CALIBRATED SUBMANIFOLDS

LORENZO FOSCOLO

Calibrated submanifolds are a special class of volume minimizing submanifolds in Riemannian manifolds endowed with special closed differential forms called calibrations. The prototypical examples of calibrated submanifolds are holomorphic submanifolds of Kähler manifolds. More generally, Riemannian manifolds with special holonomy naturally carry calibrations: besides holomorphic submanifolds, examples of calibrated submanifolds include special Lagrangian submanifolds in Calabi-Yau manifolds and associative and coassociative submanifolds in G2 manifolds. Aspects of the theory we will discuss include:

1. Introduction to calibrated submanifolds
2. Examples of calibrated submanifolds in flat Euclidean space
3. Singularities and calibrated cones
4. Moduli spaces of calibrated submanifolds; calibrated fibrations

A particular focus of the course will be on special Lagrangian submanifolds.

Contents

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1. Calibrations

- (M^n, g) Riemannian manifold
- $\phi \in \Omega^k(M)$ is a calibration if
  (i) ϕ is closed: $d\phi = 0$
  (ii) ϕ has comass 1: for all $x \in M$ and k-dimensional plane $\Pi \subset T_xM$, $\phi_x|_\Pi \leq \mathrm{vol}_\Pi$
  i.e. for one (and therefore any) o.n. basis $\{e_1, \dots, e_k\}$ of Π, $\phi_x(e_1, \dots, e_k) \leq 1$.
- $\Pi \subset T_xM$ calibrated if $\phi_x|_\Pi = \mathrm{vol}_\Pi$
- Σ^k smooth manifold, $\iota\colon \Sigma \to M$
  - immersion: $d_x\iota\colon T_x\Sigma \to T_xM$ injective for all $x \in \Sigma$
  - embedding: an injective immersion
- submanifold Σ is ϕ–calibrated if $\iota(T_x\Sigma) \subset T_{\iota(x)}M$ is calibrated by ϕ_x for all $x \in \Sigma$

Exercise 1

Theorem 1.1 (Fundamental Theorem)**.** *ϕ a calibration on (M, g), Σ compact submanifold without boundary. If Σ is ϕ–calibrated then it is volume minimizing in its holomogy class,* i.e. *for any other submanifold Σ' in the same homology class as Σ we have*

$$\mathrm{Vol}(\Sigma) \leq \mathrm{Vol}(\Sigma').$$

Moreover, equality holds iff Σ' is also ϕ–calibrated.

Proof. By Stokes' Theorem:

$$\mathrm{Vol}(\Sigma) = \int_\Sigma \mathrm{vol}_\Sigma = \int_\Sigma \phi = \int_{\Sigma'} \phi \leq \int_{\Sigma'} \mathrm{vol}_{\Sigma'} = \mathrm{Vol}(\Sigma') \qquad \square$$

Exercise 2

1.1. A prototype: the Kähler case.

- $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ with standard flat metric g_0, complex structure J_0 and holomorphic coordinates $(z_1, \dots, z_n)$
- standard Kähler form

$$\omega_0 = \tfrac{i}{2}(dz_1 \wedge d\overline{z}_1 + \cdots + dz_n \wedge d\overline{z}_n)$$

  - closed (parallel!)
  - $\omega_0(u, v) = g_0(J_0u, v)$
  - $\mathrm{Sp}(2n, \mathbb{R}) \cap \mathrm{SO}(2n) = \mathrm{SO}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{GL}(n, \mathbb{C}) \cap \mathrm{Sp}(2n, \mathbb{R}) = \mathrm{U}(n)$

Theorem 1.2 (Wirtinger's Inequality)**.** *For all $1 \leq p \leq n$, $\frac{1}{p!}\omega_0^p$ has comass 1 and its calibrated planes are the complex p–dimensional subspaces of $\mathbb{C}^n$.*

Proof. Given any $2p$–plane Π we can assume that the adapted unitary basis of Exercise 3 is the standard one. Then:

$$\tfrac{1}{p!}\omega_0^p|_\Pi = \cos\theta_1 \dots \cos\theta_p, \qquad \tfrac{1}{p!}\omega_0^p|_\Pi = \cos\theta_1 \dots \cos\theta_q$$

if, respectively, $2p \leq n$ or $2q = 2(n-p) \leq n$. $\square$

Exercises 3, 4 and 5

- almost Hermitian manifold (M^{2n}, g, ω, J)
  - g Riemannian metric
  - ω non-degenerate 2-form, *i.e.* for all $x \in M$ there is a linear isomorphism $u\colon T_xM \xrightarrow{\sim} \mathbb{C}^n$ with $u^*\omega_0 = \omega_x$
  - J almost complex structure: $J\colon TM \to TM$ with $J^2 = -1$

  with $\omega(\,\cdot\,,\,\cdot\,) = g(J\,\cdot\,,\,\cdot\,)$
- almost Hermitian manifold (M^{2n}, g, ω, J) is Kähler if $d\omega = 0$ and J is an integrable complex structure

Exercises 6 and 7

1.2. Calibrations and minimal submanifolds.

- Submanifold $\iota\colon \Sigma^k \to (M^n, g)$ is minimal if it is a critical point of the volume functional $\mathrm{Vol}(\Sigma, \iota) = \int_\Sigma \mathrm{vol}_{\iota^* g}$ (with respect to compactly supported variations)
- $\iota_s\colon \Sigma \to M$ variation generated by vector field $X \in C^\infty_0(\Sigma; \iota^* TM)$
- second fundamental form $\mathrm{II}(u, v) = (\nabla^g_u v)^\perp$: a symmetric bilinear form on $T\Sigma$ with values in the normal bundle $\iota(T\Sigma)^\perp$
  - X normal $\Rightarrow \partial_s(\iota_s^* g)(u, v)|_{s=0} = -g(\mathrm{II}(u, v), X)$ for all $u, v \in T\Sigma$
- mean curvature $H = \mathrm{tr}_{\iota^* g}\mathrm{II}$: a normal vector
- $\iota_s\colon \Sigma \to M$ variation generated by vector field $X \in C^\infty_0(\Sigma; \iota^* TM) \rightsquigarrow$

$$\frac{d}{ds}\mathrm{Vol}(\Sigma, \iota_s)|_{s=0} = -\int_\Sigma \langle H, X\rangle_{\iota^* g}\, \mathrm{vol}_{\iota^* g}$$

Exercises 8 and 9

Proposition 1.3. *A minimal hypersurface in $\mathbb{R}^{n+1}$ written as a graph of a smooth function of the first n coordinates is area minimising.*

Proof. Write the hypersurface as the graph a smooth function $u\colon \Omega \to \mathbb{R}$ for Ω an open subset of $\mathbb{R}^n$, *i.e.* $\iota_u\colon \Omega \to \mathbb{R}^{n+1}$ defined by $\iota_u(x) = (x, u(x))$ is our minimal immersion. Define the following n–form on $\Omega \times \mathbb{R}$:

$$\phi_u = \frac{dx_1 \wedge \cdots \wedge dx_n - (\nabla u \lrcorner (dx_1 \wedge \cdots \wedge dx_n)) \wedge dx_{n+1}}{\sqrt{1 + |\nabla u|^2}}.$$

Then ϕ_u is a calibration that calibrates ι_u. □

Exercise 10

- Π_1, Π_2 oriented n-planes in $\mathbb{R}^{2n}$ intersecting transversely at the origin
- existence of an adapted basis

$$\Pi_1 = \mathrm{span}\{e_1, \ldots, e_n\}, \qquad \Pi_2 = \{\cos\theta_1 e_1 + \sin\theta_1 e_{n+1}, \ldots, \cos\theta_n e_{2n-1} + \sin\theta_p e_{2n}\}$$

for angles $0 \leq \theta_1 \leq \cdots \leq \theta_n \leq \pi - \theta_{n-1}$

Theorem 1.4 (The Angle Criterion). *$\Pi_1 \cup \Pi_2$ is volume minimising iff $\theta_n \leq \theta_1 + \cdots + \theta_{n-1}$.*

Proof. We prove the if statement. The only if statement requires of the "Lawlor necks" in the next section.

Assume the adapted basis of $\mathbb{R}^{2n}$ is the standard one. For unit imaginary quaternions $u_1, \ldots, u_n$ define

$$\phi = \mathrm{Re}\left((dx_1 + u_1 dy_1) \wedge \cdots \wedge (dx_n + u_n dy_n)\right)$$

Then one shows that

(i) ϕ is a calibration
(ii) Π_1 is ϕ-calibrated and Π_2 is ϕ-calibrated iff

$$e^{\theta_1 u_1} \ldots e^{\theta_n u_n} = 1$$

The main step to prove (i) is to show that, because of the special structure of ϕ, it is enough to check the comass 1 conditions on n-planes of the same special form of Π_2 in terms of angles $\theta_1, \ldots, \theta_n$ (F. Morgan's Torus Lemma).

Now, since

$$u_i = \frac{\mathrm{Im}\, e^{\theta_i u_i}}{|\mathrm{Im}\, e^{\theta_i u_i}|}, \qquad \mathrm{Real}\, e^{\theta_i u_i} = \cos\theta_i,$$

if we set

$$e^{\theta_1 u_1} = w_1 \overline{w}_2 \qquad \ldots \qquad e^{\theta_n u_n} = w_n \overline{w}_n$$

for unit imaginary quaternions $w_1, \dots, w_n$, the condition in (ii) is automatically satisfied, the first condition above defines $u_1, \dots, u_n$ in terms of $w_1, \dots, w_n$ and the second conditions is equivalent to the fact the points $w_1, \dots, w_n \in S^2 \subset \operatorname{Im}\mathbb{H}$ are the vertices of a geodesic polygon with side-lengths $\theta_1, \dots, \theta_n$. □

Exercise 11

1.3. **Exercises.**

Exercise 1. Let Π be a k-dimensional linear subspace of $\mathbb{R}^n$. Show that there exists a calibration ϕ that calibrates Π.

