GRAVITATIONAL INSTANTONS AND DEGENERATIONS OF RICCI-FLAT METRICS ON THE K3 SURFACE

LORENZO FOSCOLO

Abstract. The study of degenerations of metrics with special holonomy is an important theme unifying the study of convergence of Einstein metrics, the study of complete non-compact manifolds with special holonomy and the construction of spaces with special holonomy by singular perturbation methods. We survey three constructions of degenerating sequences of hyperkähler metrics on the (smooth 4-manifold underlying a complex) K3 surface—the classical Kummer construction, Gross–Wilson's work on collapse along the fibres of an elliptic fibration, and the author's construction of sequences collapsing to a 3-dimensional limit—describing how they fit into the general theory and highlighting the role played in each construction by gravitational instantons, *i.e.* complete non-compact hyperkähler 4-manifolds with decaying curvature at infinity.

1. Hyperkähler metrics in dimension 4

Hyperkähler 4-manifolds are the lowest dimensional non-flat examples of manifolds with special holonomy.

Definition 1.1. A Riemannian 4-manifold (M^4, g) is *hyperkähler* if the holonomy $\mathrm{Hol}(g)$ is contained in $\mathrm{SU}(2)$.

Despite its integro-differential definition in terms of parallel transport, the holonomy reduction to $\mathrm{SU}(2)$ can be recast in terms of a PDE for a triple of 2-forms satisfying special algebraic properties at each point [17]. Recall that the space of 2-forms on an oriented 4-dimensional vector space carries a natural non-degenerate bilinear form of signature $(3,3)$.

Definition 1.2. Let (M^4, μ_0) be an oriented 4-manifold with volume form μ_0. A *definite triple* is a triple $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ of 2-forms on M such that $\mathrm{span}(\boldsymbol{\omega}) = \mathrm{span}(\omega_1, \omega_2, \omega_3)$ is a 3-dimensional positive definite subspace of $\Lambda^2 T^*_x M$ at every point $x \in M$.

Given a triple $\boldsymbol{\omega}$ of 2-forms on (M, μ_0) we consider the matrix $Q \in \Gamma(M, \mathrm{Sym}^2(\mathbb{R}^3))$ defined by

$$\tfrac{1}{2}\, \omega_i \wedge \omega_j = Q_{ij}\, \mu_0. \tag{1.3}$$

$\boldsymbol{\omega}$ is a definite triple if and only if Q is a positive definite matrix. To every definite triple $\boldsymbol{\omega}$ we associate a volume form $\mu_{\boldsymbol{\omega}}$ by

$$\mu_{\boldsymbol{\omega}} = (\det Q)^{\frac{1}{3}}\, \mu_0 \tag{1.4}$$

and the new matrix $Q_{\boldsymbol{\omega}} = (\det Q)^{-\frac{1}{3}}\, Q$ which satisfies (1.3) with $\mu_{\boldsymbol{\omega}}$ in place of μ_0. Note that the volume form $\mu_{\boldsymbol{\omega}}$ and the matrix $Q_{\boldsymbol{\omega}}$ are independent of the choice of volume form μ_0.

Now, let (M^4, μ_0) be an oriented 4-dimensional manifold. The choice of a 3-dimensional positive definite subspace of $\Lambda^2 T^*_x M$ for all $x \in M$ is equivalent to the choice of a conformal class on M, see for example [19, §1.1.5]. Thus every definite triple defines a Riemannian metric $g_{\boldsymbol{\omega}}$ by requiring that $\mathrm{span}(\boldsymbol{\omega})|_x = \Lambda^+ T^*_x M$ for all $x \in M$ and $\mathrm{dv}_{g_{\boldsymbol{\omega}}} = \mu_{\boldsymbol{\omega}}$.

Definition 1.5. A definite triple $\boldsymbol{\omega}$ is said to be
- (i) *closed* if $d\omega_i = 0$ for $i = 1, 2, 3$;
- (ii) an $\mathrm{SU}(2)$-*structure* if $Q_{\boldsymbol{\omega}} \equiv \mathrm{id}$;
- (iii) *hyperkähler* if it is both closed and an $\mathrm{SU}(2)$-structure.

A closed definite triple is also called a *hypersymplectic* triple. The metric $g_{\boldsymbol{\omega}}$ associated to a hyperkähler triple is hyperkähler in the sense of Definition 1.1.

Let $(M, \boldsymbol{\omega})$ be a hyperkähler 4-manifold. We now make a choice of direction in $\mathbb{R}^3$. Up to rotations we can assume that the chosen direction is e_1. We write $\omega = \omega_1$, $\omega_c = \omega_2 + i\omega_3$ and $\overline{\omega}_c = \omega_2 - i\omega_3$. The complex 2-form ω_c defines an almost complex structure $J = J_1$ on M by declaring a complex 1-form α of type $(1,0)$ if and only if $\alpha \wedge \omega_c = 0$. Since $d\omega_c = 0$ the differential ideal generated by the $(1,0)$–forms is closed and therefore the almost complex structure J is integrable by the Newlander–Nirenberg Theorem. Moreover, ω_c and ω are, respectively, a holomorphic $(2,0)$–form and a real $(1,1)$–form with respect to J. Since ω is closed and non-degenerate (M, ω, J) is a Kähler surface, with g the induced Kähler metric. Moreover, by the expression for the Ricci curvature in Kähler geometry, *cf.* for example [31, §4.6], $\omega^2 = \frac{1}{2}\omega_c \wedge \overline{\omega}_c$ implies that g is *Ricci-flat.* Since the choice of direction in $\mathbb{R}^3$ was arbitrary, we see that hyperkähler metrics are Kähler with respect to a 2-sphere of compatible integrable complex structures—this might be the definition of hyperkähler manifolds the reader is already familiar with.

The K3 surface. Beside the 4-torus endowed with a flat metric, the only other compact 4-manifold carrying hyperkähler metrics is the *K3 surface*. In this note *the* K3 surface is the smooth 4-manifold M underlying any simply connected complex surface (M, J) with trivial canonical bundle. The fact that all such complex surfaces are diffeomorphic to each other was proved by Kodaira [33, Theorem 13]. We say that (M, J) is a *complex K3 surface* if we make a choice of complex structure. As above, every simply connected hyperkähler 4-manifold is in particular a complex surface (M, J) with trivial canonical bundle (trivialised by ω_c). Conversely, every complex K3 surface is Kähler [46] and therefore admits a Kähler Ricci-flat metric by Yau's Theorem [51]. Since M is simply connected any Kähler Ricci-flat metric has holonomy contained in $\mathrm{SU}(2)$ and therefore is hyperkähler. Examples of complex K3 surfaces (M, J) are smooth quartics in $\mathbb{CP}^3$, complete intersections of a cubic and quadric in $\mathbb{CP}^4$ and the double cover of $\mathbb{CP}^2$ branched along a sextic.

Note also that every Einstein metric on the K3 surface must be hyperkähler [29, Theorem 1]. Indeed, given any metric g the Chern–Gauss–Bonnet and Signature Formulas are

$$8\pi^2\chi(M) = \int_M \tfrac{1}{24}\mathrm{Scal}^2 + |W|^2 - \tfrac{1}{2}|\mathring{\mathrm{Ric}}|^2, \qquad 12\pi^2\tau(M) = \int_M |W_+|^2 - |W_-|^2, \tag{1.6}$$

where Scal is the scalar curvature, $\mathring{\mathrm{Ric}}$ the traceless Ricci tensor and $W = W_+ + W_-$ is the Weyl tensor of g, decomposed into its self-dual and anti-self-dual parts. We deduce that every Einstein metric g on the K3 surface M must be Ricci-flat and anti-self-dual since

$$\frac{1}{2\pi^2}\int_M \tfrac{1}{48}\mathrm{Scal}^2 + |W_+|^2 = 2\chi(M) + 3\tau(M) = 0.$$

Indeed the Betti numbers of the K3 surface are $b_0 = 1$, $b_1 = 0$, $b_+ = 3$ and $b_- = 19$. Furthermore, the Weitzenböck formula on Λ^+ is

$$\triangle_{\Lambda^+} = \nabla^*\nabla - 2W_+ + \tfrac{1}{3}\mathrm{Scal} = \nabla^*\nabla.$$

Since $b_+ = 3$, we deduce that (M, g) carries a 3-dimensional space of parallel self-dual 2-forms and therefore the holonomy of g reduces to $\mathrm{SU}(2)$.

Let $\mathcal{M}$ be the moduli space of Ricci-flat metrics of volume 1 on the K3 surface M. The deformation theory of Einstein metrics is governed by an index zero elliptic problem and therefore moduli spaces of Einstein metrics are in general singular. In contrast, metrics with special holonomy often form smooth moduli spaces. This is the case for hyperkähler metrics and thus $\mathcal{M}$ is a smooth manifold. In fact we also know what this manifold is. Let $\mathrm{Gr}^+(3,19) = \mathrm{SO}(3,19)/\mathrm{SO}(3) \times \mathrm{SO}(19)$ be the Grassmannian of positive 3-planes in $\mathbb{R}^{3,19} \simeq H^2(M;\mathbb{R})$ and Γ be the automorphism of the lattice $H^2(M;\mathbb{Z})$ endowed with the intersection form (equivalently Γ is the quotient of the group of diffeomorphisms of M by the subgroup of diffeomorphisms acting trivially on cohomology). The period map

$$\mathcal{P}\colon\ \mathcal{M} \to \mathrm{Gr}^+(3,19)/\Gamma \tag{1.7}$$

associates to each metric the positive definite subspace $\mathrm{span}[\boldsymbol{\omega}] = \mathrm{span}([\omega_1],[\omega_2],[\omega_3]) \subset H^2(M,\mathbb{R})$. The Local Torelli Theorem [33, Theorem 17] implies that $\mathcal{P}$ is a local diffeomorphism.

The period map $\mathcal{P}$ in (1.7) is *not* surjective: smooth hyperkähler metrics correspond to triples $[\boldsymbol{\omega}] \in H^2(M,\mathbb{R})$ such that

$$[\boldsymbol{\omega}](\Sigma) \neq \mathbf{0} \in \mathbb{R}^3 \text{ for all } \Sigma \in H_2(M,\mathbb{Z}) \text{ such that } \Sigma\cdot\Sigma = -2, \tag{1.8}$$

cf. [31, Theorem 7.3.16]. Thus the image of $\mathcal{P}$ is the complement of codimension-3 "holes" in $\mathrm{Gr}^+(3,19)/\Gamma$. In the next section we describe hyperkähler metrics approaching this excluded codimension-3 locus and explain the significance of (1.8).

2. Non-collapsed limits

The Kummer construction. We begin with a prototypical example. Soon after Yau's proof of the Calabi Conjecture [51] implied that the K3 surface carries hyperkähler metrics, physicists and mathematicians alike have been interested in finding a more explicit description of these Ricci-flat metrics. Gibbons and Pope [22] suggested the construction of explicit approximately Ricci-flat metrics on Kummer surfaces.

Let $\Lambda \simeq \mathbb{Z}^4$ be a lattice in $\mathbb{R}^4$ and consider the flat 4-torus $T^4 = \mathbb{R}^4/\Lambda$. Consider the $\mathbb{Z}_2$–action on T^4 induced by the involution $x \mapsto -x$ of $\mathbb{R}^4$. Then $T^4/\mathbb{Z}_2$ is a flat 4-orbifold which is singular at the 16 points of the half-lattice $\frac{1}{2}\Lambda$. Each singular point is modelled on $\mathbb{R}^4/\mathbb{Z}_2$. If we identify $\mathbb{R}^4$ with $\mathbb{C}^2$ then T^4 becomes a complex manifold and by blow-up we can resolve $T^4/\mathbb{Z}_2$ to a complex surface (M,J) which is simply connected and satisfies $c_1(M,J)=0$ and therefore is a complex K3 surface. The blow-up replaces each singularity with a holomorphic $\mathbb{CP}^1$ with self-intersection -2. Thus a tubular neighbourhood of each $\mathbb{CP}^1 \simeq S^2$ in M is identified with a disc bundle in the $\mathbb{R}^2$–bundle T^*S^2 over S^2 with Euler class -2.

Gibbons and Pope suggested that Ricci-flat metrics can be brought into this resolution picture. The missing ingredient is a model Ricci-flat metric on T^*S^2 that is asymptotic at infinity to the flat metric on $\mathbb{R}^4/\mathbb{Z}_2$. Such a metric is explicit and is called the *Eguchi–Hanson metric* [20].

