The Surface Limit of Brownian Motion in Tubular Neighbourhoods of an embedded Riemannian Manifold

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Abstract

We construct the surface measure on the space $C([0,1],M)$ of paths in a compact Riemannian manifold M without boundary embedded into $\mathbb{R}^n$ which is induced by the usual flat Wiener measure on $C([0,1],\mathbb{R}^n)$ conditioned to the event that the Brownian particle does not leave the tubular ε-neighborhood of M up to time 1. We prove that the limit as $\varepsilon \to 0$ exists, the limit measure is equivalent to the Wiener measure on $C([0,1],M)$, and we compute the corresponding density explicitly in terms of scalar and mean curvature.

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1 Introduction

In the study of heat flow on a manifold M it is a natural idea to embed the manifold into Euclidean space via Nash's theorem and to compare some properties of the flat Brownian motion in the surrounding space with corresponding properties of the motion on M. In particular, since the Wiener measures $\mathbb{W}$ and $\mathbb{W}_M$ on the path spaces $C(\mathbb{R}^n)$ and $C(M)$ are the canonical measures and $C(M)$ is a submanifold of $C(\mathbb{R}^n)$, one could expect in analogy to the finite dimensional situation that $\mathbb{W}_M$ is just the surface measure induced by $\mathbb{W}$. However we show that this is generally false but that the surface measure is equivalent to $\mathbb{W}_M$ and that the density depends both on the (intrinsic) scalar curvature and the mean curvature of the embedding. The same surface measure has previously been identified by a completely different method based on discrete time approximation in [1].

It would be interesting to find a direct proof of the equivalence of the two approaches. The formula of the density given below allows to interpret certain geometric curvature characteristics as an additional 'effective' potential in the Schrödinger equation (in the spirit of [2]) on the manifold. The particular geometric potential in our main result appears also in the context of the study of holonomic constraints in quantum mechanics in [3], p. 500. The methods of [3] could be used to give an alternative (but not simpler) proof of results similar to ours except that our statement of weak convergence also implies uniform tightness of the family $(\mathbb{W}_\varepsilon)$ introduced below. A more detailed exposition of this background and an extension to more general surrounding manifolds will be given in [4]. We also note that our result can be used to get alternative proofs for Girsanov type theorems for a Brownian motion on M.

Let us now sketch the statement and the idea of the proof of our main theorem. Let $M \subset \mathbb{R}^n$ be a smooth compact m-dimensional Riemannian manifold without boundary. Denote by $k = n - m$ the codimension of M and by $\mathbb{M}_\varepsilon$ the tubular neighborhood of M, which consists of all points in $\mathbb{R}^n$ such that their distance to the manifold is less or equal ε. Let $a_0 \in M$ be the starting point of all stochastic processes considered below. In the sequel, we will always write $C(X)$ instead of $C_{a_0}([0,1], X)$. Denote by $\mathbb{W}$ the Wiener measure on $C(\mathbb{R}^n)$ and by $\mathbb{W}_M$ the Wiener measure on $C(M)$. Let $\mathbb{W}_\varepsilon$ be the normalized restriction of the Wiener measure $\mathbb{W}$ to $C(\mathbb{M}_\varepsilon)$

$$\mathbb{W}_\varepsilon = \frac{\mathbb{W}|_{C(\mathbb{M}_\varepsilon)}}{\mathbb{W}(C(\mathbb{M}_\varepsilon))},$$

i.e., the conditional law of a Brownian particle conditioned not to leave the ε-neighbourhood of the manifold.

The main result of the paper is the theorem in the last section, where we prove that the conditional laws $\mathbb{W}_\varepsilon$ converge weakly to a probability measure $\mathbb{W}_0$ on $C(M)$ which is absolutely continuous with respect to the Wiener measure $\mathbb{W}_M$ on the manifold, and the density is given by

$$\frac{d\mathbb{W}_0}{d\mathbb{W}_M}(\omega) = \frac{\exp\left\{-\frac{1}{4}\int_0^1 R(\omega_t)dt + \frac{1}{8}\int_0^1 ||\sigma||^2(\omega_t)dt\right\}}{\mathbb{E}_{\mathbb{W}_M}\exp\left\{-\frac{1}{4}\int_0^1 R(\omega_t)dt + \frac{1}{8}\int_0^1 ||\sigma||^2(\omega_t)dt\right\}}, \tag{1}$$

where $R(a)$ is the scalar curvature and $\sigma(a)$ is the tension field of M at $a \in M$. The measure $\mathbb{W}_0$ is called the surface measure on the path space of the manifold M generated by the Wiener measure in $\mathbb{R}^n$. In particular, if M has constant scalar and mean curvature then the surface measure $\mathbb{W}_0$ (and hence also the measure constructed in [1]) coincides with the Wiener measure $\mathbb{W}_M$, as was already announced in [5].

We remark that this result also yields another construction of the Wiener measure $\mathbb{W}_M$ which complement the classical ones (cf. e.g. [6]): Let φ denote any continuous extension of the density given by formula (1) to the space $C(\mathbb{M}_\varepsilon)$ which is also bounded away from 0. Then the measures $\varphi^{-1}\mathbb{W}_\varepsilon$ converge to $\mathbb{W}_M$.

We use the following notation. We assume that ε_0 is small enough and the orthogonal projection $\pi : \mathbb{M}_{\varepsilon_0} \to M$ is well defined (and we consider $\varepsilon < \varepsilon_0$). We denote by T_aM the tangent space of M at $a \in M$ and by N_aM the normal space of M at $\pi(a)$, for $a \in \mathbb{M}_{\varepsilon_0}$. In the sequel, we identify these spaces with the corresponding subspaces in $\mathbb{R}^n$. We also use the Einstein summation convention: an index occurring twice in a product is to be summed from one up to the space dimension.

First, we construct a special vector field v on $\mathbb{R}^n$. To do that, we notice that there are two natural measures $\lambda_{\mathbb{R}^n}$ and $\lambda_\oplus$ on $\mathbb{M}_{\varepsilon_0}$. The first one is induced from $\mathbb{R}^n$ by the embedding $\mathbb{M}_{\varepsilon_0} \subset \mathbb{R}^n$. The second one is called the reference measure, it reflects the natural product structure in the normal bundle NM and is defined by

$$\lambda_\oplus(A) = \int_{\pi(A)} \lambda_{\mathbb{R}^k}(A_x)d\lambda_M(x),$$

where $A_x = \pi^{-1}(x)$ and $\lambda_{\mathbb{R}^k}$ and λ_M are the Lebesgue measures on $\mathbb{R}^k$ and M, respectively. The reference measure $\lambda_\oplus$ is equivalent to $\lambda_{\mathbb{R}^n}$ and the vector field v is then defined by

$$v(a) = \mathrm{pr}_{N_yM}\left[\nabla \log \frac{d\lambda_\oplus}{d\lambda_{\mathbb{R}^n}}\right], a \in \mathbb{M}_{\varepsilon_0}.$$

Finally, we extend v to a smooth vector field with compact support in $\mathbb{R}^n$.

Further, we consider the stochastic process (y_t) in $\mathbb{R}^n$, which is a weak solution of the stochastic differential equation

$$dy_t = db_t + \frac{1}{2}v(y_t)dt, \;\; y_0 = a_0.$$

We prove in Proposition 15 that the surface measure corresponding to the process (y_t) is just the Wiener measure on the manifold. The idea of the proof is based on Fermi decomposition of the process (y_t), which is constructed in Section 4. Namely, we represent the process (y_t) by a pair of processes (x_t) and (z_t), where (x_t) is a process in M and (z_t) is a process in $\mathbb{R}^k$. The first one is just the projection (to the manifold) of (y_t) stopped while leaving $\mathbb{M}_{\varepsilon_0}$. To construct the second process, we fix an orthonormal basis in $N_{a_0}M$ and move it by stochastic parallel translation along the semimartingale (x_t) to the point x_t. So we get an orthonormal basis in $N_{x_t}M$ and we define z_t by the coordinates of $y_t - x_t \in N_{x_t}M$ with respect to this basis up to the exit time of $\mathbb{M}_{\varepsilon_0}$. Then we prove that (z_t) is a k-dimensional Brownian motion independent of the m-dimensional Brownian motion driving the process (x_t). Using this fact, we show that the distribution of (x_t) under the condition that $||z_t|| \leq \varepsilon$ for all $0 \leq t \leq 1$ converges to the Wiener measure on the manifold.

