Darboux property of Gâteaux derivatives of functions on $\mathbb{R}^n$

Marianna Csörnyei and Petr Holický*

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Abstract

D. Preiss proved that the graph of the derivative of a continuous Gâteaux-differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ is always connected. We show that this is no longer true in higher dimensions: we construct a continuous, Gâteaux-differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$ for which the range of its gradient mapping $\{\nabla f(x) : x \in \mathbb{R}^3\}$ is disconnected. We also give an example of an approximately differentiable continuous function on $\mathbb{R}^2$ such that the range of its gradient mapping is disconnected.

1 Introduction

As a generalization of the classical Darboux property of derivatives, J. Malý proved in [5] that the range of the derivative of any Fréchet-differentiable function defined on a connected open subset of a Banach space is connected. T. Mátrai showed in [6] that the graph of the derivative of a Gâteaux-differentiable Lipschitz function is connected if X^* is endowed with the w^*-topology and X is separable. But the analogue of Malý's result does not hold for Gâteaux derivatives: R. Deville and P. Hájek proved in [4, Theorem 1] that for every infinite dimensional separable Banach space X there exists a Gâteaux-differentiable Lipschitz function $f : X \to \mathbb{R}$ (with bounded support) whose derivative has range containing an isolated point (although

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the Lipschitz property is not explicitly stated, one may easily check that the norm of the derivative at each point is less than 4 in the proof of Theorem 1 in [4]).

The proof of the Darboux property of derivatives for functions $f : \mathbb{R} \to \mathbb{R}$ is based on Rolle's theorem: if a differentiable function $f : \mathbb{R} \to \mathbb{R}$ vanishes at two points of $\mathbb{R}$, then there is always a point between these two points at which the derivative of f is zero. This statement easily extends to finite dimensional spaces: if f is continuous and Gâteaux-differentiable on $\mathbb{R}^n$ and f vanishes on the boundary of a bounded open set U, then of course at the point $x \in U$ where $|f|$ attains its maximum we have $\nabla f(x) = 0$. The analogous statement is no longer true in infinite dimensional Banach spaces (see, e.g., in [2]). But since Rolle's theorem holds in $\mathbb{R}^n$, one can expect that the range of the derivative of a continuous Gâteaux-differentiable function on $\mathbb{R}^n$ is connected.

An unpublished result of D. Preiss says that indeed the range of the Gâteaux derivative is connected in $\mathbb{R}^2$. For sake of completeness in this paper we recall his proof. However, it turns out that Darboux property does not hold for derivatives in higher dimensional spaces: the main result of our paper is the construction of a continuous Gâteaux-differentiable function on $\mathbb{R}^3$ whose derivative has range containing an isolated point.

Another weakening of the Fréchet derivative in finite dimensional spaces is the so-called approximate derivative: a function $f : \mathbb{R}^n \to \mathbb{R}$ is approximately differentiable at a point x and its approximate derivative is L if there exists a set A such that x is a Lebesgue density point of A, and $f|_A$ is differentiable with derivative L (note that there is no difference between approximate Gâteaux and Fréchet derivatives, i.e. the differentiability of $f|_A$ in the definition may be equivalently understood in the Fréchet or in the Gâteaux sense). Equivalently, the approximate derivative of f is L at the point x if for each $\varepsilon > 0$, x is a dispersion point of the set $\{a : |f(x) - f(a) - L(x-a)| > \varepsilon\|x-a\|\}$. One can check that the graph of the derivative of a continuous approximately differentiable function $f : \mathbb{R} \to \mathbb{R}$ has connected graph (see e.g. [3, Chapter 10, §2 and Theorem 1.1]). We show that this is no longer true for functions even on $\mathbb{R}^2$.

We always consider the Euclidean norm $\|\cdot\|$ in $\mathbb{R}^n$. We denote by $B(x, r)$ the open ball, centered at $x \in \mathbb{R}^n$ and with radius $r > 0$.

2 Gâteaux derivatives on the plane

D. Preiss proved the following.

Theorem 1. *Let G be an arbitrary open connected set in $\mathbb{R}^2$, and let f be a Gâteaux-differentiable continuous function $f: G \to \mathbb{R}$. Then the graph of its derivative $\{(x, \nabla f(x)) : x \in G\} \subset \mathbb{R}^4$ is connected. In particular, the range of the derivative $\{\nabla f(x) : x \in G\} \subset \mathbb{R}^2$ is connected.*

We recall his proof.

Proof of Theorem 1 Suppose that $\{(x, \nabla f(x)) : x \in G\}$ is not connected. First we show that there exists an open ball B and a point $v \in \partial B$ such that $\nabla f(v)$ does not belong to the closure of the set $\{\nabla f(u) : u \in B\}$ (a stronger result was proved e.g. in [6]; for sake of completeness we give a short proof in our particular case).

If $\{(x, \nabla f(x)) : x \in G\}$ is not connected then it can be covered by disjoint non-empty open sets H_1, H_2, and since the mapping $x \mapsto (x, \nabla f(x))$ is Baire-1, therefore there are G_δ sets G_1, G_2 covering G such that $\{(x, \nabla f(x)) : x \in G_i\} \subset H_i$ for $i = 1, 2$. Since G is connected therefore $F \stackrel{\text{def}}{=} G \setminus (\operatorname{int} G_1 \cup \operatorname{int} G_2)$ is a nonempty relatively closed subset of G. By Baire category theorem, it is not possible that both G_1 and G_2 are dense in F. Therefore we may assume that there are $z \in F$ and $r > 0$ such that $B(z, r) \subset G$ and $B(z, r) \cap F$ is disjoint from G_1. Since $z \in F$ therefore $z \in \partial G_1 \cap \partial G_2$, in particular, both G_1 and G_2 intersect $B(z, r)$; $B(z, r) \cap G_1$ is a nonempty open and $B(z, r) \cap G_2$ is a nonempty relatively closed set in $B(z, r)$. We choose an open ball $B \subset B(z, r) \cap G_1$, so that it has a point v on its boundary with $v \in \partial B \cap G_2$. Then $(v, \nabla f(v)) \in H_2$, and $(u, \nabla f(u)) \in H_1$ for all $u \in B$, in particular, $(v, \nabla f(v))$ is not in the closure of $\{(u, \nabla f(u)) : u \in B\}$. By choosing a sub-ball of B if necessary, our claim is proved.

Let $C \subset \mathbb{R}^2$ be a convex cone with vertex v such that the interior of some neighbourhood of v in C is a subset of B. We have

$$\inf\{\|\nabla f(v) - \nabla f(u)\| : u \in C \cap B\} > 0.$$

Without loss of generality, let us suppose that $v = 0$, $f(0) = 0$, $\nabla f(0) = 0$, and for some $c > 0$,

$$C \cap B \supset \{(t, th) \in \mathbb{R}^2 : t > 0, |h| < c\} \cap B(0, 1),$$

$$\inf\{\|\nabla f(u)\| : u \in C \cap B\} > c. \tag{1}$$

Put $D = \{u \in C : \|u\| = 1\}$, and for each $\delta \in (0,1)$ consider the set

$$F_\delta = \{u \in D : |f(tu)| \le ct/4 \text{ if } 0 \le t \le \delta\}.$$

Each F_δ is closed in D since f is continuous, and since $\nabla f(0) = 0$ and f is Gâteaux-differentiable at 0, the union of the sets F_δ contains D. Hence by Baire category theorem there is a δ and there are distinct elements u^1 and u^2 in D such that the closed arc D^* between u^1, u^2 belongs to F_δ. Let $2u^*$ be the midpoint of this arc D^*, and let ρ be the distance of u^* from the line segments $[0, u^1]$, $[0, u^2]$.

Choose $\delta' < \delta$ so small that $|f(tu^i)| < ct\rho/2$ for $t \in (0, \delta')$, $i = 1, 2$, and let $D' = \delta' D^*$, $C' = \operatorname{conv}\{0, D'\}$, $u' = \delta' u^*$. Then

$$|f(x)| \le c \operatorname{dist}(x, u')/2 \quad \forall x \in \partial C'. \tag{2}$$

Indeed, $|f(x)| \le c\delta'\rho/2 \le c \operatorname{dist}(x, u')/2$ for all $x \in [0, \delta' u^i]$, $i = 1, 2$. And since $D^* \subset F_\delta$, $|f(x)| \le c\delta'/4 \le c \operatorname{dist}(x, u')/2$ if $x \in D'$.

We may and shall suppose without loss of generality that $f(u') \ge 0$. Let

$$g(x) = f(x) - c \cdot \operatorname{dist}(x, u').$$

Then $g(u') \ge 0$, and (2) shows that g is negative on $\partial C'$. Therefore g attains its maximum on C' at a point $x \in \operatorname{int}(C')$, and from the definition of g it follows that $\|\nabla f(x)\| \le c$. This contradicts (1) and proves Theorem 1. ∎

3 Gâteaux derivatives on $\mathbb{R}^3$

Our main result is the following theorem.

Theorem 2. *There is a continuous function $f : \mathbb{R}^3 \to \mathbb{R}$ which is*

(i) C^∞ *on* $\mathbb{R}^3 \setminus \{0\}$;

(ii) *Gâteaux-differentiable at the origin;*

(iii) $\nabla f(0) = 0$ *and* $\|\nabla f(x)\| > 1$ *for every* $x \neq 0$.

Proof of Theorem 2 First we state three technical lemmas about the existence of three functions of three variables from which Theorem 2 will follow easily. Lemma 3 is the most important part of our construction. Lemmas 4 and 5 will enable us to extend the function from Lemma 3 to the whole space $\mathbb{R}^3$. We use the notation $r = \sqrt{x^2 + y^2}$ for $(x, y, z) \in \mathbb{R}^3$.