Note that T^*S^2 can be identified with the total space of the holomorphic line bundle $\mathcal{O}(-2)$ over $\mathbb{CP}^1$. This identification endows T^*S^2 with a complex structure J. In fact, the blow-down of the zero-section $\pi\colon\ \mathcal{O}(-2) \to \mathbb{C}^2/\mathbb{Z}_2$ exhibits $\mathcal{O}(-2)$ as a *crepant resolution* of $\mathbb{C}^2/\mathbb{Z}_2$: the standard holomorphic $(2,0)$–form $dz_1 \wedge dz_2$ on $\mathbb{C}^2$ descends to $\mathbb{C}^2/\mathbb{Z}_2$ by $\mathbb{Z}_2$–invariance and its pull-back to $\mathcal{O}(-2)$ extends to a nowhere-vanishing holomorphic $(2,0)$–form ω_c^{eh} on $\mathcal{O}(-2)$. We now define a hyperkähler triple $\boldsymbol{\omega}^{\mathrm{eh}}$ on T^*S^2 by $\omega_2^{\mathrm{eh}} = \mathrm{Re}\,\omega_c^{\mathrm{eh}}$, $\omega_3^{\mathrm{eh}} = \mathrm{Im}\,\omega_c^{\mathrm{eh}}$ and ω_1^{eh} the Kähler form defined outside the zero-section by

$$\omega_1^{\mathrm{eh}} = \tfrac{i}{2}\partial\overline{\partial}\varphi^{\mathrm{eh}}, \qquad \varphi^{\mathrm{eh}} = \sqrt{1+r^4} + 2\log r - \log\Big(1+\sqrt{1+r^4}\Big). \tag{2.1}$$

Here we identify the complement of the zero-section in T^*S^2 with the complement of the origin in $\mathbb{C}^2/\mathbb{Z}_2$ via π and set $r = \sqrt{|z_1|^2+|z_2|^2}$. One can check that ω_1^{eh} extends to a smooth Kähler form on the whole of T^*S^2. Note that as $r \to \infty$, ω_1^{eh} approaches the flat metric $\frac{i}{2}\partial\overline{\partial}\varphi_0$, $\varphi_0 = r^2$, up to terms that decay as r^{-4}.

Now, Gibbons and Pope suggested to remove neighbourhoods of the 16 singular points of $T^4/\mathbb{Z}_2$ and replace them with 16 copies of a disc bundle in $T^*S^2 \to S^2$. This cut-and-paste construction of the smooth 4-manifold M can be promoted to the construction of a hypersymplectic triple on M by patching together the flat hyperkähler triple $\hat{\boldsymbol{\omega}}$ on $T^4/\mathbb{Z}_2$ with 16 copies of the rescaled Eguchi–Hanson hyperkähler triple. We now provide more details of this construction.

We first need to "prepare" the Eguchi–Hanson metric to be "grafted" into $T^4/\mathbb{Z}_2$. Following [8, §1.1], fix $t>0$ and consider a cut-off function $\chi=\chi_t$ such that $\chi(r)=1$ for $r\leq \frac{1}{\sqrt{t}}$ and $\chi(r)=0$ for $r\geq \frac{2}{\sqrt{t}}$. Define a new triple $\boldsymbol{\omega}^{\text{eh},t}$ by $\omega_i^{\text{eh},t}=\omega_i^{\text{eh}}$ for $i=2,3$ and $\omega_1^{\text{eh},t}=\frac{i}{2}\partial\overline{\partial}\tilde{\varphi}_t^{\text{eh}}$, where

$$\tilde{\varphi}_t^{\text{eh}}(r)=t^2\tilde{\varphi}^{\text{eh}}\left(t^{-1}r\right),\qquad \tilde{\varphi}^{\text{eh}}=\chi\,\varphi^{\text{eh}}+(1-\chi)\,\varphi_0.$$

The triple $\boldsymbol{\omega}^{\text{eh},t}$ coincides with $t^2\boldsymbol{\omega}^{\text{eh}}$ for $r\leq\sqrt{t}$ and with the flat hyperkähler triple $\boldsymbol{\omega}_0$ on $\mathbb{C}^2/\mathbb{Z}_2$ for $r\geq 2\sqrt{t}$. In the annulus $\sqrt{t}\leq r\leq 2\sqrt{t}$, $\boldsymbol{\omega}^{\text{eh},t}$ differs from $\boldsymbol{\omega}_0$ by terms of order $O(t^2)$. If t is sufficiently small $\boldsymbol{\omega}^{\text{eh},t}$ is a closed definite triple which is approximately hyperkähler in the sense that $Q_{\boldsymbol{\omega}^{\text{eh},t}}-\text{id}=O(t^2)$.

Let $p_1,\dots,p_{16}$ denote the singular points of $T^4/\mathbb{Z}_2$. We construct a smooth 4-manifold M by replacing (disjoint) balls $B_{3\sqrt{t}}(p_i)$ in $T^4/\mathbb{Z}_2$ with copies of the region $\{r\leq 3\sqrt{t}\}\subset T^*S^2$. Since $\boldsymbol{\omega}^{\text{eh},t}$ coincides with the flat triple $\boldsymbol{\omega}_0$ for $r\geq 2\sqrt{t}$, M comes equipped with a natural hypersymplectic triple $\boldsymbol{\omega}^t$ obtained by gluing $\boldsymbol{\omega}^{\text{eh},t}$ with the flat hyperkähler triple $\hat{\boldsymbol{\omega}}$ of $T^4/\mathbb{Z}_2$. Then $\boldsymbol{\omega}^t$ is an approximate hyperkähler triple in the sense that $Q_{\boldsymbol{\omega}^t}-\text{id}=O(t^2)$.

The question now is to deform the approximate hyperkähler triple $\boldsymbol{\omega}^t$ into an exact solution. A first rigorous proof of such a perturbation was given by LeBrun–Singer [37] (following an earlier attempt by Topiwala [50]); it uses twistor theory and we will not say anything about it. A different approach exploits the fact that a complex structure J on M with $c_1(M,J)=0$ can be readily constructed by blow-up $\pi\colon M\to T^4/\mathbb{Z}_2$ of the complex orbifold $T^4/\mathbb{Z}_2$: $\pi^*\hat{\omega}_c$, where $\hat{\omega}_c=\hat{\omega}_2+i\hat{\omega}_3$ is the holomorphic $(2,0)$–form on $T^4/\mathbb{Z}_2$, extends to a nowhere vanishing holomorphic $(2,0)$–form on M. Indeed, we can arrange our gluing so that $\omega_c^t=\omega_2^t+i\omega_3^t$ is closed and satisfies $\omega_c^t\wedge\omega_c^t=0$ and $\omega_c^t\wedge\overline{\omega}_c^t\neq 0$. Then the problem of perturbing $\boldsymbol{\omega}^t$ to an exact hyperkähler triple reduces to solving the complex Monge–Ampère equation

$$\left(\omega_1^t+i\partial\overline{\partial}u\right)^2=\tfrac{1}{2}\omega_c^t\wedge\overline{\omega}_c^t. \tag{2.2}$$

Since $(t,u)=(0,0)$ is a solution one can hope to solve this equation for small $t>0$ by the Implicit Function Theorem. The main issue is that $(0,0)$ correspond to a singular solution to the equation and therefore care is needed in applying the Implicit Function Theorem. This was done by Donaldson [18] exploiting the conformal equivalence between the cone metric $dr^2+r^2g_{\mathbb{RP}^3}$ (the model for the singularities of $T^4/\mathbb{Z}_2$ and for the geometry at infinity of the Eguchi–Hanson metric) and the cylindrical metric $dt^2+g_{\mathbb{RP}^3}$. This conformal rescaling allows one to control constants in the application of the Implicit Function Theorem since the cylindrical metric has bounded geometry. Alternatively, one could work with weighted Banach spaces as in analogous constructions of complete non-compact hyperkähler 4-manifolds by Biquard–Minerbe [8].

The result is a family of Kähler Ricci-flat metrics on the K3 surface that develop 16 orbifold singularities modelled on $\mathbb{R}^4/\mathbb{Z}_2$ in the limit $t\to 0$. Each singularity is associated with a 2-sphere of self-intersection -2 which shrinks to zero size as $t\to 0$. Furthermore, appropriate rescalings of the family close to each singular point converge to the Eguchi–Hanson metric.

We can also introduce further parameters in the construction to recover a full 58-dimensional family of hyperkähler metrics on the K3 surface close to the singular limit $T^4/\mathbb{Z}_2$. Indeed, when gluing the scaled Eguchi–Hanson metric to the flat metric in a neighbourhood of the point p_i we have the choice of an isometric identification between the tangent cone at a singularity of $T^4/\mathbb{Z}_2$ and $\mathbb{R}^4/\mathbb{Z}_2$. Since the Eguchi–Hanson metric is $\mathrm{U}(2)$–invariant, this choice lives in $\mathrm{SO}(4)/\mathrm{U}(2)\simeq S^2$. In other words, at each singular point we can choose a direction in $\text{span}(\hat{\omega}_1,\hat{\omega}_2,\hat{\omega}_3)$ to be identified with the direction of $\omega_1^{\text{eh},t}$ in $\text{span}(\omega_1^{\text{eh},t},\omega_2^{\text{eh},t},\omega_3^{\text{eh},t})$. In the previous situation, where we define a complex structure J on M by blow-up, we make the same choice of direction at each singular point $p_1,\dots,p_{16}$. If different choices are made at different points then M does not come equipped with an integrable complex structure and instead of solving a complex Monge–Ampère equation we need to glue hyperkähler triples directly. This can be done as follows.

Let $\boldsymbol{\omega}^t$ be the closed definite triple on M obtained by gluing 16 copies of $\omega^{\mathrm{eh},t}$ with the flat orbifold triple $\hat{\boldsymbol{\omega}}$. We know that $\|Q_{\boldsymbol{\omega}^t} - \mathrm{id}\|_{C^0} = O(t^2)$. We look for a triple of closed 2-forms $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)$ on M such that

$$\tfrac{1}{2}\left(\omega_i^t + \eta_i\right) \wedge \left(\omega_j^t + \eta_j\right) = \delta_{ij}\, \mu_{\boldsymbol{\omega}^t}. \tag{2.3}$$

Decompose $\boldsymbol{\eta}$ into self-dual and anti-self dual parts $\boldsymbol{\eta} = \boldsymbol{\eta}^+ + \boldsymbol{\eta}^-$ with respect to $g_{\boldsymbol{\omega}^t}$. The self-dual part can be written in terms of a $M_{3\times 3}(\mathbb{R})$–valued function A by

$$\eta_i^+ = \sum_{j=1}^{3} A_{ij}\, \omega_j.$$

Denote by $\boldsymbol{\eta}^- * \boldsymbol{\eta}^-$ the symmetric (3×3)–matrix with entries $(\frac{1}{2}\, \eta_i^- \wedge \eta_j^-)/\mu_{\boldsymbol{\omega}^t}$. Then we can rewrite (2.3) as

$$Q_{\boldsymbol{\omega}^t} + Q_{\boldsymbol{\omega}^t} A^T + A\, Q_{\boldsymbol{\omega}^t} + A\, Q_{\boldsymbol{\omega}^t} A^T + \boldsymbol{\eta}^- * \boldsymbol{\eta}^- = \mathrm{id}. \tag{2.4}$$

Now, consider the map

$$M_{3\times 3}(\mathbb{R}) \longrightarrow Sym^2(\mathbb{R}^3); \qquad A \longmapsto Q_{\boldsymbol{\omega}^t} A^T + A\, Q_{\boldsymbol{\omega}^t} + A\, Q_{\boldsymbol{\omega}^t} A^T$$

and its differential $A \mapsto Q_{\boldsymbol{\omega}^t} A^T + A\, Q_{\boldsymbol{\omega}^t}$. Since $Q_{\boldsymbol{\omega}^t}$ is arbitrarily close to the identity as $t \to 0$, this linear map induces an isomorphism $Sym^2(\mathbb{R}^3) \to Sym^2(\mathbb{R}^3)$ for t sufficiently small. We can therefore define a smooth function $\mathcal{F}\colon Sym^2(\mathbb{R}^3) \to Sym^2(\mathbb{R}^3)$ such that $Q_{\boldsymbol{\omega}^t} A^T + A\, Q_{\boldsymbol{\omega}^t} + A\, Q_{\boldsymbol{\omega}^t} A^T = S$ if and only if $A = \mathcal{F}(S)$. Hence we reformulate (2.4) as

$$\boldsymbol{\eta}^+ = \mathcal{F}\left((\mathrm{id} - Q_{\boldsymbol{\omega}^t}) - \boldsymbol{\eta}^- * \boldsymbol{\eta}^-\right). \tag{2.5}$$

Now, let $\mathcal{H}^+_{\boldsymbol{\omega}^t}$ be the 3-dimensional space of self-dual harmonic 2-forms with respect to $g_{\boldsymbol{\omega}^t}$. Since $\omega_1^t, \omega_2^t, \omega_3^t$ are closed and self-dual (therefore harmonic) and linearly independent (since $\boldsymbol{\omega}^t$ is a definite triple) we deduce that $\mathcal{H}^+_{\boldsymbol{\omega}^t}$ consist of constant linear combinations of $\omega_1, \omega_2, \omega_3$. By Hodge theory with respect to $g_{\boldsymbol{\omega}^t}$ we can finally rewrite (2.5) as the *elliptic* equation

$$d^+\boldsymbol{a} + \boldsymbol{\zeta} = \mathcal{F}\left((\mathrm{id} - Q_{\boldsymbol{\omega}^t}) - \boldsymbol{\eta}^- * \boldsymbol{\eta}^-\right), \qquad d^*\boldsymbol{a} = 0, \tag{2.6}$$

for a triple $\boldsymbol{a}$ of 1-forms on M and a triple $\boldsymbol{\zeta} \in \mathcal{H}^+_{\boldsymbol{\omega}^t} \otimes \mathbb{R}^3$. Here $2\, d^+a = da + *da$ is the self-dual part of da.