Exercise 2. Show that there are no closed (*i.e.* compact without boundary) calibrated submanifolds of $\mathbb{R}^n$. (Hint: more generally, show that there are no k-dimensional closed calibrated submanifold of a manifold M with kth Betti number $b_k(M) = 0$.)

Exercise 3. Let Π be a $2p$–dimensional linear subspace of $\mathbb{R}^{2n} \simeq \mathbb{C}^n$. Then there exists a unitary basis $\{e_1, \dots, e_n\}$ of $\mathbb{C}^n$, *i.e.* a basis related to the standard one by a matrix in $\mathrm{U}(n)$, such that

- if $2p \leq n$, then
$$\Pi = \operatorname{span}\{e_1, \cos\theta_1\, Je_1 + \sin\theta_1\, e_2, \dots, e_{2p-1}, \cos\theta_p\, Je_{2p-1} + \sin\theta_p\, e_{2p}\}$$
for angles $0 \leq \theta_1 \leq \dots \leq \theta_{p-1} \leq \frac{\pi}{2}$ and $\theta_{p-1} \leq \theta_p \leq \pi$;
- if $2p > n$, then, with $r = 2p - n$ and $q = n - p$,
$$\Pi = \operatorname{span}\{e_1, Je_1, \dots, e_r, Je_r, e_{r+1}, \cos\theta_1\, Je_{r+1} + \sin\theta_1\, e_{r+2}, \dots, e_{n-1}, \cos\theta_q\, Je_{n-1} + \sin\theta_q\, e_n\}$$
for angles $0 \leq \theta_1 \leq \dots \leq \theta_{q-1} \leq \frac{\pi}{2}$ and $\theta_{q-1} \leq \theta_q \leq \pi$.

(Hint: first observe that if $2p \geq n$ then Π must necessarily contain an r–dimensional complex subspace, so you can reduce the case $2p > n$ to the case $2p \leq n$; the proof in the latter case is by induction on p and proceeds by maximising $\omega_0(u, v)$ for all orthonormal pairs of vectors in Π; the non-trivial observation for the inductive step is that if u, v is such a maximising pair then differentiating $t \mapsto \omega_0(u, \cos t\, v + t\, w)$ at $t = 0$ shows that if $w \perp \{u, v\}$, then $w \perp \{Ju, Jv\}$ also.)

Exercise 4. Conclude the proof of the Wirtinger's Inequality.

Exercise 5. Let $(u, v)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map. Show that the map $\iota\colon \mathbb{R}^2 \to \mathbb{C}^2$ defined by $(x, y) \mapsto (x + iy, u + iv)$ is ω_0–calibrated if and only if u, v satisfy the Cauchy–Riemann equations $u_x - v_y = 0 = v_x + u_y$.

Exercise 6. In this exercise you show that complex projective space $\mathbb{CP}^n$ and its holomorphic submanifolds are Kähler.

(i) Show that $\omega_{\mathrm{FS}} = i\partial\overline{\partial} \log(1 + |z|^2)$ is a Kähler form on $\mathbb{C}^n$. Here $|z|^2 = |z_1|^2 + \dots + |z_n|^2$.
(ii) Show that if $\varphi(z_1, \dots, z_n) = z_1^{-1}(1, z_2, \dots, z_n)$ then $\varphi^*\omega_{\mathrm{FS}} = \omega_{\mathrm{FS}}$.
(iii) Show that the charts $(\mathcal{U}_i, \varphi_i)$ defined by
$$\varphi_i\colon \mathcal{U}_i = \{[z_0 : \dots : z_n] \mid z_i \neq 0\} \to \mathbb{C}^n, \qquad \varphi_i([z_0 : \dots : z_n]) = z_i^{-1}(z_0, \dots, \check{z}_i, \dots, z_n),$$
where $\check{z}_i$ means that we drop the ith coordinates, form a holomorphic atlas on $\mathbb{CP}^n$.
(iv) Show that the formula $\omega = \varphi_i^*\omega_{\mathrm{FS}}$ over $\mathcal{U}_i$ defines a Kähler form on $\mathbb{CP}^n$, called the Fubini–Study Kähler form.
(v) Let M be a complex submanifold of $\mathbb{CP}^n$. Show that M is Kähler.
(Hint: consider the restriction of the Fubini–Study Kähler form to M.)

Exercise 7. A plane conic is the zero-locus of a homogeneous equation $az_0^2 + bz_1^2 + cz_2^2 + dz_0z_1 + ez_1z_2 + fz_2z_0$ of degree 2 in the homogeneous coordinates $[z_0 : z_1 : z_2]$ of $\mathbb{CP}^2$. Smooth conics are ω_{FS}–calibrated submanifolds of $\mathbb{CP}^2$: they are images of holomorphic embeddings $\iota\colon \mathbb{CP}^1 \to \mathbb{CP}^2$ of degree 2, *i.e.* such that $\iota^*\omega_{\mathrm{FS},\mathbb{CP}^2} = 2\omega_{\mathrm{FS},\mathbb{CP}^1}$. Show that smooth conic curves are parametrised by an open complex manifold $\mathbb{CP}^5 \setminus S$, where S is a certain subvariety of $\mathbb{CP}^5$ you can describe explicitly. (Hint: smooth conics correspond to non-degenerate symmetric bilinear forms up to homotheties.)

Exercise 8. Prove the First Variation Formula for the volume of submanifolds

$$\frac{d}{ds}\mathrm{Vol}(\Sigma, \iota_s)|_{s=0} = -\int_\Sigma \langle H, X\rangle_{\iota^* g}\, \mathrm{vol}_{\iota^* g}.$$

Exercise 9. The catenoid is the surface of revolution in $\mathbb{R}^3 \simeq \mathbb{C}\times\mathbb{R}$ defined by

$$\{(\kappa^{-1}\cosh(\kappa t)\, e^{i\theta}, t) \,|\, t\in\mathbb{R}, e^{i\theta}\in S^1\}.$$

(i) Show that the catenoid is a minimal surface.
(ii) For a parameter $d>0$ consider the area of the portion of the catenoid with $|t|\leq d$ and the area of the union of the two discs in the planes $\{t=\pm d\}$ and radius such .
(iii) Deduce that the union of two discs does not minimise area amongst surfaces with the same boundary for d sufficiently small. (In fact, the catenoid is the minimiser in this case.)

Exercise 10. In the notation of Proposition 1.3,

(i) show that ϕ_u has comass 1;
(ii) show that $\iota_u\colon \Omega\to\mathbb{R}^{n+1}$ is a minimal immersion if and only if $d\phi_u = 0$;
(iii) deduce Proposition 1.3.

Exercise 11. Conclude the proof of the "if" statement in the Angle Criterion theorem by solving the spherical trigonometry exercise to which the statement hass been reduced to.

1.4. **Bibliographical notes.** Our presentation is mostly based on [1, Chapter 7].

2. The special Lagrangian calibration

- $\mathbb{C}^n$ with holomorphic coordinates $(z_1, \dots, z_n)$, flat metric g_0 and standard Kähler form ω_0
- standard holomorphic volume form $\Omega_0 = dz_1 \wedge \cdots \wedge dz_n$

Theorem 2.1. *Fix a constant phase $e^{i\theta} \in S^1$.*

(i) $\mathrm{Re}(e^{-i\theta}\Omega_0)$ *is a calibration.*

(ii) *An n-plane Π is calibrated by $\mathrm{Re}(e^{-i\theta}\Omega_0)$ iff $\Pi = A(\mathbb{R}^n)$, where $\mathbb{R}^n \subset \mathbb{R}^n \oplus i\mathbb{R}^n$ is the real plane and A is a unitary matrix with $\det A = e^{i\theta}$.*

(iii) *An n-dimensional submanifold $L \subset \mathbb{C}^n$ is calibrated by $\pm\mathrm{Re}(e^{-i\theta}\Omega_0)$ iff $\omega_0|_L = 0 = \mathrm{Im}(e^{-i\theta}\Omega_0)|_L$.*

Proof. Let Π be an n-plane spanned by an o.n. basis $u_1, \dots, u_n$. Define a matrix $A \in \mathrm{GL}(n, \mathbb{C})$ by $u_i = Ae_i$ for the standard unitary basis $\{e_1, \dots, e_n\}$ of $\mathbb{C}^n$. Then $\Omega_0|_\Pi = \det_{\mathbb{C}} A$. Moreover,

$$|\det_{\mathbb{C}} A|^2 = \det_{\mathbb{R}} A = |A(e_1 \wedge Je_1 \wedge \cdots \wedge e_n \wedge Je_n)| = |u_1 \cdots \wedge u_n \wedge Ju_1 \wedge \cdots \wedge Ju_n| \leq 1$$

with equality iff $u_1, Ju_1, \dots, u_n, Ju_n$ is an o.n. basis of $\mathbb{R}^n$. □

Exercise 12

- $\mathrm{SU}(n)$–structure (ω, Ω) on M^{2n}
  - ω non-degenerate 2-form
  - Ω complex volume form (*i.e.* a locally decomposable complex n-form)
  - $\omega \wedge \Omega = 0$ and $\frac{1}{n!}\omega^n = c_n\, \Omega \wedge \overline{\Omega}$

