RIEMANNIAN HOLONOMY GROUPS

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Contents

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1. Berger's list

1.1. **Holonomy.**

- M^n connected smooth manifold
- G Lie group with Lie algebra $\mathfrak{g}$
- principal G–bundle P over M: smooth manifold P, with smooth free right action $R\colon\ P\times G\to P$ of G, and smooth projection $\pi\colon\ P\to M=P/G$ to the quotient space
- Examples: Hopf circle bundle $S^3\to S^1$, quaternionic Hopf bundle $S^7\to S^4$, frame bundle
- $\rho\colon\ G\to \mathrm{End}(V)\rightsquigarrow P\times_\rho V$, *e.g.* $\rho=\mathrm{Ad}\rightsquigarrow$ adjoint bundle $\mathrm{ad}\,P$

Exercise 1

- A connection on $\pi\colon\ P\to M$ is a G–invariant splitting $TP=\ker\pi_*\oplus H$ of
$$0\to\ker\pi_*\to TP\to\pi^*TM\to 0$$
H is called the horizontal subspace
- Since $\ker\pi_*\simeq P\times\mathfrak{g}$, a connection is a 1-form $\theta\colon\ TP\to\mathfrak{g}$ such that $R_g^*\theta=\mathrm{Ad}(g^{-1})\theta$
- horizontal lift X^H of a vector field X on M: the unique G–invariant vector field on P such that $X^H\in H\subset TP$ at every point of P and $\pi_*X^H=X$
- curvature $R^\theta\in\Omega^2(M;\mathrm{ad}\,P)$ of θ: $\pi^*R^\theta(X,Y)=[X,Y]^H-[X^H,Y^H]$

Exercises 2 and 3.

- The horizontal lift of a path $\gamma\colon\ [0,1]\to M$ is $\widetilde{\gamma}\colon\ [0,1]\to P$ such that $\pi\circ\widetilde{\gamma}=\gamma$ and $\widetilde{\gamma}'(t)\in H_{\gamma(t)}$ for all $t\in[0,1]$
- Standard ODE theory $\Rightarrow$ existence of unique horizontal lift for each choice of $u\in\pi^{-1}(\gamma(0))$
- $H^\theta(u)=\{g\in G\,|\,u\cdot g=\widetilde{\gamma}(1)\text{ for some loop }\gamma\text{ on }M\}$
- Holonomy group $H(\theta)$ of θ: conjugacy class of $H^\theta(u)$
- restricted holonomy $H^0(\theta)$ if only consider horizontal lifts of contractible loops in M
- $H^0(\theta)$ is a connected Lie subgroup of G

Exercise 4

- X,Y commuting vector fields on M, $[X,Y]=0\rightsquigarrow$ contractible loops γ_s obtained by composition of flows of X and Y
- $\Longrightarrow\widetilde{\gamma}_s(1)=u\cdot g_s$ and so
$$\pi^*R^\theta(X,Y)=-[X^H,Y^H]=\frac{d}{ds}(u\cdot g_s)_{|s=0}\in\mathfrak{h}^\theta$$
- Ambrose–Singer (1953): $\mathfrak{h}^\theta$ is generated by elements of the form $\pi^*R^\theta(X,Y)_{|u'}$ for $u'=\widetilde{\gamma}(1)$ endpoint of horizontal lift with $\widetilde{\gamma}(0)=u$

1.2. **Riemannian holonomy.**

- Riemannian manifold (M^n,g)
- orthogonal frame bundle $P=\{\text{isometry } u\colon\ T_xM\to\mathbb{R}^n\}$
- tautological $\mathbb{R}^n$–valued 1-form ω, $\omega(v)_{|u}=u(\pi_*v)$
- Levi-Civita connection: $\exists\,!$ connection θ on P such that $d\omega+\theta\wedge\omega=0$
- $\mathrm{Hol}(g)=H^\theta$

Exercises 5 and 6

Classification of Riemannian holonomy groups.

- first Bianchi identity $(d\theta+\frac{1}{2}[\theta,\theta])\wedge\omega=0$
  $\Rightarrow$ curvature function $R\colon\ P\to\Lambda^2(\mathbb{R}^n)^*\otimes\mathfrak{so}(n)$ takes values at $u\in P$ in
$$K(\mathfrak{h})=\ker\left(\Lambda^2(\mathbb{R}^n)^*\otimes\mathfrak{h}\to\mathbb{R}^n\otimes\Lambda^3(\mathbb{R}^n)^*\right)$$
  $\rightsquigarrow\mathfrak{h}$ cannot strictly contain an ideal $\mathfrak{h}'$ such that $K(\mathfrak{h})\subseteq\Lambda^2(\mathbb{R}^n)^*\otimes\mathfrak{h}'$

- 2nd Bianchi identity & symmetric spaces
  - symmetric spaces G/H: $\mathrm{Hol} = H$ and curvature function is constant, $\nabla R \equiv 0$
  - $\nabla R \in K'(\mathfrak{h}) = \ker\left(K(\mathfrak{h}) \otimes (\mathbb{R}^n)^* \to \mathfrak{h} \otimes \Lambda^3(\mathbb{R}^n)^*\right)$

  $\Rightarrow$ if (M, g) non-symmetric then $K'(\mathfrak{h}) \neq 0$
- reducible Riemannian manifolds
  - $H = \mathrm{Hol}(g)$ acts on $\mathbb{R}^n$ reducibly $\Rightarrow H = H_1 \times \cdots \times H_k$ and $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$ with $H_i \subseteq \mathrm{O}(n_i)$ acting irreducibly on $\mathbb{R}^{n_i}$
  - de Rham Theorem: (M^n, g) complete and simply connected and $\mathrm{Hol}(g) = H_1 \times \cdots \times H_k$ as above $\Rightarrow (M^n, g) \simeq (M_1^{n_1}, g_1) \times \cdots \times (M_k^{n_k}, g_k)$ with $\mathrm{Hol}(g_i) = H_i$

Theorem 1.1 (Berger, 1955). *If (M^n, g) is non-symmetric and irreducible then* $\mathrm{Hol}^0(g)$ *is one of*

$$\mathrm{SO}(n), \qquad \mathrm{U}(\tfrac{n}{2}), \qquad \mathrm{SU}(\tfrac{n}{2}), \qquad \mathrm{Sp}(\tfrac{n}{4}), \qquad \mathrm{Sp}(\tfrac{n}{4})\mathrm{Sp}(1), \qquad \mathrm{G}_2, \qquad \mathrm{Spin}_7.$$

1.3. **How to analyse the groups in Berger's list.**

- (M^n, g) simply connected with $\mathrm{Hol}(g) = H$
- Holonomy Principle: H–invariant $(T_xM)^{\otimes r} \otimes (T_x^*M)^{\otimes s} \xleftrightarrow{1:1}$ parallel tensors on M
  e.g. generic holonomy $H = \mathrm{SO}(n)$: g, dv_g only parallel tensors
- Reduction of structure group to H: $P_H \subseteq P$
  - intrinsic torsion τ: $\theta_{\mathrm{LC}} = \theta_H + \tau \in \mathfrak{h} \oplus \mathfrak{h}^\perp = \mathfrak{so}(n)$
  - H–structure torsion-free, $\tau \equiv 0 \iff P_H$ preserved by holonomy $\iff \mathrm{Hol}(g) \subseteq H$

The Kähler case. $H = \mathrm{U}(m)$ $(n = 2m)$

- metric g
- almost complex structure J
- Kähler form $\omega(\,\cdot\,,\,\cdot\,) = g(J\,\cdot\,,\,\cdot\,)$

Proposition 1.2. *(M^{2m}, g) with g–orthogonal almost complex structure J and associated Kähler form ω. TFAE:*

(i) $\mathrm{Hol}(g) \subseteq \mathrm{U}(m)$
(ii) $\nabla J = 0$
(iii) $\nabla \omega = 0$
(iv) *J is integrable and $d\omega = 0$*

Proof. We only indicate why intrinsic torsion $\tau \equiv 0$ iff (iv).

- $V = \mathbb{R}^{2m}$ as a $\mathrm{U}(m)$–rep
- $\Lambda^k V^* \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}$
- $[\![\Lambda^{p,q}]\!]$ the real representation such that $[\![\Lambda^{p,q}]\!] \otimes \mathbb{C} = \Lambda^{p,q} \oplus \overline{\Lambda^{p,q}} = \Lambda^{p,q} \oplus \Lambda^{q,p}$
- $\Lambda^2 V^* = [\![\Lambda^{1,1}]\!] \oplus [\![\Lambda^{2,0}]\!] = \mathfrak{u}(n) \oplus \mathfrak{u}(n)^\perp$
- $\Longrightarrow \tau \in \Lambda^1 V^* \otimes [\![\Lambda^{2,0}]\!] = [\![\Lambda^{1,0} \otimes \Lambda^{2,0}]\!] \oplus [\![\Lambda^{2,1}]\!]$
- projection on first factor is N_J via $\Lambda^{1,0} \otimes \Lambda^{2,0} \simeq \overline{T^{1,0}} \otimes \Lambda^{2,0} \simeq T^{0,1} \otimes \Lambda^{2,0}$
- if $N_J \equiv 0$ then $d\omega \in [\![\Lambda^{2,1}]\!]$ is exactly the second factor $\square$

Exercises 7 and 8

Special holonomy and Ricci curvature.