Lemma 3. *There is a continuous function $f_1 : G_1 \cup \{0\} \to \mathbb{R}$, where*

$$G_1 = \{(x, y, z) : 3r^2 e^{-r} < z\}$$

such that

(3a) $f_1(x, y, z) = y$ *on the set* $\{(x, y, z) : 3r^2 e^{-r} < z < 4r^2\}$;

(3b) $f_1(0, 0, 0) = 0$, *and the directional derivatives of f_1 at the origin in directions (x, y, z) are zero if $z > 0$;*

(3c) $f_1 \in C^\infty(G_1)$ *and there is a $\nabla_1 > 0$ such that $\|\nabla f_1\| > \nabla_1$ on G_1.*

Lemma 4. *There is a continuous function $f_2 : G_2 \cup \{0\} \to \mathbb{R}$, where*

$$G_2 = \{(x, y, z) : -2r^2 e^{-r} < z < 2r^2 e^{-r}\}$$

such that

(4a) $f_2(x, y, z) = r$ *if* $r^2 e^{-r} < z$;

(4b) $f_2(x, y, -z) = -f_2(x, y, z)$, *in particular,* $f_2(x, y, z) = 0$ *if* $z = 0$;

(4c) $f_2 \in C^\infty(G_2)$ *and there is a $\nabla_2 > 0$ such that $\|\nabla f_2\| > \nabla_2$ on G_2.*

Lemma 5. *There is a continuous function $f_3 : G_3 \cup \{0\} \to \mathbb{R}$, where*

$$G_3 = \{(x, y, z) : r^2 e^{-r} < z < 4r^2 e^{-r}\}$$

such that

(5a) $f_3(x, y, z) = r$ *if* $z < 2r^2 e^{-r}$ *and* $f_3(x, y, z) = y$ *if* $z > 3r^2 e^{-r}$;

(5b) $f_3(0, 0, 0) = 0$;

(5c) $f_3 \in C^\infty(G_3)$ *and there is a $\nabla_3 > 0$ such that $\|\nabla f_3\| > \nabla_3$ on G_3.*

Let f_1, f_2, f_3 be functions that satisfy Lemmas 3-5. We define $f(0,0,0) = 0$, $f = f_1$ on G_1, $f = f_2$ on G_2, $f = f_3$ on G_3, and $f(x,y,z) = -f(x,y,-z)$ if $(x,y,-z) \in G_1 \cup G_2 \cup G_3$. The function f is well defined by (3a)-(3b), (4a)-(4b) and (5a)-(5b) on the whole space $\mathbb{R}^3$, and it is continuous. Using (3b) and (4b) we see that the Gâteaux derivative of f at the origin is zero. By (3c), (4c) and (5c), $f \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ and $\|\nabla f\| > \min\{\nabla_1, \nabla_2, \nabla_3\} > 0$ on $\mathbb{R}^3 \setminus \{0\}$. Replacing f by a sufficiently large constant multiple of f, the conditions of Theorem 2 are satisfied. ■

Proof of Lemma 4 Let $g : (-2,2) \to [-1,1]$ be such that

(4α) $g(-\zeta) = -g(\zeta)$ and $g(\zeta) = 1$ for $\zeta > 1$;

(4β) $g \in C^\infty(-2,2)$, $|g'| \geq 1$ on $[-\frac{1}{2}, \frac{1}{2}]$, and $|g| \geq \frac{1}{2}$ if $|\zeta| \geq \frac{1}{2}$,

and let

$$f_2(x,y,z) = rg\left(\frac{z}{r^2e^{-r}}\right) \text{ for } |z| < 2r^2e^{-r} \quad \text{and} \quad f(0,0,0) = 0.$$

Then f_2 is continuous at the origin since $|f_2| \leq r$, (4a)-(4b) follow from (4α), and $f \in C^\infty(G_2)$ follows from (4β). To prove Lemma 4 it remains to estimate the norm of the gradient of f_2 from below.

Let (x,y,z) be an arbitrary point in G_2. Then

$$\frac{\partial}{\partial z} f_2(x,y,z) = g'\left(\frac{z}{r^2e^{-r}}\right)\frac{e^r}{r}.$$

Suppose that $|\frac{\partial}{\partial z} f_2(x,y,z)|$ is less or equal to some positive ∇_2. Then

$$\left|g'\left(\frac{z}{r^2e^{-r}}\right)\right| \leq \frac{r\nabla_2}{e^r} \leq \frac{\nabla_2}{e}. \tag{3}$$

Hence for $\nabla_2 < e$, from (4β) we obtain

$$\left|g\left(\frac{z}{r^2e^{-r}}\right)\right| \geq \frac{1}{2}. \tag{4}$$

Differentiating the function $f_2(r,z) = rg(\frac{z}{r^2e^{-r}})$ with respect to the variable r, using (4) and (3) we get

$$\begin{aligned} \left|\frac{\partial}{\partial r} f_2(r,z)\right| &= \left|g\left(\frac{z}{r^2e^{-r}}\right) + rg'\left(\frac{z}{r^2e^{-r}}\right) z\,\frac{e^r(r^2-2r)}{r^4}\right| \\ &\geq \frac{1}{2} - r\cdot\frac{r\nabla_2}{e^r}\cdot 2r^2e^{-r}\cdot\frac{e^r(r^2-2r)}{r^4} \\ &= \frac{1}{2} - 2\nabla_2 e^{-r}(r^2-2r). \end{aligned}$$

Using that

$$|e^{-r}(r^2 - 2r)| \leq 1, \tag{5}$$

we get $|\frac{\partial}{\partial r} f_2(r,z)| > \nabla_2$ for $\nabla_2 < 1/6$. (To check (5) we may, e.g., easily verify that $0 \leq e^{-r}r^2 \leq e^{-2}2^2 \leq 1$ and $0 \leq e^{-r}2r \leq e^{-1}2 \leq 1$.) Finally, note that

$$\|\nabla f_2(x,y,z)\| \geq \left|\frac{\partial}{\partial r} f_2(r,z)\right| \cdot \left\|\left(\frac{x}{r}, \frac{y}{r}\right)\right\| > \nabla_2,$$

and Lemma 4 is proved. ■

Proof of Lemma 5 Let $g : (1,4) \to [0,1]$ be such that

(5α) $g(\xi) = 1$ if $\xi \geq 3$ and $g(\xi) = 0$ if $\xi \leq 2$;

(5β) $g \in C^\infty(1,4)$, $g'(\xi) \geq 1$ for $\xi \in [\frac{9}{4}, \frac{11}{4}]$, $|g(\xi)| \geq \frac{3}{4}$ if $\xi \geq \frac{11}{4}$, and $|g(\xi)| \leq \frac{1}{4}$ if $\xi \leq \frac{9}{4}$,

and let

$$f_3(x,y,z) = yg\left(\frac{z}{r^2e^{-r}}\right) + r\left(1 - g\left(\frac{z}{r^2e^{-r}}\right)\right) \text{ on } G_3 \quad \text{and} \quad f_3(0,0,0) = 0.$$

Then f_3 is continuous at the origin since $|f_3(x,y,z)| \leq |y| + r$, (5a) follows from ($5\alpha$), and it is obvious that $f_3 \in C^\infty(G_3)$. Let $\nabla_3 \in (0, 1/40)$ be fixed. If

$$\left|\frac{\partial}{\partial z} f_3(x,y,z)\right| = \left|(y-r)g'\left(\frac{z}{r^2e^{-r}}\right)\frac{e^r}{r^2}\right| \leq \nabla_3,$$

then either

$$\frac{(r-y)e^r}{r^2} \leq {\nabla_3}^{1/2}, \tag{6}$$

or

$$\left|g'\left(\frac{z}{r^2e^{-r}}\right)\right| \leq \nabla_3^{1/2}. \tag{7}$$

We can also calculate with the help of (5)

$$\begin{aligned}
\left|\frac{\partial}{\partial y} f_3(x,y,z)\right| &= \left|\frac{y}{r} + \left(1-\frac{y}{r}\right) g\left(\frac{z}{r^2e^{-r}}\right) + (y-r)g'\left(\frac{z}{r^2e^{-r}}\right) z\,\frac{e^r(r^2-2r)}{r^4}\,\frac{y}{r}\right| \\
&\geq \left|\frac{y}{r} + \left(1-\frac{y}{r}\right) g\left(\frac{z}{r^2e^{-r}}\right)\right| - \frac{r^2\nabla_3}{e^r}\cdot 4r^2e^{-r}\cdot\frac{e^r(r^2-2r)}{r^4}\cdot 1 \\
&\geq \left|\frac{y}{r} + \left(1-\frac{y}{r}\right) g\left(\frac{z}{r^2e^{-r}}\right)\right| - 4\nabla_3, \text{ and} \\
\left|\frac{\partial}{\partial x} f_3(x,y,z)\right| &= \left|\frac{x}{r} - \frac{x}{r}\cdot g\left(\frac{z}{r^2e^{-r}}\right) + (y-r)g'\left(\frac{z}{r^2e^{-r}}\right) z\,\frac{e^r(r^2-2r)}{r^4}\,\frac{x}{r}\right| \\
&\geq \left|\frac{x}{r} - \frac{x}{r}\cdot g\left(\frac{z}{r^2e^{-r}}\right)\right| - 4\nabla_3.
\end{aligned}$$

If (6) holds then we have

$$1-\frac{y}{r} \leq \frac{r{\nabla_3}^{1/2}}{e^r} \leq \frac{{\nabla_3}^{1/2}}{e},$$

and we may estimate

$$\begin{aligned}
\left|\frac{\partial}{\partial y} f_3(x,y,z)\right| &\geq \frac{y}{r} + \left(1-\frac{y}{r}\right) g\left(\frac{z}{r^2e^{-r}}\right) - 4\nabla_3 \\
&\geq \left(1-\frac{{\nabla_3}^{1/2}}{e}\right) - 4\nabla_3 \geq 1 - 5{\nabla_3}^{1/2} > {\nabla_3}^{1/2} > \nabla_3
\end{aligned}$$

since $\nabla_3 < \frac{1}{36}$. If (7) holds then, according to (5β) (and since $\nabla_3 < 1$), there are two possibilities.

First, if $|g(\frac{z}{r^2e^{-r}})| \geq \frac{3}{4}$, then

$$\begin{aligned}
\left|\frac{\partial}{\partial y} f_3(x,y,z)\right| &\geq g\left(\frac{z}{r^2e^{-r}}\right) + \frac{y}{r}\ \left(1 - g\left(\frac{z}{r^2e^{-r}}\right)\right) - 4\nabla_3 \\
&\geq \frac{3}{4} - \frac{1}{4} - 4\nabla_3 > \nabla_3
\end{aligned}$$

since $\nabla_3 < \frac{1}{10}$. In the remaining case we have $g(\frac{z}{r^2e^{-r}}) \leq \frac{1}{4}$. Then

$$\begin{aligned}
\left|\frac{\partial}{\partial y} f_3(x,y,z)\right| + \left|\frac{\partial}{\partial x} f_3(x,y,z)\right| &\geq \left|\frac{y}{r}\right| + \left|\frac{x}{r}\right| - \left(\left|1-\frac{y}{r}\right| + \left|\frac{x}{r}\right|\right)\cdot\frac{1}{4} - 8\nabla_3 \\
&\geq 1 - \frac{3}{4} - 8\nabla_3 > 2\nabla_3
\end{aligned}$$

since $\nabla_3 < 1/40$.

Summarizing the above considerations, we obtain that $\|\nabla f_3(x, y, z)\| > \nabla_3$ on G_3. ■

The rest of this section is devoted to the proof of the existence of the function f_1 from Lemma 3. Our proof has the following structure. In Subsection 3.1 we show how one can get the required function f_1, provided that we have a function h whose properties are described by Lemma 6. Then, in Subsection 3.2, we show how to obtain h of Lemma 6, using the mappings H (defined in Lemma 9) and J (defined in a straightforward way using σ from Lemma 10). While it is not difficult to verify the existence of J, the proof of the existence of H will need further steps: in Subsection 3.3 we will describe a preliminary mapping H^0 in Lemma 16, and then we will get a mapping H^∞ by a "successive implementation" of a modified H^0 in Lemma 17. Finally, using H^∞, we will construct H.