  $\rightsquigarrow$ Riemannian metric $g = g_{\omega,\Omega}$
- the $\mathrm{SU}(n)$–structure is Calabi–Yau if $d\omega = 0 = d\Omega$
  in this case $\mathrm{Ric}_g = 0$ and $\mathrm{Hol}(g) \subseteq \mathrm{SU}(n)$
- special Lagrangian submanifolds of phase $e^{i\theta}$ of (M, ω, Ω) are the submanifolds calibrated by $\mathrm{Re}(e^{-i\theta}\Omega)$

Exercises 13, 14 and 15

2.1. Lawlor necks.

Proposition 2.2. *Let N be an isotropic (*i.e. $\omega|_N \equiv 0$*) real analytic $(n-1)$–dimensional submanifold of $\mathbb{C}^n$. Then for each phase $e^{i\theta}$ there exists a unique (connected) special Lagrangian submanifold of phase $e^{i\theta}$ containing N.*

Exercise 16

Want to apply this to ellipsoid

$$N = \{(\beta_1 e^{i\psi_1(0)} x_1, \dots, \beta_n e^{i\psi_n(0)} x_n) \,|\, (x_1, \dots, x_n) \in S^{n-1} \subset \mathbb{R}^n\}$$

- Ansatz:

$$L = \{(w_1(t)x_1, \dots, w_n(t)x_n) \,|\, t \in I, (x_1, \dots, x_n) \in S^{n-1} \subset \mathbb{R}^n\}, \qquad w_j(0) = \beta_j e^{i\psi_j(0)}$$

- Step 1: up to reparametrisation, L sLag of phase $e^{i\theta}$ iff

$$\overline{w}_j \dot{w}_j = e^{i\theta}\overline{w}_1 \dots \overline{w}_n$$

- Step 2: conserved quantity $\mathrm{Im}(e^{-i\theta} w_1 \dots w_n) = -A \in \mathbb{R}$
  WLOG assume $A > 0$
- Step 3: reformulate ODE system in terms of $w_j(t) = \sqrt{\beta_j^2 + u(t)}\, e^{i\psi_j(t)}$

$$\dot{u} = 2\sqrt{Q(u)} \cos(\theta - \psi), \qquad (\beta_j^2 + u)\,\dot{\psi}_j = \sqrt{Q(u)} \sin(\theta - \psi) = A$$

  where $Q(u) = \prod_{j=1}^n (\beta_j^2 + u)$ and $\psi = \psi_1 + \cdots + \psi_n$

- Step 5: there exists a unique t_* such that $\dot{u}(t_*) = 0$ (corresponding to the unique solution u_* of $Q(u) = A^2$) $\Longrightarrow$ reparametrisation
$$w_j(y) = \sqrt{a_j^{-1} + y^2}, \qquad \psi_j(y) = \psi_j' + \int_0^y \frac{a_j\, ds}{(1 + a_j s^2)\sqrt{P(s)}}$$
where $a_j^{-1} = \beta_j^2 + u_*$ (so that $a_1 \dots a_n = A^{-2}$) and
$$P(y) = \frac{1}{y^2}\left(\prod_{j=1}^n (1 + a_j y^2) - 1\right)$$
- Step 6: observe
$$\lim_{y\to\infty} \int_0^y \frac{a_j\, ds}{(1 + a_j s^2)\sqrt{P(s)}} = \overline{\psi}_j$$
$\Longrightarrow$ L is asymptotic to planes $\pm\Pi^\pm$ as $y \to \pm\infty$, where
$$\pm\Pi^\pm = \{(e^{i(\psi_1' \pm \overline{\psi}_1)}\, s_1, \dots, e^{i(\psi_n' \pm \overline{\psi}_n)}\, s_n) \,|\, (s_1, \dots, s_n) \in \mathbb{R}^n\}$$

Exercise 17

2.2. Special Lagrangian submanifolds and symmetries.

Exercise 18

- (M^{2n}, ω, Ω) Calabi–Yau manifold
- $G \subseteq \mathrm{Aut}(M, \omega, \Omega)$ a connected Lie group of automorphisms, Lie algebra $\mathfrak{g}$
- moment map $\mu\colon M \to \mathfrak{g}^*$
  $\mathfrak{g} \ni v \mapsto V \in \mathfrak{aut}(M, \omega, \Omega) \rightsquigarrow \langle d\mu(X), v\rangle = \omega(X, V)$ for all $X \in TM$
- $\xi \in \mathfrak{g}^*$ fixed by coadjoint action of $G \rightsquigarrow$ (generally singular) Kähler quotient (M_ξ, ω_ξ)
  $M_\xi = \mu^{-1}(\xi)/G$
- holomorphic volume form $\Omega_\xi = (V_1 \wedge \dots \wedge V_k) \lrcorner\, \Omega$, where $\{v_1, \dots, v_n\}$ o.n. basis of $\mathfrak{g}$
- in general $(M_\xi, \omega_\xi, \Omega_\xi)$ not Calabi–Yau
  $\omega_\xi^{n-k} \neq c_{n-k}\, \Omega_\xi \wedge \overline{\Omega}_\xi$
- can still define special Lagrangian submanifold L_ξ by $\omega_\xi|_{L_\xi} \equiv 0 \equiv \mathrm{Im}\, \Omega_\xi|_{L_\xi}$

Proposition 2.3. *Let $L \subset M$ be a G–invariant Lagrangian submanifold. Then there exists $\xi \in \mathfrak{g}^*$ fixed by the coadjoint action such that $L \subset \mu^{-1}(\xi)$. If L is moreover special Lagrangian then $L_\xi = L/G$ is a special Lagrangian submanifold of $(M_\xi, \omega_\xi, \Omega_\xi)$ whenever it is smooth.*

- $G = T^{n-1}$ maximal torus of $\mathrm{SU}(n)$ acting on $(\mathbb{C}^n, \omega_0, \Omega_0)$
- moment map $\mu(z_1, \dots, z_n) = (|z_1|^2 - |z_n|^2, \dots, |z_{n-1}|^2 - |z_n|^2)$
- holomorphic coordinate $z = i^{n-1} z_1 \dots z_n$ on M_ξ and $\Omega_\xi = dz$
- T^{n-1}–invariant special Lagrangian submanifolds are contained in the level sets of
$$(z_1, \dots, z_n) \longmapsto \Big(|z_1|^2 - |z_n|^2, \dots, |z_{n-1}|^2 - |z_n|^2, \mathrm{Im}(i^{n-1} z_1 \dots z_n)\Big)$$
  Assume $n = 3$ for concreteness:
  - submersion over $\mathbb{R}^3 \setminus Y$
  $$Y = \{(-t, -t, 0) \,|\, t > 0\} \cup \{(t, 0, 0) \,|\, t > 0\} \cup \{(0, t, 0) \,|\, t > 0\}$$
  $\rightsquigarrow$ smooth special Lagrangians $L_\xi \simeq \mathbb{R} \times T^2$ for $\xi \in \mathbb{R}^3 \setminus Y$
  - $\xi \in Y \setminus \{0\} \rightsquigarrow L_\xi = L_\xi^+ \cup L_\xi^-$, $L_\xi^\pm \simeq \mathbb{R}^2 \times S^1$ smooth, $L_\xi^+ \cap L_\xi^- \simeq S^1$
  - $L_0 = L_0^+ \cup L_0^-$, $L_0^\pm = \mathrm{C}(T^2)$ and $L_0^+ \cap L_0^- = \{0\}$

Proposition 2.4. *Let (M^n, g) be a Riemannian manifold endowed with a parallel calibration ϕ and let Σ be a ϕ–calibrated submanifold. Then for every $f \in C^\infty(M)$ the Laplacian of the restriction of f to Σ is given by (∇f is the gradient of f on M)*
$$\triangle_\Sigma f = d(\nabla f \lrcorner\, \phi)|_\Sigma.$$

Proof. Recall that on a Riemannian manifold (M, g) the Hessian $\mathrm{Hess}(f)$ of a function is the symmetric endomorphism $\mathrm{Hess}(f)\colon TM \to TM$ defined by $\mathrm{Hess}(f) = \nabla df$, *i.e.*

$$\mathrm{Hess}(f)(X) = \nabla_X \nabla f = [X, \nabla f] + \nabla_{\nabla f} X.$$

The Laplacian $\triangle f = d^* df$ is then given in terms of the trace of the Hessian by

$$\triangle f = -\mathrm{tr}\,\mathrm{Hess}(f) = -\sum_{i=1}^{n} g(\mathrm{Hess}(f)(e_i), e_i)$$

for any o.n. basis $\{e_1, \dots, e_n\}$ of TM. The relation between the Levi–Civita connections of a submanifold and of the ambient manifold via the second fundamental form II then implies that, if Σ is a submanifold of M, the Hessian on Σ of the restriction of f is given by

$$\mathrm{Hess}_\Sigma(f)(u, v) = \mathrm{Hess}(f)(u, v) - \mathrm{II}(u, v) \cdot f$$

for all $u, v \in T\Sigma$. In particular,

$$\triangle_\Sigma f = -\mathrm{tr}_{T\Sigma}\mathrm{Hess}(f) + H \cdot f,$$

where H is the mean curvature vector. In particular, if Σ is a minimal submanifold, *i.e.* $H = 0$, then $\triangle_\Sigma f = -\mathrm{tr}_{T\Sigma}\,\mathrm{Hess}(f)$.

On the other hand, for each endomorphism of the tangent bundle $A\colon TM \to TM$ and k-form ϕ define

$$A_* \phi(u_1, \dots, u_k) = -\phi(Au_1, u_2, \dots,_k) - \dots - \phi(u_1, \dots, u_{k-1}, Au_k),$$

i.e. we extend the action of A to $\Lambda^k T^* M$ as a derivation. Since $\mathrm{Hess}(f)$ acts on vector fields as $-\mathcal{L}_\nabla f + \nabla_{\nabla f}$, we find

$$\mathrm{Hess}(f)_* \phi = d(\nabla f \lrcorner \phi) - \nabla_{\nabla f} \phi.$$

Note the last term drops if ϕ is parallel, *i.e.* $\nabla \phi = 0$.

