i \cdot (-1) \cdot (-1) = i.$$

Problem 4 (Homogenous equations with constant coefficients)**.** Find the general solution of each of the following differential equations

1. $y'' + y' + y = 0$,
2. $y'' + y' - 2y = 0$,
3. $y'' + 2y' + y = 0$.

Solution. These are second order homogenous equations with constant coefficients. In all cases, the first step is to find roots of the characteristic polynomial.

(1) In the first case, the characteristic polynomial is

$$\chi(\lambda) = \lambda^2 + \lambda + 1.$$

The roots are complex

$$\lambda_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \quad \text{and} \quad \lambda_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

The complex solutions are $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$. The general real solution is a linear combination of the real and imaginary part of one of the complex solutions. We have

$$\begin{aligned} \operatorname{Re}(e^{\lambda_1 t}) &= e^{-t/2}\cos(\sqrt{3}t/2) \\ \operatorname{Re}(e^{\lambda_1 t}) &= e^{-t/2}\sin(\sqrt{3}t/2) \end{aligned}$$

so the general solution is

$$y(t) = C_1 e^{-t/2}\cos(\sqrt{3}t/2) + C_2 e^{-t/2}\sin(\sqrt{3}t/2).$$

(2) In the second case, the roots of the characteristic polynomial

$$\chi(\lambda) = \lambda^2 + \lambda - 2$$

are

$$\lambda_1 = -2 \quad \text{and} \quad \lambda_1 = 1.$$

The roots are distinct and real. Therefore, the general solution is

$$y(t) = C_1 e^{-2t} + C_2 e^{t}.$$

(3) In the third case, the characteristic polynomial

$$\chi(\lambda) = \lambda^2 + 2\lambda + 1$$

has a repeated root $\lambda_1 = -1$. Therefore, the general solution is

$$y(t) = C_1 e^{-t} + C_2 t e^{-t}.$$

Problem 5 (Non-homogenous equation with constant coefficients)**.** Find the general solution of the following differential equation

$$y'' + 2y' = 6e^{-3t}.$$

Solution. First we find a particular solution. Following the general rule, we look for a solution of the form

$$y(t) = ae^{-3t}.$$

We have

$$y'(t) = -3ae^{-3t},$$
$$y''(t) = 9ae^{-3t}.$$

Plugging this to the left-hand side of the equation we get

$$y'' + 2y' = 9ae^{-3t} - 6ae^{-3t} = 3ae^{-3t}.$$

In order for this to be equal to $6e^{-3t}$, we need $3a = 6$, so $a = 2$. We have found a particular solution

$$y(t) = 2e^{-3t}.$$

The general solution is the sum of this particular solution and the general solution to the homogenous equation

$$y'' + 2y' = 0.$$

The roots of the characteristic polynomial $\chi(\lambda) = \lambda^2 + 2\lambda$ of this equation are $\lambda_1 = 0$ and $\lambda_2 = -2$. Therefore, the general solution of the homogenous equation is

$$C_1 + C_2e^{-2t}.$$

We conclude that the general solution to the nonhomogenous equation is

$$C_1 + C_2e^{-2t} + 2e^{-3t}.$$

Alternative versions.

$$y'' + y' = (A + Bt)e^t$$

Look for a particular solution of the form $y(t) = (a + bt)e^t$.

$$y'' + 2y' + y = A + Bt + Ct^2$$

Look for a particular solution of the form $y(t) = a + bt + ct^2$.

$$y'' + 2y' = (A + Bt)\sin(3t)$$

Find a particular solution to the complex equation $y'' + 2y' = (A + Bt)e^{3it}$ of the form $y(t) = (a + bt)e^{3it}$ with a and b complex. Then the imaginary part is a particular solution to the original equation.

Midterm 2 solutions

Problem 1. State the existence and uniqueness theorem for second order differential equations. Which of the following initial value problems satisfy the conditions of the theorem and therefore admit a unique solution for t close to the initial time? In each case, justify your answer.

1. $y'' - (\sin t)y^3 + \cos t = 0$ with initial condition $y(0) = 1$, $y'(0) = 0$.
2. $y'' + 2ty' - 3y^4 = 0$ with initial condition $y(0) = 1$.

Solution.