To prevent possible misunderstanding, we point out that by $\lim_{z\to\infty}$ we always mean $\lim_{z\to+\infty}$. For a given point $(x, y, z) \in \mathbb{R}^3$, we will often use, without further explanation, the notation $r = \sqrt{x^2 + y^2}$.

3.1 Construction of f_1 using h

Using a transform of the half space $\{(x, y, z) \in \mathbb{R}^3 : z > 0\}$, we will show that in order to prove Lemma 3 it is enough to construct a function h with the properties listed in the following lemma.

Lemma 6. *There is $h \in C^\infty(\mathbb{R}^3)$ such that*

(6a) $h(x, y, z) = y$ *if* $z \le r^2$*;*

(6b) $|h(x, y, z)| \le 9r + 1$ *on* $\mathbb{R}^3$*;*

(6c) $\lim_{z\to\infty} h(x, y, z) = 0$ *for* $(x, y) \in \mathbb{R}^2$*;*

(6d) $\|(\frac{\partial}{\partial x}h, \frac{\partial}{\partial y}h)\| > \nabla_h$ *on* $\mathbb{R}^3$ *for some* $\nabla_h > 0$.

Proof of Lemma 3 Let h be a function satisfying Lemma 6, and define

$$f_1(x, y, z) = \frac{z}{2} \cdot h\left(\frac{2x}{z}, \frac{2y}{z}, \frac{1}{z}\right), \quad z > 0.$$

Set $f_1(0,0,0)=0$. If $\frac{1}{z} \leq \|(\frac{2x}{z}, \frac{2y}{z})\|^2$, i.e., if $z \leq 4r^2$, then

$$f_1(x,y,z) = \frac{z}{2} \cdot \frac{2y}{z} = y$$

by (6a), and (3a) is verified. The function f_1 is continuous at the origin (with respect to $\{0\} \cup G_1$), since

$$\left| \frac{z}{2} \cdot h\left(\frac{2x}{z}, \frac{2y}{z}, \frac{1}{z}\right) \right| \leq \frac{z}{2} \cdot \left(9 \cdot \frac{2r}{z} + 1\right) = 9r + \frac{z}{2}.$$

Using (6c) for some $(x,y,z) \in \mathbb{R}^2 \times (0,\infty)$, we get

$$\lim_{\lambda \to 0^+} \frac{f_1(\lambda x, \lambda y, \lambda z)}{\lambda} = \frac{z}{2} \cdot \lim_{\lambda \to 0^+} h\left(\frac{2x}{z}, \frac{2y}{z}, \frac{1}{\lambda z}\right) = 0,$$

hence (3b) holds. We may estimate

$$\left\| \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) f_1(x,y,z) \right\| = \frac{z}{2} \cdot \frac{2}{z} \cdot \left\| \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) h\left(\frac{2x}{z}, \frac{2y}{z}, \frac{1}{z}\right) \right\| > \nabla_h,$$

and Lemma 3 is proved. ■

3.2 Construction of h using J and H

The function h of Lemma 6 will be defined as the projection of $\mathbb{R}^3$ to its second coordinate composed with a diffeomorphism $H \circ J$ of $\mathbb{R}^3$ onto itself, where $H, J : \mathbb{R}^3 \to \mathbb{R}^3$ are diffeomorphisms such that they do not change the third coordinate of any point (x,y,z).

Notation 7. We will use the notation H_z, J_z for the diffeomorphisms H, J restricted to the horizontal plane of points whose third coordinate equals to z, respectively.

Notation 8. By (closed) *intervals* in $\mathbb{R}^d$ we understand subsets of $\mathbb{R}^d$ which are Cartesian products of compact intervals in $\mathbb{R}$. For an interval $R = [x_1 - a_1, x_1 + a_1] \times [x_2 - a_2, x_2 + a_2] \subset \mathbb{R}^2$ and $c > 0$, we use $c * R$ to denote the interval $[x_1 - ca_1, x_1 + ca_1] \times [x_2 - ca_2, x_2 + ca_2]$. We denote $S = [-1,1]^2$, and $D_k = (3^k * S) \setminus (3^{k-1} * S)$ for $k \in \mathbb{Z}$.

Lemma 9. *There is a* $\nabla_H > 0$, *a* C^∞*-diffeomorphism* H *of* $\mathbb{R}^3$ *onto itself that does not change the third coordinate of any point, and there are finite sets* $I(k,n)$ *and closed intervals* $P_k^n(z,i) \subset D_k$, $z \in \mathbb{R}$, $k \in \mathbb{Z}$, $n \in \mathbb{N}$, $i \in I(k,n)$, *such that*

(9a) H *is the identity mapping on the set* $\{(x,y,z) : z \le r^2\}$;

(9b) $\|\frac{\partial}{\partial y}H\| < (\nabla_H)^{-1} < \infty$ *everywhere;*

(9c) *for each* $z \in \mathbb{R}$ *and* $k \in \mathbb{Z}$, H_z *maps* D_k *onto itself (consequently,* $H_z(0,0) = (0,0)$ *for each* $z \in \mathbb{R}$*);*

(9d) *for each* $k \in \mathbb{Z}$ *and* $n \in \mathbb{N}$, *if* $z \in \mathbb{R}$ *is large enough, then* $H_z(\mathbb{R} \times [-3^{-n}, 3^{-n}]) \supset D_k \setminus P_k^n(z)$, *where* $P_k^n(z) := \bigcup_{i \in I(k,n)} P_k^n(z,i)$;

(9e) $P_k^n(z_1,i) \subset P_k^n(z_2,i)$ *if* $0 \le z_2 < z_1$, $k \in \mathbb{Z}$, $n \in \mathbb{N}$, $i \in I(k,n)$;

(9f) $\lim_{z\to\infty} \operatorname{area}(P_k^n(z)) = 0$ *for every* $n \in \mathbb{N}$ *and* $k \in \mathbb{Z}$.

We postpone the proof of Lemma 9; in this section we show how one can get the function h of Lemma 6, provided that we have a mapping H satisfying the statements of Lemma 9.

We shall use a perturbation J of the identity mapping on $\mathbb{R}^3$ of the form $J(x,y,z) = (x + \sigma(x,y,z), y + \sigma(x,y,z), z)$. First we describe the function σ:

Lemma 10. *For any given functions* $\delta_k^n : \mathbb{R} \to (0,\infty)$, *if* $\lim_{z\to\infty} \delta_k^n(z) = 0$ *for each* $k \in \mathbb{Z}$ *and* $n \in \mathbb{N}$, *then there is a function* $\sigma \in C^\infty(\mathbb{R}^3, [0, \frac{2}{3}))$ *such that*

(10a) $\sigma(x,y,z) = 0$ *if* $z \le r^2$;

(10b) $\lim_{z\to\infty} \sigma(x,y,z) = 0$ *for every* $(x,y) \in \mathbb{R}^2$;

(10c) *if* $k \in \mathbb{Z}$, $(x,y) \in D_k$, *and* $n \in \mathbb{N}$, *then* $\sigma(x,y,z) > \delta_k^n(z)$ *for every sufficiently large* z;

(10d) $\|(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\sigma\| \le \frac{1}{3}$ *everywhere.*

Proof of Lemma 10 Set

$$\delta_N(z) = \max\{\delta_k^n(z) : k \in \{-N, \ldots, N\},\ n \in \{1, \ldots, N\}\}$$

for $N \in \mathbb{N}$ and $z \in \mathbb{R}$. Let $\varphi \in C^\infty(\mathbb{R}^3, [0, \infty))$ be such that $\int_{\mathbb{R}^3} \varphi = 1$ and $\varphi = 0$ outside the unit ball in $\mathbb{R}^3$. Denote by I the integral $\int_{\mathbb{R}^3} \|(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\varphi\|$. We choose an increasing sequence of positive numbers z_N tending to infinity such that

(10α) $\operatorname{dist}(\{(x, y, z) : z \le r^2\}, U) > 2$, where $U = \bigcup_{N=1}^{\infty}(3^N * S) \times [z_N, z_{N+1}]$,

(10β) $\delta_N(z) < \frac{1}{3(N+I)}$ for $N \in \mathbb{N}$ and $z \in [z_N, \infty)$.

We are going to find σ which fulfils the above properties (10a)-(10d) with (10c) improved to

(10c') $\sigma(x, y, z) > \delta_N(z)$ if $(x, y) \in 3^N * S$ and $z \ge z_N$.

Then (10c) holds by choosing $N = \max(k, n)$ and $z \ge z_N$.

For each N, set

$$\sigma_0(x, y, z) = \max\left\{\frac{1}{3(N+I)} : z \in [z_N - 1, z_{N+1} + 1]\right\},$$

if $\operatorname{dist}((x, y, z), U) \le 1$, and set $\sigma_0(x, y, z) = 0$ otherwise.

Then the convolution $\sigma = \sigma_0 * \varphi$ has the required properties. Property (10a) follows from the equality $\sigma = 0$ for all (x, y, z) which have distance at most one from the set $\{(x, y, z) : z \le r^2\}$. Property (10b) holds even uniformly in (x, y) by the corresponding property of σ_0. Property (10c') follows since $\sigma_0(x, y, z) \ge \delta_N(z)$ on the neighbourhood of $(3^N * S) \times [z_N, z_{N+1}]$ with radius one. Finally, (10d) holds since the choice of σ_0 ensures that $\sigma_0 \le \frac{1}{3I}$. ■

Proof of Lemma 6 Let $\delta_k^n(z)$ be the maximum of the lengths of the shorter sides of $P_k^n(z, i)$, $i \in I(k, n)$, described in Lemma 9. It follows from (9f) that $\lim_{z\to\infty} \delta_k^n(z) = 0$. We choose a function $\sigma \in C^\infty(\mathbb{R}^3, [0, \frac{2}{3}))$ as in Lemma 10 and we define

$$J(x, y, z) = (x + \sigma(x, y, z), y + \sigma(x, y, z), z).$$

We put

$$h = y \circ H^{-1} \circ J,$$

where $y : \mathbb{R}^3 \to \mathbb{R}$ is the projection to the second coordinate $(x, y, z) \mapsto y$. Recall that J_z denotes the restriction of J to the horizontal plane with third coordinate z, so

$$J_z(x, y) = (x, y) + (\sigma(x, y, z), \sigma(x, y, z)).$$

Property (6a) follows from (9a) and (10a). For $(x, y) \in 3^k * S$, $k = -1, 0, 1, \ldots$, and any z, we have $J_z(x, y) \in 3^{k+1} * S$ since $\sigma < \frac{2}{3}$. As H_z^{-1} maps each D_k, $k \in \mathbb{Z}$, to itself, we get $|h(x, y, z)| \leq 9 \max\{|x|, |y|\} \leq 9r$ if $(x, y) \in D_k$ for some $k = -1, 0, 1, \ldots$, or $|h(x, y, z)| \leq 1$, otherwise. So $|h(x, y, z)| \leq 9r + 1$ for all x, y, z, i.e. (6b) holds.