Finally, if Π is a ϕ–calibrated plane then one easily checks (using an o.n. frame $\{e_1, \dots, e_k\}$ for Π) that

$$(\mathrm{Hess}(f)_* \phi)\,|_\Pi = -\mathrm{tr}_\Pi\,\mathrm{Hess}(f).$$

Putting everything together, if Σ is ϕ–calibrated (so in particular it is minimal) and ϕ is parallel, we have

$$\triangle_\Sigma f = -\mathrm{tr}_{T\Sigma}\mathrm{Hess}(f) = (\mathrm{Hess}(f)_* \phi)\,|_{T\Sigma} = d(\nabla f \lrcorner \phi)|_\Sigma. \qquad \square$$

- (M, ω, Ω) Calabi–Yau manifold
- X automorphic vector field with corresponding moment map $\mu\colon M \to \mathbb{R}$
- $X = J\nabla\mu$
- $\nabla\mu \lrcorner \mathrm{Re}\,\Omega = (J\nabla\mu) \lrcorner \mathrm{Im}\,\Omega = X \lrcorner \mathrm{Im}\,\Omega \Longrightarrow d(\nabla\mu \lrcorner \mathrm{Re}\,\Omega) = 0$
- restriction of μ to a special Lagrangian submanifold is a harmonic function

Theorem 2.5. *Let* C *be the special Lagrangian cone in* $\mathbb{C}^3$ *defined by*

$$\mathrm{C} = \{(z_1, z_2, z_2) \,|\, |z_1|^2 = |z_2|^2 = |z_3|^2, z_1 z_2 z_3 \in \mathbb{R}_{\geq 0}\}$$

and for each $t > 0$ *let* $L_i(t)$ *be the special Lagrangian submanifold defined by*

$$L_i(t) = \{(z_1, z_2, z_2) \,|\, |z_i|^2 - t^2 = |z_j|^2 = |z_k|^2, z_1 z_2 z_3 \in \mathbb{R}_{\geq 0}\},$$

where (ijk) *is a cyclic permutation of* (123).

Suppose that L *is a one-ended special Lagrangian submanifold asymptotic to* C *in the following sense: there exists* $R > 0$ *and a diffeomorphism* $\Upsilon\colon L \cap \{|z| > R\} \to \mathrm{C} \cap \{|z| > R\}$ *with*

$$|z - \Upsilon(z)| = O(|z|^{-1})$$

for all $z \in L \cap \{|z| > R\}$. *Then there exist* $i \in \{1, 2, 3\}$ *and* $t > 0$ *such that* $L = L_i(t)$.

Exercises 19 and 20.

2.3. Octonions and calibrations.

- (M^6, ω, Ω) Calabi–Yau 3-fold
- on $M \times \mathbb{R}$ consider

$$\varphi = dt \wedge \omega + \operatorname{Re}\Omega, \qquad \psi = -dt \wedge \operatorname{Im}\Omega + \tfrac{1}{2}\omega^2$$

- on $M \times \mathbb{R}^2$ consider

$$\Phi = dt_1 \wedge dt_2 \wedge \omega + dt_1 \wedge \operatorname{Re}\Omega - dt_2 \wedge \operatorname{Im}\Omega + \tfrac{1}{2}\omega^2$$

Proposition 2.6. *φ, ψ and Φ are calibrations, whose calibrated submanifolds are called, respectively, associative, coassociative and Cayley submanifolds.*

Proof. It suffices to consider $M = \mathbb{C}^3$ with its standard Calabi–Yau structure (ω_0, Ω_0).

We identify $\mathbb{R}^8$ with the octonions $\mathbb{O}$. Using octonionic multiplication we define a double, triple and quadruple cross product by

$$u \times v = \operatorname{Im} \overline{v} u,$$
$$u \times v \times w = \tfrac{1}{2}\left((u\overline{v})\, w - (w\overline{v})\, u\right),$$
$$x \times u \times v \times w = \tfrac{1}{4}\left((u \times v \times w)\overline{x} - (v \times w \times x)\overline{u} + (w \times x \times u)\overline{v} - (x \times u \times v)\overline{w}\right).$$

One shows that they are all alternating and satisfy

$$|u \times v| = |u \wedge v|, \qquad |u \times v \times w| = |u \wedge v \wedge w|, \qquad |x \times u \times v \times w| = |x \wedge u \wedge v \wedge w|.$$

Now, on $\mathbb{R}^8 \simeq \mathbb{O}$ define a real 4-form Φ_0 and an $\mathbb{R}^8$–valued 4-form X_0 by

$$\Phi_0(x, u, v, w) = \operatorname{Re}(x \times u \times v \times w) = \langle x, u \times v \times w\rangle,$$
$$X_0(x, u, v, w) = \operatorname{Im}(x \times u \times v \times w).$$

Thus Φ_0 is a calibration.

Restrict this now to $\mathbb{R}^7 = \operatorname{Im}\mathbb{O}$. Define a 3-form φ_0, an $\mathbb{R}^7$–valued 3-form (the associator) χ_0 and a 4-form ψ_0 by

$$\varphi_0(u, v, w) = \operatorname{Re}(u \times v \times w) = \langle u \times v, w\rangle,$$
$$\chi_0(u, v, w) = \operatorname{Im}(u \times v \times w),$$
$$\psi_0(x, u, v, w) = \langle u \times v \times w, x\rangle = \langle \chi_0(u, v, w), x\rangle = \Phi_0(x, u, v, w).$$

Then φ_0 and ψ_0 are calibrations.

Note that these expressions imply that $\Phi_0 = dt_1 \wedge \varphi_0 + \psi_0$, where we write $\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$ with coordinate t_1 on the first factor.

Finally, if we write $\mathbb{R}^7 = \mathbb{R} \times \mathbb{R}^6$ with coordinate t_2 on the first factor, note that

$$u \longmapsto \tfrac{\partial}{\partial t_2} \times u$$

defines a complex structure on $\mathbb{R}^6$. Identifying it with the standard complex structure J_0 on $\mathbb{C}^3 \simeq \mathbb{R}^3$, from this one can immediately derive that $\varphi_0 = dt_2 \wedge \omega_0 + \rho$, with ω_0 the standard Kähler form on $\mathbb{C}^3$. The fact that $\rho = \operatorname{Re}\Omega_0$ and the expression $\psi_0 = -dt_2 \wedge \operatorname{Im}\Omega_0 + \frac{1}{2}\omega_0^2$ can then be checked in a basis. □

Exercises 21, 22 and 23

2.4. Exercises.

Exercise 12. Prove part (iii) of Theorem 2.1.

Exercise 13. Let (M, ω, Ω) be a Calabi–Yau manifold. An antiholomorphic involution of (M, ω, Ω) is a diffeomorphism $i \colon M \to M$ such that $i^2 = \mathrm{id}$, $i^*\omega = -\omega$ and $i^*\Omega = \overline{\Omega}$. Show that the fixed locus of i is a special Lagrangian submanifold.

Exercise 14. Yau's proof of the Calabi Conjecture states that given a closed complex manifold Z of complex dimension n, a class $\alpha \in H^2(Z;\mathbb{R})$ containing Kähler metrics and a closed form $\sigma \in \Omega^{2n}(Z)$ with $[\sigma] = \alpha^n$, there is a unique Kähler form $\omega \in \alpha$ such that $\omega^n = \sigma$. In this exercise you will use this theorem and some complex geometry to prove the existence of closed Calabi–Yau manifolds.

Let X be a complex manifold of complex dimension $n+1$, $n \geq 2$, such that the anticanonical bundle K_X^{-1} is ample. By the Kodaira Embedding Theorem, this is equivalent to the existence of a Hermitian metric on K_X^{-1} with curvature F_h that is a Kähler form on X. Consider a smooth anticanonical divisor $M \in |-K_X|$.

(i) Show that X has finite fundamental group. (Hint: you can use without proof the fact that Yau's Theorem can be used to show that X admits a Kähler metric with positive Ricci curvature and then use the Bochner Formula for harmonic 1-forms.)
(ii) Since K_X^{-1} is ample, the Kodaira Vanishing Theorem says that $h^{p,0}(X) = 0$ for all $p > 1$ and therefore the holomorphic Euler characteristic $\chi(X, \mathcal{O}_X) := \sum_{p=0}^{m+1} (-1)^p h^{0,p}(X) = 1$. Deduce that X is simply connected. (Hint: look at how the holomorphic characteristic behaves under finite coverings.)
(iii) Use the Lefschetz Hyperplane Theorem to deduce that M is also simply connected.
(iv) Use the Adjunction Formula to show that K_M is holomorphically trivial.
(v) Deduce that M admits a Calabi–Yau structure (ω, Ω).
(vi) Justify the fact that a hypersurface of degree $n+2$ in $\mathbb{CP}^{n+1}$ admits a Calabi–Yau structure.
(vii) Consider the quintic 3-fold $M = \{z_0^5 + \cdots + z_5^5 = 0\} \in \mathbb{CP}^5$ endowed with the Calabi–Yau structure produced by the previous discussion. Show that M contains a special Lagrangian submanifold. (Hint: use the uniqueness part of Yau's solution of the Calabi Conjecture to show that complex conjugation on $\mathbb{CP}^5$ induces an antiholomorphic involution of M with its Calabi–Yau structure.)