It follows from Girsanov's Theorem (see Lemma 17), that the distribution μ of the process (y_t) is absolutely continuous with respect to $\mathbb{W}$, and the density is given by

$$\rho = \frac{d\mathbb{W}}{d\mu} = \exp\left\{-\frac{1}{2}\int_0^1 \langle v(b_t), db_t\rangle + \frac{1}{8}\int_0^1 |v(b_t)|^2 dt\right\}.$$

If ρ were continuous and bounded we could find the density dW_0/dW_M just by the normalized restriction of ρ to $C(M)$. Since ρ is not necessarily of this kind we approximate it by a continuous bounded function ρ_0 in such a way that the approximation is quite good on the paths from $C(\mathbb{M}_{\varepsilon_0})$ (the precise definition is given in Section 8). In Proposition 22 we compute ρ_0 explicitly to

$$\rho_0 = \exp\left\{-\frac{1}{4}\int_0^1 R(x_t)dt + \frac{1}{8}\int_0^1 ||\sigma||^2(x_t)dt\right\},$$

where (x_t) is the first component of the Fermi decomposition of (b_t). Finally, we prove in the last two sections that the density dW_0/dW_M coincides with the normalized restriction of ρ_0 to $C(M)$ and obtain the formula (1).

2 Derivatives of the projection π

In this section, we compute the first derivative of the projection π at a point in the ε_0-neighborhood of M.

Let us introduce the following notation. For a given point $a \in \mathbb{M}_{\varepsilon_0}$, we consider an orthogonal coordinate system $(y^1, \ldots, y^n)$ with respect to a basis (e_i) centered at $\pi(a)$ such that its first m basis vectors form an orthonormal basis of the tangent space $T_{\pi(y)}M$. By the implicit function theorem, in this coordinate system the manifold M can be represented locally in a neighborhood of $\pi(a)$ by a system of equations $y^{s+m} = f_s(y^1, \ldots, y^m)$ or, equivalently, by a system of equations $\varphi_s(y) = 0$, where $\varphi_s(y^1, \ldots, y^n) = y^{m+s} - f_s(y^1, \ldots, y^m)$, $s \in 1, \ldots, k$. Notice that $\nabla\varphi_s(0) = e_{m+s}$, for all s.

Definition 1 *We call such a coordinate system (y^i) an orthogonal coordinate system corresponding to the point a and the functions f_s (or φ_s) the local representation of M at the point a with respect to this coordinate system.*

Further, denote by $F_s = \mathrm{Hess} f_s(0)$ the Hessian of f_s at zero and denote the last k coordinates of a in the coordinate system (y^i) by $(z^1, \ldots, z^k)$ (notice that the first m coordinates of a are equal to zero).

Lemma 2 *Let $a \in \mathbb{M}_{\varepsilon_0}$. Then the first derivative operator $D\pi(a)$ of the projection is given by the matrix*

$$D\pi(a) = \begin{bmatrix} [\mathrm{I}_{m\times m} - z^s F_s]^{-1} & 0_{m\times k} \\ 0_{k\times m} & 0_{k\times k} \end{bmatrix}.$$

in the coordinate system corresponding to a.

Proof. First notice that $\partial_{m+s}\pi(a) = 0$, for all $1 \leq s \leq k$, since the projection is constant along these directions. Therefore the both right blocks of the matrix are equal to zero.

Further, differentiating $\varphi_s \circ \pi = 0$ with respect to y_i and taking into account that $\partial_j\varphi_s(0) = \delta_{j,m+s}$ we obtain $\partial_i \pi^{m+s}(a) = 0$ for all $1 \leq i \leq n$; $1 \leq s \leq k$. This means that the left lower block is also equal to zero.

It remains to prove the formula for the remaining block, which we denote by X. Since $y - \pi(y) \in N_yM$ and $N_yM = \langle (\nabla\varphi_s \circ \pi)(y) : 1 \leq s \leq k \rangle$, we have

$$y = \pi(y) + \alpha^s(y)(\nabla\varphi_s \circ \pi)(y),$$

where α^s are some smooth functions with $\alpha^s(0) = z^s$. Differentiating with respect to y, we obtain

$$I_{n\times n} = D\pi + (\nabla\varphi_s \circ \pi)D\alpha^s + \alpha^s \mathrm{Hess}\varphi_s D\pi.$$

Taking value at the point a, we get

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} [D\alpha^s(0)] - z^s \begin{bmatrix} F_s & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix},$$

and, finally, considering the left upper block, we obtain $X = [I - z^s F_s]^{-1}$. $\square$

3 Curvature in local coordinates

In this section we compute the second fundamental form, the scalar curvature, and the norm of the tension field of the manifold M at a point $a \in M$ in terms of the local representation of M at the point a defined in the previous section.

Lemma 3 $[F_s]_{ij} = -l_{\partial_{m+s}}(\partial_i, \partial_j)|_{\pi(a)}$*, where* $l_\nu(\cdot, \cdot)$ *denotes the second fundamental form of* M *with respect to* $\mathbb{R}^n$ *and* (∂_i) *is the orthonormal basis of* $\mathbb{R}^n$ *corresponding to the local coordinate system* $(y_1, \dots, y_n)$ *at* $a \in \mathbb{M}_{\varepsilon_0}$*.*

Proof. Let $u = u^p \partial_p$, $w = w^q \partial_q$, and $\nu = \nu^r \partial_r$ be vector fields defined on a neighborhood of $\pi(a)$ in $\mathbb{M}_{\varepsilon_0}$ such that

1) $u(x), w(x) \in T_x M$ and $\nu(x) \in N_x M$ for all $x \in M$;

2) $u(\pi(a)) = \partial_i$, $w(\pi(a)) = \partial_j$, and $\nu(\pi(a)) = \partial_{m+s}$.

Notice that we can take $\nu = \nabla \varphi_s$. By the definition of the second fundamental form we obtain

$$\begin{aligned} l_\nu(u, w)|_{\pi(a)} &= \langle \nabla_u^{\mathbb{R}^n} \nu, w \rangle|_{\pi(a)} = \langle \nabla_{u^p \partial_p}^{\mathbb{R}^n} \nu^r \partial_r, w^q \partial_q \rangle|_{\pi(a)} = u^p \partial_p \nu^r w^q \langle \partial_r, \partial_q \rangle|_{\pi(a)} \\ &= \partial_i \nu^j|_{\pi(a)} = \partial_{ij} \varphi_s(0) = -[F_s]_{ij}, \end{aligned}$$

which completes the proof. $\square$

Lemma 4 *The scalar curvature R(a) of the manifold M at the point a is given by*

$$R(a) = \sum_{s=1}^{k} \left[(\mathrm{tr} F_s)^2 - \mathrm{tr}(F_s)^2 \right]$$

in the orthogonal coordinate system (y^i) *corresponding to a.*

Proof. We consider $(y_1, \dots, y_m)$ as local coordinates in M in a neighborhood of a. Then $g_{ij}(0) = \delta_{ij}$ and by the definition of the scalar curvature $R(a) =$

$R_{ijij}(0)$. Now by the Gauss equation (see [7]) and by Lemma 3 we obtain

$$\begin{aligned} R(a) &= \langle R(\partial_i, \partial_j)\partial_j, \partial_i \rangle|_a \\ &= \sum_{s=1}^{k} [l_{\partial_{m+s}}(\partial_j, \partial_j) l_{\partial_{m+s}}(\partial_i, \partial_i) - l_{\partial_{m+s}}(\partial_i, \partial_j) l_{\partial_{m+s}}(\partial_i, \partial_j)]_a \\ &= \sum_{s=1}^{k} [(F_s)_{jj}(F_s)_{ii} - (F_s)_{ij}(F_s)_{ij}] = \sum_{s=1}^{k} [(\mathrm{tr} F_s)^2 - \mathrm{tr}(F_s)^2], \end{aligned}$$

and the formula is proved. □

Lemma 5 *The norm of the tension field σ of M at the point $a \in M$ is given by*

$$||\sigma(a)||^2 = \sum_{s=1}^{k} (\mathrm{tr} F_s)^2. \tag{2}$$

Proof. By [8] $||\sigma|| = m||\kappa||$, where κ denotes the mean curvature vector of M at the point a. The first coordinates of $\kappa(a)$ are equal to zero since the mean curvature vector belongs to the normal space. By definition of κ (see [7]) and using the previous lemma, we have

$$\kappa_{m+s}(a) = \frac{1}{m} \sum_{i=1}^{m} l_{\partial_{m+s}}(\partial_i, \partial_i) = -\frac{1}{m} \mathrm{tr} F_s,$$

which implies (2). □

4 Fermi decomposition of a stochastic process

Let (y_t) be a stochastic process in $\mathbb{M}_{\varepsilon_0}$ starting at a_0. In this section, we construct a decomposition of (y_t) into two processes (x_t) and (z_t), where the first one is just the projection

$$x_t = \pi(y_t)$$

and the second one is a process in $\mathbb{R}^k$ starting at zero and describing the orthogonal component $(y_t - x_t)$ of the process (y_t).