Theorem (Existence and uniqueness for second order equations)**.** *Given numbers t_0, y_0, y_0' and a function f of three variables, consider the initial value problem*

$$\begin{cases} y'' = f(t, y, y') \\ y(t_0) = y_0, \\ y'(t_0) = y_0'. \end{cases}$$

If the functions f, $\partial f/\partial y$ and $\partial f/\partial y'$ (by this we mean the partial derivatives of f with respect to the second and third variable) are continuous in a neighborhood of the point (t_0, y_0, y_0'), then there exists a solution $y(t)$ to the initial value problem defined for t from some interval $(t_0 - \epsilon, t_0 + \epsilon)$ containing t_0. Moreover, the solution is unique in that interval.

In the second case, we do not specify the value of y' at $t_0 = 0$, so the theorem does not apply: in an initial value problem for second order equations we have prescribe both $y(t_0)$ and $y'(t_0)$ to get uniqueness. In the first case, the initial value problem prescribes both $y(t_0)$ and $y'(t_0)$. Moreover, the function

$$f(t, y, y') = (\sin t)y^3 + \cos t$$

is composed of sums of products of the polynomial y^3 and trigonometric functions $\sin t$, $\cos t$. From calculus we know that these functions are everywhere continuous, therefore so is f. The same is true for the partial derivatives

$$\partial f/\partial y = 3(\sin t)y^2 \quad \text{and} \quad \partial f/\partial y' = 0.$$

We conclude that the theorem applies in this case. (Alternatively, we can argue that sums and products of functions which are continuous and have continuous derivatives are continuous and have continuous derivatives. If we use this fact, we don't have to compute partial derivatives since we know that polynomials and $\sin t$, $\cos t$ have continuous derivatives.)

Problem 2. Verify that the functions

$$y_1(t) = t^{1/2} \quad \text{and} \quad y_2(t) = t^{-1}$$

are both solutions of the linear differential equation

$$2t^2y'' + 3ty' - y = 0.$$

Compute their Wronskian for $t > 0$. What is the general solution of this differential equation defined for $t > 0$? Justify your answer.

Solution. Let us verify that these functions solve the equation. We have

$$y_1'(t) = \frac{1}{2}t^{-1/2} \quad \text{and} \quad y_1''(t) = -\frac{1}{4}t^{-3/2}.$$

We compute

$$2t^2y_1'' + 3ty_1' - y_1 = -\frac{1}{2}t^{1/2} + \frac{3}{2}t^{1/2} - t^{1/2} = 0,$$

so y_1 is a solution. Similarly,

$$y_2'(t) = -t^{-2} \quad \text{and} \quad y_2''(t) = 2t^{-3}.$$

We compute

$$2t^2y_2'' + 3ty_2' - y_2 = 4t^{-1} - 3t^{-1} - t^{-1} = 0.$$

so y_2 is a solution. The Wronskian is

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = -t^{1/2}t^{-2} - \frac{1}{2}t^{-1/2}t^{-1} = -\frac{3}{2}t^{-3/2}.$$

Since the Wronskian is non-zero for $t > 0$, the general theorem about solutions to homogenous second order equations discussed in class tells us that any other solution defined for $t > 0$ is a linear combination of y_1 and y_2. That is, the general solution for $t > 0$ is

$$y(t) = C_1t^{1/2} + C_2t^{-1}.$$

Problem 3. Write the following complex numbers in the form $a + bi$, with a and b real:

1. $\overline{(1+3i)(2+i)}$,
2. $(1-i)^{-1}$,
3. $e^{i(1+i)}$,
4. i^4.

Solution

$$\overline{(1+3i)(2+i)} = \overline{2+i+6i-3} = \overline{-1+7i} = -1-7i.$$

$$\frac{1}{1-i} = \frac{1+i}{(1-i)(1+i)} = \frac{1+i}{1-i^2} = \frac{1+i}{2} = \frac{1}{2} + \frac{i}{2}.$$

$$e^{i(1+i)} = e^{-1+i} = e^{-1}(\cos 1 + i \sin 1).$$

$$i^4 = i^2 i^2 = (-1)(-1) = 1.$$

Problem 4. Find the general solution of each of the following differential equations

1. $y'' + 4y' + 5y = 0$,
2. $y'' - 10y' + 25y = 0$.