Exercise 15. A hyperkähler manifold $(M, \omega_1, \omega_2, \omega_3)$ is a $4n$–manifold endowed with a triple of closed non-degenerate 2-forms $\omega_i \in \Omega^2(M)$ such that for every $x \in M$ there is a linear isomorphism $u\colon T_xM \to \mathbb{H}^n$ sending the $\operatorname{Im}\mathbb{H}$–valued triple $\omega_1\, i_1 + \omega_2\, i_2 + \omega_3\, i_3$ to

$$\operatorname{Im}(d\overline{q}_1 \wedge dq_1 + \cdots + d\overline{q}_n \wedge dq_n),$$

where $(q_1, \ldots, q_n)$ are quaternionic coordinates on $\mathbb{H}^n$.

(i) Show that $\omega = \omega_1$ and $\Omega = \frac{1}{n!}(\omega_2 + i\omega_3)^n$ defines a Calabi–Yau structure on M. In particular, for $n = 1$ a hyperkähler structure is the same as a Calabi–Yau structure. (The last statement is equivalent to the isomorphism of Lie groups $\mathrm{SU}(2) \simeq \mathrm{Sp}(1)$.)

The induced Riemannian metric $g = g_{\omega,\Omega}$ is independent of the choice of direction in $\mathbb{R}^3$, *i.e.* it is the same if we set $\omega = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$ for $(a_1, a_2, a_3) \in S^2 \subset \mathbb{R}^3$ and define Ω accordingly. In particular, g admits a whole S^2 of complex structures with respect to which it is Kähler, a fact that explains the name "hyperkähler". Denote by J_i the complex structure arising from the choice $\omega = \omega_i$, $i = 1, 2, 3$.

Let now L be an n-dimensional submanifold of M. We say that L is an A brane for the complex structure J_i if it is Lagrangian with respect to ω_i. We say that L is a B brane for the complex structure J_i if it is a complex submanifold of (M, J_i). An (A, A, A) brane is a submanifold which is Lagrangian with respect to $\omega_1, \omega_2, \omega_3$, etc.

(ii) Show that there are no (A, A, A) nor (A, B, B) branes in M.
(iii) Show that (A, A, B) and (A, B, A) branes are special Lagrangian submanifolds (of phases you will determine) with respect to the choice of Calabi–Yau structure given in (i).
(iv) For $n = 1$, show that every Lagrangian submanifold with respect to the choice of Calabi–Yau structure given in (i) is in fact holomorphic with respect to a different complex structure in the hyperkähler 2-sphere of complex structures.

Exercise 16. Let N be an isotropic $(n-1)$–dimensional linear subspace of $\mathbb{C}^n$. Show that there exists a unique special Lagrangian plane (for any phase) containing N.

(Note: the proof of Proposition 2.2 uses this results and Exterior Differential Systems.)

Exercise 17. Let $\pm\Pi^{\pm}$ be the two n-planes in $\mathbb{C}^n$ defined by

$$\pm\Pi^{\pm} = \{(e_1^{\pm i\overline{\psi}_1}, \ldots, e_1^{\pm i\overline{\psi}_n}) \,|\, (s_1, \ldots, s_n) \in \mathbb{R}^n\}$$

for angles $0 \leq \overline{\psi}_1 \leq \cdots \leq \overline{\psi}_n \leq \frac{\pi}{2} - \overline{\psi}_{n-1}$.

(i) Reformulate the Angle Criterion Theorem 1.4 as follows: $\Pi^+ \cup \Pi^-$ is volume minimising if and only if $\overline{\psi}_1 + \cdots + \overline{\psi}_n \geq \frac{\pi}{2}$.

(ii) Consider the Lawlor necks of Section 2.1. Show that, up to the action of a constant unitary matrix, the asymptotic planes $\pm\Pi^{\pm}$ of a Lawlor neck can be put in the above form with $\overline{\psi}_1 + \cdots + \overline{\psi}_n = \frac{\pi}{2}$.

(iii) Show that for planes $\pm\Pi^{\pm}$ satisfying $\overline{\psi}_1 + \cdots + \overline{\psi}_n < \frac{\pi}{2}$ you can find a Lawlor neck L (*i.e.* you can choose $a_1, \ldots, a_n$ in the construction of Section 2.1) such that $L \cap (\Pi^+ \cup \Pi^-) \neq \emptyset$.

(Hint: show that, for each fixed $y > 0$ and $\psi \in (0, \frac{\pi}{2})$, the map

$$(a_1, \ldots, a_n) \longmapsto \left(\int_0^y \frac{a_1\, ds}{(1 + a_1 s^2)\sqrt{P(s)}}, \ldots, \int_0^y \frac{a_n\, ds}{(1 + a_n s^2)\sqrt{P(s)}} \right)$$

defines a surjection

$$\{(a_1, \ldots, a_n) \in \mathbb{R}^n_{\geq 0} \,|\, (1+a_1 y^2)^{-1} \ldots (1+a_n y^2)^{-1} = \cos^2 \psi\} \longrightarrow \{(\psi_1, \ldots, \psi_n) \in \mathbb{R}^n_{\geq 0} \,|\, \psi_1 + \cdots + \psi_n = \psi\}.$$

You can show this by induction on $n \geq 1$ and a degree argument, *i.e.* the fact that the restriction of the map on the boundary has the same degree as the map itself.)

(iv) Conclude the proof of the Angle Criterion Theorem 1.4.

Exercise 18. Let X be a Killing vector field, *i.e.* a vector field such that $\mathcal{L}_X g = 0$. Equivalently ∇X is a skew-symmetric $(1,1)$ tensor.

(i) Show that

$$\triangle \left(\tfrac{1}{2}|X|^2 \right) = -|\nabla X|^2 + \mathrm{Ric}(X, X).$$

(ii) Show that a closed Riemannian manifold (M, g) with $\mathrm{Ric} < 0$ does not carry any non-trivial Killing field.

(iii) Show that on a closed Riemannian manifold (M, g) with $\mathrm{Ric} \leq 0$ every Killing field X is parallel and satisfies $\mathrm{Ric}(X, X) = 0$.

Exercise 19. Prove Theorem 2.5.

(Hint: the main point is to show that if f is any of the moment maps for the T^2–action on $\mathbb{C}^3$, then f must be constant on L because of the maximum principle. For this, note that for every smooth function f on $\mathbb{C}^3$ and $z \in L \cap \{|z| > R\}$ we have

$$|f(z) - f(\Upsilon(z))| \leq |df(z')|\, |z - \Upsilon(z)|$$

for a point z' contained in the segment $tz + (1-t)\Upsilon(z)$. Apply this observation to any of the moment maps, which has quadratic growth and vanishes on C, to show that its restriction to L is bounded.)

Exercise 20. Let L be a Lagrangian submanifold of $(\mathbb{C}^n, \omega_0)$ and consider the Liouville 1-form $\lambda_0 = \frac{i}{2}\sum_{j=1}^n z_j d\overline{z}_j - \overline{z}_j dz_j$ on $\mathbb{C}^n$, which satisfies $d\lambda_0 = \omega_0$.

(i) Show that $\lambda_0|_L$ is a closed form on L and therefore defines a cohomology class $[\lambda_0|_L] \in H^1(L; \mathbb{R})$.

(ii) For the special Lagrangian submanifolds $L_i(t)$ of Theorem 2.5 calculate $[\lambda_0|_{L_i(t)}] \in H^1(L_i(t);\mathbb{R})$, construct explicit diffeomorphisms $\Upsilon_i(t)\colon L_i(t) \cap \{|z| > R\} \to \mathrm{C} \cap \{|z| > R\}$ as in the statement of the theorem and compare the images of $[\Upsilon_i(t)_*(\lambda_0|_{L_i(t)})]$ in $H^1(\mathrm{C}\setminus\{0\};\mathbb{R}) \simeq H^1(T^2;\mathbb{R}) \simeq \mathbb{R}^2$ for $i = 1, 2, 3$.

Exercise 21. Let φ_0, ψ_0 and Φ_0 be the calibrations defined in the proof of Proposition 2.6.

(i) Show that a 3-plane Π in $\mathbb{R}^7$ can be given an orientation so that it is calibrated by φ_0 if and only if $\chi_0|_\Pi \equiv 0$.
(ii) Show that a 4-plane Π in $\mathbb{R}^7$ can be given an orientation so that it is calibrated by ψ_0 if and only if $\varphi_0|_\Pi \equiv 0$.
(iii) Show that a 4-plane Π in $\mathbb{R}^8$ can be given an orientation so that it is calibrated by Φ_0 if and only if $X_0|_\Pi \equiv 0$.

Exercise 22. Let φ_0, ψ_0 and Φ_0 be the calibrations defined in the proof of Proposition 2.6.

(i) Use the expressions $\varphi_0 = dt \wedge \omega_0 + \operatorname{Re}\Omega_0$ and $\psi_0 = -dt \wedge \operatorname{Im}\Omega_0 + \frac{1}{2}\omega_0^2$ on $\mathbb{R}^7 = \mathbb{R} \times \mathbb{C}^3$ to show that
  (a) $\mathbb{R} \times C$, for $C^2 \subset \mathbb{C}^3$, is an associative submanifold if and only if C is a holomorphic curve in $\mathbb{C}^3$;
  (b) L, for $L^3 \subset \mathbb{C}^3$, is an associative submanifold if and only if L is a special Lagrangian submanifold of phase 1 in $\mathbb{C}^3$;
  (c) $\mathbb{R} \times L$, for $L^3 \subset \mathbb{C}^3$, is a coassociative submanifold if and only if L is a special Lagrangian submanifold of phase i in $\mathbb{C}^3$;
  (d) S, for $S^4 \subset \mathbb{C}^3$, is a coassociative submanifold if and only if S is a holomorphic surface in $\mathbb{C}^3$.
(ii) Discuss a similar analysis for Cayley submanifolds of $\mathbb{R}^8 = \mathbb{R}^2 \times \mathbb{C}^3$ with $\Phi_0 = dt_1 \wedge dt_2 \wedge \omega_0 + dt_1 \wedge \operatorname{Re}\Omega_0 - dt_2 \wedge \operatorname{Im}\Omega_0 + \frac{1}{2}\omega_0^2$.
(iii) Let $\mathbb{R}^8 = \mathbb{C}^4$ endowed with its standard Calabi–Yau structure (ω_0, Ω_0). Show that we can write $\Phi_0 = \frac{1}{2}\omega_0^2 + \operatorname{Re}\Omega_0$ and show that holomorphic surfaces and special Lagrangian submanifolds of $\mathbb{C}^4$ are both Cayley submanifolds.