First, let us define stochastic parallel translation of a vector $\nu \in \mathbb{R}^n$ along the M-valued semimartingale (x_t) due to [9]. First, we fix an orthonormal basis $(e_1, \ldots, e_n)$ in $\mathbb{R}^n$ such that $(e_1, \ldots, e_m)$ span $T_{a_0}M$. Further, for each $x \in M$, let P_x denote the orthogonal projection of $\mathbb{R}^n$ onto T_xM and $Q_x = Id - P_x$ denote the orthogonal projection of $\mathbb{R}^n$ onto N_xM. Then P and Q are smooth functions from M to the vector space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ (see [9, Lemma 1.24]). Since we have fixed an orthonormal basis in $\mathbb{R}^n$ we can also say that P and Q are smooth functions from M to $gl(n)$, where $gl(n)$ is the linear space of all $n \times n$-matrices with real elements. In the sequel, we will always identify matrices with corresponding linear operators if there is no risk of confusion. For $x \in M$ and $w \in T_xM$, we define

$$\Gamma_x(w) = dQ_x(w)P_x + dP_x(w)Q_x \in gl(n).$$

Definition 6 *Given $\nu \in \mathbb{R}^n$, let $\nu_t = u_t\nu$, where (u_t) solves the Stratonovich stochastic differential equation*

$$\delta u_t + \Gamma_{x_t}(\delta x_t)u_t = 0 \quad \textit{with} \quad u_0 = I \in gl(n). \tag{3}$$

Then (ν_t) is called stochastic parallel translation of ν along (x_t).

Lemma 7 *1) The system of vectors $(u_te_1, \dots, u_te_n)$ is an orthonormal basis in $\mathbb{R}^n$ such that the first m vectors form an orthonormal basis in $T_{x_t}M$ and the last k vectors form an orthonormal basis in $N_{x_t}M$;*

2) u^T solves the equation $\delta u_t^T = u_t^T\Gamma_{x_t}(\delta x_t)$ with $u_0^T = I$;

Proof. 1) By Theorem 3.18 from [9], the process (u_t) is orthogonal for all t and satisfies $P_{x_t}u_t = u_tP_{x_0}$. Hence $P_{x_t}u_te_i = u_tP_{x_0}e_i = u_te_i$, for all $i \le m$ and therefore $u_te_i \in T_{x_t}M$ for all $i \le m$ and for all t.

2) It follows from (3) that $\delta u_t^T = -[\Gamma_{x_t}(\delta x_t)u_t]^T = u_t^T(-\Gamma_{x_t}^T(\delta x_t))$. Further, using Lemma 2.39 from [9], which says that $\Gamma^T = -\Gamma$, we get $\delta u_t^T = u_t^T\Gamma_{x_t}(\delta x_t)$. □

Thus, for each t, the coordinate system (u_te_i) is an orthogonal coordinate system corresponding to x_t in the sense of the Definition 1. This allows us to construct the processes $(F_s(x_t, u_t))$ which are defined by F_s with respect to the coordinate system (u_te_i) at the point x_t. In this case F_s are smooth function from $M \times o(n)$, where $o(n) \subset gl(n)$ is the set of the orthogonal matrices.

Further, let $\mathrm{pr}_1 : \mathbb{R}^n \to \mathbb{R}^m$ (respectively, $\mathrm{pr}_2 : \mathbb{R}^n \to \mathbb{R}^k$) be the linear operator that maps $u \in \mathbb{R}^n$ to its first m (respectively, to its last k) coordinates. Denote by $\mathrm{pr}_1^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ (respectively, $\mathrm{pr}_2^{-1} : \mathbb{R}^k \to \mathbb{R}^n$) the right inverse to pr_1 (respectively, to pr_2) such that $\mathrm{pr}_1^{-1}\mathrm{pr}_1 = P_{a_0}$ (respectively, $\mathrm{pr}_2^{-1}\mathrm{pr}_2 = Q_{a_0}$). We define linear operators $I_t : \mathbb{R}^n \to \mathbb{R}^m$ and $J_t : \mathbb{R}^n \to \mathbb{R}^k$ by

$$I_t = \mathrm{pr}_1 u_t^T \quad \text{and} \quad J_t = \mathrm{pr}_2 u_t^T.$$

Definition 8 *Let (z_t) be the orthogonal component of (y_t) precisely defined by*

$$z_t = J_t(y_t - x_t).$$

We call the pair of the processes (x_t) and (z_t) (with values in M and $\mathbb{R}^k$, respectively) Fermi decomposition of the process (y_t).

5 Construction and properties of the vector field v

In order to define the vector field v mentioned in the introduction notice that there are two natural measures $\lambda_{\mathbb{R}^n}$ and $\lambda_{\oplus}$ on $\mathbb{M}_{\varepsilon_0}$. The first one is inherited from $\mathbb{R}^n$ since $\mathbb{M}_{\varepsilon_0} \subset \mathbb{R}^n$. The second one is defined by

$$\lambda_{\oplus}(A) = \int_{\pi(A)} \lambda_{\mathbb{R}^k}(A_x) d\lambda_M(x), \quad A \subset \mathbb{M}_{\varepsilon_0} \text{ Borel},$$

where $A_x = \pi^{-1}(x)$ and $\lambda_{\mathbb{R}^k}$ and λ_M are the Lebesgue measures on $\mathbb{R}^k$ and M, respectively. We have used here the fact that $A_x \subset N_xM$ and that there is a linear isometry between N_xM and $\mathbb{R}^k$. Moreover, the Lebesgue measure $\lambda_{\mathbb{R}^k}$ is independent of the choice of such an isometry and hence $\lambda_{\oplus}$ is well-defined.

Lemma 9 *$\lambda_{\oplus}$ is equivalent to $\lambda_{\mathbb{R}^n}$, and the density is given by*

$$\frac{d\lambda_{\oplus}}{d\lambda_{\mathbb{R}^n}}(a) = \det[\mathrm{I} - z^s F_s]^{-1},$$

where z and (F_s) are from some local representation corresponding to $a \in \mathbb{M}_{\varepsilon_0}$.

Proof. Let $a \in \mathbb{M}_{\varepsilon_0}$, (y^i) be an orthogonal coordinate system corresponding to a, and f (and so (F_s) and z) be from the local representation of M at the point a.

Let $V \subset M$ be a neighborhood of $\pi(a)$ and $U \subset \mathbb{R}^m$ be a neighborhood of zero such that the mapping $\psi_0 : U \to V$ given by

$$\psi_0(x) = (x, f(x))$$

is a bijection. Let $\nu_s : U \to \mathbb{R}^n$ be smooth functions such that $(\nu_s(x))$ is an orthonormal basis of $T_{\psi_0(x)}M$ and $\nu_s(0)$ is the $(m+s)$-th basis vector of the coordinate system corresponding to u. Consider now the mapping $\psi : U \times B(\varepsilon_0) \to \pi^{-1}(V)$ given by

$$\psi(x, z) = \psi_0(x) + z^s \nu_s(x).$$

We have

$$D\psi(0, z) = \begin{pmatrix} \mathrm{I} - z^s F_s & 0 \\ * & \mathrm{I} \end{pmatrix}, \tag{4}$$

where the star denotes some $k \times m$ matrix. In fact,

$$\partial_{z^s} \psi(0, z) = \nu_s(0)$$

and therefore two right blocks of $D\psi(0, z)$ are 0 and I, respectively. In order to compute the left upper block notice that $\nu_s = \alpha_s^p \eta_p$, where $\alpha_s^p : U \to \mathbb{R}$ are some smooth functions such that $\alpha_s^p(0) = \delta_s^p$ and

$$\eta_p(x) = (\nabla \varphi_p) \circ \psi_0.$$

Then, for $i \le m$, we have using $\eta_p^i(0) = 0$

$$\begin{aligned} \partial_{x^j} \psi_i(0, z) &= \delta_{ij} + z^s \partial_{x^j} \alpha_s^p \eta_p^i|_0 + z^s \alpha_s^p \partial_{x^j} \eta_p^i|_0 \\ &= \delta_{ij} + z^s \delta_s^p \partial_{ij} \varphi_p(0) = \delta_{ij} - z^s (F_s)_{ij}. \end{aligned}$$

It remains to notice that by definition $\lambda_\oplus$ is locally the image measure of $\lambda_M \otimes \lambda_{\mathbb{R}^k}$ under the mapping ψ. Therefore

$$\frac{d\lambda_\oplus}{d\lambda_{\mathbb{R}^n}}(a) = \det[D\psi(0, z)]^{-1} = \det[\mathrm{I} - z^s F_s]^{-1},$$

and the statement is proved. □

Let us now define the vector field v on $\mathbb{M}_{\varepsilon_0}$ by

$$v(a) = Q_{\pi(a)} \left[\nabla \log \frac{d\lambda_\oplus}{d\lambda_{\mathbb{R}^n}} \right],$$

Lemma 10 *For $a \in \mathbb{M}_{\varepsilon_0}$, $v(a)$ is given by*

$$v(a) = \Big(0, \ldots, 0, \mathrm{tr}(F_1 [I - z^p F_p]^{-1}), \ldots, \mathrm{tr}(F_k [I - z^p F_p]^{-1})\Big),$$

in an orthogonal coordinate system corresponding to a.

