A Quick Glimpse of Symplectic Topology

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Definition of SYMPLECTIC

Popularity: Bottom 20% of words

1 : relating to or being an intergrowth of two different minerals (as in ophicalcite, myrmekite, or micropegmatite)

2 : relating to or being a bone between the hyomandibular and the quadrate in the mandibular suspensorium of many fishes that unites the other bones of the suspensorium

Why *Symplectic*?

On page 165 of his book "The Classical Groups" Hermann Weyl starts a chapter on the Symplectic Group. In a footnote he writes:

> * The name "complex group" formerly advocated by me in allusion to line complexes, as these are defined by the vanishing of antisymmetric bilinear forms, has become more and more embarrassing through collision with the word "complex" in the connotation of complex number. I therefore propose to replace it by the corresponding Greek adjective "symplectic." Dickson calls the group the "Abelian linear group" in homage to Abel who first studied it.
>
> 165

Some Definitions

Definition 1:

A symplectic form on a real vector space V is a non-degenerate antisymmetric bilinear form

$$\omega \colon V \times V \longrightarrow \mathbb{R}$$

Exercise: Such a thing exists if and only if the dimension of V is even! Here is (the) one on $\mathbb{R}^4$:

$$\omega((v_1, v_2, v_3, v_4), (w_1, w_2, w_3, w_4)) = v_1 w_2 - v_2 w_1 + v_3 w_4 - v_4 w_3$$

Definition 2:

A symplectic structure on a manifold M is a 2-form ω, which is everywhere non-degenerate and closed. A symplectic manifold is a manifold with a fixed symplectic structure.

Some Examples

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- Complex projective varieties
- many more...

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Why I like it:

- *Flexibility* vs. *rigidity*
- Employs differential geometry, analysis, homological algebra, sheaf theory...

Why "topology"?

Darboux Theorem

Let (M^{2n}, ω) be a symplectic manifold and $p \in M$ be a point. Then there exists an open set $U \subseteq M$, containing p, an open set $V \subseteq \mathbb{R}^{2n}$ and a diffeomorphism $\varphi \colon V \to U$ such that

$$\varphi^*\omega = d\,x_1 \wedge d\,y_1 + \cdots + d\,x_n \wedge d\,y_n.$$

Lagrangian submanifolds

Definition 3

A Lagrangian subspace of (V^{2n}, ω) is an $n-$dimensional subspace $L \leq V$ such that for every $v, w \in L$ one has $\omega(v, w) = 0$.

Definition 4

A Lagrangian submanifold of (M^{2n}, ω) is an $n-$dimensional submanifold $L \subseteq M$ such that for every $x \in L$ the tangent space $T_x L$ is a Lagrangian subspace of $T_x M$.

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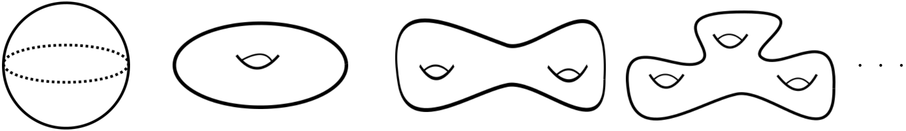

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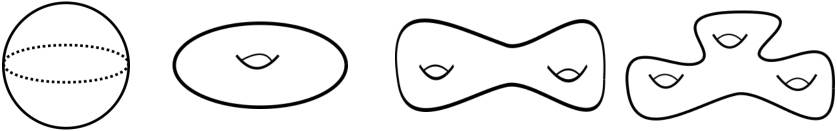
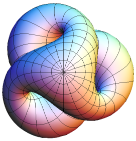


. . .

   b) Non-orientable

$\mathbb{R}P^2$



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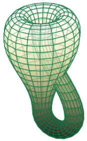

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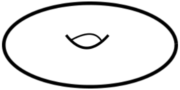

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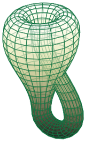

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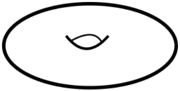

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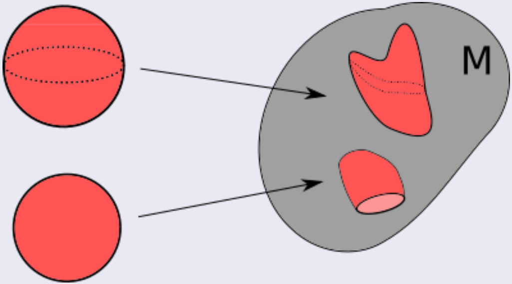
Definition 5

An almost complex structure on a manifold M is a section $J \in \Gamma(End(TM))$ such that $J^2 = -1$. A pseudoholomorphic curve in (M, J) is a map $f : (\Sigma, j) \to M$ from a surface Σ to M which satisfies the equation

$$df \circ j = J \circ df$$



These things are good, but when J is not *integrable* they can be quite wild. But if there is a symplectic structure ω compatible with J (i.e. such that $g_J = \omega(\cdot, J\cdot)$ is a Riemannian metric) things work out great!

mainly because...

$$\begin{aligned}
\int_\Sigma f^*\omega &= \int \omega(\partial_s f, \partial_t f)\, ds\, dt \\
&= \int \sqrt{\omega(\partial_s f, J\partial_s f)\omega(\partial_t f, J\partial_t f)}\, ds\, dt \\
&= \int \sqrt{\det(\{g_J(\partial_i f, \partial_j f)\}_{i,j})}\, ds\, dt \\
&= \int_\Sigma dVol_{f^* g_J}
\end{aligned}$$

that is...

Symplectic area is actual area!

Applications

Gromov's non-squeezing theorem

Suppose there exists an embedding $F\colon B^{2n}(R) \hookrightarrow B^2(r) \times \mathbb{R}^{2n-2}$ which respects the standard symplectic structures. Then $R \leq r$.

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Lagrangians in $\mathbb{R}^{2n}$

There are no compact simply-connected Lagrangians in $\mathbb{R}^{2n}$.

Input from many walks of mathematics

For the past 30 years people have been working hard and have come up with things like: Quantum Cohomology, Floer homology, Symplectic homology, Fukaya categories, Symplectic Field Theory.... All these things are machines which take information about the complicated ways in which pseudoholomorphic curves bend and twist and repackage it into algebra. Then one can do some homological algebra and deduce more and more surprising rigidity facts.

More recently people have applied microlocal sheaf theory to try and answer the same questions without pseudoholomorphic curves.

Open Problems

$Symp(M, \omega)$

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Arnold's Nearby Lagrangian Conjecture

Is it true that every compact exact Lagrangian in the cotangent bundle of a compact manifold can be mapped to the zero section by a Hamiltonian isotopy?

a weaker question...

Is it true that if M and N are compact manifolds and there exists a diffeomorphism $F\colon (T^*M, \omega_{can}) \to (T^*N, \omega_{can})$ which respects the symplectic structures, then M is diffeomorphic to N?

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a rephrasing...

Is all of differential topology just a part of symplectic topology?

Disclaimer!

This last thing was a *massive* piece of propaganda and exaggeration! This is a good time for you to stop listening to me!

Thank you!