Exercise 23. Let $(\omega_1, \omega_2, \omega_3)$ be the standard hyperkähler structure on $\mathbb{R}^4 = \mathbb{H}$.

(i) Show that

$$\varphi_0 = dt_1 \wedge dt_2 \wedge dt_3 - dt_1 \wedge \omega_1 - dt_2 \wedge \omega_2 - dt_3 \wedge \omega_3,$$
$$\psi_0 = \tfrac{1}{6}(\omega_1^2 + \omega_2^2 + \omega_3^2) - dt_2 \wedge dt_3 \wedge \omega_1 - dt_3 \wedge dt_1 \wedge \omega_2 - dt_1 \wedge dt_2 \wedge \omega_3$$

on $\mathbb{R}^7 = \operatorname{Im}\mathbb{H} \times \mathbb{H}$.
(ii) Show that the subgroup of $\mathrm{SO}(7)$ that preserves φ_0, ψ_0 and the splitting $\mathbb{R}^7 = \operatorname{Im}\mathbb{H} \times \mathbb{H}$ is $\mathrm{SO}(4)$, where the action of $\mathrm{SO}(4)$ on $\operatorname{Im}\mathbb{H} \times \mathbb{H}$ is given in quaterionic notation by

$$(q_1, q_2) \cdot (x, y) = (q_1 x \overline{q}_1, q_2 y \overline{q}_1).$$

Here q_1, q_2 are unit quaternions, *i.e.* $(q_1, q_2) \in \mathrm{SU}(2)^2$. Moreover, the action is not effective since $(-1, -1)$ acts trivially, so that we have an effective action of $\mathrm{SU}(2)^2/\mathbb{Z}_2 \simeq \mathrm{SO}(4)$.
(iii) Show that $\operatorname{Im}\mathbb{H} \times \{0\} \subset \operatorname{Im}\mathbb{H} \times \mathbb{H}$ is an associative plane and $\{0\} \times \mathbb{H} \subset \operatorname{Im}\mathbb{H} \times \mathbb{H}$ is a coassociative plane.
(iv) The group of automorphisms of $\mathbb{O}$ is the exceptional Lie group G_2 and can be equally defined as the subgroup of $\mathrm{GL}(7, \mathbb{R})$ preserving φ_0. It turns out that G_2 acts transitively on $S^6 \subset \mathbb{R}^7$. Deduce that G_2 acts transitively on the Grassmannian of associative planes. (Hint: let u, v, w and use the action of G_2 to set $u = e_1$ and $v = e_2$.)
(v) Deduce that the Grassmannian of (co)associative planes is the homogeneous space $\mathrm{G}_2/\mathrm{SO}(4)$.

2.5. **Bibliographical notes.** Our exposition is based on [2, Chapter III] and [1, Chapter 7]. The last section about calibrated geometries associated with the octonions is based on [2, Chapter IV].

Proposition 2.4 is proven in a more recent paper by Harvey and Lawson [3, Sections 1 and 2] (following earlier work). The aim of that paper is to derive a "pluripotential theory" for calibrations analogous to the pluripotential theory of several complex variables. Indeed, note that on a Kähler manifold one has $d(\nabla f \lrcorner \omega) = 2i\partial\overline{\partial} f$.

The uniqueness in Theorem 2.5 is explained (in a more general context) in the recent paper [4, Section 4]. The analogous uniqueness statement for the Lawlor necks of Section 2.1 is also known [5], but the proof is even less trivial as it uses in an essential way Lagrangian Floer homology.

3. Deformation theory and applications

- (M^n, g) with k-calibration ϕ
- $\iota_0\colon \Sigma_0 \to M$ ϕ-calibrated
- $\Sigma^k \subset M$ C^1–close to $\Sigma_0 \longleftrightarrow \Sigma = \operatorname{graph} v$ for small $v \in C^1(\Sigma_0; \nu(\Sigma_0))$
  - deformations of the immersion ι_0 = sections of $\iota_0^* TM$
    restriction to sections of normal bundle $\nu(\Sigma_0)$ to work modulo reparametrisations
  - replace C^1 with Hölder space $C^{1,\alpha}$, $\alpha \in (0,1)$, to work with Banach spaces
    Implicit Function Theorem
  - elliptic regularity: $C^{1,\alpha} \rightsquigarrow C^\infty$
- moduli space $\mathcal{M}(\Sigma_0)$ of ϕ-calibrated deformations of Σ_0
  - $\Sigma_v = \operatorname{Graph} v$ ϕ-calibrated iff $F(v) = 0$ for continuously differentiable Fredholm
$$F\colon\ B_\epsilon(0) \subset C^{1,\alpha}(\Sigma_0; \nu(\Sigma_0)) \longrightarrow Z$$
  - linearisation d_0F with finite-dimensional kernel $\mathcal{I}$ and cokernel $\mathcal{O}$
  - Kuranishi model for $\mathcal{M}$: $\pi\colon \mathcal{I} \to \mathcal{O}$ with $\pi(0) = 0 = \pi'(0)$ and $\mathcal{M}$ and open neighbourhood of the origin in $\pi^{-1}(0)$
  - in particular $\mathcal{M}$ smooth submanifold of dimension $\operatorname{index} d_0F$ if d_0F surjective

Exercise 24

Theorem 3.1. *(Lagrangian Neighbourhood Theorem) Let L be a closed Lagrangian submanifold in a symplectic manifold (M, ω). Then there exists a neighboruhood $\mathcal{U}$ of L in M, a neighbourhood $\mathcal{U}_0$ of the 0-section $\mathbf{0}_L$ in T^*L and a diffeomorphism $\Upsilon\colon \mathcal{U}_0 \to \mathcal{U}$ such that $\Upsilon^*\omega = \omega_{\rm can}$ and $L = \Upsilon(\mathbf{0}_L)$. In particular, Lagrangian submanifolds of (M, ω) C^1–close to L are identified by Υ with graphs of closed 1-forms on L.*

Proof. Fix compatible almost complex structure and induced Riemannian metric g on a neighbourhood of L in M. Use the exponential map of g to define an initial diffeomorphism Υ' such that $\omega' = (\Upsilon')^*\omega$ satisfies $\omega' \equiv \omega_{\rm can}$ on $T\mathcal{U}_0$ along $\mathbf{0}_L$. Consider the 1-parameter family of symplectic forms $\omega_t = (1-t)\omega_{\rm can} + t\omega'$, write $\omega' - \omega = d\beta$ for a 1-form β that vanishes along Σ and consider the time dependent flow φ_t generated by the vector fields X_t defined by $X_t \lrcorner \omega_t = \beta$. Then set $\Upsilon = \Upsilon' \circ \varphi_1$. □

Theorem 3.2. *Let L be a closed sLag submanifold of (M, ω, Ω). The moduli space $\mathcal{M}(L)$ of special Lagrangian deformations (of the same phase) of L in M is a smooth manifold of dimension $b_1(L)$.*

Proof. C^1–close special Lagrangian submanifolds are the graphs $\iota_\gamma\colon\ L \to M$ of 1-forms γ on L such that $F(\gamma) = (\iota_\gamma^*\omega, \iota_\gamma^* \operatorname{Im}\Omega) = 0$. By the Lagrangian Neighbourhood Theorem we have $\iota_\gamma^*\omega = d\gamma$. Moreover, one computes
$$\iota_\gamma^* \operatorname{Im}\Omega = d * \gamma - Q(\gamma)$$
for a non-linear first-order map Q depending at least quadratically on γ. Hence $d_0F(\gamma) = (d\gamma, d^*\gamma)$ has kernel $\mathcal{I} = H^1(L)$ and cokernel that can also be identified cohomologically via Hodge theory on L. Note however that the non-linear equation $F(\gamma) = 0$ we want to solve is equivalent to
$$d\gamma = 0, \qquad d * \gamma = Q(\gamma).$$
Since $[\iota_\gamma^* \operatorname{Im}\Omega] = [\iota_0^* \operatorname{Im}\Omega]$ and $\iota_0\colon\ L \to M$ is special Lagrangian, we conclude that $d\gamma - Q(\gamma)$ and therefore $Q(\gamma)$ is exact for every γ. Hence the range of F is contained in the image of d_0F, *i.e.* $\mathcal{O} = \{0\}$. □

Exercises 25, 26, 27 and 28

Theorem 3.3. *Let (M, g, φ, ψ) be a* G_2*–manifold,* i.e. *a smooth Riemannian 7-manifold endowed with a closed 3-form φ and a closed 4-form $\psi = *\varphi$ pointwise equivalent to the forms φ_0 and ψ_0 of Proposition 2.6. Recall that a 3-dimensional submanifold $P \subset M$ is said to be associative if it* *is φ–calibrated. Let $P \subset M$ be a closed associative submanifold. Then the moduli space $\mathcal{M}(P)$ of associative deformations of P has virtual dimension 0 and infinitesimal deformations are given by elements in the kernel of a naturally defined Dirac-type operator $\not{D}\colon C^\infty(P;\nu(P)) \to C^\infty(P;\nu(P))$.*

Proof. If P is associative, the cross product $(u,v) \mapsto u \times v$ defined by the G_2–structure has the property that $u \times v \in \nu(P)$ whenever $u \in TP$ and $v \in \nu(P)$. This defines a Clifford multiplication $TP \otimes \nu(P) \to \nu(P)$ and therefore a Dirac-type operator $\not{D}v = \sum_{i=1}^3 e_i \times \nabla^\perp_{e_i} v$, which is an elliptic self-adjoint operator, hence of Fredholm index 0.