Solution. In the first case, the characteristic polynomial is

$$\chi(\lambda) = \lambda^2 + 4\lambda + 5.$$

There are two complex roots

$$\lambda_1 = -2 + i \quad \text{and} \quad \lambda_2 = -2 - i.$$

We have complex solutions $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$. The real and imaginary part (of, say, the first one) are

$$\begin{aligned} \mathrm{Re}(e^{\lambda_1 t}) &= e^{-2t}\cos t, \\ \mathrm{Im}(e^{\lambda_1 t}) &= e^{-2t}\sin t. \end{aligned}$$

The general real solution is their linear combination

$$y(t) = C_1 e^{-2t}\cos t + C_2 e^{-2t}\sin t.$$

In the second case, the characteristic polynomial is

$$\chi(\lambda) = \lambda^2 - 10\lambda + 25.$$

There is one repeated root

$$\lambda_1 = 5.$$

Therefore, the general solution is

$$y(t) = C_1 e^{5t} + C_2 t e^{5t}.$$

Problem 5. Find a particular solution of the differential equation

$$y'' + y' + y = 1 + t + t^2.$$

Without doing computations, explain in one sentence how, after finding a particular solution, you would find the general solution of this equation.

Solution. Since the right-hand is a polynomial of degree two, we are looking for a particular solution which is also a polynomial of degree two, that is

$$y(t) = a + bt + ct^2.$$

We have

$$y'(t) = b + 2ct \quad \text{and} \quad y''(t) = 2c.$$

Therefore

$$y'' + y' + y = (2c + b + a) + (2c + b)t + ct^2.$$

We want this to be equal to

$$1 + t + t^2.$$

Comparing coefficients next to each power of t, we find that

$$\begin{cases} 2c + b + a = 1, \\ 2c + b = 1, \\ c = 1. \end{cases}$$

Therefore, $a = 0$, $b = -1$, and $c = 1$. The function

$$y(t) = -t + t^2.$$

Is a particular solution.

To find the general solution, we first find the general solution of the homogenous equation $y'' + y' + y = 0$ using the method from the previous problem. The general solution of the non-homogenous equation is then the sum of the general solution of the homogenous equation and a particular solution of the non-homogenous equation.

Midterms 1 and 2: second approach

30 April 2021

Problem 1. Solve the initial value problem

$$\begin{cases} y' = (1+y)^2 \\ y(0) = 0 \end{cases}$$

and determine the maximal interval of existence of the solution $y = y(t)$.

Problem 2. Find the general solution to the differential equation

$$y' - 3y = e^t.$$

Problem 3. State the existence and uniqueness theorem for first order differential equations. Consider two initial value problems for the same differential equation but with different initial values:

$$\begin{cases} y' = y^{1/2}(y+t) \\ y(1) = 0. \end{cases}$$

and

$$\begin{cases} y' = y^{1/2}(y+t) \\ y(1) = 5. \end{cases}$$

In which of these two cases can we use the existence and uniqueness theorem to conclude that there exists a solution $y(t)$ defined for t in some interval containing the initial time $t = 1$, and that the solution is unique in that interval? In each case, explain why we can or cannot apply the theorem.

Problem 4. Find equilibria of the autonomous differential equation

$$y' = y(y-2).$$

Draw the direction field and integral curves. Determine whether the equilibria are asymptotically stable, asymptotically unstable, or neither of these.

Problem 5. Explain:

1. what is an autonomous system of differential equations,
2. what it means for an autonomous system to be exact,
3. what it means for an autonomous system to be closed,
4. what is the relation between exact and closed systems.

(You can restrict yourself to two-dimensional systems, that is systems consisting of two differential equations for two unknown functions.)

Verify that the following autonomous system is closed and find an equation describing its integral curves:

$$\begin{cases} x' = -xe^y, \\ y' = e^y + 3x^2. \end{cases}$$

Problem 6. Find the general solution of the following differential equations

$$y'' - 4y' + 13y = 0,$$

and

$$y'' - 10y' + 25y = 0.$$

Problem 7. Find the general solution of the differential equation

$$y'' - y - 12y = 8e^{5x}.$$

Problem 8. Compute the Wronskian $W[y_1, y_2]$ for the following pairs of functions:

- $y_1(t) = e^{at}$, $y_2(t) = e^{bt}$,
- $y_1(t) = \sin(at)$, $y_2(t) = \cos(at)$,
- $y_1(t) = e^{at}$, $y_2(t) = te^{at}$.

Here a and b are constant. Why is this computation relevant to solving second order differential equations?