- $R \in K(\mathfrak{so}(n)) \subset \Lambda^2(\mathbb{R}^n)^* \otimes \mathfrak{so}(n)$
- as $\mathrm{SO}(n)$-rep $K(\mathfrak{so}(n)) = \mathrm{Sym}^2(\mathbb{R}^n) \oplus \mathcal{W} = \mathbb{R} \oplus \mathrm{Sym}^2_0(\mathbb{R}^n) \oplus \mathcal{W}$ $(\mathcal{W} = \mathcal{W}^+ \oplus \mathcal{W}^-$ if $n = 4)$
- $R = \mathrm{Ric} + W = \frac{1}{n}\mathrm{Scal}\,\mathrm{id}_{\mathbb{R}^n} + \mathring{\mathrm{Ric}} + W$, where $\mathrm{Ric}(X, Y) = \mathrm{tr}\,\langle R(x,\,\cdot\,)\,\cdot\,, y\rangle = \sum_{i=1}^n \langle R(x, e_i)e_i, y\rangle$
- g Einstein: $\mathring{\mathrm{Ric}} = 0 \iff \mathrm{Ric} = \lambda g$ for some $\lambda \in \mathbb{R}$
- the most natural PDE for a Riemannian metric:
  - $\mathrm{Ric}_g = \triangle_g g + h.o.t.$ in harmonic coordinates

  - g Einstein $\iff$ g critical point of Einstein–Hilbert functional $\mathcal{S}(g) = \int_M \mathrm{Scal}_g \, \mathrm{dv}_g$ restricted to metrics of fixed volume
- $H \neq \mathrm{SO}(n), \mathrm{U}(\frac{n}{2})$ in Berger's list $\Longrightarrow K(\mathfrak{h}) \subset \{\mathring{\mathrm{Ric}} = 0\}$
- $H = \mathrm{SU}(\frac{n}{2}), \mathrm{Sp}(\frac{n}{4}), \mathrm{G}_2, \mathrm{Spin}_7 \Longrightarrow K(\mathfrak{h}) \subset \{\mathrm{Ric} = 0\}$

Exercises 9 and 10

The Calabi–Yau case. $H = \mathrm{SU}(m)$ $(n = 2m)$
- Kähler form ω
- complex volume form $\Omega \rightsquigarrow$ almost complex structure J
- $\omega \wedge \Omega = 0$ and $\frac{1}{n!}\omega^n = \frac{(-1)^{\frac{m(m-1)}{2}} i^m}{2^m}\, \Omega \wedge \overline{\Omega}$
- $\mathrm{SU}(m)$–structure (ω, Ω) torsion-free iff $d\omega = 0 = d\Omega$
- every CY metric is Ricci-flat (*cf.* later)

The hyperKähler case. $H = \mathrm{Sp}(m)$ $(n = 4m)$
- triple of Kähler forms $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ modelled on $\sum_{i=1}^m d\overline{q}_i \wedge dq_i$ on $\mathbb{H}^m$

$\rightsquigarrow$ triple of almost complex structures J_1, J_2, J_3 such that $J_1 J_2 = J_3$ etc
- $\mathrm{Sp}(m)$–structure $(\omega_1, \omega_2, \omega_3)$ torsion-free iff $d\omega_i = 0$ for all $i = 1, 2, 3$
- every HK metric is Ricci-flat

Exercises 11 and 12

The quaternionic Kähler case. $H = \mathrm{Sp}(m)\mathrm{Sp}(1)$ $(n = 4m \geq 8)$
- triple $\omega_1, \omega_2, \omega_3$ rotated by $\mathrm{Sp}(1)$–factor (acting on the right on $\mathbb{H}$)

$\rightsquigarrow$ only $\mathrm{Span}(\omega_1, \omega_2, \omega_3) \subset \Lambda^2(\mathbb{R}^{4m})^*$ and 4-form $\Phi = \frac{1}{2}(\omega_1^2 + \omega_2^2 + \omega_3^2)$ well defined
- every QK metric is Einstein and is HK if Ricci-flat

Remark. Since $\mathrm{Sp}(1)\mathrm{Sp}(1) = \mathrm{SO}(4)$ quaternionic Kähler metrics in dimension 4 do not make much sense. However, quaternionic Kähler metrics turn out to have many common features with self-dual Einstein 4-manifolds. See Exercise 13.

The exceptional cases. $H = \mathrm{G}_2$ if $n = 7$ and $H = \mathrm{Spin}(7)$ if $n = 8$
- $\mathrm{G}_2 = \mathrm{Aut}(\mathbb{O})$
  - G_2 = stabiliser in $\mathrm{GL}(7, \mathbb{R})$ of $\varphi_0(u, v, w) = \langle u \times v, w\rangle$ on $\mathbb{R}^7 \simeq \mathrm{Im}\,\mathbb{O}$
  - G_2–structure φ on M^7 induces g_φ: $g_\varphi(u, v)\, \mathrm{dv}_{g_\varphi} = \frac{1}{6}(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi$

  e.g. $\mathrm{SU}(3)$–structure (ω, Ω) on $Y^6 \rightsquigarrow \mathrm{G}_2$–structure $\varphi = dt \wedge \omega + \mathrm{Re}\,\Omega$ on $X^7 = \mathbb{R} \times Y^6$

  e.g. $\mathrm{SU}(2)$–structure $(\omega_1, \omega_2, \omega_3)$ on $Z^4 \rightsquigarrow \mathrm{G}_2$–structure on $X^7 = \mathbb{R}^3 \times Z^4$

$$\varphi = dt_1 \wedge dt_2 \wedge dt_3 - dt_1 \wedge \omega_1 - dt_2 \wedge \omega_2 - dt_3 \wedge \omega_3$$

- G_2–structure (M^7, φ) torsion-free (or (M^7, φ) G_2–manifold) $\iff d\varphi = 0 = d *_\varphi \varphi$
- G_2–manifold $(M^7, \varphi) \Longrightarrow \mathrm{Ric}(g_\varphi) = 0$
- $\mathrm{Spin}(7)$–structure on M^8: choice of an "admissible" 4-form Φ

  e.g. φ on $Y^7 \rightsquigarrow \mathrm{Spin}(7)$–structure $\Phi = dt \wedge \varphi + *_\varphi \varphi$ on $X^8 = \mathbb{R} \times Y^7$
- $\mathrm{Spin}(7)$–structure (M^8, Φ) torsion-free (or (M^8, Φ) $\mathrm{Spin}(7)$–manifold) $\iff d\Phi = 0$
- $\mathrm{Spin}(7)$–manifold $(M^8, \Phi) \Longrightarrow \mathrm{Ric}(g_\Phi) = 0$

Exercise 14

Special holonomy and parallel spinors. An example: CY 3-folds
- $\mathrm{SU}(3) \subset \mathrm{SU}(4) \simeq \mathrm{Spin}(6)$ stabiliser of a non-zero vector in $\mathbb{C}^4$

$\Rightarrow$ $\mathrm{SU}(3)$–structures on $M^6 \xleftrightarrow{1:1}$ spin structures on M + nowhere-vanishing spinor ψ
- more explicitly: under $\$(M) \otimes \$(M)^* \simeq \mathrm{Cl}(M) \simeq \Lambda^\bullet T^* M$

$$8\, \psi \otimes \psi^* = 1 + \mathrm{Re}\,\Omega - \tfrac{1}{2}\omega^2$$

- $\mathrm{SU}(3)$–structure torsion-free $\iff \nabla \psi = 0$

- Clifford multiplication by dv = complex structure on spinor bundle: 2 parallel spinors $\psi, \mathrm{dv} \cdot \psi$ and Hol = SU(3) is the space of parallel spinors is exactly 2-dimensional

Exercise 15

1.4. Exercises.

Exercise 1. In this exercise we study principal bundles on spheres.

(i) Show that isomorphism classes of principal G–bundle on the sphere S^n are in 1:1 correspondence with $\pi_{n-1}(G)$.
(ii) Specialise part (i) to principal U(1)–bundles on S^2.
(iii) What is the non-trivial SO(3)–bundle on S^2?
(iv) Show that every SU(2)–bundle on S^3 is trivial, that SU(2)–bundles on S^4 are classified by an integer and that there are only two principal SU(2)–bundles on S^5. Can you describe the non-trivial SU(2)–bundle on S^5?

Exercise 2. Let $\pi\colon P \to M$ be a principal G–bundle with connection θ and let $E = P \times_\rho V$ be an associated vector bundle.

(i) Show that there is a 1:1 correspondence between sections $s \in C^\infty(M;E)$ of E and G–equivariant V–valued functions $\widetilde{s} \in C^\infty(P;V)$ on P.
(ii) Show that the formula $s \mapsto d\widetilde{s}|_H$ defines a covariant derivative on E, *i.e.* an $\mathbb{R}$–linear map $\nabla^\theta\colon C^\infty(M;E) \to C^\infty(M;T^*M \otimes E)$ satisfying the Leibniz rule $\nabla(fs) = df \otimes s + f\nabla s$ for all $f \in C^\infty(M)$ and $s \in C^\infty(M;E)$.

Exercise 3. Let θ be a connection on a principal bundle and identify the curvature R^θ with its pull-back to P as a 2-form with values in the Lie algebra $\mathfrak{g}$.

(i) Show that $R^\theta = d\theta + \frac{1}{2}[\theta,\theta]$.
(ii) Deduce the Bianchi identity $dR^\theta + [\theta, R^\theta] = 0$ from part (i) and the Jacobi identity in $\mathfrak{g}$.

Exercise 4. Show that $H(\theta)$ is well-defined. This involves the following steps:

(i) $H^\theta(u)$ is a subgroup of G;
(ii) $H^\theta(u \cdot g) = \mathrm{Ad}(g^{-1})H^\theta(u)$;
(iii) for all $u, u' \in P$ there exists $g \in G$ such that $H^\theta(u') = \mathrm{Ad}(g^{-1})H^\theta(u)$.

Exercise 5. For $\alpha \in \mathbb{R}/\mathbb{Z}$ consider the 3-manifold $M_\alpha = \mathbb{R}^3/\mathbb{Z}$, where $\mathbb{Z}$ acts by $n \cdot (t,z) = (t+n, e^{2\pi i n\alpha}z)$. Here we indentified $\mathbb{R}^3 \simeq \mathbb{R} \times \mathbb{C}$. Endow M_α with the flat metric g_α induced by the standard flat metric on $\mathbb{R}^3$.

(i) Show that M_α is diffeomorphic to $S^1 \times \mathbb{R}^2$.
(ii) Calculate the holonomy of g_α and deduce that (M_α, g_α) is not isometric to $(M_{\alpha'}, g_{\alpha'})$ if $\alpha \neq \alpha'$.

Exercise 6. Suppose that (M^n, g) has holonomy $\mathrm{Hol}(g)$ with Lie algebra $\mathfrak{h} \subseteq \mathfrak{so}(n)$. Regard the curvature of (M,g) as the curvature function $R\colon P \to \Lambda^2(\mathbb{R}^n)^* \otimes \mathfrak{so}(n)$, where P is the orthogonal frame bundle of (M,g).

(i) Use the Ambrose–Singer Theorem to show that the curvature function takes values at $u \in P$ in $\Lambda^2(\mathbb{R}^n)^* \otimes \mathfrak{h}$ where $\mathfrak{h}$ is the Lie algebra of $H^{\theta_{\mathrm{LC}}}(u)$.
(ii) Use the symmetries of the curvature operator to deduce that the curvature function takes value in $Sym^2(\mathfrak{h}) \subset \mathfrak{h} \otimes \mathfrak{h} \subseteq \mathfrak{so}(n) \otimes \mathfrak{h} \simeq \Lambda^2(\mathbb{R}^n)^* \otimes \mathfrak{h}$. Here we use the metric to identify 2-forms and skew-symmetric matrices.