Proof. The first m coordinates of $v(a)$ are equal to zero by the definition of v. Further, by Lemma 9 we have

$$\begin{aligned} v(a) = Q_{\pi(a)} \left[\nabla \log \frac{d\lambda_\oplus}{d\lambda_{\mathbb{R}^n}} \right] &= Q_{\pi(a)} \left(\nabla \log \det[I - z^s(a, u) F_s(a, u)]^{-1} \right) \\ &= Q_{\pi(a)} \left(-\mathrm{tr} \nabla \log[I - z^s(a, u) F_s(a, u)] \right) \end{aligned}$$

and therefore

$$v^{m+s}(a) = -\mathrm{tr}\ \partial_{z^s} \log[I - z^p F_p] = \mathrm{tr}(F_s [I - z^p F_p]^{-1}),$$

for all $1 \le s \le k$. □

Lemma 11 *For $a \in M$,* div $v(a) = R(a)$*, where $R(a)$ is the scalar curvature of M at the point a.*

Proof. Since div$v(a)$ is independent of the choice of orthogonal coordinates let us compute it in the coordinates (y^i) corresponding to a. Using the fact that $v(y) \in N_{\pi(y)}M$ for all y, we have

$$v(y) = \alpha^s(y)\eta_s(x(y)),$$

where α^s are smooth functions with $\alpha^s(0) = \mathrm{tr}F_s$ (since we have $v^{m+s}(0) = \mathrm{tr}F_s$ by Lemma 10) and $x^i(y) = (\psi^{-1})^i(y)$, $1 \le i \le m$. It follows from (4) that

$$(D\psi)^{-1}(0,0) = \begin{pmatrix} \mathrm{I} & 0 \\ * & \mathrm{I} \end{pmatrix}$$

and therefore $\partial_{y^i}x^j(0) = \delta_i^j$. Then using $\partial_i f_s(0) = 0$ and the definition of η_i we obtain

$$\sum_{i=1}^m \partial_i v^i(0) = -\sum_{i,s,j} \alpha^s \partial_{ij} f_s \partial_{y^i} x^j|_0 = -\sum_{i,s} \mathrm{tr}F_s (F_s)_{ii} = -\sum_{s=1}^k (\mathrm{tr}F_s)^2,$$

$$\sum_{s=1}^k \partial_{m+s} v^{m+s}(0) = \sum_{s=1}^k \partial_{z^s} \mathrm{tr}(F_s[I - z^p F_p]^{-1})(0) = \sum_{s=1}^k \mathrm{tr}(F_s)^2.$$

Finally, by Lemma 4 we get

$$\mathrm{div}\, v(a) = \sum_{i=1}^m \partial_i v^i(0) + \sum_{s=1}^k \partial_{m+s} v^{m+s}(0) = -\sum_{s=1}^k (\mathrm{tr}F_s)^2 + \sum_{s=1}^k \mathrm{tr}(F_s)^2 = -R(a)$$

which proves the assertion. □

6 Construction and properties of the shifted process (y_t)

Let us extend the vector field v from $\mathbb{M}_{\varepsilon_0}$ to $\mathbb{R}^n$ in such a way that the extension is smooth and has compact support (the choice of the extension is not essential for further considerations). We denote such an extension also by v. Then there exists a unique weak solution (y_t) of the stochastic differential equation

$$\begin{cases} dy_t = db_t + \frac{1}{2}v(y_t)dt, \\ y_0 = a_0. \end{cases} \tag{5}$$

Let τ be the exit time of (y_t) from $\mathbb{M}_{\varepsilon_0}$. Consider the stopped process $(y_{t\wedge\tau})$ and denote by $((x_t),(z_t))$ its Fermi decomposition. Further, consider the process

$$\tilde{b}_t = \int_0^t u_s^T db_s.$$

It is also an n-dimensional Brownian motion by the Lévy's characterization theorem since it is a continuous local martingale with $d\tilde{b}_t^i d\tilde{b}_t^j = \delta_{ij}dt$ by the orthogonality of u_s for all s. Denote by $\tilde{b}_t'$ (respectively, by $\tilde{b}_t''$) the first m (respectively, the last k) components of $\tilde{b}_t$.

Lemma 12 *The Itô differential of the process (x_t) up to time τ is given by*

$$dx_t = u_t \operatorname{pr}_1^{-1}[I - z_t^s F_s(x_t,u_t)]^{-1} d\tilde{b}_t' + \frac{1}{2}\Delta\pi(y_t)dt.$$

Proof. By Lemma 2 we can compute $D\pi(y_t)$ in the coordinate system corresponding to the basis $(u_t e_i)$. Hence the formula for $D\pi(y_t)$ with respect to the original coordinate system (e_i) is given by

$$D\pi(y_t) = u_t \operatorname{pr}_1^{-1}[I - z_t^s F_s(x_t,u_t)]^{-1}\operatorname{pr}_1 u_t^T.$$

Now by Itô's formula and using $v(y_t)\in N_{x_t}M$ we obtain up to time τ

$$\begin{aligned}
dx_t = d\pi(y_t) &= D\pi(y_t)dy_t + \frac{1}{2}DD\pi(y_t)dy_t dy_t \\
&= u_t \operatorname{pr}_1^{-1}[I - z_t^s F_s(x_t,u_t)]^{-1}\operatorname{pr}_1 u_t^T(db_t + \frac{1}{2}v(y_t)dt) + \frac{1}{2}\Delta\pi(y_t)dt \\
&= u_t \operatorname{pr}_1^{-1}[I - z_t^s F_s(x_t,u_t)]^{-1} d\tilde{b}_t' + \frac{1}{2}\Delta\pi(y_t)dt,
\end{aligned}$$

which completes the proof. □

Lemma 13 $z_t = \tilde{b}_t''$ *up to time τ.*

Proof. Let us show that the Stratonovich differentials of the processes (z_t) and $(\tilde{b}_t'')$ coincide. Recall that for two continuous semimartingales a and b holds $a\delta b = adb + \frac{1}{2}\delta a\delta b$. Then, using the equalities $dQP = -QdP$ and $dPQ =$

$-PdQ$, the definition of Γ, and Lemma 7 we obtain

$$\begin{aligned}
\delta\tilde{b}''_t &= d\tilde{b}''_t = \mathrm{pr}_2 u_t^T db_t \\
&= J_t\delta b_t - \frac{1}{2}\mathrm{pr}_2 \delta u_t^T \delta b_t \\
&= J_t\delta b_t - \frac{1}{2}\mathrm{pr}_2 u_t^T \Gamma_{x_t}(\delta x_t)\delta b_t \\
&= J_t\delta b_t - \frac{1}{2}J_t dQ_{x_t}(\delta x_t)P_{x_t}\delta b_t - \frac{1}{2}J_t dP_{x_t}(\delta x_t)Q_{x_t}\delta b_t \\
&= J_t\delta b_t + \frac{1}{2}J_t Q_{x_t} dP_{x_t}(\delta x_t)\delta b_t + \frac{1}{2}J_t P_{x_t} dQ_{x_t}(\delta x_t)\delta b_t \\
&= J_t\delta b_t + \frac{1}{2}J_t dP_{x_t}(\delta x_t)\delta b_t.
\end{aligned}$$

Analogously, using $y_t - \pi(y_t) \in N_{x_t}M$ and Lemma 2, which implies that $\mathrm{Im}[D\pi(y)] = T_{\pi(y)}M$, we compute

$$\begin{aligned}
dz_t &= \mathrm{pr}_2 \delta u_t^T (y_t - \pi(y_t)) + \mathrm{pr}_2 u_t^T \delta(y_t - \pi(y_t)) \\
&= \mathrm{pr}_2 u_t^T \Gamma_{x_t}(\delta x_t)(y_t - \pi(y_t)) + \mathrm{pr}_2 u_t^T \delta y_t - \mathrm{pr}_2 u_t^T D\pi(y_t)\delta y_t \\
&= J_t(dQ_{x_t}(\delta x_t)P_{x_t} + dP_{x_t}(\delta x_t)Q_{x_t})(y_t - \pi(y_t)) + J_t\delta y_t \\
&= J_t\delta y_t - J_t P_{x_t} dQ_{x_t}(\delta x_t)(y_t - \pi(y_t)) \\
&= J_t\delta y_t \\
&= J_t\delta b_t + \frac{1}{2}J_t v(y_t)dt.
\end{aligned}$$

It remains to show that $J_t dP_{x_t}(\delta x_t)\delta b_t = J_t v(y_t)dt$ or, equivalently, that the last k coordinates of of the vectors $dP_{x_t}(\delta x_t)\delta b_t$ and $v(y_t)dt$ with respect to the coordinate system $(u_t e_i)$ coincide. We compute them using Lemma 2, Lemma 10, and Lemma 14 below.