ORDINARY DIFFERENTIAL EQUATIONS (MATH 2030)

Practice for the final exam

Exam date: Thu 04/22/21 9:00am-12:00pm

TOPICS

The exam will test your knowledge of the entire course. List of topics:

- Examples of differential equations
  1. Harmonic oscillator
  2. Damped and forced vibrations
  3. Radioactive decay
  4. Population growth
  5. Logistic model
- First order equations
  1. Existence and uniqueness theorems for first order equations, maximal interval of existence
  2. Separable equations
  3. First order linear equations
  4. Direction fields, vector fields, integral curves
  5. Autonomous differential equations and equilibria
  6. Closed and exact autonomous systems
- Second and higher order equations
  1. Higher order equations as systems of first order equations
  2. Existence and uniqueness theorem for second order equations
  3. Second order linear equations, Wronskian, Abel's formula
  4. Complex numbers and finding roots of polynomials
  5. Homogenous and nonhomogenous second order linear equations with constant coefficients
  6. Higher order linear equations with constant coefficients
- Linear algebra and systems of differential equations
  1. Existence and uniqueness theorem for systems of first order differential equations
  2. Vectors and matrices, systems of linear algebraic equations
  3. Vector spaces, linear dependence, bases

4. Determinants, eigenvectors and eigenvalues
5. Linear systems of differential equations with constant coefficients

- Nonlinear equations and stability

1. Equilibria of autonomous systems, stability, long-time behavior
2. Equilibria of linear autonomous systems
3. Linearization around equilibrium
4. Predator-prey equations

PRACTICE PROBLEMS

The problems on the final exam will be very similar to the ones from the following list (for the topics discussed before Midterm 2) and the ones listed in this document (for the topics discussed after Midterm 2).

- Homework 1: 3, 4, 5.1, 6
- Homework 2: 1, 2, 6
- Homework 3: 3, 4
- Midterm 1 practice

  [Instead of alternative versions see: Examples 1-5 from Section 3.5 in Boyce-DiPrima and/or Examples 1-6 in Section 2.5 of Braun.]
- Midterm 1
- Midterm 1 retake
- Homework 4: 1, 2, 3, 4, 5
- Homework 5: 1, 2, 3, 4, 5
- Homework 6: 1, 2, 3, 4, 5
- Midterm 2 practice
- Midterm 2
- Homework 7: 1, 2, 3, 4, 5, 6, 7
- Homework 8: 1, 2, 3, 4, 5, 6
- Homework 9: 1, 2, 3

Problem 1 (Matrices and vectors)**.** Let

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 1 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 4 & -2 & 3 \\ -1 & 5 & 0 \\ 6 & 1 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

Compute $2\mathbf{A} + \mathbf{B}$, $\mathbf{AB}$, $\mathbf{BA}$, and $\mathbf{Ax}$.

Solution. This is a straightforward calculation. For rules of multiplying matrices and multiplying vectors by matrices see section 7.2 in Boyce–DiPrima.

Problem 2 (Vector spaces)**.** Which of the following sets form a vector space with the standard operations of addition and multiplication by scalars? In each case, justify your answer:

1. the set of solutions of the differential equation $y'' = y' + y$,
2. the set of solutions of the differential equation $y' = y^2$,
3. the set of vectors $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ satisfying $x_1^2 - x_2^2 = 0$,
4. the set of polynomials of degree at most 3.

Solution. (1). The equation $y'' - y' - y = 0$ is a linear homogenous differential equation. We proved in the lecture that the set of solutions to such equations is a vector space (that is: a linear combination of solutions is also a solution).

(2) The equation $y' = y^2$ is nonlinear. To see that the set of solutions is not a vector space, we can solve the equation since it is separable. The general solution is $y(t) = 1/(c-t)$ or $y(t) = 0$. For example, $y(t) = -1/t$ is a solution. But $2y(t) = -2/t$ is not a solution, so we have found an element such that when we multiply it by 2 we don't get an element of the same set. (We could alternatively observe that in general the sum of two solutions is not a solution.)

(3) This set is not a vector space. Interestingly, the set does contain the zero vector and for every vector $\mathbf{x}$ in the set, its scalar multiple $\lambda\mathbf{x}$ is also in the set. However, in general the sum of two vectors in the set is not in the set. For example, vectors $\mathbf{x} = (1,1)$ and $\mathbf{y} = (1,-1)$ are in this set but their sum $\mathbf{x} + \mathbf{y} = (2,0)$ is not.

(4) This is a vector space because: the zero polynomial is in this set, the sum of two polynomials of degree at most 3 is also a polynomial of degree at most 3, and a multiple of a polynomial of degree at most 3 is a polynomial of the same degree.