Exercise 7. Complete the proof of Proposition 1.2.

Exercise 8. Show that there exists a metric g on $\mathbb{R}^4$ which is flat outside the disjoint union of two balls, every point $x \in \mathbb{R}^4$ has a neighbourhood U_x such that $g|_{U_x}$ is Kähler, but $\mathrm{Hol}(g) = \mathrm{SO}(4)$. *(This shows that the holonomy of a metric cannot be determined locally in general. However the holonomy reduction to one of the other special holonomy groups in Berger's list can be detected* *locally, since the metric must be Einstein in all these cases and therefore it is real analytic in harmonic coordinates.)*

Exercise 9. Let M be a closed manifold and denote by $\mathfrak{Met}(M)$ (respecticely, $\mathfrak{Met}_1(M)$) the space of Riemannian metrics (with unit volume) on M. In this exercise you are going to show that critical points of the Einstein–Hilbert functional $\mathcal{S}(g) = \int_M \mathrm{Scal}_g \, \mathrm{dv}_g$ restricted to $\mathfrak{Met}_1(M)$ are Einstein.

(i) Let $\{g_t\}_{t\in(-\epsilon,\epsilon)} \subset \mathfrak{Met}(M)$ be a 1-parameter family of metrics on M depending smoothly on t and set $h = \frac{d}{dt} g_t|_{t=0}$. Show that $\frac{d}{dt}\mathrm{dv}_{g_t}|_{t=0} = \frac{1}{2}(\mathrm{tr}_g h)\,\mathrm{dv}_g$. Deduce that $\int_M \mathrm{tr}_g h \, \mathrm{dv}_g = 0$ if $\{g_t\}_{t\in(-\epsilon,\epsilon)} \subset \mathfrak{Met}_1(M)$.

(ii) The L^2–gradient $\mathrm{grad}_g\mathcal{S}$ of the Einstein–Hilbert functional $\mathcal{S}$ at $g \in \mathfrak{Met}(M)$ is defined as follows: if $\{g_t\}_{t\in(-\epsilon,\epsilon)} \subset \mathfrak{Met}(M)$ is a 1-parameter family of metrics on M with $g_t|_{t=0} = g$ and $\frac{d}{dt} g_t|_{t=0} = h$ then

$$\frac{d}{dt}\mathcal{S}(g_t)|_{t=0} = \langle \mathrm{grad}_g\mathcal{S}, h\rangle_{L^2} = \int_M \langle \mathrm{grad}_g\mathcal{S}, h\rangle_g \, \mathrm{dv}_g.$$

Show that

$$\mathrm{grad}_g\mathcal{S} = -\left(\mathrm{Ric}_g - \tfrac{1}{2}\mathrm{Scal}_g\, g\right).$$

(Hint: you can take for granted the following formula: if g_t is a smooth path in $\mathfrak{Met}(M)$ starting at g in the direction of h then

$$\frac{d}{dt}\mathrm{Scal}_{g_t}|_{t=0} = \triangle(\mathrm{tr}_g h) + d^*(\delta h) - \langle \mathrm{Ric}_g, h\rangle.)$$

(iii) Show that $g \in \mathfrak{Met}_1(M)$ is a critical point of $\mathcal{S}|_{\mathfrak{Met}_1(M)}$ if and only if there exists a function $\lambda \in C^\infty(M)$ such that $\mathrm{Ric} = \lambda g$.

(iv) For a 1-form ξ let $\delta^*\xi$ denote the symmetrisation of $\nabla\xi$, *i.e.*

$$\delta^*\xi(X,Y) = \tfrac{1}{2}\left((\nabla_X\xi)(Y) + (\nabla_Y\xi)(X)\right)$$

for every pair X, Y of vector fields.

(a) Show that $\delta^*\xi = -\frac{1}{2}\mathcal{L}_{\xi^\sharp} g$.

(b) Let $\delta\colon C^\infty(M;\mathrm{Sym}^2T^*M) \to \Omega^1(M)$ denote the formal L^2–adjoint of $\delta^*\colon \Omega^1(M) \to C^\infty(M;\mathrm{Sym}^2T^*M)$. Show that $\delta(ug) = -du$ for every function u.

(v) Use the invariance under diffeomorphisms of the Hilbert–Einstein functional to deduce that

$$\delta\mathrm{Ric} + \tfrac{1}{2}d\mathrm{Scal} = 0,$$

and use this fact to show that if $\mathrm{Ric} = \lambda g$ then λ is constant. (Hint: invariance of $\mathcal{S}$ under diffeomorphisms implies that $\delta(\mathrm{grad}_g\mathcal{S}) = 0$.)

Exercise 10. Let M be a compact reductive homogeneous space, *i.e.* $M = G/K$ for a compact Lie group G and a closed subgroup K of G and moreover there is a K–invariant decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of G in terms of the Lie algebra $\mathfrak{k}$ of K and a complementary subspace $\mathfrak{p}$. The induced action of K on $\mathfrak{p}$ is called the isotropy representation of M. You are going to study G–invariant Einstein metrics on M.

(i) Suppose that M is isotropy irreducible, *i.e.* $\mathfrak{p}$ is an irreducible representation of K. Show that there exists a unique G–invariant metric on M up to scale, which must therefore be Einstein. Apply this fact to $G = \mathrm{SU}(n+1)$ and $K = \mathrm{U}(n)$ (embedded in SU(n+1) as the stabiliser of a vector in $\mathbb{C}^{n+1}$) to construct an Einstein metric on $\mathbb{CP}^n$. (Hint: use the fact that G–invariant Riemannian metrics on M are in 1-to-1 correspondence with K–invariant positive definite inner products on $\mathfrak{p}$ and Schur's Lemma.)

(ii) Suppose there exists a closed subgroup H of G which contains K as a closed subgroup and so that $B = G/H$ and $F = H/K$ are both isotropy irreducible reductive homogeneous spaces. Then the projection $G/K \to G/H$ exhibits M as the total space of a fibre bundle over B with fibre F. Let g_B and g_F be G–invariant (Einstein) metrics on B and F and denote by s_B and

s_F their respective scalar curvatures. Further assume that any G–invariant metric on M can be written up to scale as

$$g_t = g_B + tg_F, \qquad t \in \mathbb{R}_+.$$

(This happens if and only if the two summands $\mathfrak{p}_F$ and $\mathfrak{p}_B$ in the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_B = (\mathfrak{k} \oplus \mathfrak{p}_F) \oplus \mathfrak{p}_B$ are non-isomorphic irreducible K–representations.) Restricting the normalised Hilbert–Einstein functional to G–invariant metrics we obtain a function $\mathcal{S}\colon \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\frac{\mathcal{S}(g_t)}{\mathrm{Vol}(M,g_t)^{\frac{n-2}{n}}} \propto t^{\frac{\dim F}{n}} \left(\tfrac{1}{t}\mathrm{s}_F + \mathrm{s}_B - t|\mathrm{curv}|^2\right) =: \mathcal{S}(t).$$

Here curv is the curvature of the connection on the principal H–bundle $G \to B$ induced by the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_B$. By Palais' Principle of Symmetric Criticality, g_t is Einstein if and only if $\mathcal{S}'(t) = 0$.

You are going to apply this facts to produce an Einstein metric on S^7 that does not have constant curvature. Consider $K \subset H \subset G$ with $K = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$, $H = \mathrm{Sp}(1)$ and $G = \mathrm{Sp}(2)$.

(a) Show that $M = S^7$, $B = S^4$ and $F = S^3$. (Hint: you might want to use the double covers $\mathrm{Sp}(2) \to \mathrm{SO}(5)$ and $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(4)$.)

(b) Verify that B and F are isotropy irreducible homogeneous spaces. Normalise the resulting Einstein (constant curvature) metrics so that $s_B = 12$ and $s_F = 6$.

(c) Calculate $|\mathrm{curv}|^2$. (Hint: you can use the fact that g_1 is the standard round metric on $\mathbb{S}^7$ with scalar curvature 42.)

(d) Deduce the existence of a critical point $t_* \neq 1$ of $\mathcal{S}(t)$.

(e) Show that the Einstein metric g_{t_*} does not have constant curvature.

(f) Show that g_1 and g_{t_*}, normalised to have the same volume, cannot be connected by a path of Einstein metrics. (Hint: compare the values of the Hilbert–Einstein functional.)

Exercise 11. Show that $\mathrm{Sp}(m)$ is a compact, connected and simply connected Lie group. Calculate its dimension. (Hint: use the fact that $\mathrm{Sp}(1) = \mathrm{SU}(2) \simeq S^3$ and $S^{4m-1} = \mathrm{Sp}(m)/\mathrm{Sp}(m-1)$.)

Exercise 12. This exercise is about hyperkähler structures.

(i) Let $(\omega_1, \omega_2, \omega_3)$ be an $\mathrm{Sp}(m)$–structure on M^{4m}. Observe that $\mathrm{Sp}(m) \subseteq \mathrm{SU}(2m)$ (with equality only if $m = 1$) and use this to explain why the triple $(\omega_1, \omega_2, \omega_3)$ determines (g, J_1, J_2, J_3). (Hint: what about ω_1 and the complex $2m$–form $\Omega = \frac{1}{m!}(\omega_2 + i\omega_3)^m$?)

(ii) Suppose that $(M, \omega_1, \omega_2, \omega_3)$ is torsion-free. For every $\boldsymbol{a} = (a_1, a_2, a_3)$ define $\omega_{\boldsymbol{a}} = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$ and $J_{\boldsymbol{a}} = a_1 J_1 + a_2 J_2 + a_3 J_3$. Show that $(M, g, J_{\boldsymbol{a}}, \omega_{\boldsymbol{a}})$ is Kähler.

Exercise 13. Let V be a 4–dimensional vector space endowed with a positive definite inner product and a volume form $\mathrm{dv} \in \Lambda^4 V^*$.

(i) Using dv and the wedge product define a non-degenerate pairing q on $\Lambda^2 V^*$. Show that q has signature $(3,3)$. Let $\Lambda^\pm V^*$ be maximal positive/negative subspaces of $(\Lambda^2 V^*, q)$.