$$\begin{aligned}
(dP_{x_t}(\delta x_t)\delta b_t)^{m+s} &= (\partial_i P_{x_t})_{m+s,k}(\mathrm{pr}_1 u_t^T \delta x_t)^i (u_t^T \delta b_t)^k \\
&= \partial_{ik} f_s(x_t)(\mathrm{pr}_1 u_t^T D\pi(y_t)\delta y_t)^i (u_t^T \delta b_t)^k \\
&= \partial_{ik} f_s(x_t)(\mathrm{pr}_1 u_t^T u_t \mathrm{pr}_1^{-1}[I - z_t^p F_p(x_t,u_t)]^{-1}\mathrm{pr}_1 u_t^T \delta b_t)^i (u_t^T \delta b_t)^k \\
&= \partial_{ik} f_s(x_t)([I - z_t^p F_p(x_t,u_t)]^{-1}\mathrm{pr}_1)_{ir}(u_t^T)_{rq}(\delta b)^q (u_t^T)_{kl}(\delta b)^l \\
&= \partial_{ik} f_s(x_t)[I - z_t^p F_p(x_t,u_t)]^{-1}_{ik} dt \\
&= \mathrm{tr}(F_s(x_t,u_t)[I - z_t^p F_p(x_t,u_t)]^{-1})dt \\
&= v^{m+s}(y_t)dt.
\end{aligned}$$

This implies $\delta z_t = \delta\tilde{b}''_t$ and $z_t = \tilde{b}''_t$ since this stochastic differential equation is exact. □

Lemma 14 *dP_x is given by formulae*

$$\partial_i P_x = \begin{bmatrix} 0 & \partial_i Df^T \\ \partial_i Df & 0 \end{bmatrix}_0$$

in an orthogonal coordinate system at $x \in M$.

Proof. Let $x \in M$. Consider an orthogonal coordinate system (y^i) at x and the local representation (f_s) of M at the point x. Notice that

$$P \begin{bmatrix} \mathrm{I} & -Df^T \\ Df & \mathrm{I} \end{bmatrix} = \begin{bmatrix} \mathrm{I} & 0 \\ Df & 0 \end{bmatrix}.$$

since the first m columns of the matrix on the left hand side generate the tangent space $T_{(y,f(y))}M$ and the last k columns generate the normal space $N_{(y,f(y))}M$. Differentiating with respect to y_i, we obtain

$$\partial_i P \begin{bmatrix} \mathrm{I} & -Df^T \\ Df & \mathrm{I} \end{bmatrix} + P \begin{bmatrix} 0 & -\partial_i Df^T \\ \partial_i Df & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \partial_i Df & 0 \end{bmatrix}.$$

Evaluating at zero and using $Df(0) = 0$, we get

$$\partial_i P_x = \begin{bmatrix} 0 & 0 \\ \partial_i Df & 0 \end{bmatrix}_0 - \begin{bmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\partial_i Df^T \\ \partial_i Df & 0 \end{bmatrix}_0 = \begin{bmatrix} 0 & \partial_i Df^T \\ \partial_i Df & 0 \end{bmatrix}_0$$

which proves the lemma. □

7 Surface measure corresponding to the process (y_t)

Let μ be the distribution of the process (y_t). Denote by μ_ε the normalized restriction of μ to $C(\mathbb{M}_\varepsilon)$

$$\mu_\varepsilon = \frac{\mu|_{C(\mathbb{M}_\varepsilon)}}{\mu(C(\mathbb{M}_\varepsilon))}.$$

Proposition 15 *The family μ_ε converges weakly to $\mathbb{W}_M$, i.e., the surface measure corresponding to the process (y_t) is just the Wiener measure on $C(M)$.*

Proof. We need to prove that the conditional law of the process (y_t), given that the (y_t) does not leave $\mathbb{M}_\varepsilon$ before time 1 converges to $\mathbb{W}_M$ as ε tends to zero.

Consider the Fermi decomposition $((x_t), (z_t))$ of the stopped process $(y_{t\wedge\tau})$. By Lemma 12 and Lemma 13 the process (z_t) is just a k-dimensional Brownian motion independent of the m-dimensional Brownian motion driving the process (x_t).

Consider now any probability space on which there is an n-dimensional Brownian motion (b_t^*) and, moreover, there is a family (z_t^ε) of processes such that each (z_t^ε) has the same law as (z_t) under μ_ε and the whole family $\{(z_t^\varepsilon) : \varepsilon < \varepsilon_0\}$ is independent of (b_t^*). On this probability space we consider the filtration $(\mathcal{F}_t)$ generated by $\{b_s^* : s \leq t\}$ and all the processes (z_t^ε), $0 \leq t \leq 1$. Then (b_t^*) is an n-dimensional $(\mathcal{F}_t)$-Brownian motion, $(\mathrm{pr}_1 b_t^*)$ is an m-dimensional $(\mathcal{F}_t)$-Brownian motion and the coefficients in the system of the stochastic differential equations

$$\begin{cases} \delta u_t^\varepsilon + \Gamma_{x_t^\varepsilon}(\delta x_t^\varepsilon) u_t^\varepsilon = 0, \\ dx_t^\varepsilon = u_t^\varepsilon \mathrm{pr}_1^{-1}[I - (z_t^\varepsilon)^s F_s(x_t^\varepsilon)]^{-1} \mathrm{pr}_1 db_t^* + \frac{1}{2}\Delta\pi(x_t^\varepsilon + u_t^\varepsilon \mathrm{pr}_2^{-1} z_t^\varepsilon) dt, \\ u_0^\varepsilon = I, \\ x_0^\varepsilon = a_0 \end{cases}$$

are adapted. It follows from $||z_t^\varepsilon|| \leq \varepsilon$ that the coefficients are also bounded and hence there is a unique solution $(u_t^\varepsilon, x_t^\varepsilon)$ of this system for each ε and the law of (x_t^ε) is the same as the law of (x_t) under μ_ε. Moreover, on this probability space the processes (z_t^ε) converge uniformly to zero. It follows now from Lemma 16 below that $(u_t^\varepsilon, x_t^\varepsilon)$ converges locally uniformly in probability to the solution $(\bar{u}_t, \bar{x}_t)$ of the system of the stochastic differential equation

$$\begin{cases} \delta \bar{u}_t + \Gamma_{\bar{x}_t}(\delta \bar{x}_t) \bar{u}_t = 0, \\ d\bar{x}_t = \bar{u}_t P_{a_0} db_t^* + \frac{1}{2}\Delta\pi(\bar{x}_t) dt, \\ \bar{u}_0 = I, \\ \bar{x}_0 = a_0. \end{cases}$$

It remains to show that the process $(\bar{x}_t)$ is a Brownian motion on M. Since $\bar{u}_t$ is orthogonal for all t we have $\bar{u}_t P_{a_0} db_t^* = P_{\bar{x}_t} \bar{u}_t db_t^* = P_{\bar{x}_t} db_t^{**}$, where (b_t^{**}) is another n-dimensional Brownian motion starting in a_0. Further, notice that $P_x = D\pi(x)$ for $x \in M$ by Lemma 2, and the Itô differential equation for the process $(\bar{x}_t)$ now looks like

$$\begin{cases} d\bar{x}_t = D\pi(\bar{x}_t) db_t^{**} + \frac{1}{2}\Delta\pi(\bar{x}_t) dt, \\ \bar{x}_0 = a_0. \end{cases}$$

Due to [10, Th.30.14] the drift c of the Stratonovich stochastic differential equation at the point $x \in M$ can be computed in local coordinates corresponding to x, and using $\partial_q \pi^j(a) = \delta_q^j$ for $j, q \leq m$ and $\partial_q \pi^j(a) = 0$ otherwise, we obtain

$$2c^i = \Delta\pi^i(x,0) - \sum_{q=1}^{n} \partial_q\pi^j\partial_j(\partial_q\pi^i)(x) = \Delta\pi^i(x,0) - \sum_{j=1}^{m} \partial_{jj}\pi^i(x,0)$$
$$= \sum_{j=m+1}^{n} \partial_{jj}\pi^i = 0$$

as π is constant in the normal directions. Now the Stratonovich stochastic differential equation for the process $(\bar{x}_t)$ looks like

$$\begin{cases} \delta\bar{x}_t = P_{\bar{x}_t}\delta b_t^{**}, \\ \bar{x}_0 = a_0. \end{cases}$$

Hence $\bar{x}_t$ is a Brownian motion on M. □

Lemma 16 $(x^\varepsilon, u^\varepsilon) \to (\bar{x}, \bar{u})$ *locally uniformly in probability.*