Problem 3 (Linear transformations)**.** Let V be the vector space of polynomials of degree at most 3. Define a map $L\colon V \to V$ by

$$L(f) = \frac{\mathrm{d}f}{\mathrm{d}x} - f$$

for any polynomial $f(x)$ of degree at most 3. Verify that L is a linear transformation and find its matrix with respect to the basis of V given by the polynomials

$$1,\ x,\ x^2,\ x^3.$$

Solution The map $f \mapsto \frac{\mathrm{d}f}{\mathrm{d}x}$ is a linear transformation from V to V because: the derivative of a polynomial of degree $k \leq 3$ is a polynomial of degree $k-1 \leq 3$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}(0) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}x}(f+g) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x}, \quad \frac{\mathrm{d}(\lambda f)}{\mathrm{d}x} = \lambda \frac{\mathrm{d}f}{\mathrm{d}x}.$$

Similarly, the map $f \mapsto -f$ is a linear transformation from V to V because for any vector space multiplication by a scalar (in this case, -1) is a linear transformation. Since L is the sum of these two transformations, it is a linear transformation. To find the matrix of L, we compute

$$L(1) = -1, \quad L(x) = 1 - x, \quad L(x^2) = 2x - x^2, \quad L(x^3) = 3x^2 - x^3$$

so in the basis given by $1, x, x^2, x^3$, L is given by the matrix

$$L = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Problem 4 (Linear dependence)**.** Determine if the following collections of vectors are linearly independent:

1. polynomials $f(x) = 1 - x$, $g(x) = x^2$, $h(x) = -x^2 + 3x - 3$ in the vector space of polynomials of degree at most 4,
2. the following vectors in $\mathbb{R}^2$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -7 \\ 3 \end{bmatrix},$$

3. the following vectors in $\mathbb{R}^3$:

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}.$$

Solution. (1) Observe that

$$h(x) = -g(x) - 3f(x),$$

so these three polynomials are linearly dependent.

(2) $\mathbb{R}^2$ is a vector space of dimension two, so any collection of three vectors in $\mathbb{R}^2$ is linearly dependent.

(3) We will show that these vectors are linearly independent. Suppose that there are numbers $\lambda_1, \lambda_2, \lambda_3$ such that

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0.$$

That means that

$$\begin{aligned} \lambda_1 + 2\lambda_2 &= 0 \\ -\lambda_1 - 3\lambda_3 &= 0 \\ -\lambda_2 &= 0. \end{aligned}$$

From these equations we immediately see that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. That means that there is no nontrivial linear combination of v_1, v_2, v_3 which is zero, i.e. these vectors are linearly independent.

Problem 5 (Eigenvectors)**.** Find eigenvalues and eigenvectors of the matrices

$$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution. See Examples 4 and 5 in section 7.3 of Boyce–DiPrima.

Problem 6 (Homogenous linear systems with constant coefficients)**.** Find the general solutions of the following systems of differential equations

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{y}$$

$$\mathbf{y}' = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} \mathbf{y}$$

$$\mathbf{y}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{y}.$$

Solution. See Example 3 in section 7.5, Example 2 in section 7.6, and Example 2 in section 7.8 of Boyce–DiPrima; we will also discuss these problems in class.

SAMPLE FINAL EXAMS

Here are some examples of what the final exam might look like. At the actual exam, the problems will not be exactly the same as the ones from homework and midterms, but they will be similar.

Example 1

1. Problem 5.1 from Homework 1
2. Problem 2 from Practice Midterm 1
3. Problem 4 from Homework 2
4. Problem 4 from Midterm 2
5. Problem 2 from Homework 6
6. Problem 7 parts 1–2 from Homework 7
7. Problem 6 from Homework 8
8. Problem 3 from Homework 9

Example 2

1. Problem 2 from Homework 2
2. Problem 4 from Midterm 1
3. Problem 3 from Homework 3
4. Problem 4 from Homework 5
5. Problem 3 from this file
6. Problem 4 from this file
7. Problem 6 from Homeowkr 8
8. Problem 1 from Homework 9

Example 3

1. Problem 6 from Homework 1
2. Problem 2 from Midterm 1
3. Problem 3 from Midterm 1
4. Problem 2 from Midterm 2 practice
5. Problem 7 parts 3–5 from Homework 7
6. Problem 2 from Homework 8
7. Problem 6 from this file
8. Problem 2 from Homework 9

Final exam

Thu 04/22/21 9:00am-12:00pm

Problem 1 (10 points). For every y_0, solve the initial value problem

$$\begin{cases} y' = y(1-y), \\ y(0) = y_0. \end{cases}$$

What is the maximal interval of existence of the solution, depending on y_0? (Hint: consider separately the cases: $y_0 < 0$, $0 \le y_0 \le 1$, and $y_0 > 1$.)

Find equilibria of this equation, sketch integral curves, and determine whether each equilibrium is stable or unstable.