Instead of the Monge–Ampère equation (2.2), one must now solve (2.6) applying the Implicit Function Theorem close to the singular limit $t \to 0$ to deform $\boldsymbol{\omega}^t$ into an exact hyperkähler triple. Assuming this can be done, if we now count parameters in the construction we find

(i) 10 moduli of the flat metric on T^4;
(ii) the choice of scale t of the Eguchi–Hanson metric and gauge $\psi \in \mathrm{SO}(4)/\mathrm{U}(2) \simeq S^2$ for each singular point.

Thus we have $10 + 3 \times 16 = 58$ parameters in total, exactly the dimension of the moduli space of Ricci-flat metrics (without any normalisation for the volume) on the K3 surface.

Orbifold singularities. From a broader perspective the Kummer construction furnishes the prototypical example of the appearance of orbifold singularities in non-collapsing sequences of Einstein 4-manifolds. By work of Anderson [1, Theorem C], Nakajima [42, Theorem 1.3] and Bando–Kasue–Nakajima [7, Theorem 5.1], we know that a sequence of Einstein 4-manifolds (M_i, g_i) with a uniform lower bound on volume and upper bounds on diameter and Euler characteristic converges (after passing to a subsequence) to an Einstein 4-orbifold M_∞ with finitely many singular points. The formation of orbifold singularities is modelled on complete Ricci-flat ALE spaces which appear as rescaled limits, or "bubbles", of the sequence (M_i, g_i) around points that approach one of the singularities of the orbifold M_∞. We now provide a more detailed description of these results.

Theorem 2.7. *Fix $\Lambda, C, V, D > 0$ and let (M_i^4, g_i) be a sequence of Einstein 4-manifolds satisfying*

(i) $|\mathrm{Ric}(g_i)| \leq \Lambda$;
(ii) $\chi(M_i) \leq C$;
(iii) $\mathrm{Vol}(M_i, g_i) \geq V$;
(iv) $\mathrm{diam}(M_i, g_i) \leq D$.

Then a subsequence converges to an Einstein orbifold (M_∞, g_∞) with finitely many isolated singular points $\{x_1, \dots, x_n\}$ with $n \leq n(\Lambda, C, V, D)$. More precisely, (M_i, g_i) converges to (M_∞, g_∞) in the Gromov–Hausdorff sense and there are smooth embeddings $f_i \colon M_\infty \setminus \{x_1, \dots, x_n\} \to M_i$ such that $f_i^* g_i$ converges to the smooth Einstein metric $g_\infty|_{M_\infty \setminus \{x_1, \dots, x_n\}}$ in C^∞ over compact sets of $M_\infty \setminus \{x_1, \dots, x_n\}$.

Here are some ingredients in the proof of the theorem. First of all, there exists a subsequence that converges to a compact metric space (M_∞, d_∞) in Gromov–Hausdorff topology and one has to understand the structure of M_∞. The Bishop–Gromov volume comparison and hypotheses (iii) and (iv) imply the *non-collapsing* condition $\mathrm{Vol}\,(B_1(p)) \geq v$ for all $p \in M_i$ and all i and some uniform $v > 0$. Moreover, the hypotheses of the Theorem guarantee that we have uniform control on the Sobolev constant of (M_i, g_i). Since the Einstein equation implies the differential inequality $\triangle |\mathrm{Rm}_{g_i}| + c\,|\mathrm{Rm}_{g_i}|^2 \geq 0$, Moser iteration now yields the following *ε–regularity* result: there exists $\varepsilon > 0$, $C > 0$, $r_0 > 0$ such that for all $0 < r < r_0$

$$\int_{B_{2r}(p)} |\mathrm{Rm}_{g_i}|^2 \,\mathrm{dv}_{g_i} < \varepsilon \Longrightarrow \sup_{B_r(p)} |\mathrm{Rm}_{g_i}| \leq C r^{-2} \left(\int_{B_{2r}(p)} |\mathrm{Rm}_{g_i}|^2 \,\mathrm{dv}_{g_i} \right)^{\frac{1}{2}}. \tag{2.8}$$

Given (i), the bound (ii) is equivalent to a global bound $\|\mathrm{Rm}_{g_i}\|_{L^2} \leq C'$ by the Gauss–Chern–Bonnet Formula (1.6). Then (2.8) fails only for a definite number of balls. Together with a bootstrap argument using the Einstein equation, we conclude that (M_∞, g_∞) is a smooth Einstein manifold away from a definite number of points $x_1, \dots, x_n$. A first step in analysing the structure of these singular points is to study their *tangent cone.* Fix $a = 1, \dots, n$. Consider a sequence $r_i \to 0$ and consider the sequence of pointed manifolds $(M_\infty, r_i^{-2} g_\infty, x_a)$. The pointed Gromov–Hausdorff limit (Y_a, o^*) of a subsequence $i_k \to \infty$ is called a tangent cone to M_∞ at x_a. A priori it depends on the sequence of rescaling r_i. Now, since $\|\mathrm{Rm}_{g_\infty}\|_{L^2}$ is bounded by the lower continuity of the energy, we have $\int_{B_{2r}(x_a) \setminus B_r(x_a)} |\mathrm{Rm}_{g_i}|^2 \,\mathrm{dv}_{g_i} \to 0$ as $r \to 0$. Then using (2.8) one can show that the annulus $B_2(o^*) \setminus B_1(o^*)$ in Y_a is flat. In fact Y_a is a flat cone $Y_a = \mathrm{C}(\mathbb{S}^3/\Gamma_a)$ which is smooth outside of its vertex o^*.

Not only does the available theory characterise the singularities of non-collapsed limits of Einstein 4-manifolds; it also explains *how* these singularities arise. The key notion is the one of *ALE (asymptotically locally Euclidean)* manifolds.

Definition 2.9. A complete Riemannian 4-manifold (W^4, h) is ALE of rate $\nu < 0$ if there exists a finite group $\Gamma \subset \mathrm{SO}(4)$ acting freely on $\mathbb{R}^4 \setminus \{0\}$, a compact set $K \subset W$, $R > 0$ and a diffeomorphism $f \colon \left(\mathbb{R}^4 \setminus B_R(0)\right)/\Gamma \to W \setminus K$ such that

$$|\nabla^k (f^* h - h_{\mathbb{R}^4/\Gamma})| = O(r^{\nu - k}).$$

Here the norm and covariant derivative are computed using the flat metric $h_{\mathbb{R}^4/\Gamma}$.

Theorem 2.10. *In the same notation and in addition to the statements of Theorem 2.7, for each $a = 1, \dots, n$ there exist $x_{a,i} \in M_i$ and $r_i \to \infty$ such that, up to subsequences,*

(i) *$B(x_{a,i}, \delta) \subset M_i$ converges to $B(x_a, \delta) \subset M_\infty$ for all $\delta > 0$ sufficiently small;*
(ii) *$(M_i, r_i^2 g_i, x_{a,i})$ converges to a Ricci-flat ALE 4-manifold $(W_a, g_a, x_{a,\infty})$ of rate -4 in the following sense: for each $R > 0$ there exists maps $f_{a,i} \colon B(x_{a,\infty}, R) \to M_i$ such that $f_{a,i}^*(r_i^2 g_i)$ converges in C^∞ to h_a on $B(x_{a,\infty}, R) \subset W_a$.*

The points $x_{a,i}$ and scales r_i are chosen so that $|\mathrm{Rm}_{g_i}|(x_{a,i}) = r_i^2$ is essentially the maximum of $|\mathrm{Rm}_{g_i}|$ in a small ball that is converging to a neighbourhood of x_a in the Gromov–Hausdorff topology. The existence of a limit (W_a, h_a) which is a complete Ricci-flat manifold with finite energy and maximal volume growth, *i.e.* $\|\mathrm{Rm}_{h_a}\|_{L^2} < \infty$ and $\lim_{r\to\infty} r^{-4}\mathrm{Vol}(B_r(x_{a,\infty})) > 0$, follows by arguments based on (2.8) as before. The fact that any such manifold is ALE of rate $\nu = -4$ follows from [7, Theorem 1.5].

In fact, the tangent cone at infinity of W_a might not match the tangent cone at the orbifold singularity $x_a \in M_\infty$ and a series of blow-ups at different scales might be necessary to capture the full picture of the degeneration of M_i to M_∞. Such bubbling-off of a "bubble-tree" of ALE Ricci-flat orbifolds was made precise by Bando [6] and Anderson–Cheeger [3]. We will see later some explicit examples of this phenomenon, *cf.* Remark 3.3.

ALE gravitational instantons. Theorem 2.10 provides the motivation for the study and ideally the *classification* of all Ricci-flat ALE 4-manifolds. It is here that the hyperkähler case differs dramatically from the more general Ricci-flat case: ALE hyperkähler 4-manifolds were constructed and classified by Kronheimer [34, 35] following earlier work of Eguchi–Hanson, Gibbons–Hawking and Hitchin (the classification was extended to non-simply connected Kähler Ricci-flat ALE 4-manifolds by Suvaina [47]); in contrast, not a single example of an ALE Ricci-flat 4-manifold with generic holonomy SO(4) is currently known and the question of whether Ricci-flat ALE 4-manifolds must have special holonomy is wide open.

A *gravitational instanton* is a complete non-compact hyperkähler 4-manifold with finite energy $\|\mathrm{Rm}\|_{L^2}$. We will see later that often stronger assumptions of curvature decay have to be imposed to obtain better control of the asymptotic geometry at infinity. Note that since every hyperkähler manifold is in particular Ricci-flat, gravitational instantons have constrained volume growth: the volume of a geodesic ball of radius r grows at most as r^4 and at least linearly in r. By the result of Bando–Kasue–Nakajima [7, Theorem 1.5] mentioned above, gravitational instantons of maximal volume growth are ALE hyperkähler 4-manifolds in the sense of Definition 2.9.

We now state Kronheimer's results. Let Γ be a finite subgroup of SU(2) that acts freely on $\mathbb{C}^2 \setminus \{0\}$. Such groups are classified by simply-laced Dynkin diagrams, *i.e.* the Dynkin diagrams of type ADE. The Kleinian (or Du Val) singularity $\mathbb{C}^2/\Gamma$ admits a (unique) *minimal resolution* $\pi\colon X_\Gamma \to \mathbb{C}^2/\Gamma$: X_Γ is a smooth complex surface, π is an isomorphism outside of $\pi^{-1}(0)$ and X_Γ does not contain any rational curve with self-intersection -1 (which could be blown-down to produce another smooth resolution). The exceptional locus $\pi^{-1}(0)$ is a configuration of rational curves with self-intersection -2 that intersects according to the Dynkin diagram of Γ. Finally, X_Γ has trivial canonical bundle, *i.e.* it admits a nowhere vanishing holomorphic $(2,0)$-form ω_c that outside of $\pi^{-1}(0)$ restricts to the pull-back of the standard complex volume form $dz_1 \wedge dz_2$ on $\mathbb{C}^2/\Gamma$. In the following theorem we forget the complex structure and regard X_Γ as a smooth 4-manifold.

Theorem 2.11. *Let Γ be a finite subgroup of* SU(2) *that acts freely on $\mathbb{C}^2 \setminus \{0\}$ and X_Γ be the smooth 4-manifold underlying the minimal resolution of $\mathbb{C}^2/\Gamma$.*

(i) *Let $\boldsymbol{\alpha} \in H^2(X_\Gamma, \mathbb{R}) \otimes \mathbb{R}^3$ satisfy*

$$\boldsymbol{\alpha}(\Sigma) \neq \mathbf{0} \in \mathbb{R}^3 \text{ for all } \Sigma \in H_2(X_\Gamma, \mathbb{Z}) \text{ such that } \Sigma \cdot \Sigma = -2. \tag{2.12}$$

Then there exists an ALE hyperkähler structure $\boldsymbol{\omega}$ on X_Γ with $[\boldsymbol{\omega}] = \boldsymbol{\alpha}$.