(ii) Show that the induced action of $\mathrm{SL}(V) \simeq \mathrm{SL}(4,\mathbb{R})$ (*i.e.* the matrices that preserve dv) on $\Lambda^2 V^*$ defines a double cover $\mathrm{SL}(4,\mathbb{R}) \to \mathrm{SO}(3,3)$. Restricting to compact subgroups, we see that $\mathrm{SO}(4) \to \mathrm{SO}(3)^+ \times \mathrm{SO}(3)^-$ is a double-cover; here $\mathrm{SO}(3)^\pm$ is the induced action of $\mathrm{SO}(4)$ on $\Lambda^\pm V^*$.

(iii) Identify V with the quaternions $\mathbb{H}$ and $\mathrm{SU}(2)$ with the unit sphere $\mathbb{S}^3 \subset \mathbb{H}$. Define a map $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbb{H}$, by $(q_1, q_2, x) \mapsto q_1 x \overline{q_2}$. Show that this defines a double cover $\mathrm{SU}(2)^+ \times \mathrm{SU}(2)^- \to \mathrm{SO}(4)$.

(iv) Show that this induces a double cover of $\mathrm{U}(1) \times \mathrm{SU}(2)^- \to \mathrm{U}(2)$, where $\mathrm{U}(1) \subset \mathrm{SU}(2)^+$ is the subgroup of diagonal matrices.

(v) Show that $\mathrm{U}(2)$ acts on $\Lambda^- V^*$ as $\mathrm{SO}(3)^-$ and on $\Lambda^+ V^*$ as the subgroup $\mathrm{SO}(2) \subset \mathrm{SO}(3)^+$ preserving the standard Kähler form ω_1 on $\mathbb{H} \simeq \mathbb{C}^2$.

(vi) Deduce that on a Kähler surface (M, ω), $\Lambda^+ M = [\![\Lambda^{2,0} M]\!] \oplus \mathbb{R}\omega$ and $\Lambda^- M = [\![\Lambda^{1,1}_0 M]\!]$, where $\Lambda^{1,1}_0 M$ are the $(1,1)$-forms on $T_x M$ orthogonal to ω.

Exercise 14. Work out all the possibilities for the holonomy group of a simply connected Ricci-flat manifold of real dimension 8. (Hint: the only simply connected Ricci-flat symmetric space is $\mathbb{R}^n$.)

Exercise 15. Let (M^6, ω, Ω) be a Calabi–Yau 3-fold and denote by ψ the defining parallel spinor.

(i) Show that $(f, g, \gamma) \mapsto f\psi + g\,\mathrm{dv}\cdot\psi + \gamma\cdot\psi$ identifies the spinor bundle of M with $\Lambda^0 T^*M \oplus \Lambda^0 T^*M \oplus \Lambda^1 T^*M$.

(ii) Show that
$$\not{D}(f\psi + g\,\mathrm{dv}\cdot\psi) = (df + Jdg)\cdot\psi$$
where the complex structure J on 1-forms is defined by $(J\gamma)\cdot\psi = \gamma\cdot\mathrm{dv}\cdot\psi = -\,\mathrm{dv}\cdot\gamma\cdot\psi$. (Hint: observe that $\not{D}\psi = 0 = \not{D}(\mathrm{dv}\cdot\psi)$.)

(iii) Show that $\not{D}(\gamma\cdot\psi) = d\gamma\cdot\psi + (d^*\gamma)\,\psi$.

(iv) It remains to understand the action of the 2-form $d\gamma$ on ψ via Clifford multiplication. This requires various steps. Consider the decomposition of 2-forms $\Lambda^2 = \Lambda^2_1 \oplus \Lambda^2_6 \oplus \Lambda^2_8$ into irreducible $\mathrm{SU}(3)$–representations, where
$$\Lambda^2_1 = \mathbb{R}\omega, \qquad \Lambda^2_6 = \{X \lrcorner \operatorname{Re}\Omega \mid X \text{ a vector field}\},$$
and Λ^2_8 consists of primitive $(1,1)$–forms, *i.e.* 2-forms σ such that $\sigma\wedge\omega^2 = 0 = \sigma\wedge\operatorname{Re}\Omega$ or equivalently $\sigma\cdot\psi = 0$.

(a) Show that $\omega\cdot\psi = 3\,\mathrm{dv}\cdot\psi$. (Hint: do the computation in an orthonormal coframe $\{e_1, Je_1, \dots, e_3, Je_3\}$ such that $\omega = \sum_{i=1}^3 e_i \wedge Je_i$.)

(b) Show that $\frac{1}{2} d\gamma\wedge\omega^2 = -d^*(J\gamma)$, *i.e.* the projection of $d\gamma$ in Λ^2_1 is $-\frac{1}{3}d^*(J\gamma)\,\omega$.

(c) Show that a 2-form $\sigma = X\lrcorner\operatorname{Re}\Omega \in \Lambda^2_6$ acts on ψ by $\sigma\cdot\psi = *(\sigma\wedge\operatorname{Re}\Omega)\cdot\psi$. (Hint: first argue that $\sigma\cdot\psi$ must be of the form $\eta\cdot\psi$ for some 1-form η; next using Schur's Lemma deduce that $\eta = c * (\sigma\wedge\operatorname{Re}\Omega)$ for some constant c independent of σ and (M, ω, Ω); finally calculate c by considering the standard Calabi–Yau structure on $\mathbb{C}^3$ and X the first vector in the standard orthonormal basis.)

(v) Deduce that the Dirac operator $\not{D}$ of M can be identified with the operator
$$\not{D}(f, g, \gamma) = (d^*\gamma, d^*(J\gamma), \operatorname{curl}\gamma + df + Jdg)\,,$$
where $\operatorname{curl}\gamma = *(d\gamma\wedge\operatorname{Re}\Omega)$ for every 1-form γ.

1.5. **Bibliographical notes.** Our presentation of Riemannian holonomy groups and Berger's list is based on [4, Chapter 10], [9], [13] and the survey paper [5]. For background on spin structures and the Dirac operator see [2] and [10].

2. Kähler Ricci-flat (KRF) metrics

2.1. The Calabi–Yau Theorem: the existence and uniqueness problem.

Ricci-curvature in Kähler geometry. Want to explain why

$$\tfrac{1}{n!}\,\omega^n = \frac{(-1)^{\frac{m(m-1)}{2}}\,i^m}{2^m}\,\Omega\wedge\overline{\Omega} \tag{2.1}$$

$\Longrightarrow \mathrm{Ric}(g_{\omega,\Omega})=0$

- Chern connection: $E\to M$ holomorphic vector bundle over complex manifold (M,J) + h Hermitian metric on $E \rightsquigarrow \exists!$ connection ∇ on E such that $\nabla^{0,1}=\overline{\partial}_E$ and $\nabla h=0$
  Proof. (in the case where $E=L$ is a line bundle)
  Over $U\subset X$ open choose local trivialising holomorphic section s
  Write $\nabla=d+\alpha$ and $\|s\|_h^2=e^{2\varphi}s\overline{s}$
    - $\alpha=\alpha^{1,0}$
    - $d\varphi=\frac{1}{2}(\alpha+\overline{\alpha})$

  $\Longrightarrow \alpha=2\partial\varphi$ □

Exercise 16

- first Chern class: $c_1(E)=c_1(\det E)=\frac{i}{2\pi}[\mathrm{tr}(F_\nabla)]\in H^2(X;\mathbb{R})$
- Prescribing curvature of a line bundle
  - fix (L,h_0) and closed $(1,1)$-form $\rho\in -2\pi i c_1(L)$
  - $h=e^{2\varphi}h_0$ with $F_h=\rho \Longleftrightarrow 2\overline{\partial}\partial\varphi=\rho-F_{h_0}$
  - on a compact Kähler manifold can always solve this thanks to $\overline{\partial}\partial$–Lemma: σ exact $(1,1)$–form $\Longrightarrow \sigma=\overline{\partial}\partial u$ for some function u
- Also by $\overline{\partial}\partial$–Lemma: every Kähler metric in a fixed Kähler class $[\omega]\in H^{1,1}(M;\mathbb{R})$ is of the form $\omega_u=\omega+i\overline{\partial}\partial u$

Exercises 17 and 18

- Ricci-form: $(1,1)$–form $\rho_\omega(X,Y)=\mathrm{Ric}(JX,Y)$
- $T^{1,0}M$ holomorphic bundle + Hermitian metric $h=g+i\omega$

$\rightsquigarrow R=$ curvature of Chern connection and $\mathrm{Tr}\,R$ curvature of K_M^{-1}

- $\rho_\omega=i\mathrm{Tr}\,R$

Exercise 19

The Calabi Conjecture.

Theorem 2.2 (Yau, 1978)**.** *Let (M^{2m},g,J,ω) be a closed Kähler manifold. Then for every $f\in C^\infty(M)$ such that $f>0$ and $\int_M f\,\omega^m=\int_M\omega^m$ there exists $u\in C^\infty(M)$, unique up to the addition of a constant, such that $\omega+i\partial\overline{\partial}u$ is a Kähler form in the same Kähler class as ω and*

$$\left(\omega+i\partial\overline{\partial}u\right)^m=f\,\omega^m.$$

Corollary 2.3. *Let (M^{2m},J) be a closed complex manifold with $c_1(M,J)=0$ and admitting Kähler metrics. Then in every Kähler class there exists a unique Kähler metric that is Ricci-flat.*

Exercises 20 and 21

2.2. The Beauville–Bogomolov decomposition: when is $\mathbf{Hol}(g)=\mathbf{SU}(m)$ or $\mathbf{Sp}(m)$?

The Cheeger–Gromoll Splitting Theorem.

Exercise 22

- ray: unit-speed geodesic $\gamma\colon[0,\infty)\to M$ such that $\mathrm{dist}_g(\gamma(t),\gamma(s))=|s-t|$ for all $t,s\geq 0$
- line: unit-speed geodesic $\gamma\colon\mathbb{R}\to M$ such that $\mathrm{dist}_g(\gamma(t),\gamma(s))=|s-t|$ for all $t,s\in\mathbb{R}$
- (M^n,g) complete non-compact: for every point $p\in M$ $\exists$ ray γ with $\gamma(0)=p$
- (M^n,g) complete non-compact and $\pi_0(M\setminus K)\geq 2$ for connected compact K: $\exists$ line

Exercise 23

Theorem 2.4 (Cheeger–Gromoll, 1971)**.** *Let (M^n, g) be a complete Riemannian manifold with* $\mathrm{Ric}(g) \geq 0$*. If M contains a line then $(M, g) \simeq (N^{n-1}, g_N) \times (\mathbb{R}, dt^2)$.*

Corollary 2.5. *Let (M^n, g) be a closed Riemannian manifold with* $\mathrm{Ric}(g) \geq 0$*. Then the universal cover $(\widetilde{M}, \widetilde{g})$ of (M, g) splits isometrically as the product of $(\mathbb{R}^k, g_{Eucl})$ and a compact Riemannian manifold (N^{n-k}, g_N). If* $\mathrm{Ric}(g) = 0$ *then there exists a finite cover M' that splits isometrically as the product of a flat k-torus T^k and the compact manifold N.*

Proof. There are two main steps in the proof.