Solution. This is a separable equation. (In fact, this is the logistic model which we discussed in class.) We separate y and t:

$$\frac{y'}{y(1-y)} = 1$$

and integrate with respect to t

$$\int \frac{y'}{y(1-y)} \mathrm{d}t = t + C.$$

By substitution $y = y(t)$ the integral on the left-hand side is

$$\begin{aligned} \int \frac{y'}{y(1-y)} \mathrm{d}t = \int \frac{1}{y(1-y)} \mathrm{d}y = \int \left(\frac{1}{y} + \frac{1}{1-y} \right) \mathrm{d}y \\ = \ln|y| - \ln|1-y| + C = \ln\left| \frac{y}{1-y} \right| + C. \end{aligned}$$

We get

$$\ln\left| \frac{y}{1-y} \right| = t + C.$$

Taking the exponential of both sides,

$$\left| \frac{y}{1-y} \right| = e^C e^t.$$

Here e^C is any positive constant. If we drop the absolute value, we get that

$$\frac{y}{1-y} = Ce^t$$

for constant C of any sign (as usual, this is a different C than before, as we name any constant expression C). ($C = 0$ is also allowed because the constant function $y(t) = 0$ is a solution.) Solving for y we get, again for a different constant C,

$$y(t) = \frac{e^t}{e^t + C}.$$

This is the general solution of the equation. Plugging $y_0 = y(0)$ we get that

$$C = \frac{1}{y_0} - 1.$$

This gives us the solution of the initial value problem.

We see that the solution goes to infinity as the denominator goes to zero, that is when $e^t = -C$. The maximal interval of existence is the interval which contains 0 and has as one of the endpoints the solution t to the equation $e^t = -C$. When $y_0 < 0$, then $C < -1$, so $-C > 1$ and $t = \ln(-C)$ is positive. So the interval $(-\infty, \ln(-C))$ contains 0 and this is the maximal interval of existence. When $0 \leq y_0 \leq 1$, then $C \leq 0$ and the equation $e^t = -C$ has no solution, so the denominator is never zero and the maximal interval of existence is $(-\infty, \infty)$. When $y_0 > 1$, then $0 < -C < 1$ so $t = \ln(-C)$ is negative and the interval $(\ln(-C), \infty)$ contains 0, so this is the maximal interval of existence. We see that solutions with $y_0 < 0$ have infinite past but go to $-\infty$ in finite future, solutions with $0 \leq y_0 \leq 1$ have infinite past and infinite future, and solutions with $y_0 > 1$ have infinite future but goto ∞ in finite past.

There are two equilibria $y = 0$ and $y = 1$. We sketch the integral curves as usual, see, for example, Midterm 1. We find that $y = 0$ is unstable and $y = 1$ is stable.

Problem 2 (5 points)**.** Describe the general method of solving first order linear equations

$$y' + a(t)y = b(t).$$

Solution. This is the method from Lecture 3.

1. Multiply the equation $y' + a(t)y = b(t)$ by a function $\mu(t)$.
2. Observe that this equation is equivalent to $(\mu y)' = \mu(t)b(t)$ if μ satisfies the homogenous equation $\mu' = a(t)\mu$.
3. Solve the homogenous equation $\mu' = a(t)\mu$. This is a separable equation and has solution $\mu(t) = \exp \int a(t)\mathrm{d}t$.
4. Find μy by integrating both sides of $(\mu y)' = \int \mu(t)b(t)\mathrm{d}t$.
5. Finally, divide by $\mu(t)$ to get a formula for y:

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)b(t)\mathrm{d}t \quad \text{with } \mu(t) = \exp \int a(t)\mathrm{d}t.$$

Problem 3 (5 points). State the existence and uniqueness theorem for first order differential equations. Write two examples of initial value problems: one which satisfies the assumptions of the theorem and one which doesn't. In each case, explain why the assumptions are or are not satisfied. You can use examples from lectures and homework, or you can come up with your own examples.

Solution. The existence and uniqueness theorem was stated in Lecture 4. It asserts that the initial value problem

$$\begin{cases} y' = f(t,y), \\ y(t_0) = y_0. \end{cases} \tag{0.1}$$

has a unique solution defined for t from the interval $(t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$, provided that f and $\partial_y f$ are both continuous in a neighborhood of (t_0, y_0). Here are two simple examples. Consider the initial value problems

$$\begin{cases} y' = y, \\ y(0) = 0. \end{cases}$$

and

$$\begin{cases} y' = y^{1/2}, \\ y(0) = 0. \end{cases}$$

In the first case, the function $f(t,y) = y$ is a polynomial so it is continuous and has continuous partial derivative in y at $(t_0, y_0) = (0,0)$, so it satisfies the assumption of the theorem. (In fact, in this case we know that the solution is $y(t) = e^t$.) In the second case, the function $f(t,y) = y^{1/2}$ is continuous but it does not have a partial derivative in y at $(t_0, y_0) = (0,0)$, so the theorem does not apply. (In fact, we proved in class that there are two solutions of this initial value problem, so the uniqueness statement fails.)