(ii) *If $(X, \boldsymbol{\omega})$ is an ALE hyperkähler 4-manifold asymptotic to $\mathbb{C}^2/\Gamma$ then X is diffeomorphic to X_Γ and $[\boldsymbol{\omega}]$ satisfies* (2.12)*. Moreover, if $(X, \boldsymbol{\omega})$ and $(X', \boldsymbol{\omega}')$ are two such manifolds and there exists a diffeomorphism $f\colon X \to X'$ such that $[f^*\boldsymbol{\omega}'] = [\boldsymbol{\omega}]$ then $(X, \boldsymbol{\omega})$ and $(X', \boldsymbol{\omega}')$ are isomorphic hyperkähler manifolds.*

The hyperkähler structures in (i) are obtained by the so-called *hyperkähler quotient* construction. For example, the Eguchi–Hanson metric can be described as the hyperkähler quotient of $\mathbb{H}^2$ by U(1) acting by $e^{i\theta} \cdot (q_1, q_2) = (e^{i\theta} q_1, e^{-i\theta} q_2)$. The hyperkähler moment map for this U(1)–action is $\boldsymbol{\mu}(q_1, q_2) = \overline{q}_1 i q_1 - \overline{q}_2 i q_2 \in \mathrm{Im}\,\mathbb{H} \simeq \mathbb{R}^3$. Given $\boldsymbol{\zeta} \in \mathbb{R}^3$, the hyperkähler quotient construction guarantees that, when smooth, $\boldsymbol{\mu}^{-1}(\boldsymbol{\zeta})/\mathrm{U}(1)$ is a hyperkähler manifold. When $\boldsymbol{\zeta} = \mathbf{0}$ we have the flat metric on $\mathbb{C}^2/\mathbb{Z}_2$ and when $\boldsymbol{\zeta} \neq \mathbf{0}$ we have the Eguchi–Hanson hyperkähler structure on T^*S^2 scaled and rotated so that $[\boldsymbol{\omega}](S^2) = \boldsymbol{\zeta}$.

For the classification result in (ii) Kronheimer exploits twistor theory and the natural "1-point" conformal compactification of an ALE gravitational instanton to an anti-self-dual 4-orbifold. More recently, Conlon–Hein [16, Corollary D] have obtained a different proof of this result that does not use twistor theory: with respect to any complex structure, an ALE gravitational instanton asymptotic to $\mathbb{C}^2/\Gamma$ must be the crepant resolution of a member of the versal $\mathbb{C}^*$–deformation of the Kleinian singularity $\mathbb{C}^2/\Gamma$; every such deformation has a unique crepant resolution and the latter admit a unique ALE Kähler Ricci-flat metric in each Kähler class.

3. Codimension one collapse

If we include hyperkähler orbifolds with finitely many isolated singularities, the period map (1.7) can be extended as a map from the completion of the moduli space $\mathcal{M}$ of Einstein metrics on the K3 surface with unit volume in the Gromov–Hausdorff topology onto $\mathrm{Gr}^+(3,19)/\Gamma$ [2, Theorem IV]. However $\mathrm{Gr}^+(3,19)/\Gamma$ is non-compact so we must still consider sequences of hyperkähler metrics that do not converge in Gromov–Hausdorf topology. This amounts to understanding *collapsing* sequences of hyperkähler metrics on the K3 surface.

Let (M,g_i) be a sequence of unit-volume hyperkähler metrics with $\mathrm{diam}(M,g_i) \to \infty$. Then $\mathrm{Vol}_{g_i}(B_1(p)) \to 0$ as $i \to \infty$ for all $p \in M$, since otherwise we would bound the diameter of (M,g_i) in terms of the total volume [45, Theorem I.4.1]. Under these assumptions, Anderson [2, Theorem II] showed that (M,g_i) collapses in the sense of Cheeger–Gromov outside finitely many points $x_1,\dots,x_n$, where the number n is controlled by the Euler characteristic $\chi(M)$. This means that for $x \in M \setminus \{x_1,\dots,x_n\}$ the injectivity radius $\mathrm{inj}_{g_i}(x)$ converges to zero and that we control the curvature after rescaling the metric so that the injectivity radius stays bounded: $\mathrm{inj}_{g_i}(x)^2|\mathrm{Rm}_{g_i}|_{g_i}(x) \leq \epsilon_0$, for a universal constant $\epsilon_0 > 0$. In fact, Cheeger and Tian [10, Theorems 0.1 and 0.8] have proven the much stronger result that the collapse occurs with *bounded* curvature away from a definite number of points.

Cheeger–Tian's result implies that Cheeger–Fukaya–Gromov's theory of collapse with bounded curvature [9] can be applied outside of finitely many points. The most important feature of this theory in our discussion is that the limiting geometry acquires continuous symmetries. Here we describe these symmetries only at the level of the local geometry around each point in the region that collapses with bounded curvature, referring to [9] for the globalisation of this local picture. Let (M_i^n, g_i) be a sequence of manifolds with sectional curvature bounded by a uniform constant $K > 0$. If $p_i \in M_i$ and $3r \in (0, \frac{1}{\sqrt{K}})$ then we can consider the sequence of Riemannian metrics $\hat{g}_i = \exp_{p_i}^* g_i$ on the ball $B_{3r}(0) \subset \mathbb{R}^n \simeq T_{p_i}M_i$. For each i there exists a pseudo-group Γ_i of local isometries of $(B_r(0), \hat{g}_i)$ whose action induces the equivalence relation $x \sim_{\Gamma_i} y$ if and only if $\exp_{p_i}(x) = \exp_{p_i}(y) \in M_i$. Up to passing to a subsequence, $(B_r(0), \hat{g}_i)$ converges in $C^{1,\alpha}$ to $(B_r(0), \hat{g}_\infty)$ (the limit and the convergence are smooth if we control higher order derivatives of the curvature, as in the Einstein case) and the pseudogroups Γ_i converge to a pseudogroup Γ_∞ of isometries of $(B_r(0), \hat{g}_\infty)$. The Gromov–Hausdorff limit of $(B_r(p_i), g_i)$ is $(B_r(0), \hat{g}_\infty)/\Gamma_\infty$. Since Γ_i acts in an increasingly dense fashion, Γ_∞ contains continuous isometries: in fact, a neighbourhood of the identity in Γ_∞ is isomorphic to a neighbourhood of the identity in a nilpotent Lie group.

Now, this general theory of Riemannian collapse with bounded curvature motivates us to study hyperkähler metrics in dimension 4 with a *triholomorphic* Killing field, *i.e.* a Killing field that preserves the hyperkähler triple as well as the metric, as models for regions that collapse with bounded curvature. Thought experiments based on the Kummer construction suggest that we should study gravitational instantons with *non-maximal volume growth* as models for regions that collapse with unbounded curvature. Indeed, consider the Kummer construction of Ricci-flat metrics on the K3 surface along a family of split tori $T^4 = T^{4-k} \times T^k_\epsilon$ with a T^k–factor of volume $\epsilon^k \to 0$. We can then think of the 2-spheres arising in the resolution of the 16 singularities of $T^4/\mathbb{Z}_2$ as coming in 2^k–tuples aligned along the collapsing k-torus over each of the 2^{4-k} singular points of $T^{4-k}/\mathbb{Z}_2$. If we now rescale the sequence of Kähler Ricci-flat metrics on the K3 surface by ϵ^{-2} around one of these 2^k–tuples, in the limit $\epsilon \to 0$ we should obtain a complete hyperkähler metric asymptotic to $(\mathbb{R}^{4-k} \times T^k)/\mathbb{Z}_2$. In the case $k = 1$, the appearance of gravitational instantons asymptotic to $(\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$ as rescaled limits was suggested by Page [44]. Hyperkähler metrics asymptotic to $(\mathbb{R}^{4-k} \times T^k)/\mathbb{Z}_2$ for $k = 1, 2, 3$ have been constructed by Biquard–Minerbe [8] using a non-compact version of the Kummer construction (earlier Hitchin [30] used twistor methods in the case $k = 1$).

The Gibbons–Hawking Ansatz. The Gibbons–Hawking ansatz [23] describes 4-dimensional hyperkähler metrics with a triholomorphic S^1–action (or more generally metrics with a triholomorphic Killing field).

Let U be an open set of $\mathbb{R}^3$ and $\pi\colon\ P \to U$ be a principal U(1)–bundle. Suppose that there exists a positive harmonic function h on U such that $*dh$ is the curvature $d\theta$ of a connection θ on P. Then the metric

$$g^{\mathrm{gh}} = h\,\pi^* g_{\mathbb{R}^3} + h^{-1}\theta^2 \tag{3.1a}$$

on P is hyperkähler. Indeed, we can exhibit an explicit hyperkähler triple $\boldsymbol{\omega}^{\mathrm{gh}}$ that induces the metric g^{gh}. Fix coordinates (x_1, x_2, x_3) on $U \subset \mathbb{R}^3$ and define

$$\omega_i^{\mathrm{gh}} = dx_i \wedge \theta + h\, dx_j \wedge dx_k, \tag{3.1b}$$

where (ijk) is a cyclic permutation of (123). One can check that $\boldsymbol{\omega}^{gh}$ is an SU(2)–structure inducing the Riemannian metric g^{gh}. Moreover, the requirement that $\boldsymbol{\omega}^{gh}$ is also closed is equivalent to the abelian *monopole equation*

$$* \, dh = d\theta. \tag{3.2}$$

The fibre-wise circle action on P preserves $\boldsymbol{\omega}^{\mathrm{gh}}$ and π is nothing but a hyperkähler moment map for this action. Conversely, every 4-dimensional hyperkähler metric with a triholomorphic circle action is locally described by (3.1).

The basic example of the Gibbons–Hawking construction is given in terms of so-called Dirac monopoles on $\mathbb{R}^3$. Fix a set of distinct points $p_1, \ldots, p_n$ in $\mathbb{R}^3$ and consider the harmonic function

$$h = m + \sum_{j=1}^{n} \frac{k_j}{2|x - p_j|},$$

where $m \geq 0$ and $k_1, \ldots, k_n$ are constants. Since $\mathbb{R}^3 \setminus \{p_1, \ldots, p_n\}$ has non-trivial second homology, we must require $k_j \in \mathbb{Z}$ for all j in order to be able to solve (3.2). If these integrality constraints are satisfied then $*dh$ defines the curvature $d\theta$ of a connection θ (unique up to gauge transformations) on a principal U(1)–bundle P over $\mathbb{R}^3 \setminus \{p_1, \ldots, p_n\}$ which restricts to the principal U(1)–bundle associated with the line bundle $\mathcal{O}(k_j) \to S^2$ on a small punctured neighbourhood of p_j. The pair (h, θ) is a solution of (3.2) which we call a *Dirac monopole* with singularities at $p_1, \ldots, p_n$.

The Gibbons–Hawking ansatz (3.1) associates a hyperkähler metric g^{gh} to every Dirac monopole on the open set where $h > 0$. When $k_j > 0$ then g^{gh} is certainly defined on the restriction of P to a small punctured neighbourhood of p_j. By a change of variables one can check that g^{gh} can be extended to a smooth orbifold metric modelled on $\mathbb{C}^2/\mathbb{Z}_{k_j}$ by adding a single point.

Remark 3.3. By considering clusters of points $p_1, \ldots, p_n$ coalescing together at different rates one can easily construct sequences of (non-compact) hyperkähler metrics developing orbifold singularities modelled on bubble-trees of ALE spaces.

In particular g^{gh} is a complete metric whenever $m \geq 0$ and $k_j = 1$ for all $j = 1, \ldots, n$. When $m = 0$ one can check that g^{gh} is an ALE metric in the sense of Definition 2.9. When $m > 0$ (by scaling we can then assume that $m = 1$) g^{gh} has a drastically different asymptotic geometry called *ALF (asymptotically locally flat)*.

Definition 3.4. A gravitational instanton (M,g) is called ALF if there exists a compact set $K \subset M$ such that the following holds. The (unique) end $M \setminus K$ is the total space of a circle fibration $\pi\colon\ M\setminus K \to (\mathbb{R}^3 \setminus B_R)/\Gamma$, where $R>0$ and Γ is a finite subgroup of O(3) acting freely on S^2. Passing to a Γ–cover we can always assume that π is a principal circle bundle. Define a model metric g_∞ on $M\setminus K$ by choosing a connection θ on (the Γ–cover of) π and setting $g_\infty = \pi^* g_{\mathbb{R}^3} + \theta^2$. Then we have

$$|\nabla^k_{g_\infty}(g-g_\infty)|_{g_\infty} = O(r^{\nu-k}) \tag{3.5}$$

for some $\nu<0$ and all $k\geq 0$.