(i) If M is compact and its universal cover $\widetilde{M}$ is non-compact, then $\widetilde{M}$ contains a line: one starts with a ray γ and constructs a line as a limit of translations $f_i \cdot \gamma(t_i + \cdot)$ for $t_i \to \infty$ and elements $f_i \in \pi_1(M)$.

(ii) If $\widetilde{M}$ splits as an isometric product $\widetilde{M} = N \times \mathbb{R}^k$ with N containing no lines, then $\mathrm{Isom}(\widetilde{M}) = \mathrm{Isom}(N) \times \mathrm{Isom}(\mathbb{R}^k)$ (since isometries must preserve lines).

Theorem 2.4 implies that $\widetilde{M}$ splits as in (ii). If by contraddicition N were non-compact then the construction in (i) would yield a line in $\widetilde{M}$ starting from a ray in N, which would then be contained in N by (ii).

For the final statement, we need to observe that a simply connected closed Riemannian manifold with $\mathrm{Ric} \leq 0$ has no Killing fields (since any Killing field X satisfies $\nabla^*\nabla X - 2\mathrm{Ric}(X, X) = 0$ and therefore the dual 1-form is harmonic), so the subgroup of $\mathrm{Isom}(\mathbb{R}^k)$ given by the kernel of $\pi_1(M) \to \mathrm{Isom}(N)$ must act cocompactly on $\mathbb{R}^k$: by Bieberbach Theorem it must therefore contain a full rank lattice as a finite index subgroup. ☐

Exercise 24

The Kähler case.

Theorem 2.6 (Beauville, 1983)**.** *Let (M^{2m}, g, J, ω) be a closed Kähler manifold with* $\mathrm{Ric}(g) = 0$*.*

(i) *The universal cover $\widetilde{M}$ is isomorphic as a Kähler manifold to the product $\widetilde{M} = \mathbb{C}^k \times \prod_i X_i \times \prod_j Y_j$, where X_i is a compact simply connected manifold with holonomy* $\mathrm{SU}(m_i)$ *and Y_j is a compact simply connected manifold with holonomy* $\mathrm{Sp}(n_j)$*. The decomposition is unique up to reordering the factors.*

(ii) *There exists a finite cover M' of M isomorphic as a Kähler manifold to the product $M' = \mathbb{T} \times \prod_i X_i \times \prod_j Y_j$, where $\mathbb{T}^k$ is a compact complex torus of complex dimension k. In particular, $\pi_1(M)$ is an extension of a finite group by $\mathbb{Z}^{2k}$.*

Exercises 25 and 26

2.3. **Moduli spaces.**

- (M^{2m}, ω, Ω) Calabi–Yau m-fold, *i.e.* $\mathrm{Hol}(g_{\omega,\Omega}) \subseteq \mathrm{SU}(m)$
- Deformations of (ω, Ω) as a CY structure

Complex-structure deformations: Bogomolov 1978, Tian 1988, Todorov 1989

Theorem 2.7. *Let (M^{2m}, J, ω) is a closed Kähler manifold with $c_1(M, J) = 0$. Then the local universal deformation space of (M, J) is isomorphic to an open set in $H^1(M; T^{1,0}M)$.*

Combined with Corollary 2.2 and a result of Kodaira that guarantees that complex-structure deformations of a Kähler manifold remain Kähler $\Longrightarrow$

Corollary 2.8. *Let (M^{2m}, g, J, ω) be a closed Kähler Ricci-flat manifold. Then the moduli space of Kähler Ricci-flat metrics on M is smooth of dimension*

$$h^{1,1}(M) + 2 \dim_{\mathbb{C}} H^1(M; T^{1,0}M) - 2h^{0,2}(M).$$

Note: K_M trivial $\Longrightarrow \dim_{\mathbb{C}} H^1(M; T^{1,0}M) = h^{1,m-1}(M)$ via $X \mapsto X \lrcorner \Omega$

Exercise 27

The case $m=2$: deforming hyperkähler triples.

- (M^4,μ_0) oriented 4-manifold $\rightsquigarrow$ quadratic form on $\Lambda^2 T^*_xM$ for all $x\in M$
- $\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3)$ definite triple if $\mathrm{Span}(\boldsymbol{\omega}_x)\subset\Lambda^2T^*_xM$ positive definite
  $\boldsymbol{\omega}$ definite $\Longleftrightarrow Q>0$, where $Q_{ij}\mu_0=\frac{1}{2}\omega_i\wedge\omega_j$
- normalisation: $(Q_{\boldsymbol{\omega}},\mu_{\boldsymbol{\omega}})$ so that $Q_{\boldsymbol{\omega}}\mu_{\boldsymbol{\omega}}=Q\mu_0$ and $\det Q_{\boldsymbol{\omega}}\equiv 1$
- $g_{\boldsymbol{\omega}}$ unique metric with $\Lambda^+T^*M=\mathrm{Span}(\boldsymbol{\omega})$ and volume form $\mu_{\boldsymbol{\omega}}$
- $g_{\boldsymbol{\omega}}$ hyperkähler $\Longleftrightarrow Q_{\boldsymbol{\omega}}\equiv\mathrm{id}$ and $d\omega_i=0$ for all $i=1,2,3$
- deform hyperkähler triple $\boldsymbol{\omega}\rightsquigarrow\boldsymbol{\omega}+\boldsymbol{\eta}$ such that $d\boldsymbol{\eta}=0$ and, for some $v>0$,

$$\frac{1}{2}(\omega_i+\eta_i)\wedge(\omega_j+\eta_j)=v\,\delta_{ij}\,\mu_{\boldsymbol{\omega}} \tag{2.9}$$

- decompose $\boldsymbol{\eta}=\boldsymbol{\eta}^++\boldsymbol{\eta}^-$
- $\eta_i^+=\sum_{j=1}^3 A_{ij}\,\omega_j$ for matrix-valued function A
- $\boldsymbol{\eta}^-*\boldsymbol{\eta}^-$ the symmetric (3×3)–matrix with entries $(\frac{1}{2}\,\eta_i^-\wedge\eta_j^-)/\mu_{\boldsymbol{\omega}}$
- rewriting of (2.9)

$$A^T+A+AA^T+\boldsymbol{\eta}^-*\boldsymbol{\eta}^-=0.$$

- smooth function $\mathcal{F}\colon\ Sym^2(\mathbb{R}^3)\to Sym^2(\mathbb{R}^3)$ such that $A^T+A+AA^T=S$ iff $A=\mathcal{F}(S)$
  3-dimensional kernel of linearisation of $A\mapsto A^T+A+AA^T$ at $0\overset{1:1}{\longleftrightarrow}$ HK rotations
- reformulation of (2.9)

$$\boldsymbol{\eta}^+=\mathcal{F}\left((\mathrm{id}-Q_{\boldsymbol{\omega}})-\boldsymbol{\eta}^-*\boldsymbol{\eta}^-\right)$$

- $\mathcal{H}^+_{\boldsymbol{\omega}}=\mathrm{Span}(\boldsymbol{\omega})$

$\Longrightarrow$ elliptic equation

$$d^+\boldsymbol{a}+\boldsymbol{\zeta}=\mathcal{F}\left((\mathrm{id}-Q_{\boldsymbol{\omega}})-\boldsymbol{\eta}^-*\boldsymbol{\eta}^-\right)\qquad d^*\boldsymbol{a}=0$$

for a triple $\boldsymbol{a}$ of 1-forms on M and a triple $\boldsymbol{\zeta}\in\mathcal{H}^+_{\boldsymbol{\omega}}\otimes\mathbb{R}^3$

- linearisation

$$(D\oplus\mathrm{id})\otimes\mathbb{R}^3\colon\ \Big(\Omega^1(M)\oplus\mathcal{H}^+_{\boldsymbol{\omega}}\Big)\otimes\mathbb{R}^3\to\Big(\Omega^0(M)\oplus\Omega^+(M)\Big)\otimes\mathbb{R}^3$$

$$D\colon\ (\boldsymbol{a},\boldsymbol{\zeta})\longmapsto(d^*\boldsymbol{a},d^+\boldsymbol{a}+\boldsymbol{\zeta})$$

surjective map with kernel harmonic 1-forms

Conclusion

- M^4 closed, simply connected
- $\mathfrak{C}(M)=\{\boldsymbol{\omega}\text{ hyperkähler triple}\}$
- $\mathfrak{Diff}_0(M)$ acts on $\mathfrak{C}(M)$
- $\mathfrak{M}=\mathfrak{C}(M)/\mathfrak{Diff}_0(M)$

Theorem 2.10. *If non-empty $\mathfrak{M}$ is a smooth manifold of dimension $3b_2^-(M)+3+1$ and the projection $\pi\colon\ \mathfrak{M}\to H^2(M)\otimes\mathbb{R}^3$ induced by $\boldsymbol{\omega}\mapsto[\boldsymbol{\omega}]$ is an immersion.*

- M^4 HK simply connected $\rightsquigarrow M$ diffeomorphic to quartic surface in $\mathbb{CP}^3$, the "K3 manifold"
- fix isomorphism $H^2(M;\mathbb{Z})\simeq(\Lambda,q)$ unimodular lattice of signature $(3,19)$
- $\lambda\in\Lambda$ is a root $\Longleftrightarrow q(\lambda,\lambda)=-2$
- $\mathcal{Q}=\{\boldsymbol{\alpha}\in\Lambda\otimes\mathbb{R}^3\,|\,q(\alpha_i,\alpha_j)=v\,\delta_{ij}\text{ for some }v>0\text{ and }q(\boldsymbol{\alpha},\lambda)\neq\mathbf{0}\text{ for all root }\lambda\}$
- Torelli Theorem: $\pi\colon\ \mathfrak{M}\to\mathcal{Q}$ is a diffeomorphism

Exercises 28 and 29

2.4. **Exercises.**

Exercise 16. Let L be a holomorphic line bundle over a complex manifold M. Let h be a Hermitian metric on L and denote by F_h the curvature of the Chern connection of (L,h).