Proof. Denote the processes $(x_t^\varepsilon, u_t^\varepsilon)$ and $(\bar{x}_t, \bar{u}_t)$ by (a_t^ε) and $(\bar{a}_t)$, respectively. Then the processes (a_t^ε) and $(\bar{a}_t)$ satisfy the stochastic differential equations

$$da_t^\varepsilon = f_1(a_t^\varepsilon, z_t^\varepsilon)db_t^* + f_2(a_t^\varepsilon, z_t^\varepsilon)dt \quad \text{and} \quad d\bar{a}_t = f_1(\bar{a}_t, 0)db_t^* + f_2(\bar{a}_t, 0)dt$$

respectively, with the same initial conditions, where f_i are short notations for the coefficients. It can be easily seen that $f_i(a,z) \to f_i(a,0)$ as $z \to 0$ uniformly in a and the functions $f_i(x,0)$ are Lipschitz. Now, let

$$\varphi_\varepsilon(t) = \mathbb{E}\sup_{s\le t} ||a_s^\varepsilon - \bar{a}_s||^2.$$

It is sufficient to show that $\varphi_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. Let $\delta > 0$. According to the uniform convergence of f_i, choose ε' such that for all $x \in M$ and for all z with $||z|| \le \varepsilon'$ hold $||f_i(a,z) - f_i(a,0)|| < \delta$ (it does not matter what norm we use since they are equivalent). Then we have for $||z|| < \varepsilon'$ and for all a_1, a_2

$$||f_i(a_1,z) - f_i(a_2,0)||^2 \le 2||f_i(a_1,z) - f_i(a_1,0)||^2 + 2||f_i(a_1,0) - f_i(a_2,0)||^2$$
$$\le 2\delta^2 + 2c||a_1 - a_2||^2,$$

where $c^{1/2}$ is a Lipschitz constant for all $f_i(x,0)$ simultaneously. Then, by Corollary 11.2.2 from [11], we obtain

$$f_\varepsilon(t) = \mathbb{E}\sup_{s\le t} ||\int_0^s [f_1(a_u^\varepsilon, z_u^\varepsilon) - f_1(\bar{a}_u, 0)]db_u^* + \int_0^s [f_2(a_u^\varepsilon, z_u^\varepsilon) - f_2(\bar{a}_u, 0)]du||^2$$
$$\le c_1\mathbb{E}\int_0^t ||f_1(a_u^\varepsilon, z_u^\varepsilon) - f_1(\bar{a}_u, 0)||^2 du + c_1\mathbb{E}\int_0^t ||f_2(a_u^\varepsilon, z_u^\varepsilon) - f_2(\bar{a}_u, 0)||^2 du$$
$$\le c_2\delta^2 + c_3\mathbb{E}\int_0^t ||a_u^\varepsilon - \bar{a}_u||^2 du$$
$$\le c_2\delta^2 + c_3\int_0^t f_\varepsilon(u)du, \text{ for all } t \text{ and } \varepsilon < \varepsilon',$$

where c_1, c_2, and c_3 are positive constants independent of δ. Now by Gronwall's lemma $f_\varepsilon(t) \le c_2 \delta^2 \varepsilon^{c_3 t}$ and in particular $f_\varepsilon(1) \le c_2 \delta^2 \varepsilon^{c_3}$ for all $\varepsilon < \varepsilon'$. Hence $f_\varepsilon(1) \to 0$ and the processes $(x_t^\varepsilon, u_t^\varepsilon)$ converge to $(\bar{x}_t, \bar{u}_t)$ locally uniformly in probability. □

8 Absolute continuity of $\mathbb{W}$ with respect to μ, formula for the corresponding density, and its approximation

Now we study the relation between the families $\mathbb{W}_\varepsilon$ and μ_ε. It can be derived from the relation between the measures $\mathbb{W}$ and μ. We prove in the following lemma that these two measures are equivalent and compute the corresponding density in terms of the vector field v.

Lemma 17 *$\mathbb{W}$ is equivalent to μ and the density ρ is given by*

$$\rho = \frac{d\mathbb{W}}{d\mu} = \exp\left\{ -\frac{1}{2} \int_0^1 \langle v(b_t), db_t \rangle + \frac{1}{8} \int_0^1 |v(b_t)|^2 dt \right\} \tag{6}$$

Proof. Recall that μ is the distribution of the process (y_t), which solves the stochastic differential equation (5). Hence the process (y_t) satisfies also

$$db_t = dy_t - \frac{1}{2} v(y_t) dt.$$

It follows now from Girsanov's theorem that the distribution $\mathbb{W}$ of (b_t) is equivalent to the distribution μ of (y_t) and the corresponding density is given by (6). □

It is easy to see that the density ρ is not necessarily continuous and bounded. In order to prove weak convergence of the family $\mathbb{W}_\varepsilon$ we will approximate ρ by a continuous and bounded function in such a way that the approximation is quite good on the paths staying in $\mathbb{M}_\varepsilon$ for the whole time. In the definition below we describe the type of the approximation we need, in Lemma 21 we investigate the approximation of stochastic differentials with respect to dt and dx_t, where $((x_t), (z_t))$ is the Fermi decomposition of the process (b_t) stopped while leaving $\mathbb{M}_{\varepsilon_0}$. This enables us to find the approximation for the density ρ (Proposition 22) since the stochastic integrals in (6) partly can be reduced to integrals with respect to dt and dx and partly are approximated directly.

Definition 18 *Let $\xi : C(\mathbb{R}^n) \to \mathbb{R}$ be a measurable function. We say that ξ is $O(\varepsilon)$ if there exists $c > 0$ such that $\mathbb{E}_{\mu_\varepsilon} |\xi|^p \le (pc\varepsilon)^p$ for all $p \in \mathbb{N}$. We say that a stochastic differential is $O(\varepsilon)$ if the corresponding stochastic integral from zero to one is $O(\varepsilon)$.*

It follows from Minkowski's inequality that the sum of two $O(\varepsilon)$ is again $O(\varepsilon)$.

Lemma 19 *Let $\xi = O(\varepsilon)$, then $\mathbb{E}_{\mu_\varepsilon}|e^\xi - 1| \to 0$ as $\varepsilon \to 0$.*

Proof. We have by the definition of $O(\varepsilon)$

$$\mathbb{E}_{\mu_\varepsilon}|e^\xi - 1| \leq \mathbb{E}_{\mu_\varepsilon} \sum_{p=1}^{\infty} \frac{|\xi|^p}{p!} = \sum_{p=1}^{\infty} \frac{\mathbb{E}_{\mu_\varepsilon}|\xi|^p}{p!} \leq \sum_{p=1}^{\infty} \frac{p^p}{p!}(c\varepsilon)^p \to 0,$$

since the radius of convergence of the power series $\sum_{p=1}^{\infty} \frac{p^p}{p!} z^p$ is positive. □

Lemma 20 *Let $((x_t),(z_t))$ be the Fermi decomposition of (b_t) stopped while leaving $\mathbb{M}_{\varepsilon_0}$ There are continuous bounded functions g_{xx}, g_{ux} from the product space $o(n) \times M \times B_k(\varepsilon_0)$ to $gl(n)$ and to $\mathbb{R}^n$, respectively, such that*

1) $(\delta x_t)(\delta x_t)^T = g_{xx}(u_t, x_t, z_t)dt$;

2) $\delta u_t \delta x_t = g_{ux}(u_t, x_t, z_t)dt$;

3) $(\delta x_t)(\delta z_t)^T = 0_{n\times k}dt$.

Proof. Using Lemma 2 and the computations for the process (b_t) analogous to the computations in Lemma 12 and Lemma 13 for the process (y_t), we obtain

$$\begin{aligned}(\delta x_t)(\delta x_t)^T =& D\pi(b_t)\delta b_t(\delta b_t)^T D\pi(b_t)^T = g_{xx}(u_t, x_t, z_t)dt;\\ (\delta x_t)(\delta z_t)^T =& u_t \mathrm{pr}_1^{-1}[I - z_t^s F_s(x_t, u_t)]^{-1}\delta\tilde{b}_t'(\delta\tilde{b}_t'')^T = 0_{n\times k}dt;\\ \delta u_t \delta x_t =& -\Gamma_{x_t}(\delta x_t)u_t\delta x_t = g_{ux}(u_t, x_t, z_t)dt\end{aligned}$$

which proves the statement. □

Lemma 21 *Let $f : o(n) \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ be a smooth bounded function. Then*

$$\begin{aligned} f(u_t, x_t, z_t)dt &= f(u_t, x_t, 0)dt + O(\varepsilon),\\ f(u_t, x_t, z_t)dx_t^i &= f(u_t, x_t, 0)dx_t^i + O(\varepsilon),\\ f(u_t, x_t, z_t)\delta x_t^i &= f(u_t, x_t, 0)\delta x_t^i + O(\varepsilon),\end{aligned}$$

for all i.

Proof. The first assertion is obvious. It suffices to prove the last two formulas for the case when $f(u, x, z) = 0$ for all $x \in M$ and $u \in o(n)$.

Let (ω_t) be the coordinate process on $(C(\mathbb{R}^n), \mathcal{F}_t)$ and τ be its exit time from $\mathbb{M}_{\varepsilon_0}$. Since (ω_t) is a semimartingale with respect to μ its projection $x_t = \pi(\omega_{t\wedge\tau})$ is also a semimartingale with respect to μ and therefore the Fermi decomposition $((x_t),(z_t))$ of the stopped process $(\omega_{t\wedge\tau})$ is well-defined.