We are going to prove (6c) now. Since $\lim_{z\to\infty} \sigma(0, 0, z) = 0$ and H_z^{-1} maps each D_k to itself, we can see from the definition of h that $\lim_{z\to\infty} h(0, 0, z) = 0$. Fix a point $(x, y) \in \mathbb{R}^2 \setminus \{0\}$ and fix a number $n \in \mathbb{N}$. Choose $k \in \mathbb{Z}$ such that $(x, y) \in D_k$. There is a z_0 such that $\sigma(x, y, z) < 2 \cdot 3^{k-2}$ for every $z \geq z_0$, and thus

$$J_z(x, y) \in D_{k-1} \cup D_k \cup D_{k+1} \quad \text{for every } z \geq z_0.$$

We are going to show that $J_z(x, y) \notin \bigcup_{m=1}^n (P_{k-1}^m(z) \cup P_k^m(z) \cup P_{k+1}^m(z))$ for sufficiently large $z \geq z_0$.

Since $\bigcup_{m=1}^n (P_{k-1}^m(z_0) \cup P_k^m(z_0) \cup P_{k+1}^m(z_0))$ is a finite union of closed intervals, therefore there is a positive number ρ such that the open ball $B((x, y), \rho)$ intersects only those intervals of this union that contain (x, y). Due to the monotonicity from (9e), $P_\kappa^m(z) \cap B((x, y), \rho) = \emptyset$ if $\kappa = k - 1, k, k + 1$, $z \geq z_0$ and $(x, y) \notin P_\kappa^m(z_0)$. Since $\lim_{z\to\infty} \sigma(x, y, z) = 0$, there is a $z_1 > z_0$ such that for every $z \geq z_1$ we have $\|(\sigma(x, y, z), \sigma(x, y, z))\| < \rho$, and thus $J_z(x, y) \notin P_\kappa^m(z)$.

Let $(x, y) \in P_\kappa^m(z, i)$ for some $1 \leq m \leq n$, $k - 1 \leq \kappa \leq k + 1$, $i \in I(\kappa, m)$ and $z \geq z_0$. The length of the smaller side of $P_\kappa^m(z, i)$ is at most $\delta_\kappa^m(z)$, and $\delta_\kappa^m(z) < \sigma(x, y, z)$ by (10c), for every sufficiently large $z \geq z_0$. Hence $J_z(x, y) \notin P_\kappa^m(z, i)$ if z is large enough.

By (9d) and the definition of h, we conclude that $|h(x, y, z)| \leq 3^{-n}$ and so $\lim_{z\to\infty} h(x, y, z) = 0$.

Let $J = (J^1, J^2, J^3)$ with $J^i : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2, 3$, and compute $\frac{\partial}{\partial x} h$ first.

$$\begin{aligned}
\frac{\partial}{\partial x}h &= \left\langle (\frac{\partial}{\partial x},\frac{\partial}{\partial y})(y\circ H^{-1}), \frac{\partial}{\partial x}(J^1,J^2)\right\rangle \\
&= \left\langle (\frac{\partial}{\partial x},\frac{\partial}{\partial y})(y\circ H^{-1}), (1+\frac{\partial}{\partial x}\sigma, \frac{\partial}{\partial x}\sigma)\right\rangle \\
&= \frac{\partial}{\partial x}(y\circ H^{-1})(1+\frac{\partial}{\partial x}\sigma)+\frac{\partial}{\partial y}(y\circ H^{-1})\frac{\partial}{\partial x}\sigma.
\end{aligned}$$

Of course, the function $(y\circ H^{-1})$ should be evaluated at $J(x,y,z)$ and the functions h and σ at (x,y,z). Interchanging the derivatives with respect to x and y, we get

$$\frac{\partial}{\partial y}h = \frac{\partial}{\partial y}(y\circ H^{-1})(1+\frac{\partial}{\partial y}\sigma)+\frac{\partial}{\partial x}(y\circ H^{-1})\frac{\partial}{\partial y}\sigma.$$

Property (9b) implies the existence of a direction in which the derivative of the y-coordinate of H^{-1} is greater than ∇_H. That is, $\|(\frac{\partial}{\partial x},\frac{\partial}{\partial y})(y\circ H^{-1})\| > \nabla_H$. Computing now the gradient of h with respect to x and y, using (10d), we get

$$\begin{aligned}
\left\|\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)h\right\| &= \Big\|\Big(\frac{\partial}{\partial x}(y\circ H^{-1})(1+\frac{\partial}{\partial x}\sigma)+\frac{\partial}{\partial y}(y\circ H^{-1})\frac{\partial}{\partial x}\sigma, \\
&\quad \frac{\partial}{\partial y}(y\circ H^{-1})(1+\frac{\partial}{\partial y}\sigma)+\frac{\partial}{\partial x}(y\circ H^{-1})\frac{\partial}{\partial y}\sigma\Big)\Big\| \\
&\geq \Big\|(\frac{\partial}{\partial x},\frac{\partial}{\partial y})(y\circ H^{-1})\Big\| - \\
&\quad \Big(|\frac{\partial}{\partial x}(y\circ H^{-1})|+|\frac{\partial}{\partial y}(y\circ H^{-1})|\Big)\Big\|(\frac{\partial}{\partial x},\frac{\partial}{\partial y})\sigma\Big\| \\
&\geq \Big\|(\frac{\partial}{\partial x},\frac{\partial}{\partial y})(y\circ H^{-1})\Big\| - \\
&\quad 2\Big\|(\frac{\partial}{\partial x},\frac{\partial}{\partial y})(y\circ H^{-1})\Big\|\cdot\Big\|(\frac{\partial}{\partial x},\frac{\partial}{\partial y})\sigma\Big\| \\
&\geq \Big\|(\frac{\partial}{\partial x},\frac{\partial}{\partial y})(y\circ H^{-1})\Big\| - 2\Big\|(\frac{\partial}{\partial x},\frac{\partial}{\partial y})(y\circ H^{-1})\Big\|\cdot\frac{1}{3} \\
&\geq \frac{\nabla_H}{3},
\end{aligned}$$

and thus (6d) holds with $\nabla_h = \frac{\nabla_H}{4}$. ■

3.3 Construction of H – Step 1: Two technical lemmas

Before we turn to the very construction of H we need two auxiliary lemmas. The first one is about the existence of a *transform* T of an interval in a z_0-plane (with fixed third coordinate z_0) to another interval in a z_1-plane. The other one, roughly speaking, is about the existence of an *expansion* E of the square $3^{-1} * S$ in the z_0-plane with $z_0 \leq 1$ to S in the z-plane, with z tending to infinity. We use the term C^∞*-homotopy* exclusively for C^∞-mappings T of $\mathbb{R}^3$ to $\mathbb{R}^2$ which are homeomorphisms when restricted to any z-plane. The corresponding mapping $(x, y, z) \mapsto (T(x, y, z), z)$ is denoted by $\widehat{T}$.

By a *canonical affine mapping* of an interval $I_1 \times I_2 \subset \mathbb{R}^2$ onto an interval $J_1 \times J_2 \subset \mathbb{R}^2$ we mean the affine mapping of $I_1 \times I_2$ to $J_1 \times J_2$ whose coordinate functions are the increasing affine surjections of I_i onto J_i, $i = 1, 2$.

Lemma 11. *Let* $R = [a, b] \times [c, d]$ *and* $R^j = [a^j, b^j] \times [c^j, d^j] \subset (a, b) \times (c, d)$ *for* $j = 0, 1$. *Let* $z_0 < z_1$. *Then there is a* C^∞*-homotopy* $T : \mathbb{R}^3 \to \mathbb{R}^2$ *with the following properties:*

(11a) $T_z = T(\cdot, \cdot, z)$ *is a* C^∞*-diffeomorphism of* $\mathbb{R}^2$ *onto itself.*

(11b) $\widehat{T}$ *coincides with the identity mapping on a neighbourhood of* $((\mathbb{R}^2 \setminus R) \times \mathbb{R}) \cup (\mathbb{R}^2 \times (-\infty, z_0])$.

(11c) $T_{z_1}|_{R^0}$ *is the canonical affine mapping of* R^0 *onto* R^1.

(11d) $T_z = T_{z_1}$ *for every* z *from some neighbourhood of* $[z_1, \infty)$.

Proof of Lemma 11 Without loss of generality we can assume that $[c^0, d^0] = [c^1, d^1]$ (in the general case, we have to apply the statement twice). We choose a C^∞-function $f : \mathbb{R} \to \mathbb{R}$ such that

- $f(a^0) = a^1 - a^0$, $f(b^0) = b^1 - b^0$;
- f is affine on $[a^0, b^0]$, and it is 0 in a neighbourhood of $\mathbb{R} \setminus [a, b]$;
- $f'(x) > -1$ for every $x \in \mathbb{R}$.

Now we find a C^∞-function $g : \mathbb{R} \to [0, 1]$ such that

- $g(y) = 1$ for $y \in [c^0, d^0]$,

- $g(y) = 0$ for y in a neighbourhood of $\mathbb{R} \setminus [c, d]$.

Finally, let $h : \mathbb{R} \to [0, 1]$ be an increasing C^∞-function such that

- $h(z) = 1$ on a neighbourhood of $[z_1, \infty)$,
- $h(z) = 0$ on a neighbourhood of $(-\infty, z_0]$.

The mapping
$$T(x, y, z) = (x + f(x)g(y)h(z), y)$$
fulfils all the requirements of Lemma 11 (with $[c^0, d^0] = [c^1, d^1]$). ■

Notation 12. Let ρ be an arbitrary increasing Lipschitz C^∞-function that maps $\mathbb{R}$ onto $[3^{-1}, 1)$ with $\rho(z) = 3^{-1}$ for z in a neighbourhood of $(-\infty, 1]$.