Now, the graph of $v \in C^\infty(P;\nu(P))$ is associative iff $\exp(v)^*\chi|_P \equiv 0$, so that the linearisation of the associative condition is the linear operator

$$v \longmapsto (\mathcal{L}_v\chi)(e_1,e_2,e_3) = (\nabla_v\chi)(e_1,e_2,e_3) + \sum_{i=1}^3 \chi(\nabla_v e_i, e_j, e_k) = \sum_{i=1}^3 \chi(\nabla_v e_i, e_j, e_k) = \not{D}v.$$

Here we used the (non-obvious) fact that the conditions $d\varphi = 0 = d\psi$ imply the a priori stronger conditions that φ, ψ (and therefore χ) are parallel with respect to g. □

Exercise 29

3.1. Calibrated fibrations. The following discussion is based on

(i) N. Hitchin, *The moduli space of special lagrangian submanifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (1997), no. 3-4, 503–515.

(ii) S. Donaldson, *Adiabatic limits of co-associative Kovalev-Lefschetz fibrations*, in Algebra, geometry, and physics in the 21st century, 1–29, Progr. Math., 324, Birkhäuser/Springer, Cham, 2017.

- (M^{2n},ω,Ω) Calabi–Yau
- $\pi\colon M \to B$ with compact special Lagrangian fibres $L_t = \pi^{-1}(t)$
  $\rightsquigarrow b_1(L_t) = n$ and L_t has n pointwise linearly independent g_t–harmonic 1-forms
  (L_t, g_t) almost flat n-torus
- work locally on base $B \simeq \mathbb{R}^n$
- cohomological data: $h = (u,v)\colon B \to H^1(L;R) \times H^{n-1}(L;\mathbb{R})$
- connection: $TM = V \oplus H$ with $V = \ker d\pi$ and $H = V^{\perp_{g_{\omega,\Omega}}} \simeq \pi^*TB$
  - $\Lambda^k T^*M = \pi^*\Lambda^i T^*B \otimes \Lambda^j V^*$
  - $d = d^v + d^H + F^H$ of degree $(+1,0),(0,+1),(+2,-1)$
    $d^2 = 0 \Rightarrow d^v \circ d^H + d^H \circ d^v = 0$
- $\omega = \omega^{1,1} = \sum \alpha_i \wedge dt_i$ with $d^v\alpha_i = 0$ and $[\alpha_i] = \partial_{t_i} u$
- $\operatorname{Im}\Omega^{1,n-1} = \sum dt_i \wedge \beta_i$ with $d^v\beta_i = 0$ and $[\beta_i] = \partial_{t_i} v$
- $\omega \wedge \operatorname{Im}\Omega = 0 \Longrightarrow \alpha_i \wedge \beta_j = \alpha_j \wedge \beta_i$
- $(\alpha_1,\dots,\alpha_n)$ and $(\beta_1,\dots,\beta_n)$ determine the $SU(n)$–structure (ω,Ω)
  - $\alpha_i \wedge \beta_j = Q_{ij}\,\alpha_1 \wedge \dots \wedge \alpha_n$, $Q = Q^T$ and $Q > 0$
  - $(\det Q)^{-\frac{1}{n-2}} Q = A^2$, $A = A^T \rightsquigarrow SU(n)$–adapted basis $\sum_l A^{li}\alpha_l, \sum_l A_{li} dt_l$
  - *e.g.* metric $g_{\omega,\Omega} = A^2_{ij} dt_i \otimes dt_j + A^{-2}_{ij} \alpha_i \otimes \alpha_j$
- Torsion-free conditions $d\omega = 0 = d\Omega$:
  - $d^v\omega = d^H\omega = F^H \cdot \omega = 0$
  - $d^H \operatorname{Re}\Omega^{2m,n-2m} = 0$, $d^v \operatorname{Re}\Omega^{2m+2,n-2m-2} + F^H \cdot \operatorname{Re}\Omega^{2m,n-2m} = 0$
  - $d^H \operatorname{Im}\Omega^{2m+1,n-2m-1} = 0$, $d^v \operatorname{Im}\Omega^{2m+3,n-2m-3} + F^H \cdot \operatorname{Im}\Omega^{2m+1,n-2m-1} = 0$
- adiabatic limit: rescale fibres $\alpha_i \rightsquigarrow \epsilon\alpha_i$, $\beta_i \rightsquigarrow \epsilon^{n-1}\beta_i$ and take formal limit $\epsilon \to 0$
  - limiting torsion-free equation as above with $F_H \equiv 0$

Proposition 3.4. *The cohomological data of a special Lagrangian fibration satisfy the following.*

(i) *The map $h\colon B \to H^1(L;R) \times H^{n-1}(L;\mathbb{R})$ is a positive definite Lagrangian immersion.*

(ii) *Choose closed representatives* $\overline{\alpha}_i, \overline{\beta}_j$ *giving a symplectic basis* $\{\overline{\alpha}_1, \ldots, \overline{\alpha}_n, \overline{\beta}_1, \ldots, \overline{\beta}_n\}$ *of* $H^1(L) \times H^{n-1}(L)$, i.e.

$$\int_L \overline{\alpha}_i \wedge \overline{\beta}_j = \delta_{ij}.$$

Consider the period matrices $\lambda, \mu \colon B \to \mathbb{R}^{n\times n}$ *defined by*

$$[\alpha_i] = \sum_l \lambda_{li}[\overline{\alpha}_l], \qquad [\beta_i] = \sum_l \mu_{li}[\overline{\beta}_l].$$

Then $\sum_l \lambda_{il} dt_l$ *and* $\sum_l \mu_{il} dt_l$ *are closed 1-forms on* B *yielding two coordinate systems* $(u_1, \ldots, u_m)$ *and* $(v_1, \ldots, v_n)$.

(iii) *We have*

$$v_i = \frac{\partial \phi}{\partial u_i}, \qquad u_i = \frac{\partial \psi}{\partial v_i},$$

for two real valued functions ϕ, ψ *on* B. *These functions satisfy the real Monge–Ampère equation* $\det \mathrm{Hess}(\phi) = c = \det \mathrm{Hess}(\psi)^{-1}$ *for* $c > 0$ *if and only if the* L^2*–metric induced by* g_t *on* $H^1(L; \mathbb{R}/\mathbb{Z})$ *(and its dual* $H^{n-1}(L_t; \mathbb{R}/\mathbb{Z})$*) is constant in* B.

(iv) *Assuming the last condition is satisfied and choosing* $H = H_0$ *the natural flat connection on the bundle on* B *with fibre* $H^1(L) \times H^{n-1}(L)$ *(for which* $\overline{\alpha}_i, \overline{\beta}_j$ *are parallel), these cohomological data determine two 1-parameter family of Calabi–Yau structures* $(\omega_\epsilon, \Omega_\epsilon)$, *one on the total space of the fibration with fibre* $H^1(L)$ *and one of the total space of the fibration with fibre* $H^{n-1}(L)$, *with the property that the special Lagrangian fibres are flat tori of volume* $c\epsilon^n$ *(and therefore called semiflat Calabi–Yau metrics).*

Proof. The first part is immediate. For part (ii), by fibrewise Hodge theory write $\alpha_i = \sum_l \lambda_{li}\overline{\alpha}_l + d^v \alpha'_i$. Then, using $d^H \circ d^v = -d^v \circ d^H$ we rewrite

$$0 = -d^H \omega = \sum_l \overline{\alpha}_l \wedge d\left(\sum_i \lambda_{li} dt_i\right) + d^v \left(\sum_i d^H \alpha'_i \wedge dt_i\right).$$

For part (iii) we now replace the given coordinates $t_1, \ldots, t_n$ on B with the coordinates $u_1, \ldots, u_m$ so that

$$[\alpha_i] = [\overline{\alpha}_i], \qquad [\beta_i] = \sum_l (\mu\lambda^{-1})_{li}[\overline{\beta}_l].$$

We then calculate

$$\mu\lambda^{-1} = \int_{L_t} Q\, \alpha_1 \wedge \cdots \wedge \alpha_n.$$

(In particular $\mu\lambda^{-1}$ is a symmetric matrix.) On the other hand we observe that $\mathrm{Re}\, \Omega^{0,n} = (\det Q)^{\frac{1}{n-2}} \alpha_1 \wedge \ldots \alpha_n$ so that we can rewrite

$$(\mu\lambda^{-1})_{ij} = \int_{L_t} (\det Q)^{-\frac{1}{n-2}} Q_{ij} (\det Q)^{\frac{1}{n-2}} \alpha_1 \wedge \cdots \wedge \alpha_n = \int_L \langle \alpha_i, \alpha_j \rangle_{g_t} \, d\mathrm{vol}_{L_t}.$$

Thus the condition in part (iii) is equivalent to $\det(\mu\lambda^{-1})$ being constant. The existence of ϕ (and ψ) follows from the fact that h is a Lagrangian immersion. One then immediately calculates $\mu\lambda^{-1} = \mathrm{Hess}\, \phi$. □

- if L is an oriented torus then there is a canonical isomorphism $L \simeq H^{n-1}(L)$ and $H^1(L)$ is the dual torus ⇝ SYX approach to Mirror Symmetry
- if B is compact one can show that the only solutions as in part (iv) are trivial, *i.e.* of the form $M = B \times T^2$ for a flat torus T^2 ⇝ necessary to introduce singular fibres
- $n = 2$: the Ooguri–Vafa metric on a neighbourhood of a pinched torus
  see `http://www.homepages.ucl.ac.uk/~ucahlfo/Talks_files/Ooguri-Vafa.pdf`

3.2. **Exercises.**

Exercise 24. Let $v\colon \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map and consider the graph L_v of v in $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$, *i.e.*

$$L_v = \{x + iv(x) \mid x \in \mathbb{R}^n\}.$$

Endow $\mathbb{C}^n$ with the standard Calabi–Yau structure (ω_0, Ω_0).