There are only two possibilities for Γ: if $\Gamma=\mathrm{id}$ we say that M is an ALF gravitational instanton of *cyclic type*; if $\Gamma=\mathbb{Z}_2$ we say that M is an ALF gravitational instanton of *dihedral type*.

Recall that gravitational instantons have constrained volume growth: $\mathrm{Vol}\big(B_r(p)\big)$ grows at least linearly in r and at most as r^4. Under the assumption of faster than quadratic curvature decay, *i.e.* $|\mathrm{Rm}| = O(r^{-2-\epsilon})$ for some $\epsilon>0$ (or a slightly weaker finite weighted energy assumption), Minerbe [39, Theorem 0.1] has shown that if we assume a uniformly submaximal volume growth, $\mathrm{Vol}\big(B_r(p)\big) \leq Cr^a$ for some $3\leq a<4$ and all p, say, then the volume growth is at most cubic, $a\leq 3$. Minerbe also described the asymptotic geometry of gravitational instantons of cubic volume growth and faster than quadratic curvature decay: they are all ALF spaces as in Definition 3.4.

ALF gravitational instantons. Now we describe the classification of ALF gravitational instantons obtained by Minerbe [40] and Chen–Chen [12] in the cyclic and dihedral case respectively.

Let H^k be the total space of the principal U(1)–bundle associated with the line bundle $\mathcal{O}(k)$ over S^2 radially extended to $\mathbb{R}^3 \setminus B_R$ for any $R>0$. Let θ_k denote the (unique up to gauge transformation) SO(3)–invariant connection on H^k. The Gibbons–Hawking ansatz (3.1) yields a hyperkähler metric

$$g_k = \left(1+\frac{k}{2r}\right)(dr^2 + r^2 g_{\mathbb{S}^2}) + \left(1+\frac{k}{2r}\right)^{-1}\theta_k^2 \tag{3.6}$$

on H^k for all $k\in\mathbb{Z}$. Here r is a radial function on $\mathbb{R}^3$. Finally, on H^{2k} we consider the $\mathbb{Z}_2$–action which is generated by the simultaneous involutions on the base $\mathbb{R}^3$ and the fibre: on $\mathbb{R}^3$ we act by the standard involution $x\mapsto -x$ and the involution on the fibre $S^1=\mathbb{R}/2\pi\mathbb{Z}$ is the one induced by the standard involution on the universal cover $\mathbb{R}$. We refer to this involution of H^{2k} as its standard involution.

Definition 3.7. Let (M^4,g) be an ALF gravitational instanton.

(i) We say that M is of type A_k for some $k\geq -1$ if there exists a compact set $K\subset M$, $R>0$ and a diffeomorphism $\phi\colon\ H^{k+1}\to M\setminus K$ such that
$$|\nabla^l_{g_{k+1}}(g_{k+1}-\phi^* g)|_{g_{k+1}} = O(r^{-3-l})$$
for every $l\geq 0$.

(ii) We say that M is of type D_m for some $m\geq 0$ if there exists a compact set $K\subset M$, $R>0$ and a double cover $\phi\colon\ H^{2m-4}\to M\setminus K$ such that the group $\mathbb{Z}_2$ of deck transformations is generated by the standard involution on H^{2m-4} and
$$|\nabla^l_{g_{2m-4}}(g_{2m-4}-\phi^* g)|_{g_{2m-4}} = O(r^{-3-l})$$
for every $l\geq 0$.

Chen–Chen [12, Theorem 1.1] have shown that every ALF gravitational instanton is either of type A_k for some $k\geq -1$ or D_m for some $m\geq 0$. The constraints $k\geq -1$ and $m\geq 0$ were derived earlier by Minerbe [38, Theorem 0.1] in the cyclic case and by Biquard–Minerbe [8, Corollary 3.2] in the dihedral case.

ALF spaces of cyclic type. We saw that gravitational instantons of type A_k can be constructed from Dirac monopoles on $\mathbb{R}^3$ with $k+1$ singularities via the Gibbons–Hawking ansatz. These are usually called *multi-Taub–NUT* metrics. The case $k=0$ is the Taub–NUT metric on $\mathbb{R}^4$ and $k=-1$ is $\mathbb{R}^3 \times \mathbb{S}^1$ with its flat metric. Minerbe [40, Theorem 0.2] has shown that every ALF space of cyclic type must be isometric to a multi-Taub–NUT metric.

ALF spaces of dihedral type. ALF metrics of dihedral type are not globally given by the Gibbons–Hawking construction and in most cases are not explicit. A number of different constructions have appeared over the past decades, but only recently Chen–Chen [12, Theorem 1.2] have shown that all these constructions yield equivalent families of ALF metrics.

- $m=0$: The D_0 ALF manifold is the moduli space of centred charge 2 monopoles on $\mathbb{R}^3$ with its natural L^2–metric, known as the *Atiyah–Hitchin manifold.* The metric admits a cohomogeneity one isometric action of $\mathrm{SO}(3)$ and is explicitly given in terms of elliptic integrals [4, Chapter 11]. The D_0 ALF metric is rigid modulo scaling.
- $m=1$: The Atiyah–Hitchin manifold is diffeomorphic to the complement of a Veronese $\mathbb{RP}^2$ in S^4 and therefore it retracts to $\mathbb{RP}^2$ and is not simply connected. The double cover of the Atiyah–Hitchin manifold is a D_1 ALF space. As a smooth manifold it is diffeomorphic to the complement of $\mathbb{RP}^2$ in $\mathbb{CP}^2$, or equivalently to the total space of $\mathcal{O}(-4)$ over S^2. This rotationally invariant D_1 ALF metric admits a 3-dimensional family of D_1 ALF deformations, sometimes referred to as the *Dancer metrics.*
- $m=2$: D_2 ALF metrics were constructed by Hitchin [30, §7] using twistor methods and by Biquard–Minerbe [8, Theorem 2.4] using a non-compact version of the Kummer construction: one considers the quotient of $\mathbb{R}^3 \times \mathbb{S}^1$ by an involution and resolves the two singularities gluing in copies of the Eguchi–Hanson metric.
- $m \geq 3$: D_m ALF metrics (for all $m \geq 1$) appeared in the work of Cherkis–Kapustin [15] on moduli spaces of singular monopoles on $\mathbb{R}^3$ and were rigorously constructed by Cherkis–Hitchin [14] using twistor methods and the generalised Legendre transform. In the case $m \geq 3$ a more transparent construction due to Biquard–Minerbe [8, Theorem 2.5] yields D_m ALF metrics by desingularising the quotient of the Taub–NUT metric by the binary dihedral group $\mathcal{D}_m$ of order $4(m-2)$ using ALE dihedral spaces. Using complex Monge–Ampère methods Auvray [5] has then constructed $3m$–dimensional families of D_m ALF metrics on the smooth 4-manifold underlying the minimal resolution of $\mathbb{C}^2/\mathcal{D}_m$.

ALF gravitational instantons and collapsing Ricci-flat metrics on the K3 surface. Despite this rich theory of ALF gravitational instantons, until recently it has remained unclear how they can appear as models for the formation of singularities in collapsing sequences of hyperkähler metrics on the K3 surface. In [21] the author exploited singular perturbation methods to construct examples of Ricci-flat metrics on the K3 surface collapsing to a 3-dimensional limit and exhibit ALF gravitational instantons as the "bubbles" appearing in the process.

Theorem 3.8. *Let $T^3=\mathbb{R}^3/\Lambda$ be a 3-torus for some lattice $\Lambda \simeq \mathbb{Z}^3$. Endow T^3 with a flat metric g_{T^3}. Let $\tau\colon T^3 \to T^3$ be the standard involution $x \mapsto -x$ and denote by $q_1, \dots, q_8$ its fixed points. Fix a τ–symmetric configuration of further $2n$ distinct points $p_1, \tau(p_1), \dots, p_n, \tau(p_n)$. Denote by T^* the complement of $\{q_1, \dots, q_8, p_1, \dots, \tau(p_n)\}$ in T^3.*

Let $m_1, \dots, m_8 \in \mathbb{Z}_{\geq 0}$ and $k_1, \dots, k_n \in \mathbb{Z}_{\geq 1}$ satisfy

$$\sum_{j=1}^{8} m_j + \sum_{i=1}^{n} k_i = 16. \tag{3.9}$$

For each $j=1,\dots,8$ fix a D_{m_j} ALF space M_j and for each $i=1,\dots,n$ an A_{k_i-1} ALF space N_i.

Then there exists a 1-parameter family of hyperkähler metrics $\{g_\epsilon\}_{\epsilon\in(0,\epsilon_0)}$ on the K3 surface with the following properties. We can decompose the K3 surface into the union of open sets $K^\epsilon \cup \bigcup_{j=1}^{8} M_j^\epsilon \cup \bigcup_{i=1}^{n} N_i^\epsilon$ such that as $\epsilon \to 0$:

(i) (K^ϵ, g_ϵ) *collapses to the flat orbifold* $T^*/\mathbb{Z}_2$ *with bounded curvature away from the punctures;*

(ii) *for each* $j = 1, \ldots, 8$ *and* $k \geq 0$, $(M_j^\epsilon, \epsilon^{-2} g_\epsilon)$ *converges in* $C^{k,\alpha}_{loc}$ *to the* D_{m_j} *ALF space* M_j*;*

(iii) *for each* $i = 1, \ldots, n$ *and* $k \geq 0$, $(N_i^\epsilon, \epsilon^{-2} g_\epsilon)$ *converges in* $C^{k,\alpha}_{loc}$ *to the* $A_{k_i - 1}$ *ALF space* N_i*.*

The metric g_ϵ is constructed by gluing methods: we first construct an approximate hyperkähler metric by patching together known models and then perturb it to an exact solution. The construction of the approximate hyperkähler metric proceeds as follows. The ALF gravitational instantons provide models for the collapsing geometry near points of curvature concentration. We aim to construct a model for the collapsing sequence of hyperkähler metrics on regions where the curvature remains bounded using the Gibbons–Hawking ansatz over the punctured 3-torus T^*. We look for a Dirac monopole (h, θ) on T^* with the following singular behaviour: h is a harmonic function on T^* with prescribed singularities at the punctures

$$h \sim \frac{2m_j - 4}{2r_j} \text{ as } r_j \to 0, \qquad h \sim \frac{k_i}{2r_i} \text{ as } r_i \to 0.$$

Here r_j and r_i denote the distance functions from the points q_j and $p_i, \tau(p_i)$ with respect to the flat metric g_{T^3}. The balancing condition (3.9) guarantees the existence of the harmonic function h. Since the weights m_j and k_i are integers and the configuration of punctures is τ–invariant, one can also show the existence of a connection θ with curvature $*dh$ on a principal circle bundle over T^*.

Fix a (small) positive number $\epsilon > 0$. The Gibbons–Hawking ansatz (3.1) yields a hyperkähler metric

$$g^{\mathrm{gh}}_\epsilon = (1 + \epsilon h)\, \pi^* g_{T^3} + \epsilon^2 (1 + \epsilon h)^{-1}\, \theta^2$$

over the region where $1 + \epsilon h > 0$. Unless h is constant (which corresponds to Page's suggestion of considering the Kummer construction starting with $T^3 \times S^1_\epsilon$ for a circle factor of length $2\pi\epsilon \to 0$) there must exists some j with $m_j = 0, 1$ and therefore the harmonic function $1 + \epsilon h$ must become negative somewhere. The key observation is that by taking ϵ sufficiently small (which geometrically corresponds to making the circle fibres have small length) it is possible to construct highly collapsed hyperkähler metrics g^{gh}_ϵ outside of an arbitrarily small neighbourhood of the punctures. More precisely, one can prove that there exists $\epsilon_0 > 0$ such that for every $\epsilon < \epsilon_0$ we have $1 + \epsilon h > \frac{1}{2}$ on the complement of $\bigcup_{j=1}^k B_{8\epsilon}(q_j)$.

Now, as we know from Definition 3.7 the asymptotic model of any ALF metric (up to a double cover in the dihedral case) can be written in Gibbons–Hawking coordinates. The configuration of punctures and weights on T^3 was chosen so that, after taking a $\mathbb{Z}_2$–quotient, we are able to glue in copies of ALF spaces to extend the Gibbons–Hawking metric g^{gh}_ϵ to an approximately hyperkähler triple $\boldsymbol{\omega}_\epsilon$: close to the fixed point q_j of the $\mathbb{Z}_2$–action on T^3 we glue in the D_{m_j} ALF space M_j (this explains why we need 8 of them in the theorem); close to the image of $p_i, \tau(p_i)$ in $T^3/\mathbb{Z}_2$ we glue in the $A_{k_i - 1}$ ALF space N_i. In this way one obtains a closed definite triple $\boldsymbol{\omega}_\epsilon$ which is approximately hyperkähler in the sense that $|Q_{\boldsymbol{\omega}_\epsilon} - \mathrm{id}| \to 0$ as $\epsilon \to 0$. The approximate hyperkähler triple is then deformed into an exact solution by solving an equation like (2.6) using the Implicit Function Theorem in appropriately chosen weighted Hölder spaces.