(i) Show that F_h is a closed $(1,1)$–form.

(ii) Show that $i[F_h]\in H^2(M;\mathbb{R})$ is independent of h. Up to a factor of 2π, $i[F_h]=2\pi c_1(L)$ is called the (real) first Chern class of L.

(iii) Think of $\mathbb{CP}^1$ as parametrising lines in $\mathbb{C}^2$. Let $\mathbb{CP}^1 \times \mathbb{C}^2 \to \mathbb{CP}^1$ be the trivial rank 2 bundle and denote by $\mathcal{O}(-1)$ the holomorphic subbundle whose fibre over $[z] \in \mathbb{CP}^1$ is the line in $\mathbb{C}^2$ parametrised by $[z]$. Calculate $c_1(\mathcal{O}(-1))$.

Exercise 17. Let (M^{2m}, J, ω) be a closed Kähler manifold. You are going to prove the $\overline{\partial}\partial$–Lemma for $(1,1)$–forms.

(i) First observe that $2i\overline{\partial}\partial u = d(Jdu)$ and therefore we need to prove that every exact $(1,1)$–form σ on M can be written in the form $\sigma = d(Jdu)$ for some function u.
(ii) Show that you can find a function u, unique up to the addition of a constant, such that $d(Jdu) \wedge \omega^{m-1} = \sigma \wedge \omega^{m-1}$. (Hint: note that $d(Jdu) \wedge \omega^{m-1} = m \,\triangle u\, \omega^m$.)
(iii) Consider now the exact primitive $(1,1)$-form $\sigma' = \sigma - d(Jdu)$. Here primitive means that $\sigma' \wedge \omega^{n-1} = 0$. Show that if $d^*\sigma' = 0$ then $\sigma' = 0$. (Hint: here you need to use the fact that M is closed.)
(iv) Show that there exists a universal constant $c_m \neq 0$ such that if σ' is a primitive $(1,1)$-form then $*\sigma' = c_m \sigma' \wedge \omega^{m-2}$. (Hint: consider the map $\sigma' \mapsto *(\sigma' \wedge \omega^{m-2})$ and argue that it must be a non-zero multiple of the identity.)
(v) Conclude the proof.

Exercise 18. Let Ω be a (local) holomorphic volume form on a Kähler manifold (M^{2m}, ω, J) (if Ω is globally defined then we must have $c_1(M,J) = 0$). We can therefore think of Ω as a (local) trivialising holomorphic section of the canonical line bundle $K_M = \Lambda^{(m,0)}T^*M$. The Kähler metric $g_{\omega,J}$ induces a Hermitian metric h_ω on K_M.

(i) Show that $\|\Omega\|^2_{h_\omega} \frac{1}{m!}\omega^m = c_m \Omega \wedge \overline{\Omega}$, where c_m is the dimensional constant of (2.1).
(ii) Deduce that equation (2.1) implies that the curvature of the Chern connection of (K_M, h_ω) vanishes (over the set where Ω is defined).
(iii) Conversely, show that if K_M is holomorphically trivial and the Chern connection on K_M is flat then (2.1) must hold for a suitably chosen trivialising holomorphic section Ω of K_M.

Exercise 19. Let (M, g, J, ω) be a Kähler manifold. Let R be the curvature of the Chern connection of $T^{1,0}M$ endowed with the Hermitian metric $h = g + i\omega$.

For computations it will be convenient to introduce a local orthonormal frame $\{E_1, JE_1, \dots, E_m, JE_m\}$ adapted to the $\mathrm{U}(m)$–structure.

As usual, we introduce the notation $\mathrm{Rm}(X,Y,Z,W) = \langle \mathrm{Rm}(X,Y)W, Z\rangle$ for the curvature of g. Recall the symmetries of the Riemannian curvature: Rm is skew-symmetric in (X,Y) and (Z,W), invariant under exchange of the pairs (X,Y) and (Z,W) and (the Bianchi identity)

$$R(X,Y,\,\cdot\,,W) + R(Y,W,\,\cdot\,,X) + R(W,X,\,\cdot\,,Y) \equiv 0.$$

Finally, the Ricci curvature of g is defined by

$$\mathrm{Ric}(X,Y) = \sum_{j=1}^m \mathrm{Rm}(E_j, X, E_j, Y) + \mathrm{Rm}(JE_j, X, JE_j, Y).$$

(i) Show that $\mathrm{Rm}(X,Y,Z,JW) = -\mathrm{Rm}(X,Y,JZ,W)$.
(ii) Show that the Ricci curvature of g satisfies

$$\mathrm{Ric}(X,Y) = \sum_{j=1}^m \mathrm{Rm}(E_j, JE_j, X, JY)$$

(Hint: use (i) to rewrite $\mathrm{Rm}(E_j, X, E_j, Y) = \mathrm{Rm}(E_j, X, JE_j, JY)$ and $\mathrm{Rm}(JE_j, X, JE_j, Y) = -\mathrm{Rm}(JE_j, X, E_j, JY)$, then rearrange terms so that JY is in the third position and finally use the Bianchi identity.)
(iii) Deduce that $\mathrm{Ric}(JX, JY) = \mathrm{Ric}(X,Y)$ and therefore $\rho_\omega(X,Y) = \mathrm{Ric}(JX,Y)$ is a $(1,1)$–form.
(iv) Show that $i\mathrm{tr}\, R(X,Y) = \sum_{i=1}^m \mathrm{Rm}(X,Y,E_j,JE_j)$. (Hint: note that R is the curvature of g thought of as a 2-form with value in the bundle of skew-Hermitian endomorphisms of TM; now

write the trace of a matrix in $\mathfrak{u}(m) \subset \mathfrak{gl}(n, \mathbb{C})$ in terms of the basis $\{E_1, JE_1, \ldots, E_m, JE_m\}$ of $\mathbb{R}^{2m} \simeq \mathbb{C}^m$.)

(v) Deduce that $\rho_\omega = i\mathrm{tr}\, R$.

Exercise 20. Prove Corollary 2.3 assuming Theorem 2.2.

Exercise 21. Let X be a complex manifold of complex dimension $m+1$, $m \geq 2$, such that the anticanonical bundle K_X^{-1} is ample. By the Kodaira Embedding Theorem, we can assume this is equivalent to the existence of a Hermitian metric on K_X^{-1} (not necessarily induced by a a Kähler metric on X) with curvature F_h that is a Kähler form on X. Consider a smooth anticanonical divisor $M \in |-K_X|$.

(i) Show that X has finite fundamental group. (Hint: use Theorem 2.2 to prove that X admits a Kähler metric with positive Ricci curvature.)

(ii) Since K_X^{-1} is ample, the Kodaira Vanishing Theorem says that $h^{p,0}(X) = 0$ for all $p > 1$ and therefore the holomorphic Euler characteristic $\chi(X, \mathcal{O}_X) := \sum_{p=0}^{m+1} (-1)^p h^{0,p}(X) = 1$. Deduce that X is simply connected. (Hint: look at how the holomorphic characteristic behaves under finite coverings.)

(iii) Use the Lefschetz Hyperplane Theorem to deduce that M is also simply connected.

(iv) Use the Adjunction Formula to show that K_M is trivial.

(vi) Use the exact sequence

$$0 \to K_X \to \mathcal{O}_X \to \mathcal{O}_M \to 0$$

and the fact that $H^i(X, K_X) = 0$ for all $i \leq m$ (by the Nakano Vanishing Theorem) to show that $h^{p,0}(M) = 0$ for all $0 < p < m$.

(v) Deduce that M admits a metric g with $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$. In fact, in view of Exercise 26 we have $\mathrm{Hol}(g) = \mathrm{SU}(m)$.

(vii) Justify the fact that a hypersurface of degree $m+2$ in $\mathbb{CP}^{m+1}$ admits a Calabi–Yau metric.

Exercise 22. In this exercise you dicuss Bochner's theorem about harmonic 1-forms on closed manifolds (M^n, g) with non-negative Ricci curvature.

(i) Show that $\triangle\gamma = \nabla^*\nabla\gamma + \mathrm{Ric}(\gamma^\sharp)^\flat$ for every 1-form γ.

(ii) (Bochner, 1948) Suppose that M is closed and $\mathrm{Ric}(g) \geq 0$. If γ is a harmonic 1-form then $X = \gamma^\sharp$ is a parallel vector field such that $\mathrm{Ric}(X) = 0$.

(iii) Assume that $\widetilde{X}$ is a parallel vector field on a simply connected manifold $(\widetilde{M}, \widetilde{g})$. Show that $(\widetilde{M}, \widetilde{g}) \simeq (N, g_N) \times (\mathbb{R}, dt^2)$.

(iv) Let (M, g) be a closed manifold with $\mathrm{Ric} \geq 0$ and $b_1(M) = k$. Show that the universal cover $(\widetilde{M}, \widetilde{g})$ of M splits isometrically as $(\widetilde{M}, \widetilde{g}) \simeq (N, g_N) \times (\mathbb{R}^k, g_{Eucl})$.

Exercise 23. Let (M^n, g) be a complete non-compact Riemannian manifold and fix $p \in M$. Show that there exists a ray $\gamma\colon [0, \infty) \to M$ such that $\gamma(0) = p$ and that if M has at least two ends then there exists a line. (Hint: for the ray consider geodesics γ_i joining p and a sequence of points $q_i \to \infty$; for the line consider geodesics joining points $p_i^-, p_i^+ \to \infty$ belonging to two distinct connected components of $M \setminus K$.)

Exercise 24. In this exercise we consider consequences of Theorem 2.4 and Corollary 2.5.

(i) Show that $S^2 \times S^1$ and $S^3 \times S^1$ cannot carry Ricci-flat metrics.

(ii) Let (M^n, g) be a closed manifold with $\mathrm{Ric}(g) \geq 0$. Show that if $\mathrm{Hol}(g)$ acts irreducibly on $\mathbb{R}^n$ then $\pi_1(M)$ is finite.

(iii) Let (M^7, φ) be a closed 7-manifold with a torsion-free G_2–structure, so that $\mathrm{Hol}(g_\varphi) \subseteq \mathrm{G}_2$. Show that $\mathrm{Hol}(g) = \mathrm{G}_2$ if and only if $\pi_1(M)$ is finite. (Hint: given the above, the only new ingredient is to establish, going through Berger's list and using the fact that the only symmetric Ricci-flat metrics are flat, that if M is simply connected and has $\mathrm{Hol}(g)$ strictly contained in G_2 then there must be a parallel 1-form.)