Lemma 13. *There is a C^∞-homotopy $E : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ such that $E_z = E(\cdot, \cdot, z)$ is Lipschitz with a uniform Lipschitz constant C_1 (independent of $z \in \mathbb{R}$) and*

(13a) *$\widehat{E}$ equals to the identity on a neighbourhood of* $(\mathbb{R}^2 \times (-\infty, 1]) \cup ((\mathbb{R}^2 \setminus [-1, 1]^2) \times \mathbb{R})$*;*

(13b) *E_z is the canonical affine mapping of* $[-3^{-1}, 3^{-1}]^2$ *onto* $[-\rho(z), \rho(z)]^2$*.*

Proof of Lemma 13 Let $h(z) = 3\rho(z) - 1$. Note that $h < 2$. We choose C^∞-functions $f : \mathbb{R}^2 \to [-3^{-1}, 3^{-1}]$ and $g : \mathbb{R} \to [0, 1]$ such that

- $f(x, z) = x$ for $x \in [-3^{-1}, 3^{-1}]$;
- $f(x, z) = 0$ for (x, y) from a neighbourhood of $(\mathbb{R} \setminus [-1, 1]) \times \mathbb{R}$;
- $\frac{\partial}{\partial x} f$ is bounded on $\mathbb{R}^2$;
- $h(z) \frac{\partial}{\partial x} f(x, z) > -1$;
- $g(y) = 1$ on $[-3^{-1}, 3^{-1}]$;
- $g(y) = 0$ for y from a neighbourhood of $\mathbb{R} \setminus [-1, 1]$.

Having such functions, it is sufficient to put

$$E(x,y,z) = (x + f(x,z)g(y)h(z), y + f(y,z)g(x)h(z)).$$

Note that (13a) and (13b) are satisfied, and E is a homotopy since E_z is identity on a neighbourhood of $\mathbb{R}^2 \setminus [-1,1]^2$ and $\frac{\partial}{\partial x}(x + f(x,z)g(y)h(z)) > 0$ everywhere. Since the functions g and h are bounded, g is Lipschitz, and $\frac{\partial}{\partial x} f$ is bounded, E_z is Lipschitz with a Lipschitz constant independent of $z \in \mathbb{R}$.

We just roughly indicate how to obtain f for $x \in [3^{-1}, 1]$ (on the interval $[-1, -3^{-1}]$ it can be obtained similarly). Let $I = (3^{-1}, 1)$.

Our aim is to find a C^∞-smooth and bounded function $\varphi(x,z) = -\frac{\partial}{\partial x} f(x,z)$ on $\mathbb{R}^2$ such that $\varphi(x,z) = -1$ for $x \le 3^{-1}$, $\varphi(x,z) = 0$ for x from a neighbourhood of $[1,\infty)$, $\int_I \varphi(x,z)\, dx = 3^{-1}$, and $h(z)\varphi(x,z) < 1$ (note that the length of the interval I is 2/3, and $h(z) < 2$).

Choose smooth functions $0 < \varepsilon(w) \to 0$ as $w \to \infty$ and $\psi : \mathbb{R}^2 \to [-1,1]$ such that $\psi(x,w) = -1$ if $x < 3^{-1} + \varepsilon(w)$, $\psi(x,w) = 0$ if $x > 1 - \varepsilon(w)$, $\max_{x \in I} \psi(x,w) \to 1/2$ as $w \to \infty$, and for each fixed $x \in I$, $\psi(x,w) \to 1/2$ as $w \to \infty$. Let η be an arbitrary smooth function with support in $[1/2, 3/4]$ of integral 1. Then

$$\varphi(x,z) = \psi(x,w) + \left(3^{-1} - \int_I \psi(t,w)\, dt\right) \eta(x)$$

satisfies our requirements, where $w = w(z)$ tends to infinity fast enough to get $h(z)\varphi(x,z) < 1$ (note that $h \ge 0$ is increasing with $\lim_{z\to\infty} h(z) = 2$). ■

3.4 Construction of H – Step 2: The function H^0

Notation 14. For any given square $Q = a * S = [-a,a]^2$, we define squares $Q^{[i]}$, $i = 0, \dots, 8$, so that they form a 3×3 regular partition of Q. To fix a unique notation, let $Q^{[0]} = \frac{1}{3} * Q$ be the middle square of this partition, let $Q^{[1]}$ denote the square $(0, -(2/3)a) + Q^{[0]}$, and choose $Q^{[1]}, Q^{[2]}, Q^{[3]}, \dots, Q^{[8]}$ surrounding $Q^{[0]}$ clockwise so that $Q^{[i+1]}$ touches $Q^{[i]}$.

If $Q = (c_1, c_2) + a * S$ and $i = 0, 1, \dots, 8$, we write $Q^{[i]}$ for $(a * S - (c_1, c_2))^{[i]} + (c_1, c_2)$.

Notation 15. We fix pairwise disjoint closed intervals $I_1, \dots, I_4 \subset (-1 + 3^{-2}, -3^{-1} - 3^{-2})$ and $I_5, \dots, I_8 \subset (3^{-1} + 3^{-2}, 1 - 3^{-2})$ such that they are ordered according to the natural order of $\mathbb{R}$. Let $I_0 = [-\frac{1}{3}, \frac{1}{3}]$, and let $R_i = I_i \times [-\frac{1}{3}, \frac{1}{3}] \subset \mathbb{R}^2$, $i = 0, 1, \dots, 8$.

We also use Notation 12.

Lemma 16. *For any given* $\varepsilon \in (0, 3^{-2}]$, *there is a* $0 < \nabla_H < 1$ *and a* C^∞*-homotopy* $H^0 : \mathbb{R}^3 \to \mathbb{R}^2$ *such that*

(16a) $\widehat{H^0}$ *is the identity mapping on the set*

$$\{(x, y, z) : z \le \varepsilon\} \cup \big((\mathbb{R}^2 \setminus (1-\varepsilon) * S) \times \mathbb{R}\big) \cup \big(((3^{-1}+\varepsilon) * S) \times \mathbb{R}\big) ;$$

(16b) H^0_z *is the canonical affine mapping of* R_i *onto* $\rho(z) * S^{[i]}$ *for* $z \ge 1$, $i = 1, \ldots, 8$*;*

(16c) $\|\frac{\partial}{\partial y} H^0\| < (\nabla_H)^{-1} < \infty$ *everywhere.*

Proof of Lemma 16 First we construct H^0 on $D_0 \times (0, 1]$. We define its value outside $((1-\varepsilon) * S \setminus (3^{-1}+\varepsilon) * S) \times \mathbb{R}$ and on $\{(x, y, z) : z \le \varepsilon\}$ according to (16a). We will also ensure that $H^0_z = H^0_1$ for all $z \in (1-\varepsilon, 1]$, where $H^0_z(\cdot, \cdot) \stackrel{\text{def}}{=} H^0(\cdot, \cdot, z)$.

To get (16b) we may apply Lemma 11 successively, e.g. in the following way. Let $\varepsilon = t_0 < t_1 < \cdots < t_{13} < t_{14} = 1-\varepsilon$ be fixed. Let $J_1, \ldots, J_8$ be closed pairwise disjoint subintervals of $(-1+\varepsilon, 1-\varepsilon)$ such that the interior of J_i contains I_i and $J_i \cap [-3^{-1}-\varepsilon, 3^{-1}+\varepsilon] = \emptyset$ for $i = 1, \ldots, 8$.

Put $P_1 = J_1 \times [-1+\varepsilon, 1-\varepsilon]$. We use Lemma 11 with $R = P_1$, $R^0 = R_1$, $R^1 = (0, -2/3) + 3^{-1} * R_1$, and $z_0 = t_0$, $z_1 = t_1$, to get H^0 on $P_1 \times [t_0, t_1]$. Let H^0_z be the identity outside P_1 for $z \in [t_0, t_1]$. In this way we transformed R_1 to $(0, -2/3) + 3^{-1} * R_1$ within P_1 and $z \in [t_0, t_1]$.

Similarly, we may transform R_4 to $(0, +2/3) + 3^{-1} * R_4$ within $P_2 = J_4 \times [-1+\varepsilon, 1-\varepsilon]$ and $z \in [t_1, t_2]$ letting H^0_z unchanged outside P_2.

Within the interval $[t_2, t_3]$ and the rectangle $P_3 = [-1+\varepsilon, 3^{-1}-\varepsilon] \times [-1+\varepsilon, -3^{-1}-\varepsilon]$, we transform $(0, -2/3) + 3^{-1} * R_1$ to $3^{-1} * S^{[1]}$.

Within the interval $[t_3, t_4]$ and the rectangle $P_4 = (1-\varepsilon) * S^{[4]}$, we transform $(0, +2/3) + 3^{-1} * R_4$ to $3^{-1} * S^{[4]}$.

Within $[t_4, t_5]$ and the rectangle $P_5 = J_2 \times [-1+\varepsilon, 3^{-1}+\varepsilon]$ we transform R_2 to $(0, -2/3) + 3^{-1} * R_2$, and then within $[t_5, t_6]$ and the rectangle $P_6 = (1-\varepsilon) * S^{[2]}$ we transform $(0, -2/3) + 3^{-1} * R_2$ to $3^{-1} * S^{[2]}$.

Finally, within $[t_6, t_7]$ and $P_7 = [-1+\varepsilon, -3^{-1}-\varepsilon] \times [-3^{-1}-\varepsilon, 3^{-1}+\varepsilon]$, we transform R_3 to $3^{-1} * S^{[3]}$.

We proceed within the next seven successive intervals of z symmetrically with respect to the origin to transform R_i to $3^{-1} * S^{[i]}$, $i = 5, \ldots, 8$.

Now we turn to the definition of H_z^0 for $z > 1$. Let $E^{[i]}$ be the expansion mapping from Lemma 13 used with the origin replaced by the center of $S^{[i]}$. Let E be the C^∞-homotopy such that $E_z = E_z^{[i]}$ on $S^{[i]}$ and E_z is the identity otherwise for every $z \in \mathbb{R}$. Thus E_z equals to the identity in a neighbourhood of $\mathbb{R}^2 \setminus D_0$, E_1 equals to the identity everywhere, and E_z, $z \in [1, \infty)$, maps in the canonical affine way the squares $\rho(1) * S^{[i]}$, $1 \le i \le 8$, to $\rho(z) * S^{[i]}$.