4. Collapse and elliptic fibrations

In this final section we describe an influential work of Gross–Wilson [26] on the behaviour of hyperkähler metrics on the K3 surface collapsing to a 2-dimensional limit along the fibres of an elliptic fibration. We will also discuss more recent work of Hein [28] and related work by Chen–Chen [11–13] on gravitational instantons with non-maximal volume growth, in which elliptic fibrations also play a key role.

The Gross–Wilson's construction. A complex surface (*i.e.* a complex manifold of complex dimension 2) (M, J) is said to be elliptic if it admits a holomorphic map $\pi\colon M \to C$ onto a smooth complex curve C such that the generic fibre is a smooth curve of genus 1. If $\pi\colon M \to C$ has a holomorphic section σ, then the generic fibre becomes a smooth elliptic curve. We say that M is a minimal elliptic surface if there are no (-1)–curves contained in the fibres.

If (M, J) is an elliptic complex K3 surface not all fibres can be smooth elliptic curves because $\chi(M) = 24$. The possible singular fibres of elliptic surfaces have been classified by Kodaira. They are distinguished by the monodromy. Work locally with a minimal elliptic surface $\pi\colon M \to \triangle$ over a disc with a section σ and assume that all fibres except possibly the one over the origin are smooth elliptic curves. Using σ, one can describe the restriction $M|_{\triangle^*}$ of M to the punctured disc as $\pi\colon (\triangle^* \times \mathbb{C})/\Lambda \to \triangle^*$, for a family of lattices $\Lambda \subset \mathbb{C}$ defined by (possibly multi-valued) holomorphic functions τ_1, τ_2 on $\triangle^*$. The monodromy is the representation of the fundamental group of $\triangle^*$ on the mapping class group of the smooth fibre. We can think of it as the conjugacy class of the matrix $A \in \mathrm{SL}(2, \mathbb{Z})$ generating the action of $\pi_1(\triangle^*)$ on the oriented pair (τ_1, τ_2). We refer to [41, Tables I.4.1 and I.4.2] for Kodaira's list and limit ourself to the example of a singular fibre of type I_1. In this case $M|_{\triangle^*}$ is isomorphic to $(\triangle^* \times \mathbb{C})/(\tau_1\mathbb{Z} + \tau_2\mathbb{Z})$ with $\tau_1 = 1$, $\tau_2 = \frac{1}{2\pi i}\log z$. Since $\tau_2(e^{i\theta}z) = \tau_2(z) + \frac{\theta}{2\pi}$, the monodromy around an I_1 fibre is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{4.1}$$

The singular fibre $\pi^{-1}(0)$ is a pinched torus, *i.e.* a 2-sphere with south and north poles identified.

Generically a complex K3 surface that admits an elliptic fibration (necessarily over $\mathbb{CP}^1$) has exactly 24 singular fibres, all of type I_1. Let $\pi\colon M \to \mathbb{CP}^1$ be such such an elliptic K3 surface with 24 singular I_1 fibres. Up to changing the complex structure of M preserving the fibration $\pi\colon M \to \mathbb{CP}^1$ we can always reduce to the case that π has a holomorphic section. Gross–Wilson studied the behaviour of Kähler Ricci-flat metrics on M as we fix this complex structure and deform the Kähler class so that the elliptic fibres of π shrink to zero size. In other words, they considered a sequence of Kähler classes converging to the class $[\pi^*\omega_{\mathrm{FS}}]$ at the boundary of the Kähler cone of M and described the behaviour of the Kähler Ricci-flat metric given by Yau's Theorem along this sequence. Here ω_{FS} is the Fubini–Study metric on $\mathbb{CP}^1$. Gross–Wilson's description of the collapsing Ricci-flat metrics is achieved by a gluing construction.

The semi-flat metric. The model for the collapsing Ricci-flat metrics away from the singular fibres is provided by a certain semi-flat metric [26, §2], *i.e.* a metric that restricts to a flat metric on each elliptic fibre.

Let $\pi\colon M \to \mathbb{CP}^1$ be an elliptic K3 surface with a section and restrict the fibration to a small disc $\triangle \subset \mathbb{CP}^1$. Fix a holomorphic coordinate z on $\triangle$. We assume that $\pi\colon M|_\triangle \to \triangle$ is a minimal elliptic fibration with a section such that all fibres are smooth. Using the given holomorphic section, we can identify $M|_\triangle$ with $(\triangle \times \mathbb{C}_w)/(\tau_1\mathbb{Z} + \tau_2\mathbb{Z})$ as before. Without loss of generality we assume that $\mathrm{Im}(\overline{\tau_1}\tau_2) > 0$. Fix a holomorphic symplectic form ω_c on M. In coordinates z, w we can assume that $\omega_c = dz \wedge dw$. Given $\varepsilon > 0$, we construct a semi-flat metric $\omega_{sf,\varepsilon}$ using the following ingredients.

(i) For each $z \in \triangle$ define a flat Kähler metric $\omega_{z,\varepsilon}$ on $\pi^{-1}(z)$ by choosing a dual basis $\xi_1(z), \xi_2(z)$ to $\tau_1(z), \tau_2(z)$ and setting $\omega_{z,\varepsilon} = \varepsilon\, \xi_1(z) \wedge \xi_2(z)$. Changing basis to $dw, d\overline{w}$ yields $\omega_{z,\varepsilon} = \frac{i}{2} W dw \wedge d\overline{w}$, with $W = \frac{\varepsilon}{\mathrm{Im}(\overline{\tau_1}\tau_2)}$.

(ii) Define $\omega_{\triangle,\varepsilon}$ as the unique Kähler metric on $\triangle$ such that the pairing $T^{1,0}\triangle \times (\triangle \times \mathbb{C}) \to \mathbb{C}$ induced by ω_c is isometric with respect to the Hermitian metrics induced by $\omega_{z,\varepsilon}$ and $\omega_{\triangle,\varepsilon}$. Explicitly, $\omega_{\triangle,\varepsilon} = \frac{i}{2} W^{-1} dz \wedge d\overline{z}$.

(iii) The family of lattices $\tau_1\mathbb{Z}+\tau_2\mathbb{Z}$ defines a flat connection on the trivial bundle $\triangle\times\mathbb{C}$ by declaring τ_1 and τ_2 to be flat sections. The associated connection 1-form is

$$\Gamma\, dz = \frac{1}{\mathrm{Im}(\overline{\tau_1}\tau_2)}\big(\mathrm{Im}(\overline{\tau_1}w)d\tau_2 - \mathrm{Im}(\overline{\tau_2}w)d\tau_1\big).$$

The semi-flat metric is then

$$\omega_{sf,\varepsilon} = \tfrac{i}{2}W^{-1}dz\wedge d\overline{z} + \tfrac{i}{2}W(dw-\Gamma dz)\wedge(d\overline{w}-\overline{\Gamma}d\overline{z}). \tag{4.2}$$

Note that the triple $(\omega_{sf,\varepsilon}, \mathrm{Re}\,\omega_c, \mathrm{Im}\,\omega_c)$ is hyperkähler and that $\omega_{sf,\varepsilon}|_{\pi^{-1}(z)}$ is the flat metric with volume ε.

The construction of the semi-flat metric can be extended to the situation where $M|_\triangle$ has a singular fibre over the origin $z=0$. We simply replace $\triangle$ with the punctured disc $\triangle^*$ and use Kodaira's normal form for $M|_{\triangle^*}$. For example, if $\pi^{-1}(0)$ is a fibre of type I_1 then $M|_{\triangle^*}$ is isomorphic to $(\triangle^*\times\mathbb{C})/(\tau_1\mathbb{Z}+\tau_2\mathbb{Z})$ with $\tau_1=1$, $\tau_2=\frac{1}{2\pi i}\log z$. Note that the assumption $\mathrm{Im}(\overline{\tau}_1\tau_2)>0$ forces $\triangle$ to be strictly contained in the unit disc in $\mathbb{C}$. The semi-flat metric (4.2) on the complement of a singular fibre of type I_1 admits a tri-holomorphic S^1–action and, following [26], we can rewrite it in Gibbons–Hawking coordinates.

First of all, since $W\,\mathrm{Im}\,(\Gamma dz) = -\,\mathrm{Im}(w)\,dW$, the imaginary part of $W(dw-\Gamma\, dz)$ is closed. Hence there exists a function $t\colon\ \triangle^*\times\mathbb{C}\to\mathbb{R}$, unique up to the addition of a constant, such that $-W^{-1}dt = \mathrm{Im}(dw-\Gamma\, dz)$. Then $\pi\colon\ (\triangle^*\times\mathbb{C})/\tau_1\mathbb{Z}\to\triangle^*\times\mathbb{R}_t$ is a principal U(1)–bundle. Explicitly, $t=-\frac{2\pi\varepsilon\mathrm{Im}(w)}{\log|z|}$. Taking the quotient by $\tau_2\mathbb{Z}$ we obtain a principal U(1)–bundle $(\triangle^*\times\mathbb{C})/(\tau_1\mathbb{Z}+\tau_2\mathbb{Z})\to\triangle^*\times\mathbb{R}/\varepsilon\mathbb{Z}$. Its Euler class evaluated on $|z|=\text{const}$ is ±1, depending on the orientation. Now set $h=W^{-1}$, $dw-\Gamma\, dz=\theta-ih\,dt$ and use polar coordinates $re^{i\psi}=z$. The semi-flat metric (4.2) can then be written in Gibbons–Hawking coordinates (3.1)

$$g_{sf,\varepsilon} = \frac{-\log r}{2\pi\varepsilon}\left(dr^2+r^2d\psi^2+dt^2\right)+\frac{2\pi\varepsilon}{-\log r}\theta^2. \tag{4.3}$$

The Ooguri–Vafa metric. The second building block in Gross–Wilson's construction is the Ooguri–Vafa metric, an explicit (incomplete) hyperkähler metric defined in a neighbourhood of a singular fibre of type I_1. This metric was first constructed in [43]. A more thorough analysis is given in [26, §3]. The Ooguri–Vafa metric is a periodic version of the Taub–NUT metric, in the sense that it can be constructed by the Gibbons–Hawking ansatz on $\mathbb{R}^2\times S^1$ with a harmonic function h with a Green's function singularity at a point. Since the Green's function of $\mathbb{R}^2\times S^1$ changes sign (we say that $\mathbb{R}^2\times S^1$ is *parabolic*), the Ooguri–Vafa metric is only defined on a small enough neighbourhood of the Green's function singularity.

Fix $\varepsilon>0$ sufficiently small so that $2\varepsilon<1$. Let $\triangle$ be the unit disc in $\mathbb{C}$ with coordinate $z=re^{i\psi}$. Let t be a periodic coordinate of period ε and consider the product $\triangle\times S^1_t$, where $S^1_t=\mathbb{R}/\varepsilon\mathbb{Z}$. By abuse of notation we denote by 0 the point with coordinates $z=0$ and $t=0\ (\mathrm{mod}\,\varepsilon\mathbb{Z})$. Consider the power series

$$h(z,t)=\frac{1}{2}\sum_{m\in\mathbb{Z}}\left(\frac{1}{\sqrt{r^2+4\pi^2(t-m\varepsilon)^2}}-a_{|m|}\right), \tag{4.4}$$

where

$$a_{|m|}=\tfrac{1}{2|m|\pi\varepsilon}\ \text{if } m\neq0 \quad\text{ and }\quad a_0=\tfrac{\log4\pi\varepsilon-2\gamma}{\pi\varepsilon}.$$

Here γ is the Euler constant, $\gamma=\lim_{n\to\infty}\sum_{k=1}^n k^{-1}-\log n$. The series converges uniformly on compact subsets of $(\triangle\times S^1_t)\setminus\{0\}$ to the Green's function of $\mathbb{R}^2\times S^1_t$ with singularity at 0. Whenever $z\neq0$, h can be expressed as

$$h(z,t)=-\frac{1}{2\pi\varepsilon}\log r+\frac{1}{2\pi\varepsilon}\sum_{m\in\mathbb{Z}^*}K_0\left(\frac{|m|r}{\varepsilon}\right)e^{\frac{2\pi mi}{\varepsilon}t},$$

where K_0 is the second modified Bessel function. In particular, due to the exponential decay of the Bessel function, for all $k \geq 0$ there exists a constant $C_k > 0$ such that

$$\left| \nabla^k \left(h(z,t) + \frac{1}{2\pi\varepsilon} \log r \right) \right| \leq \frac{C_k}{\varepsilon} e^{-\frac{r}{\varepsilon}} \tag{4.5}$$

for all $r \geq 2\varepsilon$.