Exercise 25. Prove part (i) of Theorem 2.6 using the de Rham Theorem, Berger's list and Theorem 2.5.

Exercise 26. Let (M^{2m}, g, J, ω) be a closed Kähler manifold with $\mathrm{Ric}(g) = 0$. In this exercises we develop vanishing theorems for holomorphic $(p, 0)$-forms on M and derive some consequences.

(i) Prove that every holomorphic $(1, 0)$-form is parallel. (Hint: use the fact that $\overline{\partial}^*|_{\Lambda^{p,0}} = 0$, $\triangle_{\overline{\partial}} = 2\triangle_d$ and part (i) of Exercise 22.)
(ii) Prove that the automorphism group of (M, J) is discrete and that the isometry group of (M, g) is therefore finite. (Hint: the Lie algebra of the automorphism group consists of holomorphic sections of $T^{1,0}M$; the automorphism group is the complexification of the isometry group.)
(iii) Prove part (ii) of Theorem 2.6. (Hint: every element of $\pi_1(M)$ must act on $\widetilde{M}$ as a product of isometric automorphisms of each factor in the decomposition.)
(iv) Prove that every holomorphic $(p, 0)$-form is parallel for every $p \geq 0$. (Hint: the argument is the same as in part (i), using the fact that $\triangle_d = \nabla^*\nabla$ on $\Lambda^{p,0}$ if g is Ricci-flat.)
(v) Prove that $\mathrm{Hol}(g) = \mathrm{SU}(m)$ if and only if $h^{p,0}(M) = 0$ for all $0 < p < m$ and $h^{p,0}(M) = 1$ if $p = 0, m$.
(vi) Suppose that $\mathrm{Hol}(g) = \mathrm{SU}(m)$ and $m \geq 3$. Then (M, J) is projective. (Hint: show that the Kähler cone must contain a rational class and then use the Kodaira Embedding Theorem.)
(vii) Let (M^{2m}, J) be a closed complex manifold admitting Kähler metrics. We say that M is an irreducible holomorphic symplectic variety if there exists a holomorphic $(2, 0)$–form ω_c that is non-degenerate at every point and if $h^{p,0}(M) = 0$ if p is odd, while every holomorphic $(2k, 0)$-form is a constant multiple of ω_c^k. Prove that (M^{4m}, g, J, ω) has $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ if and only if (M, J) is an irreducible holomorphic symplectic manifold.

Exercise 27. Let M be a smooth hypersurface in $\mathbb{CP}^{m+1}$ of degree $m + 2$, $m \geq 3$. It follows from Exercise 21 and Corollary 2.3 that M carries a unique Kähler Ricci-flat metric g in the cohomology class of the restriction of the Fubini–Study metric.

(i) Show that $\mathrm{Hol}(g) = \mathrm{SU}(m)$. (Hint: use Exercises 21 and 26.)
(ii) Use the Lefschetz Hyperplane Theorem to show that $b_k(M) = b_k(\mathbb{CP}^{m+1})$ for all $k \neq m$.
(iii) Use the exact sequence

$$0 \to TM \to T\mathbb{CP}^{m+1}|_M \to \mathcal{O}_{\mathbb{CP}^{m+1}}(m + 1)|_M \to 0$$

to derive the recursive formula $c_k(M) + (m + 1)h \cup c_{k-1}(M) = \binom{m+1}{k}h^k$, where h is the generator of $H^2(\mathbb{CP}^{m+1}; \mathbb{Z}) \simeq H^2(M; \mathbb{Z})$.
(iv) Specialising to $m = 3$ for simplicity, find a formula for the top Chern class $c_m(M)$ and then calculate $b_m(M)$. (Hint: the top Chern class yields the Euler characteristic of M.)
(v) Use Corollary 2.8 to calculate the dimension of the moduli space of Kähler Ricci-flat metrics on M when $m = 3$.

Exercise 28. Let M be a quartic surface in $\mathbb{CP}^3$.

(i) Consider the exact sequence

$$0 \to TM \to T\mathbb{CP}^3|_M \to \mathcal{O}(4)|_M \to 0.$$

  (a) Taking determinants, show that $c_1(M) = 0$.
  (b) Show that M has Euler characteristic $\chi(M) = 24$. (Hint: the Euler class of M is $c_2(M)$.)

(ii) Show that $b_2(M) = 22$. (Hint: M is simply connected by the Lefschetz Hyperplane Theorem.)
(iii) Show that $h^{2,0}(M) = h^{0,2}(M) = 1$ and $h^{1,1}(M) = 20$.
(iv) Show that $b^+(M) = 3$ and $b^-(M) = 19$.

Exercise 29. By Exercise 13 in dimension 4 we have a decomposition $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$. We can then decompose the curvature operator $\mathcal{R}\colon \Lambda^2 \to \Lambda^2$ of a Riemannian metric g on a 4-manifold M as follows:

$$\mathcal{R} = \left(\begin{array}{c|c} \frac{1}{12}\mathrm{Scal} + \mathrm{W}^+ & \overset{\circ}{\mathrm{Ric}} \\ \hline \overset{\circ}{\mathrm{Ric}} & \frac{1}{12}\mathrm{Scal} + \mathrm{W}^- \end{array}\right).$$

If M is closed, Chern–Weyl theory for the Riemannian curvature yields the Chern–Gauss–Bonnet Theorem, expressing the Euler characteristic $\chi(M) = \sum_i (-1)^i b_i(M)$ as

$$\chi(M) = \tfrac{1}{8\pi^2} \int_M \left(\tfrac{1}{24}\mathrm{Scal}^2 + |\mathrm{W}^+|^2 + |\mathrm{W}^-|^2 - \tfrac{1}{2} |\overset{\circ}{\mathrm{Ric}}|^2 \right) \mathrm{dv}_g,$$

and the Hirzebruch Signature Theorem, expressing the signature $\tau(M) = b_+(M) - b_-(M)$ as

$$\tau(M) = \tfrac{1}{12\pi^2} \int_M \left(|\mathrm{W}^+|^2 - |\mathrm{W}^-|^2 \right) \mathrm{dv}_g.$$

(i) Let g be an Einstein metric on a closed 4-manifold M. Show the Hitchin–Thorpe Inequality $2\chi(M) \geq 3|\tau(M)|$.

(ii) Let $M_{k,l} = k\mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}$, where $\overline{\mathbb{CP}^2}$ denotes $\mathbb{CP}^2$ with the opposite orientation. For which (k, l) can't $M_{k,l}$ admit an Einstein metric?

(iii) Let g be an Einstein metric on the K3 manifold. Show that g is hyperkähler. (Hint: use the fact that every flat metric bundle on a simply connected manifold can be trivialised by a basis of orthonormal parallel sections.)

2.5. **Bibliographical notes.** See [1] for background in Kähler geometry. There, as well as in [9], you will find a complete account of Yau's Theorem and the Calabi Conjecture. For a reference on background in complex geometry (perhaps needed to solve a couple of exercises) see [8].

Beauville's original paper [3] (in French) on the Beauville–Bogomolov Decomposition Theorem is quite readable. A complete proof of the Cheeger–Gromoll Splitting Theorem is contained in [12].

For the deformation theory of CY 3-folds we have followed the approach pioneered by [7], see also [11]. The notion of definite triples is discussed in [6].

3. Examples of complete non-compact manifolds with special holonomy, singularities and degenerations

In this section we discuss some examples of complete non-compact KRF manifolds. In contrast to the abstract existence theorem of Corollary 2.3, the non-compact setting will allow us to describe various (almost) explicit examples. As we will discuss later, complete non-compact examples are also important to understand formation of singularities in sequence of compact manifolds with special holonomy.

3.1. **Volume growth.**

(M^n, g) complete non-compact with $\mathrm{Hol}(g)$ one of the Ricci-flat holonomy groups

Theorem 2.4 $\Longrightarrow$ M has only one end unless reducible

Theorem 3.1 (Bishop–Gromov volume comparison)**.** *Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric}(g) \geq 0$. Then the function*

$$r \longmapsto \frac{\mathrm{Vol}_g\left(B\left(p, r\right)\right)}{r^n}$$

is decreasing.

Proof. (sketch) Ignoring issues with the cut-locus (which has anyway measure zero), we work in normal coordinates around p: consider $T_pM \simeq \mathbb{R}^n$ endowed with the Riemannian metric $\exp_p^* g = dr^2 + g_r$, for a 1-parameter family of metrics g_r on S^{n-1} depending smoothly on r and with $g_r \approx r^2 g_{S^{n-1}}$ as $r \to 0$.

For each $r > 0$ define a function μ_r on $\mathbb{S}^{n-1}$ by $\mathrm{dv}_{g_r} = \mu_r\, \mathrm{dv}_{S^{n-1}}$. Then $\mathrm{Vol}_g\left(B\left(p, R\right)\right) = \int_0^R \left(\int_{S^{n-1}} \mu_r\, \mathrm{dv}_{S^{n-1}}\right) dr$. The shape operator $S = \nabla \partial_r$ of the hypersurfaces $\{r = \mathrm{const}\}$ satisfies

$$\tfrac{1}{2}\partial_r g_r = g_r(S\,\cdot\,,\,\cdot\,)$$

and therefore $\partial_r \log \mu_r = \mathrm{tr}_{g_r} S$.

The shape operator satisfies the Riccati equation $\partial_r S + S^2 + \mathrm{Rm}_g(\,\cdot\,, \partial_r)\partial_r = 0$. Using Gauss' equations to express the Ricci curvature of g in terms of S and the Ricci curvature of g_r, taking traces one then finds

$$\partial_r(\mathrm{tr}_{g_r} S) + |S|^2_{g_r} \leq 0.$$

Since $|S|^2_{g_r} = \frac{1}{n-1}(\mathrm{tr}_{g_r} S)^2 + |\,\mathring{S}\,|^2_{g_r}$, where $\mathring{S}$ denotes the trace-less part of S, one deduces $\mathrm{tr}_{g_r} S \leq (n-1)r^{-1}$ from ODE comparison theory. A further integration yields the result. $\square$

Exercise 30

Corollary 3.2. *Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric}(g) \geq 0$.*

(i) *For all $p \in M$ and $r \geq 0$ we have*

$$\mathrm{Vol}_g\left(B\left(p, r\right)\right) \leq \omega_n r^n$$

where $n\omega_n$ is the volume of the unit $(n-1)$–sphere.