We extend H^0 to $D_0 \times (0, \infty)$ as a C^∞-homotopy such that, for each $z \in [1, \infty)$, $H_z^0 = E_z \circ H_1^0$. To verify (16c) we need to show that there is some constant $\nabla_H > 0$ such that $\left\| \frac{\partial}{\partial y} H^0 \right\| \le (\nabla_H)^{-1}$. Indeed, $\widehat{H^0}$ is the identity on $\{(x, y, z) : z \le 0\}$ and outside $D_0 \times \mathbb{R}$. It is a diffeomorphism on the compact set $D_0 \times [0, 1]$ and so there is a $C_2 > 0$ such that $\| \frac{\partial}{\partial y} H^0 \| = \| \frac{\partial}{\partial y} H_z^0 \| \le C_2$ outside $D_0 \times (1, \infty)$. Finally, $\| \frac{\partial}{\partial y} H_z^0 \| \le \| (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) E_z \| \cdot \| \frac{\partial}{\partial y} H_1^0 \| \le C_1 \cdot C_2$ for $z \ge 1$. Put $(\nabla_H)^{-1} = C_1 \cdot C_2 + 1$ to get (16c). ■

3.5 Construction of H – Step 3: The function H^∞

Lemma 17. *Let H^0 and $1 > \nabla_H > 0$ be as in Lemma 16 with $\varepsilon = 3^{-2}$. There is a C^∞-homotopy $H^\infty : \mathbb{R}^3 \to \mathbb{R}^2$ and there are finite sets $I(0, n)$ and closed intervals $P_0^n(z, i) \subset D_0$, $z \in \mathbb{R}$, $n \in \mathbb{N}$, $i \in I(0, n)$, such that*

(17a) $\widehat{H^\infty}$ *is the identity mapping on the set*

$$\{(x, y, z) :\ z \le 0\} \cup ((\mathbb{R}^2 \setminus D_0) \times \mathbb{R});$$

(17b) $\| \frac{\partial}{\partial y} H^\infty \| < (\nabla_H)^{-1} < \infty$ *everywhere;*

(17c) *if $n \in \mathbb{N}$ and $z \ge n$, then $H_z^\infty(\mathbb{R} \times [-3^{-n}, 3^{-n}]) \supset D_0 \setminus P_0^n(z)$, where $P_0^n(z) := \bigcup_{i \in I(0,n)} P_0^n(z, i)$;*

(17d) *$P_0^n(z_2, i) \subset P_0^n(z_1, i)$ if $0 \le z_1 \le z_2$, $n \in \mathbb{N}$, $i \in I(0, n)$;*

(17e) $\lim_{z \to \infty} \operatorname{area}(P_0^n(z)) = 0$.

Proof of Lemma 17 Let H^0 be the C^∞-homotopy from Lemma 16. Note that the difference $D_0 \setminus \bigcup_{i=1}^8 \rho(z) * S^{[i]} = S \setminus \bigcup_{i=0}^8 H_z^0(R_i)$, where $R_0 = 3^{-1} * S$

and $R_1,\dots,R_8$ are as in Notation 15, is covered by the following 8 intervals:

$$[a_j-\rho^*(z),a_j+\rho^*(z)]\times[-1,1],\ [-1,1]\times[a_j-\rho^*(z),a_j+\rho^*(z)],\ a_j=-1,-\frac{1}{3},\frac{1}{3},1,$$

where $\rho^*(z)=\frac{1-\rho(z)}{3}$ for $z\ge 1$. We denote these intervals (ordering them arbitrarily) by $P_0^1(z,i)$, $i\in I(0,1)=\{1,\dots,8\}$. Set $P_0^1(z)\stackrel{\text{def}}{=}\bigcup_{i=1,\dots,8}P_0^1(z,i)$. Note that the area of $P_0^1(z)$ tends to zero with z tending to infinity, and $P_0^1(z_2)\subset P_0^1(z_1)$ if $1\le z_1\le z_2$.

Let us recall that, for a square $Q\subset\mathbb{R}^2$, we defined $Q^{[0]},\dots,Q^{[8]}$ in Notation 14. We use the following notation for finite iterations: $Q^{[\mathbf{i}]}=(Q^{[i_1]}\dots)^{[i_k]}$ if $\mathbf{i}=(i_1,\dots,i_k)\in\{0,\dots,8\}^k$.

We define H^∞ by successive implementation of modified versions of H^0. For arbitrary intervals $V,W\subset\mathbb{R}^2$, let $\varphi_{W,V}$ denote the canonical affine mapping of V onto W. We write

$$H_z^W\stackrel{\text{def}}{=}\varphi_{W,S}\circ H_z^0\circ\varphi_{S,W},\quad H^W(\cdot,\cdot,z)\stackrel{\text{def}}{=}H_z^W.$$

Note that H_z^W is the identity on a neighbourhood of ∂W since H_z^0 is the identity on a neighbourhood of ∂S by (16a).

For each $\mathbf{i}=(i_1,\dots,i_n)\in\{0,\dots,8\}^n$ let

$$R_{\mathbf{i}}\stackrel{\text{def}}{=}\varphi_{R_{i_1},S}\circ\dots\circ\varphi_{R_{i_n},S}(S).$$

Obviously, this notation is consistent with the definition of R_i's.

By induction on n, for each $n\ge 1$ define

$$H_z^n\stackrel{\text{def}}{=}\begin{cases}H_z^{n-1} & \text{if } z\le n;\\ H_z^{n-1} & \text{outside } \bigcup_{\mathbf{i}\in\{0,\dots,8\}^n}R_{\mathbf{i}};\\ H_z^{n-1}\circ H_{z-n}^{R_{\mathbf{i}}} & \text{on each } R_{\mathbf{i}} \text{ if } z\ge n \text{ and } \mathbf{i}\in\{0,\dots,8\}^n,\end{cases}$$

and $H^n(\cdot,\cdot,z)\stackrel{\text{def}}{=}H_z^n$.

It is easy to check that each H_z^n maps $\frac{1}{3}*S$ onto itself and $\widehat{H^n}$ equals to the identity mapping in a neighbourhood of

$$\{(x,y,z):\ z\le 0\}\cup((\mathbb{R}^2\setminus S)\times\mathbb{R})\cup\left(\partial(\frac{1}{3}*S)\times\mathbb{R}\right). \qquad (8)$$

Also note that

$$\begin{aligned} H_z^n(R_{i_1,\dots,i_{n-1}}) &= H_z^{n-1}(R_{i_1,\dots,i_{n-1}} \setminus \bigcup\{R_{i_1,\dots,i_{n-1},i_n} : i_n = 0,1,\dots,8\}) \\ &\cup\ H_z^{n-1}(\bigcup\{R_{i_1,\dots,i_{n-1},i_n} : i_n = 0,1,\dots,8\} \\ &= H_z^{n-1}(R_{i_1,\dots,i_{n-1}}) \end{aligned}$$

for $z \geq n$ since $H_z^{R_{i_1,\dots,i_n}}(R_{i_1,\dots,i_n}) = R_{i_1,\dots,i_n}$. Using this inductively (with $k = 1, 2, \dots$), we get that

$$H_z^{n-1+k}(R_{i_1,\dots,i_{n-1}}) = H_z^{n-1}(R_{i_1,\dots,i_{n-1}}) \tag{9}$$

for $z \geq n$ and $k \in \mathbb{N}$.

As the mapping $H^{R_{\mathbf{i}}}$ is identity in a neighbourhood of $\partial R_{\mathbf{i}}$ in R_i, we can extend it as a C^∞-homotopy of $\mathbb{R}^2 \times (0,\infty)$ to $\mathbb{R}^2$ by defining it to be the identity outside $R_{\mathbf{i}}$. This also extends H^n to be a C^∞-homotopy of $\mathbb{R}^2 \times (0,\infty)$ to $\mathbb{R}^2$ (note that $H_{z-n}^{R_{\mathbf{i}}}$ is the identity if $z - n \in (0,\varepsilon) = (0, 3^{-2})$, so $H_z^n = H_z^{n-1}$ on this domain).

For $z \geq n+1$ and $\mathbf{i} = (i_1,\dots,i_n) \in \{0,\dots,8\}^n$ let

$$S^{[\mathbf{i}]}(z) \stackrel{\text{def}}{=} (r_{i_1}(z) r_{i_2}(z-1) \cdots r_{i_n}(z-n+1)) * S^{[\mathbf{i}]},$$

where $r_0(z) = 1$ and $r_i(z) = \rho(z)$ for $i \in \{1,\dots,8\}$. We prove by induction on n that

(17α) for each $\mathbf{i} = (i_1,\dots,i_n) \in \{0,\dots,8\}^n$ and for every $z \geq n$, H_z^{n-1} restricted to $R_{\mathbf{i}}$ is the affine mapping $\varphi_{S^{[\mathbf{i}]}(z),R_{\mathbf{i}}}$.

It follows from (16b) that for $n = 1$ we have $H_z^{n-1} = \varphi_{S^{[i]}(z),R_i}$ on R_i, for each $i \in \{0,\dots,8\}$ and $z \geq 1$. Suppose that $H_z^{n-1} = \varphi_{S^{[\mathbf{i}]}(z),R_{\mathbf{i}}}$ on $R_{\mathbf{i}}$ for all $\mathbf{i} \in \{0,\dots,8\}^n$ if $z \geq n$, and consider H_z^n on $R_{\mathbf{i}j}$, $j \in \{0,\dots,8\}$ and $z \geq n+1$. We can see that

$$H_z^{n-1} \circ \varphi_{R_{\mathbf{i}},S} = \varphi_{S^{[\mathbf{i}]}(z),R_{\mathbf{i}}} \circ \varphi_{R_{\mathbf{i}},S} = \varphi_{S^{[\mathbf{i}]}(z),S} = \varphi_{S^{[\mathbf{i}j]}(z),S^{[j]}(z-n)} \text{ on } S^{[j]}(z-n)$$

and

$$H_{z-n}^0 \circ \varphi_{S,R_{\mathbf{i}}} = H_{z-n}^0 \circ \varphi_{R_j,R_{\mathbf{i}j}} = \varphi_{S^{[j]}(z-n),R_j} \circ \varphi_{R_j,R_{\mathbf{i}j}} = \varphi_{S^{[j]}(z-n),R_{\mathbf{i}j}} \text{ on } R_{\mathbf{i}j},$$

hence indeed, on $R_{\mathbf{i}j}$,

$$\begin{aligned} H_z^n &= H_z^{n-1} \circ H_{z-n}^{R_{\mathbf{i}}} = (H_z^{n-1} \circ \varphi_{R_{\mathbf{i}},S}) \circ (H_{z-n}^0 \circ \varphi_{S,R_{\mathbf{i}}}) \\ &= \varphi_{S^{[\mathbf{i}j]}(z),S^{[j]}(z-n)} \circ \varphi_{S^{[j]}(z-n),R_{\mathbf{i}j}} = \varphi_{S^{[\mathbf{i}j]}(z),R_{\mathbf{i}j}}. \end{aligned}$$

Now we easily see by induction on n that

$$\left\| \frac{\partial}{\partial y} H^n \right\| \leq (\nabla_H)^{-1}. \tag{10}$$

Indeed, the estimate holds for H_z^0 by (16c), and either $H_z^n = H_z^{n-1}$, or $(x,y) \in R_{\mathbf{i}}$ and $H_z^{n-1} = \varphi_{S^{[\mathbf{i}]}(z),R_{\mathbf{i}}}$, by (17$\alpha$). If $H_z^n = H_z^{n-1}$ then we use the induction hypothesis. In the latter case we can further write

$$H_z^n = \varphi_{S^{[\mathbf{i}]}(z),R_{\mathbf{i}}} \circ H_{z-n}^{R_{\mathbf{i}}} = \varphi_{S^{[\mathbf{i}]}(z),R_{\mathbf{i}}} \circ \varphi_{R_{\mathbf{i}},S} \circ H_{z-n}^0 \circ \varphi_{S,R_{\mathbf{i}}} = \\ \varphi_{S^{[\mathbf{i}]}(z),S} \circ H_{z-n}^0 \circ \varphi_{S,S^{[\mathbf{i}]}(z)} \circ \varphi_{S^{[\mathbf{i}]}(z),R_{\mathbf{i}}}.$$