Problem 4 (5 points). Find the general solution of the differential equation

$$y'' + y' - 6y = 4e^t.$$

Solution. This is a second order nonhomogenous linear equation with constant coefficients. To find the general solution, we first find the general solution of the homogenous equation

$$y'' + y' - 6y = 0$$

and then find a particular solution of the nonhomogenous equation. The characteristic polynomial $\lambda^2 + \lambda - 6$ has roots -3 and 2 so the general solution is

$$Ae^{-3t} + Be^{2t}$$

for any constants A and B. To find a particular solution of the nonhomogenous equation, we look for a solution of the form Ce^t. Easy calculation shows that for $C = -1$ we get a solution. So the general solution of the nonhomogenous equation is

$$y(t) = Ae^{-3t} + Be^{2t} - e^t.$$

Problem 5 (10 points)**.** Let V be the set of solutions of the equation

$$y'' + ay' + by = 0,$$

with a, b constant.

1. Show that with the standard operations of addition and multiplication of functions by numbers, V is a vector space. Based on what we learned in class, what is the dimension of this vector space?

2. Show that if y is a solution to the equation, then so is y' (hint: differentiate both sides of the equation) and that the map $L\colon V \to V$ defined by $L(y) = y'$ is a linear transformation.

3. Show that the eigenvectors of L are solutions of the form $e^{\lambda t}$ where λ is a root of the characteristic polynomial

$$\lambda^2 + a\lambda + b = 0.$$

Solution. Suppose that y_1 and y_2 are solutions of the equation. To show that V is a vector space, we need to show that for any constant λ_1, λ_2 the linear combination $y = \lambda_1 y_1 + \lambda_2 y_2$ is also a solution. We compute

$$\begin{aligned} y'' + ay' + by &= y_1'' + y_2'' + ay_1' + ay_2' + by_1 + by_2 \\ &= (y_1'' + ay_1' + by_1) + (y_2'' + ay_2' + by_2) = 0. \end{aligned}$$

so y is a solution if y_1 and y_2 are.

To prove the second part, suppose that y' is a solution. Differentiating the equation

$$y'' + ay' + by = 0$$

we get

$$y''' + ay'' + by' = 0$$

which can be written as:

$$(y')'' + a(y')' + b(y') = 0$$

which shows that y' is also a solutions of the same differential equation. This gives us a map $L\colon V \to V$. To show that the map is linear, we need to show that for two elements $y_1, y_2 \in V$ and two constants λ_1, λ_2 we have $L(\lambda_1 y_1 + \lambda_2 y_2) = \lambda_1 L(y_1) + \lambda_2 L(y_2)$. We compute

$$\begin{aligned} L(\lambda_1 y_1 + \lambda_2 y_2) &= (\lambda_1 y_1 + \lambda_2 y_2)' = (\lambda_1 y_1)' + (\lambda_2 y_2)' \\ &= \lambda_1 y_1' + \lambda_2 y_2' = \lambda_1 L(y_1) + \lambda_2 L(y_2), \end{aligned}$$

so L is a linear transformation.

An eigenvector of L is an element y of V such that $L(y) = \lambda y$ for some λ. This means that $y' = \lambda y$ for some λ. But this implies that y is a function of the form $y(t) = Ce^{\lambda t}$. Since y has to be an element of V, it also also has to solve the differential equation:

$$y'' + ay' + by = 0$$

Plugging $y(t) = Ce^{\lambda t}$ to this equation we get that λ has to satisfy the characteristic equation $\lambda^2 + a\lambda + b = 0$.

Problem 6. (5 points) Find the general solution of the linear system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

in two cases: when $a = 0$ and $a > 0$. In both cases, sketch the integral curves of the system and determine whether solutions are stable or unstable.