One can now use the harmonic function h defined in (4.4) in the Gibbons–Hawking ansatz (3.1) to produce a hyperkähler metric—the Ooguri–Vafa metric—on a circle bundle X over $\triangle \times S^1_t$. As in the case of the multi-Taub–NUT metrics, a change of coordinates shows that the Gibbons–Hawking metric on X extends smoothly over a point corresponding to the singular points 0 of h.

By (4.5) the Ooguri–Vafa metric approaches the semi-flat metric (4.3) up to terms that decay exponentially fast as $\varepsilon \to 0$. It remains to check that the Ooguri–Vafa metric is defined on an elliptic fibration over a disc with a singular fibre of type I_1 over the origin. Choose the complex structure such that dz and $\theta - ihdt$ span the space of $(1,0)$–forms. In this complex structure the projection $\pi : X \to \triangle$ is an elliptic fibration and $\pi^{-1}(0)$ is the only singular fibre. One can identify the periods and therefore the monodromy of this elliptic fibration by integrating the $(1,0)$–form $\theta - ihdt$ over a basis $\{\gamma_1, \gamma_2\}$ of the first homology of a fibre $\pi^{-1}(z)$. If one chooses γ_1 to be an orbit for the S^1–action on the circle bundle $X \to \triangle \times S^1_t$ and γ_2 to be the circle parametrised by t in the base then one finds easily that the monodromy coincides with (4.1). Alternatively, one can identify $\pi^{-1}(0)$ with a pinched torus, since the restriction of the circle fibration X over $\{z = 0\} \times S^1_t$ degenerating at the point 0 is a 2-sphere with the two poles identified.

Behaviour of Ricci-flat metrics. For $\varepsilon > 0$ sufficiently small, Gross–Wilson now patch together the semi-flat metric (4.2) with 24 copies of the Ooguri–Vafa metric to obtain an approximate Ricci-flat metric ω_ε on the elliptic K3 surface M. The error (measured in terms of appropriate Hölder norms of the Ricci-potential of ω_ε) is of order $e^{-C/\varepsilon}$. This exponential decay is crucial for the perturbation argument to work. Indeed, by Yau's proof of the Calabi Conjecture there exists a unique function u_ε on M such that

$$(\omega_\varepsilon + i\partial\overline{\partial} u_\varepsilon)^2 = \tfrac{1}{4}\omega_c \wedge \overline{\omega}_c, \qquad \int_M u_\varepsilon \, \omega_\varepsilon^2 = 0. \tag{4.6}$$

Gross–Wilson run through Yau's proof of the existence of u_ε keeping careful track of all the constants involved (*e.g.* the Sobolev constant in the Moser iteration argument to prove the C^0–estimate). All these constants do blow-up as $\varepsilon \to 0$, but only polynomially in ε^{-1}. Since the error is exponentially small the Implicit Function Theorem can still be applied to obtain the following theorem [26, Theorems 5.6 and 6.4].

Theorem 4.7. *Let $\pi \colon (M, \omega_c) \to \mathbb{CP}^1$ be an elliptic K3 surface with a holomorphic section and 24 singular I_1 fibres. For $\varepsilon > 0$ sufficiently small let ω_ε be the Kähler metric on X constructed by gluing the semi-flat metric* (4.2) *to* 24 *copies of the Ooguri–Vafa metric. Let u_ε be the unique solution to* (4.6).

(i) *For every $k \geq 2$, $\alpha \in (0,1)$ and every simply connected open subset $U \subset \mathbb{CP}^1$ with closure contained in the complement of the* 24 *points $p_1, \dots, p_{24}$ corresponding to singular fibres there exist constants $C, c > 0$ such that $\| u_\varepsilon \|_{C^{k,\alpha}(U)} \leq Ce^{-c/\varepsilon}$.*

(ii) *(X, ω_ϵ) converges in the Gromov–Hausdorff sense to $\mathbb{CP}^1$ endowed with the distance induced by the (singular) metric ω_0 limit of the semi-flat metric* (4.2) *away from the* 24 *singular points. Away from $p_1, \dots, p_{24}$, ω_0 satisfies $\mathrm{Ric}(\omega_0) = \omega_{WP}$, where ω_{WP} is the pull-back to $\mathbb{CP}^1 \setminus \{p_1, \dots, p_{24}\}$ of the Weil–Peterson metric on the moduli space of elliptic curves.*

Similar results—convergence after rescaling to the semi-flat metric on the locus of smooth fibres and global Gromov–Hausdorff convergence to $\mathbb{CP}^1$ as in (ii)—have been obtained more recently for arbitrary elliptic K3 surfaces without a detailed picture of the collapsing hyperkähler metrics in a neighbourhood of the singular fibres, *cf.* [24, 25].

ALG and ALH gravitational instantons. In [28] Hein constructs families of gravitational instantons with quadratic and lower-than-quadratic volume growth. The metrics are constructed by applying Tian–Yau's method to a rational elliptic surface, *i.e.* a complex surface (X, J) which is birationally equivalent to $\mathbb{CP}^2$ and which admits a minimal elliptic fibration with a section. All rational elliptic surfaces can be constructed in the following way. Let C_1 be a smooth plane cubic and C_2 a second distinct cubic. The pencil $\{\lambda_1 C_1 + \lambda_2 C_2 \,|\, [\lambda_1 : \lambda_2] \in \mathbb{CP}^1\}$ has $C_1 \cdot C_2 = 9$ base points (counted with multiplicities). After blowing them up we obtain a rational elliptic surface $\pi\colon X \to \mathbb{CP}^1$; X is a minimal elliptic surface because we blew-up just enough to resolve all the tangencies of the pencil and X has at least a section given by the (-1)–curve obtained in the last blow-up. As for the K3 surface, if X is a rational elliptic surface not all fibres can be smooth elliptic curves because $\chi(X) = \chi\left(\mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2\right) = 12$.

The crucial point now is that the class of an elliptic fibre in a rational elliptic surface is an anti-canonical divisor: there exists a holomorphic symplectic form ω_c on $M = X \setminus \pi^{-1}(\infty)$ with simple poles along $\pi^{-1}(\infty)$. (Here we choose an affine coordinate on the base of the fibration $\mathbb{CP}^1$ so that the chosen elliptic fibre is the fibre over ∞.) Assuming the existence of an appropriate complete background metric ω_0 on M, Tian and Yau's method [48,49] can be applied to construct a Ricci-flat Kähler metric on M by solving the complex Monge–Ampère equation $\left(\omega_0 + i\partial\overline{\partial}u\right)^2 = \frac{1}{2}\omega_c \wedge \overline{\omega}_c$ on the complement of $\pi^{-1}(\infty)$. In order to be able to solve this Monge–Ampère equation it is necessary to assume that ω_0 is already an approximate solution at infinity, in the sense that the Ricci potential of ω_0 decays with a certain rate. Note that the choice of the background ω_0 is not obvious nor unique: the flat and Taub–NUT metrics on $\mathbb{C}^2 = \mathbb{CP}^2 \setminus \mathbb{CP}^1$ are different complete hyperkähler metrics with the same holomorphic symplectic form [36]. In the case of rational elliptic surfaces, Hein exploits the elliptic fibration to construct a good background Kähler metric ω_0 which is approximately Ricci-flat at infinity. The type of fibre $\pi^{-1}(\infty)$ removed dictates the asymptotics of the metric ω_0 using Kodaira's normal form for a neighbourhood of $\pi^{-1}(\infty)$ and a semi-flat metric as in Gross–Wilson's construction.

The simplest examples of Hein's construction are those obtained by removing a smooth elliptic fibre (a fibre of type I_0 in Kodaira's classification): in this case the metric is ALH.

Definition 4.8. A gravitational instanton (M, g) is called ALH if there exists a compact subset $K \subset M$ and a diffeomorphism $f\colon \mathbb{R}_+ \times T^3 \to M \setminus K$ such that

$$|\nabla^k_{g_{\text{flat}}}(f^*g - g_{\text{flat}})|_{g_{\text{flat}}} = O\left(e^{-\delta t}\right)$$

for all $k \geq 0$ and some $\delta > 0$. Here $g_{\text{flat}} = dt^2 + g_{T^3}$ for a flat metric g_{T^3} on T^3.

Examples of ALH metrics have also been obtained by Biquard–Minerbe [8] by desingularising the flat orbifold $(\mathbb{R} \times T^3)/\mathbb{Z}_2$ by gluing in 8 copies of the Eguchi–Hanson metric. More recently, Chen–Chen [13, Theorem 1.5] have given a complete classification of ALH gravitational instantons.

Theorem 4.9. *Let M be the smooth 4-manifold underlying the minimal resolution of $(\mathbb{R} \times T^3)/\mathbb{Z}_2$, where $T^3 = \mathbb{R}^3/(\mathbb{Z}v_1 + \mathbb{Z}v_2 + \mathbb{Z}v_3)$. For each $i = 1, 2, 3$ let F_i be the element of $H_2(T^3, \mathbb{Z})$ corresponding to $\operatorname{span}(v_j, v_k)$, where $\epsilon_{ijk} = 1$. Then $H_2(M, \mathbb{Z})$ is spanned by F_1, F_2, F_3 and the classes of the 8 (-2)–curves introduced by the resolution $M \to (\mathbb{R} \times T^3)/\mathbb{Z}_2$.*

(i) *Let $\boldsymbol{\alpha} \in H^2(M, \mathbb{R}) \otimes \mathbb{R}^3$ satisfy*

$$\begin{aligned} &\boldsymbol{\alpha}(\Sigma) \neq \mathbf{0} \in \mathbb{R}^3 \text{ for all } \Sigma \in H_2(M, \mathbb{Z}) \text{ such that } \Sigma \cdot \Sigma = -2, \\ &\text{the matrix with rows } \boldsymbol{\alpha}(F_i),\ i = 1, 2, 3, \text{ is positive definite.} \end{aligned} \tag{4.10}$$

Then there exists an ALH hyperkähler structure $\boldsymbol{\omega}$ on M with $[\boldsymbol{\omega}] = \boldsymbol{\alpha}$, unique up to triholomorphic isometries acting trivially on $H^2(M, \mathbb{R})$.

(ii) *If $(X, \boldsymbol{\omega})$ is an ALH gravitational instanton then X is diffeomorphic to M and $[\boldsymbol{\omega}]$ satisfies (4.10).*

ALH gravitational instantons can be used to produce hyperkähler metrics on the K3 surface that develop a long neck. Indeed, if $(M,\boldsymbol{\omega})$ and $(M,\boldsymbol{\omega}')$ are two ALH gravitational instantons asymptotic to the same flat cylinder $dt^2+g_{T^3}$, then one can cut off their cylindrical ends for $t\gg 1$ and glue the resulting manifolds with boundary to produce a sequence of approximately Ricci-flat metrics on the K3 surface that develop a very long neck. Alternatively, by rescaling the metrics in the sequence so that the diameter stays bounded, one produces in this way a sequence of approximately Ricci-flat metrics that collapse to a closed interval with curvature concentration at the two end points. Chen–Chen [13, §5] show that these approximate solutions can be perturbed into exact hyperkähler metrics with the same collapsing behaviour.

Hein's examples of gravitational instantons defined on the complement of a singular fibre of type I_0^*, II, III, IV in Kodaira's classification are also easily understood, in particular those examples that arise from isotrivial elliptic fibrations. Let E be a smooth elliptic curve admitting a $\mathbb{Z}_r$–subgroup of automorphisms for $r=2,3,4$ or 6. Thus E is any elliptic curve if $r=2$; a Weierstrass equation for E is $y^2=x^3+x$ if $r=4$, with $\mathbb{Z}_4$–action generated by $(x,y)\mapsto(-x,iy)$; if $r=3$ or 6 then $E\colon\, y^2=x^3+1$ and the $\mathbb{Z}_3$ and $\mathbb{Z}_6$–actions are generated by $(x,y)\mapsto(e^{2\pi i/3}x,y)$ and $(x,y)\mapsto(e^{2\pi i/3}x,-y)$ respectively. Now consider the orbifold $(\mathbb{CP}^1\times E)/\mathbb{Z}_r$, where the cyclic group $\mathbb{Z}_r$ acts diagonally on $\mathbb{CP}^1$ and E. Resolve the singularities and blow down all (-1)–curves in the fibres to obtain a rational elliptic surface with only two singular fibres over 0 and ∞ and such that all smooth fibres are isomorphic. Corresponding to $r=2,3,4,6$ this construction yields four pairs of singular fibres—(I_0^*,I_0^*), (II,II^*), (III,III^*) and (IV,IV^*) in Kodaira's notation. Unless $r=2$, the two fibres in each pair are different because the $\mathbb{Z}_r$–action on $\mathbb{CP}^1$ has different weights at 0 and ∞. By removing the fibre of non–$*$–type in each pair, one obtains a crepant resolution of $T^*E/\mathbb{Z}_r$ and the resulting semi-flat metric coincides with the flat metric on $T^*E/\mathbb{Z}_r$. In fact, in this case some of Hein's Ricci-flat metrics can also be obtained from the Kummer-type construction of Biquard–Minerbe [8], gluing rescaled ALE spaces to resolve the singularities of the flat orbifold. When we remove the fibre of $*$–type in each pair, Hein's Ricci-flat metric is asymptotic to the twisted product of a flat metric on E and of a flat 2-dimensional cone which is not a quotient of $\mathbb{C}$ [28, Theorem 1.5 (ii)].