The Jacobi matrix of the affine mapping $\varphi_{S^{[\mathbf{i}]}(z),R_{\mathbf{i}}}$ is diagonal and the derivative of the y-coordinate of this mapping with respect to y is at most 1, since $R_{\mathbf{i}}$ is an interval of height $2/3^n$ and $S^{[\mathbf{i}]}(z)$ is an interval of height at most $2/3^n$. The Jacobi matrix of $\varphi_{S,S^{[\mathbf{i}]}(z)}$ is a positive multiple of the identity matrix and the Jacobi matrix of $\varphi_{S^{[\mathbf{i}]}(z),S}$ is its inverse. Thus

$$\left\| \frac{\partial}{\partial y} H_z^n \right\| \leq \left\| \frac{\partial}{\partial y} H_{z-n}^0 \right\| \leq (\nabla_H)^{-1}.$$

We also observe that

$$H_z^{n-1}((\mathbb{R} \times [-3^{-n}, 3^{-n}])) \supset \bigcup_{\mathbf{i} \in \{0,\ldots,8\}^n} H_z^{n-1}(R_{\mathbf{i}}) = \bigcup_{\mathbf{i} \in \{0,\ldots,8\}^n} S^{[\mathbf{i}]}(z),$$

where $S^{[\mathbf{i}]}(z) \nearrow S^{[\mathbf{i}]}$ as $z \to \infty$, for each $\mathbf{i} \in \{0, \ldots, 8\}^n$. Therefore one can choose a finite set $I(0,n)$ and intervals $P_0^n(z,i)$, $i \in I(0,n)$, so that

$$H_z^{n-1}((\mathbb{R} \times [-3^{-n}, 3^{-n}])) \supset \bigcup_{\mathbf{i} \in \{0,\ldots,8\}^n} H_z^{n-1}(R_{\mathbf{i}}) \supset D_0 \setminus \bigcup_{i \in I(0,n)} P_0^n(z,i) \tag{11}$$

for $z \geq n$, and so that for each fixed i, $\{P_0^n(z,i) : z \geq n\}$ is a decreasing family of intervals as $z \to \infty$, tending to a line segment. Thus (17d) holds and also the measure of $P_0^n(z) \stackrel{\text{def}}{=} \bigcup_{i \in I(0,n)} P_0^n(z,i)$ tends to 0, which gives (17e).

Put

$$H^\infty(x,y,z) \stackrel{\text{def}}{=} \begin{cases} (x,y) & (x,y) \in \frac{1}{3} * S; \\ H^0(x,y,z) & z \leq 1 \text{ and } (x,y) \notin \frac{1}{3} * S; \\ H^n(x,y,z) & z \in [n, n+1] \text{ and } (x,y) \notin \frac{1}{3} * S. \end{cases}$$

As $H_z^{n+1} = H_z^n$ for $z \leq n+1$, and the mappings H_z^n are C^∞-diffeomorphisms and they map $\frac{1}{3} * S$ onto itself and they are identity in a neighbourhood of $\partial(\frac{1}{3} * S)$, H^∞ is a C^∞-homotopy. Due to (8) the property (17a) is fulfilled. By the estimate (10) we have $\|\frac{\partial}{\partial y} H^\infty\| \leq (\nabla_H)^{-1}$ outside $(\frac{1}{3} * S) \times \mathbb{R}$, and since $\widehat{H^\infty}$ is identity on $(\frac{1}{3} * S) \times \mathbb{R}$ and $\nabla_H < 1$, this proves (17b). By the relations (11) and (9), we have

$$H_z^\infty(\mathbb{R} \times [-3^{-n}, 3^{-n}]) \supset (D_0 \setminus P_0^n(z)) \setminus \left(\frac{1}{3} * S\right) = D_0 \setminus P_0^n(z)$$

for $z \geq n$, which proves (17c). ∎

3.6 Construction of H – Final step.

Proof of Lemma 9 We choose positive $z_0, z_1, \ldots$ so that

$$(3^k * S) \times [z_k, \infty) \subset \{(x, y, z) \in \mathbb{R}^3 : z \geq r^2\}.$$

Necessarily, $\lim_{k \to \infty} z_k = \infty$. We define

$$H(x,y,z) = \begin{cases} (3^k u, 3^k v, z) & (x,y) \in D_k, H^\infty_{z - z_{|k|}}(3^{-k}x, 3^{-k}y) = (u,v), k \in \mathbb{Z} \\ (0,0,z) & (x,y) = (0,0). \end{cases}$$

As the mapping H_z^∞ is a C^∞-diffeomorphism of each D_k onto itself for $z \in \mathbb{R}$, and the expansion of H^∞ by the identity outside of each $D_k \times \mathbb{R}$ is C^∞, H is a C^∞-diffeomorphism of $(\mathbb{R}^2 \setminus \{(0,0)\}) \times \mathbb{R}$ onto itself.

Since it is the identity on a neighbourhood of $\{(0,0)\} \times \mathbb{R}$, it is a C^∞-diffeomorphism of all $\mathbb{R}^3$. We have that H^∞ is the identity for $z \leq 0$ by (17a). So for $(x,y) \in D_k$ and $z < r^2$ we have $z < z_0$ if $k < 0$ and $z < z_k$ otherwise. In each case $z < z_{|k|}$ and so $H(x,y,z) = (x,y,z)$. As H is continuous, property (9a) is fulfilled.

It is clear from the definition that the ranges of $\frac{\partial}{\partial y} H$ and of $\frac{\partial}{\partial y} \widehat{H^\infty}$ coincide, and $\|\frac{\partial}{\partial y} \widehat{H^\infty}\| = \|\frac{\partial}{\partial y} H^\infty\|$. Thus (9b) follows from (17b).

Property (17a) of the homotopy H^∞ gives that the restriction of each H_z to any D_k is a homeomorphism of D_k. This is (9c).

Using the invariance of D_0 and its complement by any H_z^∞ due to (17a) again, we may rewrite (17c) as $H_z^\infty(D_0 \cap (\mathbb{R} \times [-3^{-n}, 3^{-n}])) \supset D_0 \setminus P_0^n(z)$, where $P_0^n(z) := \bigcup_{i\in I(0,n)} P_0^n(z,i)$ for $n \in \mathbb{N}$ and $z \geq n$. The definition of H thus gives (9d) for $k \in \mathbb{Z}$, $z \geq n + z_{|k|}$ if we define $P_k^n(z,i) = 3^k * P_0^n(z,i)$ for $z \geq 1$ and $i \in I(k,n)$.

The properties (9e) and (9f) of $P_k^n(z,i)$ follow now immediately from the corresponding properties of $P_0^n(z,i)$. ■

4 Approximate derivatives

Obviously, if a function is Gâteaux differentiable at a given point, then it is also approximately differentiable, and the Gâteaux and the approximate derivatives coincide. Thus Theorem 2 gives an example of a continuous function $f : \mathbb{R}^3 \to \mathbb{R}$ with disconnected range of the approximate derivative. Our next theorem shows that there are approximately differentiable functions on $\mathbb{R}^2$ for which the range of the approximate derivative is disconnected, and hence Gâteaux differentiability cannot be replaced by approximate differentiability in Theorem 1.

We also show that if the approximate derivative of a function is locally bounded, then the function is locally Lipschitz. Since for Lipschitz functions there is no difference between Gâteaux and Fréchet differentiability, therefore by [5], for such functions the range of the derivative is always connected.

Lemma 18. *Let f be a continuous function whose approximate derivative f'_{ap} exists and it is of norm at most 1 on a convex open set $G \subset \mathbb{R}^2$. Then $|f(u) - f(v)| \leq \|u - v\|$ for every $u, v \in G$.*

Proof of Lemma 18 Let us assume that $f(u) - f(v) > (1+\varepsilon)^2\|u - v\|$ for some $u, v \in G$ and some $\varepsilon > 0$. Without loss of generality we can assume that $v = 0$, $f(v) = 0$, and that $u = (u_0, 0)$ with $u_0 > 0$.

Since f is continuous, if δ_0 is small enough then $f(w) > (1+\varepsilon)^2\|w\|$ on the vertical segment $I \stackrel{\text{def}}{=} \{(u_0, \delta) : |\delta| \leq \delta_0\}$. We choose δ_0 so small that moreover $T \stackrel{\text{def}}{=} \operatorname{conv}\{0, I\} \subset G$, and that for any $x, z \in T$:

$$(1+\varepsilon)(\|x + z\| - \|x\|) \geq \|z\|. \tag{12}$$

Since $g(x) \stackrel{\text{def}}{=} f(x) - (1+\varepsilon)^2\|x\|$ is zero at the origin and it is positive on I, therefore it attains its minimum on T at a point x that does not belong to I. Let $T' = (x + T) \cap T$. Then $g(x) \leq g(y)$ for every $y \in T'$. This, together with (12) implies $f(y) - f(x) \geq (1+\varepsilon)^2(\|y\| - \|x\|) \geq (1+\varepsilon)\|y - x\|$ for every $y \in T'$, and hence the approximate derivative of f at x has norm at least $1+\varepsilon$, which is a contradiction. ∎

Theorem 19. *There is an approximately differentiable continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ which is*

(i) *C^∞ on $\mathbb{R}^2 \setminus \{0\}$,*

(ii) *its approximate derivative is 0 at the origin and $\|\nabla f(x)\| \geq 1/2$ for every $x \neq 0$.*

Proof of Theorem 19 Let y_n be a decreasing sequence of positive real numbers tending to zero such that $y_n - y_{n+1} \leq 2y_{n+1}^2$ and $y_1 = 1$. It is not difficult to verify that such a sequence exists, we can take for example $y_{n+1} = (y_n/2 + 1/16)^{1/2} - 1/4$. This also implies the inequality $\frac{y_n}{y_{n+1}} \leq 2y_{n+1} + 1 \leq 3$.

We define several families of subsets of $\mathbb{R}^2$. For $i = 1, 2, 3$ put

$$P^i = \{(x, y) \in \mathbb{R}^2 : |x| \leq C_i\sqrt{y},\ y \geq 0\},$$

where $C_1 = 1$, $C_2 = 2$, and $C_3 = 2 + 2C$ with a positive real number C that we will specify later. Let $L_n = \{(x, y) \in \mathbb{R}^2 : y = y_n\}$, $S_n = \mathbb{R} \times [y_{n+1}, y_n]$. We choose ε_n so small that the origin is a density point of the union

$$A = \bigcup\{R_n : n \in \mathbb{N}\} \cup \bigcup\{R_n^- : n \in \mathbb{N}\},$$

where R_n denotes the interval

$$R_n = [-\sqrt{y_n}, \sqrt{y_n}] \times [y_{n+1} + \varepsilon_n, y_n - \varepsilon_n],$$

and R_n^- is the reflected copy $R_n^- = \{(x, y) \in \mathbb{R}^2 : (x, -y) \in R_n\}$. We may easily check that our choice of y_n ensures that each R_n is a subset of the interior of $P^2 \cap S_n$ (indeed, $\sqrt{y_n} \leq \sqrt{3y_{n+1}} < 2\sqrt{y_{n+1}}$).