Denote

$$\xi(\omega) = \int_0^1 f(u_t, x_t, z_t)dx_t^i.$$

Notice that μ_ε is absolutely continuous with respect to μ and the corresponding density is given by $d\mu_\varepsilon/d\mu(\omega) = \varphi_\varepsilon(||z||)$, where $||\cdot||$ is the supremum norm on $C([0,1],\mathbb{R}^k)$ with respect to the euklidean norm $|\cdot|$ in $\mathbb{R}^k$ and $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ is given by $\varphi_\varepsilon = \mu(C(\mathbb{M}_{\varepsilon_0}))\mathbf{1}_{[0,\varepsilon]}$.

For each ε, let us approximate μ_ε by measures μ_ε^n that are not only absolute continuous but also equivalent to μ. Let (φ_ε^n) be a sequence of functions from $\mathbb{R}$ to $\mathbb{R}$ such that $\varphi_\varepsilon^n > 0$ and $\varphi_\varepsilon^n \downarrow \varphi_\varepsilon$. Denote

$$d_\varepsilon^n(\omega) = \frac{\varphi_\varepsilon^n(||z||)}{\mathbb{E}_\mu \varphi_\varepsilon^n(||z||)} > 0$$

and define μ_ε^n by the measure that is absolutely continuous with respect to μ with density d_ε^n. Then μ_ε^n is a probability measure equivalent to μ since d_ε^n is positive and $\mathbb{E}_\mu d_\varepsilon^n = 1$. Therefore μ and μ_ε^n have the same semimartingales and, in particular, the process (x_t) is a semimartingale with respect to μ_ε^n.

Let $x = m + a$ and $x = m_\varepsilon^n + a_\varepsilon^n$ be the decompositions of (x_t) into a local martingale part and a part of bounded variation with respect to μ and μ_ε^n, respectively. Let us show that these decompositions coincide, for all n and ε. Consider $(d_\varepsilon^n)_t = \mathbb{E}_\mu[d_\varepsilon^n|\mathcal{F}_t]$ and notice that $(d_\varepsilon^n)_t = \mathbb{E}_\mu[d_\varepsilon^n|\mathcal{F}'_t]$, where $\mathcal{F}'_t$ is the natural filtration of (z_t). This follows from the fact that

$$\begin{aligned}
(d_\varepsilon^n)_t &= \mathbb{E}_\mu[d_\varepsilon^n|\mathcal{F}_t] \\
&= \frac{1}{\mathbb{E}_\mu \varphi_\varepsilon^n(||z||)} \mathbb{E}_\mu[\varphi_\varepsilon^n(\max\{\sup_{0\le s\le t}|z_s|, \sup_{t\le r\le 1}|z_t + (z_r - z_t)|\})|\mathcal{F}_t] \\
&= \frac{1}{\mathbb{E}_\mu \varphi_\varepsilon^n(||z||)} \int_{C_0([0,1-t],\mathbb{R})} \varphi_\varepsilon^n(\max\{\sup_{0\le s\le t}|z_s|, \sup_{0\le r\le 1-t}|z_t + \tilde{\omega}_r|\})d\mathbb{W}(\tilde{\omega})
\end{aligned}$$

is measurable with respect to $\mathcal{F}'_t$ (the last equality is fulfilled since $z_r - z_t$ is a Brownian motion independent of $\mathcal{F}_t$). Hence (see [12]) the process d_ε^n is a stochastic integral with respect to the process z, i.e. $(d_\varepsilon^n)_t = \int_0^t h_s dz_s$ for some k-dimensional process (h_t). Then by Girsanov's theorem and by Lemma 20 (3)

$$(m_\varepsilon^n)_t = m_t - \int_0^t \frac{1}{(d_\varepsilon^n)_t} d[m_t, (d_\varepsilon^n)_t] = m_t - \int_0^t \frac{1}{(d_\varepsilon^n)_t}(dx_t)(dz_t)^T h_t^T = m_t.$$

This means that the process (x_t) has the same semimartingale decomposition with respect to the measure μ and all measures μ_ε^n.

Further we have $(dm_t^i)^2 = (dx_t^i)^2 < c_1^2 dt$ and $|da_t^i| = |\Delta\pi^i(b_t)dt|/2 < c_2 dt$, by the definition of (x_t) and by Lemma 20, with some constants c_1 and c_2. Now we can use Corollary 11.2.2 from [11] (notice that the constant c_p there can be chosen equal to $(2p)^p$)

$$\mathbb{E}_{\mu_\varepsilon^n}|\xi|^p \leq (2p)^p \mathbb{E}_{\mu_\varepsilon^n}\left[\left(\int_0^1 (dx_t^i)^2\right)^{\frac{p}{2}-1}\int_0^1 |f(u_t,x_t,z_t)|^p (dx_t^i)^2 \right.$$
$$\left. + \int_0^1 |f(u_t,x_t,z_t)|^p |da_t^i| \left(\int_0^1 |da_t^i|\right)^{p-1}\right] \leq (2p)^p c_3^p \varepsilon^p (c_1^p + c_2^p) \leq (pc\varepsilon)^p,$$

where c_3 is the Lipschitz constant for the function f with respect to z and $c = 4c_3(c_1 + c_2)$.

By the monotone convergence theorem we have $\mathbb{E}_\mu \varphi_\varepsilon^n(||z||) \to \mathbb{E}_\mu \varphi_\varepsilon(||z||) = 1$ as $n \to \infty$ and

$$\begin{aligned}\mathbb{E}_{\mu_\varepsilon}|\xi|^p &= \mathbb{E}_\mu \varphi_\varepsilon(||z||)|\xi(\omega)|^p = \lim_{n\to\infty} \frac{\mathbb{E}_\mu \varphi_\varepsilon^n(||z||)|\xi(\omega)|^p}{\mathbb{E}_\mu \varphi_\varepsilon^n(||z||)} \\ &= \lim_{n\to\infty} \mathbb{E}_\mu d_\varepsilon^n |\xi|^p = \lim_{n\to\infty} \mathbb{E}_{\mu_\varepsilon^n}|\xi|^p \leq (pc\varepsilon)^p\end{aligned}$$

by the previous estimate for $\mathbb{E}_{\mu_\varepsilon^n}|\xi|^p$. This implies $f(u_t,x_t,z_t)dx_t^i = O(\varepsilon)$. The last statement follows now from the previous ones and from Lemma 20. In fact,

$$\begin{aligned}f(u_t,x_t,z_t)\delta x_t^i &= f(u_t,x_t,z_t)dx_t^i + \frac{1}{2}\delta f(u_t,x_t,z_t)\delta x_t^i \\ &= \frac{1}{2}D_u f(u_t,x_t,z_t)\delta u_t \delta x_t^i + \frac{1}{2}D_x f(u_t,x_t,z_t)\delta x_t \delta x_t^i \\ &+ \frac{1}{2}D_z f(u_t,x_t,z_t)\delta z_t \delta x_t^i + O(\varepsilon) = O(\varepsilon),\end{aligned}$$

as $D_u f(u_t,x_t,0) = 0$, $D_x f(u_t,x_t,0) = 0$, and $\delta u_t \delta x_t^i$ and $\delta x_t \delta x_t^i$ are proportional do dt. □

Proposition 22 *The asymptotic of the density is given by* $\rho = \rho_0 \exp(O(\varepsilon))$, *where*

$$\rho_0(\omega) = \exp\left\{-\frac{1}{4}\int_0^1 R(x_t)dt + \frac{1}{8}\int_0^1 ||\sigma||^2(x_t)dt\right\},$$

$R(a)$ *is the scalar curvature, and* $\sigma(a)$ *is the tension field of* M *at the point* $a \in M$.