(i) Show that L_v is Lagrangian iff $v = df$ for a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$, *i.e.* $v = (v_1, \dots, v_n)$ with $v_i = \frac{\partial f}{\partial x_i}$ if $(x_1, \dots, x_n)$ are the standard coordinates on $\mathbb{R}^n$.

(ii) Show that $\operatorname{Im}\Omega_0|_{L_v} \equiv 0$ iff

$$\operatorname{Im}\det{}_{\mathbb{C}}\,(\mathrm{id} + i\,\mathrm{Hess}(f)) = 0.$$

(iii) In low dimensions $n = 2$ and $n = 3$ write explicitly the equation in (ii) in terms of the Laplacian $\triangle f$ and the real Monge–Ampère operator $\det \mathrm{Hess}(f)$.

(iv) Suppose that f_0 is a solution to the equation of (ii) and consider the linearisation $\mathfrak{L}$ of the map

$$f \longmapsto \operatorname{Im}\det{}_{\mathbb{C}}\,(\mathrm{id} + i\,\mathrm{Hess}(f))$$

at f_0. Show that $\mathcal{L}$ is an elliptic second-order operator.

Exercise 25. Prove the following variation of Theorem 3.2: let (ω_t, Ω_t), $t \in (-\delta_0, \delta_0)$, be a smooth 1-parameter family of Calabi–Yau structures on M. Suppose that L is a closed special Lagrangian submanifold of phase $e^{i\theta_0}$ in (M, ω_0, Ω_0). If $[\omega_t|_L] = 0$ for all t, then there exist $\delta \in (0, \delta_0)$ and a 1-parameter family of special Lagrangian submanifolds L_t of phase $e^{i\theta_t}$ in (M, ω_t, Ω_t) for $t \in (-\delta, \delta)$ and with $L_0 = L$. Moreover, if $[\operatorname{Im}(e^{-i\theta_0}\Omega_t)|_L] = 0$ for all t then $e^{i\theta_t} \equiv e^{i\theta_0}$

Exercise 26. One can show that T^*S^n admits a (complete) Calabi–Yau structure (ω, Ω) for which the antipodal map $x \mapsto -x$ in the fibres of $T^*S^n \to S^n$ is an antiholomorphic involution. Show that (T^*S^n, ω, Ω) contains a rigid (*i.e.* $\mathcal{M} = \{0\}$) special Lagrangian submanifold.

Exercise 27. This exercise is based on A. Butscher, *Deformations of minimal Lagrangian submanifolds with boundary*, Proc. Amer. Math. Soc. 131 (2002).

Let (M, ω, Ω) be a Calabi–Yau n-fold and let D be a divisor in M, *i.e.* D is a $2(n-1)$–dimensional submanifold calibrated by $\frac{\omega^{n-1}}{(n-1)!}$. Let $L \subset M$ be a special Lagrangian submanifold with boundary $\Sigma = \partial L$ on D.

(i) Observe that $\Sigma \subset (D, \omega|_D)$ is a Lagrangian submanifold and show that $\nu(D)|_\Sigma$ is a trivial $\mathbb{R}^2$–bundle. (Hint: consider the inward pointing unit normal v to Σ in L and Jv.)

This fact can be used to prove a Lagrangian Neighbourhood Theorem with Boundary which identifies a neighbourhood of L in (M, ω) with an open subset of $T^*\Sigma \times \mathbb{R}^2$ endowed with the canonical symplectic form $\omega_{\mathrm{can}} + dt \wedge ds$, where $(s, t) \in \mathbb{R}^2$, such that

$$D = T^*\Sigma \times \{(0,0)\}, \qquad L = \mathbf{0}_\Sigma \times [0, \epsilon) \times \{0\}.$$

Assume this without proof.

(ii) Reduce the deformation problem for special Lagrangian deformations of L with arbitrary Lagrangian phase and with boundary on D to the vanishing of the map

$$F\colon\ (\gamma, a) \longmapsto (d\gamma, d * \gamma - a\mathrm{vol}_L - Q(\gamma))$$

defined for all $\gamma \in \Omega^1(L)$ satisfying the Neumann boundary condition $(*\gamma)|_{\partial L} \equiv 0$ and every $a \in \mathbb{R}$.

(iii) Show that the linearisation $d_{(0,0)}F$ is surjective and that the moduli space of special Lagrangian deformations of L of arbitrary phase and with boundary on D is a smooth manifold of dimension $b_1(L)$. (Hint: for the relevant Hodge theory with manifolds with boundary see G. Schwarz, *Hodge Decomposition—A Method for Solving Boundary Value Problems*, Springer-Verlag, Berlin, 1995.)

Exercise 28. Let (M,g,φ,ψ) be a G_2–manifold, *i.e.* a smooth Riemannian 7-manifold endowed with a closed 3-form φ and a closed 4-form $\psi=*\varphi$ pointwise equivalent to the forms φ_0 and ψ_0 of Proposition 2.6. Recall that a 4-dimensional submanifold $N\subset M$ is said to be coassociative if it is ψ–calibrated.

(i) Show that the normal bundle to a coassociative submanifold $N\subset M$ is identified with $\Lambda^2_+T^*N$. (Hint: use Exercise 23 and consider the map $v\mapsto v\lrcorner\varphi$ for a normal vector v.)

(ii) By (i) we can identify a small C^1–deformation of N with the graph $\iota_\sigma\colon N\to M$ of a self-dual 2-form σ on N. Show that ι_σ is a coassociative immersion iff $d\sigma=Q(\sigma)$ for a non-linear first-order map Q vanishing quadratically at $\sigma=0$. (Hint: recall that ι_σ is a coassociative immersion iff $\iota_\sigma^*\varphi=0$.)

(iii) Suppose that N is closed. Deduce that the moduli space of coassociative deformations of N is a smooth manifold of dimension $b_2^+(N)$.

Exercise 29. This exercise is based on Section 5.3 of A. Corti, M. Haskins, J. Nordström and T. Pacini, G_2*–manifolds and associative submanifolds via semi-Fano 3-folds*, Duke Math. J. 164 (2015).

Let (M,ω,Ω) be a Calabi–Yau 3-fold and consider the G_2–manifold $(M\times S^1,\varphi=dt\wedge\omega+\mathrm{Re}\,\Omega)$. Let C be a holomorphic curve in M and consider the associative submanifold $C\times S^1\subset M\times S^1$.

(i) Show that the cross product on $M\times S^1$ restricts to give a complex antilinear map $TC\times\nu(C)\to\nu(C)$, that can be used to define a complex Dirac operator $\not{D}^c$.

(ii) Show that the kernel of $\not{D}^c$ coincides with the space of holomorphic normal vector fields on C. (Hint: use the fact that since M is Kähler and C is holomorphic, the connection on $\nu(C)$ induced by the Levi–Civita connection on M is the Chern connection on $\nu(C)$, *i.e.* its $(0,1)$–part is the Dolbeault operator $\overline{\partial}$ of the holomorphic bundle $\nu(C)$.)

(iii) Regard a normal vector field v on $P\subset M\times S^1$ as an S^1–dependent vector field on $C\subset M$. Show that
$$\not{D}v=J\tfrac{\partial v}{\partial t}+\not{D}^c v,$$
where $\not{D}$ is the Dirac-type operator defined in Theorem 3.3. (Hint: recall that the complex structure J on M coincides with the cross-product by ∂_t.)

(iv) Suppose that C is closed. Show that the kernel of $\not{D}$ coincides with the space of holomorphic normal vector fields on C. (Hint: consider the L^2–inner product $\langle\not{D}v,v\rangle_{L^2}$.)

(v) Suppose that C is a rational curve, *i.e.* $C\simeq\mathbb{CP}^1$, with normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. Show that P is a rigid unobstructed associative submanifold, *i.e.* the moduli space of associative deformations is a smooth isolated point, or in other words $\mathcal{I}=\{0\}=\mathcal{O}$.

3.3. **Bibliographical notes.** Deformations of calibrated submanifolds where first considered by McLean in [6].

References

[1] R. Harvey. *Spinors and calibrations.* Perspectives in Mathematics 9. Academic Press, Inc., Boston, MA, 1990.

[2] R. Harvey and H. B. Lawson. Calibrated geometries. *Acta Math.*, 148:47–157, 1982.

[3] R. Harvey and H. B. Lawson. An introduction to potential theory in calibrated geometry. *Amer. J. Math.*, 131(4):893–944, 2009.

[4] Yohsuke Imagi. Surjectivity of a gluing construction in special Lagrangian geometry. *Comm. Anal. Geom.*, 25(5):1019–1061, 2017.

[5] Yohsuke Imagi, Dominic Joyce, and Joana Oliveira dos Santos. Uniqueness results for special Lagrangians and Lagrangian mean curvature flow expanders in $\mathbb{C}^m$. *Duke Math. J.*, 165(5):847–933, 2016.

[6] R. C. McLean. Deformations of calibrated submanifolds. Comm. Anal. Geom. 6 (1998), 705–747.