We will find a C^∞-smooth vector field $V = (V_1, V_2) : \mathbb{R} \times (0, \infty) \to [0, \infty)^2$ such that

- $\|V(x, y)\| = 1$ for every $(x, y) \in \mathbb{R} \times (0, \infty)$;
- $V(x, y) = (1, 0)$ if $(x, y) \in \bigcup_{n\in\mathbb{N}} L_n$, if $(x, y) \notin P_2$, or if $y \geq 1$;
- $V(x, y) = (0, 1)$ on $\bigcup_{n\in\mathbb{N}} R_n$.

It is not difficult to find a C^∞-smooth real function $v_1 : \mathbb{R} \times (0, \infty) \to [0, 1]$ that coincides with the first coordinate function V_1 of V at those points (x, y) where its value, zero or one, is prescribed by the previous conditions. We can, e.g., find such a v_1 on each S_n independently, assuming that it is equal to 1 outside S_n (this can be achieved using the convolution with a suitable C^∞-function with small support). We put $v_2(x, y) = 1 - v_1(x, y)$ and $V(x, y) = \frac{(v_1, v_2)}{\|(v_1, v_2)\|}$.

We are going to use the vector field V to find a C^∞-smooth real function g on $\mathbb{R} \times (0, \infty)$ such that

(19α) $\|\nabla g(x, y)\| \geq 1$ on $\mathbb{R} \times (0, \infty)$;

(19β) $|g(x, y)| \leq y^2$ on each R_n;

(19γ) $\delta_l(y) = x - g(x, y)$ (respectively $\delta_r(y) = x - g(x, y)$) is constant and its absolute value is less than $22\sqrt{y}$ if $x \leq -2\sqrt{y}$ (respectively if $x \geq 2\sqrt{y}$) and $y > 0$ is fixed;

(19δ) $\lim\limits_{\substack{(x,y)\to 0\\ y>0}} g(x, y) = 0.$

Let $\Phi(t, x_0, y_0)$, $t \in \mathbb{R}$, $(x_0, y_0) \in \mathbb{R} \times (0, \infty)$, be the value $(x(t), y(t))$ of the unique solution of the ordinary differential equation $(x'(t), y'(t)) = V(x(t), y(t))$ with the initial condition $(x(0), y(0)) = (x_0, y_0)$. It is well-known that such a mapping Φ is well defined and C^∞-smooth on $\mathbb{R}^2 \times (0, \infty)$ (see, e.g., [1, Chapter 2, §7, Corollary 4]).

We are going to define $g = g_0$ on the set $G^{-1}(0)$, where $G : \mathbb{R}\times(0, \infty) \to \mathbb{R}$ is defined by $G(x, y) = x - 2\sqrt{y}$. For every $(x, y) \in \mathbb{R}\times(0, \infty)$ there is a unique $t = t(x, y)$ with $G(\Phi(t, x, y)) = 0$ because every integral curve $\Phi(\cdot, x, y)$ is contained in some strip S_n and its projection to the first coordinate is necessarily $\mathbb{R}$ by the facts that all L_n are integral curves of V and that $V_i \geq 0$ for $i = 1, 2$. It follows from the implicit function theorem that $t(x, y)$ is a C^∞-smooth function as the vector field V is not tangent to $G^{-1}(0)$ at any of its points ($\frac{\partial}{\partial t}(G \circ \Phi) = 1 \neq 0$ on $G^{-1}(0)$). The absolute value of $t(x, y)$, $(x, y) \in P^2$, is at most equal to the length of the part of the integral curve $\Phi(\cdot, x, y)$ contained in P^2. Since $V_i \geq 0$ and each integral curve is contained in an S_n, we get

$$|t(x, y)| \leq 4\sqrt{y_n} + (y_n - y_{n+1}) \leq 4\sqrt{3y_{n+1}} + 2y_{n+1}^2 < 10\sqrt{y} \qquad (13)$$

for $y \in [y_{n+1}, y_n]$.

If $(x,y) \in G^{-1}(0)$ is such that $(x,y) = \Phi(t_0, x_0, y_0)$ for some $(x_0, y_0) \in [-\sqrt{y_n}, \sqrt{y_n}] \times \{\frac{y_n + y_{n+1}}{2}\}$ and $t_0 = t(x_0, y_0)$, we define $g_0(x,y) = t_0$. By (13), $0 < t_0 < 10\sqrt{y}$ at those $(x,y) \in G^{-1}(0)$ for which g_0 was already defined. We now extend g_0 to $G^{-1}(0)$ as a C^∞-smooth function so that $|g(x,y)| < 10\sqrt{y}$ on $G^{-1}(0)$. Finally, for an arbitrary $(x,y) \in \mathbb{R} \times (0,\infty)$, we put $g(x,y) = g_0(\Phi(t(x,y),x,y)) - t(x,y)$.

Note that we get in this way a C^∞-smooth function on $\mathbb{R} \times (0,\infty)$ which is zero on the segments $[-\sqrt{y_n}, \sqrt{y_n}] \times \{\frac{y_n+y_{n+1}}{2}\}$, $n \in \mathbb{N}$. Moreover, since the first expression in the definition of g is constant along the integral curves of V, the derivative of g at $(x,y) \in \mathbb{R}\times(0,\infty)$ in the direction (with t increasing) of the integral curve $\{\Phi(t,x,y) : t \in \mathbb{R}\}$ of V is 1 which gives (19α). On each R_n we have $g(x,y) = y - \frac{y_n+y_{n+1}}{2}$ and so $|g(x,y)| \leq \frac{y_n - y_{n+1}}{2} \leq y_{n+1}^2 \leq y^2$ which verifies (19β). The function $x - g(x,y)$ is constant for fixed $y > 0$ on the intervals $(-\infty, -2\sqrt{y}]$ and $[2\sqrt{y}, \infty)$ since $V = (1,0)$ outside P^2. The absolute value of $2\sqrt{y} - g(2\sqrt{y}, y)$ is at most $12\sqrt{y}$ by the estimate of $g = g_0$ on $G^{-1}(0)$. It follows that $|g(x,y)| = |g_0(\Phi(t(x,y),x,y)) - t(x,y)| \leq 10\sqrt{y} + 10\sqrt{y} \leq 20\sqrt{y}$ for $(x,y) \in P^2 \cap S_n$. This concludes the proof of property (19γ). To obtain (19δ) we only need to observe that $|g(x,y)| \leq |x| + 20\sqrt{y}$ on $\mathbb{R} \times (0,\infty)$.

Our next aim is to 'correct' g by adding a C^∞-function h with support in the closure of $\mathbb{R}^2 \setminus P^2$ so that, for $y > 0$,

- $|h(x,y)| \leq 22\sqrt{y}$,
- $|\frac{\partial}{\partial x} h(x,y)| \leq 1/2$, and
- $h(x,y) = x - g(x,y)$ for $|x| \geq (2+2C)\sqrt{y}$.

Let $\varphi(x) = \exp(\frac{-1}{1-x^2})$ if $|x| < 1$ and $\varphi(x) = 0$ otherwise. Then φ is a C^∞-function with a finite positive integral $I = \int_{\mathbb{R}} \varphi(x)\, dx$. So $IC\sqrt{y} = \int_{\mathbb{R}} \varphi(\frac{x+x_0}{C\sqrt{y}})\, dx$ for any x_0 and $y > 0$.

We define

$$\begin{aligned} h(x,y) &= \frac{1}{IC\sqrt{y}} \int_x^\infty \delta_l(y)\, \varphi\left(\frac{\xi + (2+C)\sqrt{y}}{C\sqrt{y}}\right) d\xi \\ &\quad + \frac{1}{IC\sqrt{y}} \int_{-\infty}^x \delta_r(y)\, \varphi\left(\frac{\xi - (2+C)\sqrt{y}}{C\sqrt{y}}\right) d\xi. \end{aligned}$$

Then indeed h is zero on P^2. Also note that

$$|h(x,y)| \leq \max(|\delta_l(y)|, |\delta_r(y)|) \leq 22\sqrt{y} \text{ and}$$

$$\left|\frac{\partial}{\partial x} h(x,y)\right| \leq \frac{1}{IC\sqrt{y}} \cdot 44\sqrt{y}\, e^{-1} \leq 1/2$$

for sufficiently large C.

We now define $f(x,y) = g(x,|y|) + h(x,|y|)$ if $y \neq 0$, and $f(x,0) = x$. The function f is C^∞-smooth for $y \neq 0$ by the smoothness of g and h. On $\mathbb{R}^2 \setminus P^3$ we have $f(x,y) = x$ and so f is C^∞-smooth on $\mathbb{R}^2 \setminus \{(0,0)\}$. The approximate derivative of f at the origin is zero as $f = g$ on $A = \bigcup_{n\in\mathbb{N}}(R_n \cup R_n^-)$, the origin is a density point of A, and $|g| \leq y^2$ on A. The Fréchet derivative of f is at least one on P^2 since $\|g'(x,y)\| \geq 1$ and $h = 0$ for $(x,y) \in P^2$. Outside P^2 the partial derivative $\frac{\partial}{\partial x} g$ is at least one, the absolute value of the partial derivative $\frac{\partial}{\partial x} h$ is at most $1/2$, and hence (ii) of Theorem 19 follows. ■

References

[1] V.I. Arnol'd, *Ordinary Differential Equations*, Springer-Verlag, Berlin 1992.

[2] D. Azagra and M. Jiménez-Sevilla, *On the range of the derivatives of a smooth function between Banach spaces*, Math. Proc. Cambridge Philos. Soc. 134 (2003), 163-185.

[3] A. Bruckner, *Differentiability of Real Functions*, Amer. Math. Soc., Providence 1994.

[4] R. Deville and P. Hájek, *On the range of the derivative of Gâteaux-smooth functions on separable Banach spaces*, Israel J. Math. 145 (2005), 257–269.

[5] J. Malý, *The Darboux property for gradients*, Real Anal. Exchange 22 (1996/97), 167–173.

[6] T. Mátrai, *Graphs of Gâteaux derivatives are $w^\star$-connected*, Real Anal. Exchange 29 (2003/04), 291–297.

Marianna Csörnyei: Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, UK (e-mail: mari@math.ucl.ac.uk).

Petr Holický: Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis, Sokolovská 83, 186 75 Prague 8, Czech Republic (email: holicky@karlin.mff.cuni.cz).