Proof. The asymptotic of the second term in the exponents is given by Lemma 5, Lemma 10, and Lemma 21 (1)

$$\begin{aligned}|v(b_t)|^2dt &= |v(x_t)|^2dt + O(\varepsilon)\\ &= \sum_{l=1}^{k}(\mathrm{tr}F_s(x_t,u_t))^2dt + O(\varepsilon) = ||\sigma||^2(x_t)dt + O(\varepsilon).\end{aligned}$$

Consider now the first term. By the definition of the Fermi decomposition we have $b_t = x_t + z_t^s u_t e_{m+s}$ and $x_t = \pi(b_t)$ up to the exit time τ. Using Itô's formula, the transformation rule from the Itô to the Stratonovich calculus, and Lemma 11, we get

$$\begin{aligned}&\langle v(b_t), db_t\rangle = \langle v(b_t), \delta b_t\rangle - \frac{1}{2}\langle dv(b_t), db_t\rangle\\ &= \langle v(b_t), \delta(\pi(b_t) + z_t^s u_t e_{m+s})\rangle - \frac{1}{2}\langle Dv(b_t)db_t, db_t\rangle\\ &= \langle v(b_t), D\pi(b_t)\delta b_t + \delta z^s u_t e_{m+s} + z^s \delta u_t e_{m+s}\rangle - \frac{1}{2}\mathrm{div}v(b_t)dt\\ &= \delta z^s\langle v(b_t), u_t e_{m+s}\rangle - z^s\langle v(b_t), \Gamma_{x_t}(\delta x_t)u_t e_{m+s}\rangle + \frac{1}{2}R(x_t)dt + O(\varepsilon).\end{aligned}$$

We have used here the fact that $\mathrm{Im}D\pi(y) \perp N_yM$ which implies

$$\langle v(b_t), D\pi(b_t)\delta b_t\rangle = 0.$$

Now let us show that the first term is $O(\varepsilon)$. In order to do this consider the process

$$c_t = \mathrm{tr}\log[I - z_t^s F_s(x_t,u_t)].$$

By the equation of the parallel transport (3) we have $\delta F_s(x_t,u_t) = \varphi(x_t,u_t)\delta x_t$, where φ is some smooth function. By Itô's formula, Lemma 10, and Lemma 21 δc_t can be computed as

$$\begin{aligned}&- \mathrm{tr}(F_s(x_t,u_t)[I - z_t^p F_p(x_t,u_t)]^{-1})\delta z_t^s - z_t^s\mathrm{tr}[I - z_t^p F_p(x_t,u_t)]^{-1}\delta F_s(x_t,u_t)\\ =& - \delta z^s\langle v(\omega_t), u_t e_{m+s}\rangle - z_t^s\mathrm{tr}[I - z_t^p F_p(x_t,u_t)]^{-1}\varphi(x_t,u_t)\delta x_t\\ =& - \delta z^s\langle v(\omega_t), u_t e_{m+s}\rangle + O(\varepsilon),\end{aligned}$$

On the other hand $\delta c_t = O(\varepsilon)$ since $\mathrm{tr}\log I = 0$. Together we get

$$\delta z^s\langle v(\omega_t), u_t e_{m+s}\rangle = O(\varepsilon).$$

Further, notice that the second term is equal to zero. In fact, by the definition of Γ and the relation $dPQ = -PdQ$ we have

$$\begin{aligned}\langle v(b_t), \Gamma_{x_t}(\delta x_t) u_t e_{m+s}\rangle &= \langle v(b_t), (dQ_{x_t}(\delta x_t) P_{x_t} + dP_{x_t}(\delta x_t) Q_{x_t}) u_t e_{m+s}\rangle \\ &= -\langle v(b_t), P_{x_t} dQ_{x_t}(\delta x_t) u_t e_{m+s}\rangle = 0\end{aligned}$$

since e_{m+s} and $v(b_t)$ belong to $N_{x_t}M$. Finally,

$$\begin{aligned}\rho &= \exp\left\{-\frac{1}{2}\int_0^1 \langle v(b_t), db_t\rangle + \frac{1}{8}\int_0^1 |v(b_t)|^2 dt\right\} \\ &= \exp\left\{-\frac{1}{4}\int_0^1 R(x_t)dt + \frac{1}{8}\int_0^1 ||\sigma||^2(x_t)dt + O(\varepsilon)\right\},\end{aligned}$$

which completes the proof. □

9 Convergence of $\mathbb{W}_\varepsilon$ and formula for the density

In the last section we prove the main theorem. Consider the function ρ_0 introduced in Proposition 22. It is defined on $C(M)$ and is continuous and bounded. Moreover, it can be extended to a continuous bounded function on $C(\mathbb{R}^n)$. In the sequel we understand under ρ_0 this extension. It turns out that ρ_0 approximates the Girsanov density $\rho = d\mathbb{W}/d\mu$ near the manifold also in the following sense.

Lemma 23 $\mathbb{E}_{\mu_\varepsilon}|\rho - \rho_0| \to 0$ *as* $\varepsilon \to 0$.

Proof. Since ρ_0 is bounded there exists a constant c such that $|\rho_0(\omega)| \le c$ for all ω. Further, denote by ξ the $O(\varepsilon)$-part in the asymptotic representation of ρ. Then

$$\mathbb{E}_{\mu_\varepsilon}|\rho - \rho_0| = \mathbb{E}_{\mu_\varepsilon}|\rho_0(e^\xi - 1)| \le c\mathbb{E}_{\mu_\varepsilon}|e^\xi - 1| \to 0$$

by Lemma 19. □

Theorem 24 *Let $\mathbb{W}_\varepsilon$ be the normalized restriction of the flat Wiener measure $\mathbb{W}$ in $\mathbb{R}^n$ to the set of the paths that do not leave the tubular ε-neighborhood of the manifold M up to time 1. Then $\mathbb{W}_\varepsilon$ converges weakly to a measure $\mathbb{W}_0$, which is equivalent to the Wiener measure $\mathbb{W}_M$ on the manifold, and the density is given by*

$$\frac{d\mathbb{W}_0}{d\mathbb{W}_M}(\omega) = \frac{\exp\left\{-\frac{1}{4}\int_0^1 R(\omega_t)dt + \frac{1}{8}\int_0^1 ||\sigma||^2(\omega_t)dt\right\}}{\mathbb{E}_{\mathbb{W}_M}\exp\left\{-\frac{1}{4}\int_0^1 R(\omega_t)dt + \frac{1}{8}\int_0^1 ||\sigma||^2(\omega_t)dt\right\}},$$

where $R(a)$ is the scalar curvature and $\sigma(a)$ is the tension field of M at the point $a \in M$.

Proof. First, let us prove that $\rho\mu_\varepsilon \to \rho_0\mathbb{W}_M$ weakly. Let $h : C(\mathbb{R}^n) \to \mathbb{R}$ be continuous and bounded. Then

$$|\mathbb{E}_{\rho\mu_\varepsilon} h - \mathbb{E}_{\rho_0\mathbb{W}_M} h| \le ||h||_\infty \mathbb{E}_{\mu_\varepsilon}|\rho - \rho_0| + |\mathbb{E}_{\mu_\varepsilon} h\rho_0 - \mathbb{E}_{\mathbb{W}_M} h\rho_0| \to 0,$$

where the first term tends to 0 by Lemma 23 and the second term tends to 0 due to the weak convergence of μ_ε to $\mathbb{W}_M$ and since $h\rho_0$ is continuous. Now we can compute

$$\begin{aligned}\lim_{\varepsilon\to 0} \mathbb{E}_{\mathbb{W}_\varepsilon} h &= \lim_{\varepsilon\to 0} \frac{\mathbb{E}_{\mathbb{W}} \mathbf{1}_{C(\mathbb{M}_\varepsilon)} h}{\mathbb{E}_{\mathbb{W}} \mathbf{1}_{C(\mathbb{M}_\varepsilon)}} = \lim_{\varepsilon\to 0} \frac{\mathbb{E}_{\mu} \mathbf{1}_{C(\mathbb{M}_\varepsilon)} \rho h}{\mathbb{E}_{\mu} \mathbf{1}_{C(\mathbb{M}_\varepsilon)} \rho} = \lim_{\varepsilon\to 0} \frac{\mathbb{E}_{\mu} \mathbf{1}_{C(\mathbb{M}_\varepsilon)} h\rho}{\mathbb{E}_{\mu} \mathbf{1}_{C(\mathbb{M}_\varepsilon)}} \cdot \frac{\mathbb{E}_{\mu} \mathbf{1}_{C(\mathbb{M}_\varepsilon)}}{\mathbb{E}_{\mu} \mathbf{1}_{C(\mathbb{M}_\varepsilon)} \rho} \\ &= \lim_{\varepsilon\to 0} \frac{\mathbb{E}_{\mu_\varepsilon} \rho h}{\mathbb{E}_{\mu_\varepsilon} \rho} = \lim_{\varepsilon\to 0} \frac{\mathbb{E}_{\rho\mu_\varepsilon} h}{\mathbb{E}_{\rho\mu_\varepsilon} \mathbf{1}} = \lim_{\varepsilon\to 0} \frac{\mathbb{E}_{\rho_0\mathbb{W}_M} h}{\mathbb{E}_{\rho_0\mathbb{W}_M} \mathbf{1}} = \lim_{\varepsilon\to 0} \frac{\mathbb{E}_{\mathbb{W}_M} \rho_0 h}{\mathbb{E}_{\mathbb{W}_M} \rho_0} \\ &= \mathbb{E}_{\mathbb{W}_M} h \left[\frac{\rho_0}{\mathbb{E}_{\mathbb{W}_M} \rho_0} \right],\end{aligned}$$

where the last line follows from the first step of the proof. This means that $\mathbb{W}_\varepsilon$ converges weakly to a measure $\mathbb{W}_0$ that is absolutely continuous with respect to the Wiener measure $\mathbb{W}_M$ with the density ρ given above, and the theorem is proved. □

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