Use your solution to the first part of the problem to determine if the equilibrium $x = 0, y = 0$ of the following nonlinear system is stable or unstable:

$$\begin{cases} x' = x + 2y + xy - x^2 - xy^3, \\ y' = -2x + y - 2xy - y^3 + x^2 y. \end{cases}$$

Solution. When $a = 0$, there is one eigenvalue $\lambda = 1$ and any vector is an eigenvector. The general solution is

$$y(t) = e^t \begin{bmatrix} A \\ B \end{bmatrix}$$

for any constants A, B. For $a > 0$, the characteristic polynomial of the matrix,

$$\chi(\lambda) = (1 - \lambda)^2 + a^2$$

has two complex roots $1 \pm ia$. To find a complex eigenvector with eigenvalue $1 + ia$, we solve the equation

$$\begin{bmatrix} -ia & a \\ -a & -ia \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We find that $y = ix$, so for example we can take $x = 1$ and $y = i$ to get the eigenvector

$$\begin{bmatrix} 1 \\ i \end{bmatrix}.$$

This gives us a complex solution

$$y_c(t) = e^{(1+ia)t} \begin{bmatrix} 1 \\ i \end{bmatrix} = e^t(\cos(at) + i\sin(at)) \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

We compute its real and imaginary part to get real solutions

$$\mathrm{Re} y_c(t) = e^t \begin{bmatrix} \cos(at) \\ -\sin(at) \end{bmatrix}, \quad \mathrm{Im} y_c(t) = e^t \begin{bmatrix} \sin(at) \\ \cos(at) \end{bmatrix}.$$

The general solution is a linear combination of those two

$$y(t) = Ae^t \begin{bmatrix} \cos(at) \\ -\sin(at) \end{bmatrix} + Be^t \begin{bmatrix} \sin(at) \\ \cos(at) \end{bmatrix}.$$

In both cases, there is an eigenvalue with positive real part, so according to the theorem proved in class the solutions are unstable. We draw the integral curves as in the lecture: for $a = 0$ they are straight lines going from zero to infinity, and for $a > 0$ they are spirals going away from zero.

To investigate the stability of $x = 0$, $y = 0$ of the nonlinear system, observe that we can write it as

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix}$$

where the functions g_1 and g_2 satisfy

$$|g_1(x,y)| \leq c(x^2 + y^2), \quad |g_2(x,y)| \leq c(x^2 + y^2)$$

for some constants c and all x, y close to 0. Therefore, the linear system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is the linearization of the nonlinear system at the equilibrium $x = 0, y = 0$. Since the solutions of the linearization are unstable, we conclude from the theorem discussed in Lecture 24 that the equilibrium $x = 0, y = 0$ of the nonlinear system is also unstable.

ORDINARY DIFFERENTIAL EQUATIONS (MATH 2030)

Final exam: version 2

Monday 05/03/2021 10am–2:30pm

Problems 1 and 5 are worth 10 points. Problems 2, 3, 4, 6 are worth 5 points.

Problem 1 (10 points). For every y_0, solve the initial value problem

$$\begin{cases} y' = (1-y)^2, \\ y(0) = y_0. \end{cases}$$

What is the maximal interval of existence of the solution, depending on y_0?

Find equilibria of this equation, sketch integral curves, and determine whether each equilibrium is stable or unstable.

Problem 2 (5 points). Describe the general method of solving first order linear equations

$$y' + a(t)y = b(t).$$

Problem 3 (5 points). State the existence and uniqueness theorem for second order differential equations. Does the theorem apply to the following initial value problem? Justify your answer.

$$\begin{cases} y'' = -y^2 + yy' - (y')^2 \\ y(0) = 1, \\ y'(0) = 0. \end{cases}$$

Problem 4 (5 points). Find the general solution of the following differential equations

$$y'' - 4y' + 13y = 0,$$
$$y'' - 10y' + 25y = 0,$$
$$y'' - y - 12y = 8e^{5x}.$$

Problem 5 (10 points). Let k be a positive integer and let V be the set of polynomials of degree at most k.

1. Show that with the standard operations of addition and multiplication of polynomials by numbers, V is a vector space. Find any basis of V and compute the dimension of V.

2. Show that if f is a polynomial in V, then its second derivative f'' is also a polynomial in V. Show that the map $L\colon V \to V$ defined by $L(f) = f''$ is a linear transformation. Compute the matrix of L with respect to the basis of V you found in the first part of the problem.

Problem 6. (5 points) Find the general solution of the linear systems

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

In both cases, sketch the integral curves of the system and determine whether solutions are stable or unstable.

Use your solution to the first part of the problem to determine if the equilibrium $x = 0, y = 0$ of the following nonlinear system is stable or unstable:

$$\begin{cases} x' = 3x - 2y + xy - x^2 - xy^3, \\ y' = 2x - 2y - 2xy - y^3 + x^2y. \end{cases}$$