All these examples have faster than quadratic curvature decay and their asymptotic geometry is called ALG. The recent classification result of Chen–Chen [13, Theorem 1.4] states that all ALG gravitational instantons arise from (a slight improvement of) Hein's construction on the complement of a fibre of type $I_0^*, II, II^*, III, III^*, IV$ or IV^*. Furthermore, we note that constructions of sequences of Ricci-flat metrics on the K3 surface obtained by desingularising orbifolds $(E_1\times E_2)/\mathbb{Z}_r$ for a product of $\mathbb{Z}_r$–invariant elliptic curves with $\mathrm{Vol}(E_2)\to 0$ could provide examples of collapsing sequences of hyperkähler metrics with ALG spaces of type I_0^*, II, III, IV as rescaled limits.

By removing a singular fibre with infinite monodromy, Hein is also able to produce examples with more exotic asymptotic geometry, often referred to as gravitational instantons of type ALG* and ALH*. The examples of type ALG* (ALH*) have quadratic volume growth (volume growth $r^{\frac{4}{3}}$) and are obtained by removing a fibre of Kodaira type I_b^*, $b=1,\dots,4$, (I_b, $b=1,\dots,9$) from a rational elliptic surface. These examples do not have faster than quadratic curvature decay and do not fit into Chen–Chen's classification.

The asymptotic geometry of the ALG* and ALH* examples can be constructed using the Gibbons–Hawking ansatz on (the $\mathbb{Z}_2$–quotient of) $\mathbb{R}^2\times S^1$ and $\mathbb{R}\times T^2$, respectively, with a finite number of punctures. Since $\mathbb{R}^2\times S^1$ and $\mathbb{R}\times T^2$ are parabolic, the sum of Green's functions used as the harmonic function in the Gibbons–Hawking construction is only positive at infinity and the construction provides only good asymptotic models. We expect that a gluing construction as in Theorem 3.8 using Atiyah–Hitchin spaces as building blocks together with the Gibbons–Hawking construction on $\mathbb{R}^2\times S^1$ and $\mathbb{R}\times T^2$ will yield families of ALG* and ALH* gravitational instantons close to a collapsed limit $(\mathbb{R}^2\times S^1)/\mathbb{Z}_2$ and $(\mathbb{R}\times T^2)/\mathbb{Z}_2$, respectively. We also expect that extensions of Theorem 3.8 where one considers sequences of flat metrics on T^3 collapsing to T^2 and S^1 should provide examples of collapsing Ricci-flat metrics with ALG* and ALH* gravitational instantons as rescaled limits. More generally, it is expected that ALG, ALH, ALG* and ALH* gravitational instantons will play an important role in understanding relations between collapsing sequences of Ricci-flat metrics on the K3 surface and degenerations of a compatible complex structure, *cf.* for example [32]. Very recently Hein–Sun–Viaclovsky–Zhang [27] gave a general construction of families of Ricci-flat metrics on the K3 surface that collapse to a closed bounded interval with curvature concentrating at a finite number of points (always including the two endpoints). The building blocks for this gluing construction are a pair ALH* metrics bubbling off at the endpoints of the interval and an incomplete "neck" joining the two obtained from the Gibbons–Hawking ansatz on $\mathbb{R} \times T^3$.

Acknowledgements. The author is supported by a Royal Society University Research Fellowship. He would also like to thank NSF for the partial support of his work under grant DMS-1608143.

References

[1] M. T. Anderson, *Ricci curvature bounds and Einstein metrics on compact manifolds*, J. Amer. Math. Soc. **2** (1989), no. 3, 455–490.
[2] ———, *The L^2 structure of moduli spaces of Einstein metrics on 4-manifolds*, Geom. Funct. Anal. **2** (1992), no. 1, 29–89.
[3] M. T. Anderson and J. Cheeger, *Diffeomorphism finiteness for manifolds with Ricci curvature and $L^{n/2}$-norm of curvature bounded*, Geom. Funct. Anal. **1** (1991), no. 3, 231–252.
[4] M. Atiyah and N. Hitchin, *The geometry and dynamics of magnetic monopoles*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 1988.
[5] H. Auvray, *From ALE to ALF gravitational instantons*, Compos. Math. **154** (2018), no. 6, 1159–1221.
[6] S. Bando, *Bubbling out of Einstein manifolds*, Tohoku Math. J. (2) **42** (1990), no. 2, 205–216.
[7] S. Bando, A. Kasue, and H. Nakajima, *On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth*, Invent. Math. **97** (1989), no. 2, 313–349.
[8] O. Biquard and V. Minerbe, *A Kummer construction for gravitational instantons*, Comm. Math. Phys. **308** (2011), no. 3, 773–794.
[9] J. Cheeger, K. Fukaya, and M. Gromov, *Nilpotent structures and invariant metrics on collapsed manifolds*, J. Amer. Math. Soc. **5** (1992), no. 2, 327–372.
[10] J. Cheeger and G. Tian, *Curvature and injectivity radius estimates for Einstein 4-manifolds*, J. Amer. Math. Soc. **19** (2006), no. 2, 487–525.
[11] G. Chen and X. Chen, *Gravitational instantons with faster than quadratic curvature decay (I)*, 2015.
[12] ———, *Gravitational instantons with faster than quadratic curvature decay (II)*, 2015, arXiv:1508.07908. To appear in J. Reine Angew. Math.
[13] ———, *Gravitational instantons with faster than quadratic curvature decay (III)*, 2016.
[14] S. A. Cherkis and N. J. Hitchin, *Gravitational instantons of type D_k*, Comm. Math. Phys. **260** (2005), no. 2, 299–317.
[15] S. A. Cherkis and A. Kapustin, *Singular monopoles and gravitational instantons*, Comm. Math. Phys. **203** (1999), no. 3, 713–728.
[16] R. J. Conlon and H.-J. Hein, *Asymptotically conical Calabi-Yau manifolds, III*, 2014, arXiv:1405.7140.
[17] S. K. Donaldson, *Two-forms on four-manifolds and elliptic equations*, Inspired by S. S. Chern, Nankai Tracts Math., vol. 11, World Sci. Publ., Hackensack, NJ, 2006, pp. 153–172.
[18] ———, *Calabi-Yau metrics on Kummer surfaces as a model gluing problem*, Advances in geometric analysis, Adv. Lect. Math. (ALM), vol. 21, Int. Press, Somerville, MA, 2012, pp. 109–118.
[19] S. K. Donaldson and P. B. Kronheimer, *The geometry of four-manifolds*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1990, Oxford Science Publications.
[20] T. Eguchi and A. J. Hanson, *Self-dual solutions to Euclidean gravity*, Ann. Physics **120** (1979), no. 1, 82–106.
[21] L. Foscolo, *ALF gravitational instantons and collapsing Ricci-flat metrics on the K3 surface*, 2016, arXiv:1603.06315. To appear in the Journal of Diff. Geom.
[22] G. W. Gibbons and C. N. Pope, *The positive action conjecture and asymptotically Euclidean metrics in quantum gravity*, Comm. Math. Phys. **66** (1979), no. 3, 267–290.
[23] G. Gibbons and S. Hawking, *Gravitational multi-instantons*, Physics Letters B **78** (1978), no. 4, 430–432.
[24] M. Gross, V. Tosatti, and Y. Zhang, *Collapsing of abelian fibered Calabi-Yau manifolds*, Duke Math. J. **162** (2013), no. 3, 517–551.
[25] ———, *Gromov-Hausdorff collapsing of Calabi-Yau manifolds*, Comm. Anal. Geom. **24** (2016), no. 1, 93–113.

[26] M. Gross and P. M. H. Wilson, *Large complex structure limits of $K3$ surfaces*, J. Differential Geom. **55** (2000), no. 3, 475–546.

[27] H.-J. Hein, S. Sun, J. Viaclovsky, and R. Zhang, *Nilpotent structures and collapsing Ricci-flat metrics on K3 surfaces*, 2018, arXiv:1807.09367.

[28] H.-J. Hein, *Gravitational instantons from rational elliptic surfaces*, J. Amer. Math. Soc. **25** (2012), no. 2, 355–393.

[29] N. Hitchin, *Compact four-dimensional Einstein manifolds*, J. Differential Geometry **9** (1974), 435–441.

[30] ———, *Twistor construction of Einstein metrics*, Global Riemannian geometry (Durham, 1983), Ellis Horwood Ser. Math. Appl., Horwood, Chichester, 1984, pp. 115–125.

[31] D. D. Joyce, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[32] R. Kobayashi, *Ricci-flat Kähler metrics on affine algebraic manifolds and degenerations of Kähler-Einstein $K3$ surfaces*, Kähler metric and moduli spaces, Adv. Stud. Pure Math., vol. 18, Academic Press, Boston, MA, 1990, pp. 137–228.

[33] K. Kodaira, *On the structure of compact complex analytic surfaces. I*, Amer. J. Math. **86** (1964), 751–798.

[34] P. B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differential Geom. **29** (1989), no. 3, 665–683.

[35] ———, *A Torelli-type theorem for gravitational instantons*, J. Differential Geom. **29** (1989), no. 3, 685–697.

[36] C. LeBrun, *Complete Ricci-flat Kähler metrics on $\mathbf{C}^n$ need not be flat*, Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), Proc. Sympos. Pure Math., vol. 52, Amer. Math. Soc., Providence, RI, 1991, pp. 297–304.

[37] C. LeBrun and M. Singer, *A Kummer-type construction of self-dual 4-manifolds*, Math. Ann. **300** (1994), no. 1, 165–180.

[38] V. Minerbe, *A mass for ALF manifolds*, Comm. Math. Phys. **289** (2009), no. 3, 925–955.

[39] ———, *On the asymptotic geometry of gravitational instantons*, Ann. Sci. Éc. Norm. Supér. (4) **43** (2010), no. 6, 883–924.

[40] ———, *Rigidity for multi-Taub-NUT metrics*, J. Reine Angew. Math. **656** (2011), 47–58.

[41] R. Miranda, *The basic theory of elliptic surfaces*, Dottorato di Ricerca in Matematica, ETS Editrice, Pisa, 1989.

[42] H. Nakajima, *Hausdorff convergence of Einstein 4-manifolds*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **35** (1988), no. 2, 411–424.

[43] H. Ooguri and C. Vafa, *Summing up Dirichlet instantons*, Phys. Rev. Lett. **77** (1996), no. 16, 3296–3298.

[44] D. N. Page, *A periodic but nonstationary gravitational instanton*, Physics Letters B **100** (1981), no. 4, 313–315.

[45] R. Schoen and S.-T. Yau, *Lectures on differential geometry*, Conference Proceedings and Lecture Notes in Geometry and Topology, I, International Press, Cambridge, MA, 1994.

[46] Y. T. Siu, *Every $K3$ surface is Kähler*, Invent. Math. **73** (1983), no. 1, 139–150.

[47] I. Suvaina, *ALE Ricci-flat Kähler metrics and deformations of quotient surface singularities*, Ann. Global Anal. Geom. **41** (2012), no. 1, 109–123.

[48] G. Tian and S.-T. Yau, *Complete Kähler manifolds with zero Ricci curvature. I*, J. Amer. Math. Soc. **3** (1990), no. 3, 579–609.

[49] G. Tian and S.-T. Yau, *Complete Kähler manifolds with zero Ricci curvature. II*, Invent. Math. **106** (1991), no. 1, 27–60.

[50] P. Topiwala, *A new proof of the existence of Kähler-Einstein metrics on $K3$. I, II*, Invent. Math. **89** (1987), no. 2, 425–448, 449–454.

[51] S. T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), no. 5, 1798–1799.

Department of Mathematics, Heriot-Watt University
E-mail address: `l.foscolo@hw.ac.uk`