(ii) *Fix $p \in M$ and $r_0 > 1$. Then there exists $c = c(g, p, r_0)$ such that*

$$c\, r \leq \mathrm{Vol}_g\left(B\left(p, r\right)\right)$$

for all $r \geq r_0$.

Proof. Since $\lim_{r\to 0} \mathrm{Vol}_g\left(B\left(p, r\right)\right) = \omega_n r^n$, the first part of the Corollary is an immediate consequence of Theorem 3.1.

In order to prove the second part, fix a point $q \in \partial B\left(p, r\right)$, so that $B\left(p, 1\right)$ is contained in the annulus $B\left(q, r+1\right) \setminus B\left(q, r-1\right)$. Since $B\left(q, r\right) \subset B\left(p, 2r\right)$ it suffices to show that $\mathrm{Vol}_g\left(B\left(q, r\right)\right)$

grows at least linearly in $r \geq r_0$, with a constant that depends only on r_0 and $\mathrm{Vol}_g(B(p,1))$. We calculate

$$\begin{aligned}\frac{\mathrm{Vol}_g(B(p,1))}{\mathrm{Vol}_g(B(q,r))} \leq \frac{\mathrm{Vol}_g(B(p,1))}{\mathrm{Vol}_g(B(q,r-1))} &\leq \frac{\mathrm{Vol}_g(B(q,r+1)\setminus B(q,r-1))}{\mathrm{Vol}_g(B(q,r-1))}\\ &= \frac{\mathrm{Vol}_g(B(q,r+1))}{\mathrm{Vol}_g(B(q,r-1))} - 1 \leq \frac{(r+1)^n}{(r-1)^n} - 1 \leq \frac{C}{r}\end{aligned}$$

by Theorem 3.1 and Taylor's Theorem applied to the function $x \mapsto \frac{(1+x)^n}{(1-x)^n} - 1$ for $x \leq r_0^{-1}$. □

Exercise 31

3.2. 4-dimensional hyperkähler ALE spaces.

Theorem 3.3 (Bando–Kasue–Nakajima, 1989). *Let (M^4, g) be a complete Ricci-flat manifold. Suppose that there exists $p \in M$, $v, C > 0$ such that*

$$\mathrm{Vol}_g(B(p,r)) \geq vr^4 \text{ for all } r \geq 0 \qquad \text{and} \qquad \|\mathrm{Rm}_g\|_{L^2} \leq C.$$

Then (M,g) is ALE of order $\tau = 4$: there exists a compact set $K \subset M$, a finite group $\Gamma \subset \mathrm{O}(4)$ acting freely on S^3, a ball $B_R \subset \mathbb{R}^4$ and a diffeomorphism $f\colon (\mathbb{R}^4 \setminus B_R)/\Gamma \to M \setminus K$ such that

$$|\nabla^k(g_{flat} - f^*g)|_{g_{flat}} = O(r^{-\tau-k}) \text{ for all } k \geq 0.$$

Remark. There is a similar statement in any dimension, but the necessary condition $\|\mathrm{Rm}_g\|_{L^{\frac{n}{2}}} < \infty$ is less geometrically natural for $n > 4$.

All known simply connected examples are hyperkähler and in particular $\Gamma \subset \mathrm{SU}(2) \subset \mathrm{O}(4)$; Nakajima (1990) asks whether there are no further examples

Exercise 32

- $X = \mathbb{C}^2/\Gamma$ with $\Gamma \subset \mathrm{SU}(2)$ acting freely on $\mathbb{C}^2 \setminus \{0\}$
  A_n: $\Gamma = \mathbb{Z}_n$, $xy + z^{n+1} = 0$
  D_n: Γ binary dihedral group of order $4(n-2)$, $n \geq 4$: $x^2 + y^2 z + z^{n-1} = 0$
  E_n: Γ binary tetrahedral, octahedral, icosahedral group ($n = 6, 7, 8$)
- existence of minimal resolution $\pi\colon \widetilde{X} \to X$
  - $\mathrm{Exc}(\pi) = \Sigma_1, \dots, \Sigma_n$ union of rational curves
  - $(\Sigma_i \cdot \Sigma_j)_{1 \leq i,j \leq n} = -$ Cartan matrix of root system of ADE type (in particular $\Sigma_i \cdot \Sigma_i = -2$)
- $\pi\colon \widetilde{X} \to X$ is a crepant resolution: $K_{\widetilde{X}} = \pi^* K_X$ trivial $\iff \pi^*\omega_c$ extends to holomorphic symplectic form on $\widetilde{X}$

Exercise 33

Theorem 3.4 (Kronheimer, 1989). *Let M be the smooth 4-manifold underlying the minimal resolution of $\mathbb{C}^2/\Gamma$ for Γ a finite subgroup of $\mathrm{SU}(2)$ acting freely on S^3. Let $\boldsymbol{\alpha} \in H^2(M) \otimes \mathbb{R}^3$ be a triple of cohomology classes such that*

$$\boldsymbol{\alpha}(\Sigma) \neq 0 \text{ for all } \Sigma \in H_2(M;\mathbb{Z}) \text{ with } \Sigma \cdot \Sigma = -2. \tag{3.5}$$

Then there exists an ALE hyperkähler structure $\boldsymbol{\omega}$ on M with $[\boldsymbol{\omega}] = \boldsymbol{\alpha}$.

Furthermore, every ALE hyperkähler 4-manifold $(M, \boldsymbol{\omega})$ is diffeomorphic to the minimal resolution of $\mathbb{C}^2/\Gamma$ for Γ as above and $[\boldsymbol{\omega}]$ satisfies (3.5). Finally, if $(M_1, \boldsymbol{\omega}^1)$ and $(M_2, \boldsymbol{\omega}^2)$ are two ALE hyperkähler 4-manifolds and there exists a diffeomorphism $f\colon M_1 \to M_2$ such that $[f^*\boldsymbol{\omega}^2] = [\boldsymbol{\omega}^1]$ then $(M_1, \boldsymbol{\omega}^1)$ and $(M_2, \boldsymbol{\omega}^2)$ are isomorphic.

Proof. (sketch) We illustrate Kronheimer's construction with the simplest example $\Gamma = \mathbb{Z}_2$.

Let $\mathbb{H}^2$ be endowed with the flat hyperkähler structure $\boldsymbol{\omega}$ and consider the action of $\mathrm{U}(1) \subset \mathrm{Sp}(2)$ on $\mathbb{H}$ defined by $e^{i\theta} \cdot (q_1, q_2) = (e^{i\theta} q_1, e^{i\theta} q_2)$.

Let ξ be the vector field generating this U(1)–action. Then $\boldsymbol{\mu}\colon \mathbb{H}^2 \to \operatorname{Im}\mathbb{H}$ defined by $\boldsymbol{\mu}(q_1, q_2) = \overline{q}_1 i q_1 + \overline{q}_2 i q_2$ is a hyperkähler moment map for the U(1)–action, *i.e.* $d\boldsymbol{\mu} = \boldsymbol{\omega}(\xi, \cdot)$.

The hyperkähler quotient construction endows (the smooth part of) $M_{\boldsymbol{\zeta}} = \boldsymbol{\mu}^{-1}(\boldsymbol{\zeta})/\mathrm{U}(1)$ with a hyperkähler structure $\boldsymbol{\omega}_{\boldsymbol{\zeta}}$ uniquely determined by the relation $\iota_{\boldsymbol{\zeta}}^*\boldsymbol{\omega} = \pi_{\boldsymbol{\zeta}}^*\boldsymbol{\omega}_{\boldsymbol{\zeta}}$, where $\iota_{\boldsymbol{\zeta}}\colon \boldsymbol{\mu}^{-1}(\boldsymbol{\zeta}) \to \mathbb{H}^2$ is the inclusion and $\pi_{\boldsymbol{\zeta}}\colon \boldsymbol{\mu}^{-1}(\boldsymbol{\zeta}) \to M_{\boldsymbol{\zeta}}$ the projection.

One can identify $(M_{\mathbf{0}}, \boldsymbol{\omega}_{\mathbf{0}})$ with $\mathbb{H}/\mathbb{Z}_2$ endowed with its flat hyperkähler structure, while for $\boldsymbol{\zeta} \neq \mathbf{0}$ $(M_{\boldsymbol{\zeta}}, \boldsymbol{\omega}_{\boldsymbol{\zeta}})$ is a smooth hyperkähler manifold. In order to show that $\boldsymbol{\omega}_{\boldsymbol{\zeta}}$ is ALE of order 4:

(i) identify $M_{\boldsymbol{\zeta}}$ with $M_{\mathbf{0}}$ outside a compact set by identifying $\boldsymbol{\mu}^{-1}(\boldsymbol{\zeta})$ and $\boldsymbol{\mu}^{-1}(\mathbf{0})$ outside a compact set using the exponential map of $\mathbb{H}^2$;

(ii) use the homogeneity of the moment map to observe that $\boldsymbol{\omega}_{\boldsymbol{\zeta}|_{r\hat{x}}} = \boldsymbol{\omega}_{r^{-2}\boldsymbol{\zeta}|_{\hat{x}}}$, where we work in polar coordinates $x = r\hat{x}$ on $\mathbb{H}/\mathbb{Z}_2 = \mathbb{R}_+ \times (S^3/\mathbb{Z}_2)$;

(iii) identify $\partial_s \boldsymbol{\omega}_{s\boldsymbol{\zeta}|_{s=0}}$ with the curvature of the connection on the circle bundle $\boldsymbol{\mu}^{-1}(\mathbf{0}) \to M_{\mathbf{0}}$ induced by the Riemannian metric on $\boldsymbol{\mu}^{-1}(\mathbf{0}) \subset \mathbb{H}^2$;

(iv) observe that $\boldsymbol{\mu}^{-1}(\mathbf{0}) = \mathbb{H}^2 \times \mathrm{U}(1) \to \mathbb{H}^2/\mathbb{Z}_2$ is the flat non-trivial circle bundle.

Then under the identification in (i) we have an expansion

$$\boldsymbol{\omega}_{\boldsymbol{\zeta}|_{r\hat{x}}} = \boldsymbol{\omega}_{r^{-2}\boldsymbol{\zeta}|_{\hat{x}}} = \boldsymbol{\omega}_{\mathbf{0}} + O(r^{-4})$$

as $r \to \infty$. □

Exercise 34

3.3. **Remarks on complete CY 3-folds with maximal volume growth.**

Theorem 3.6 (Calabi, 1979)**.** *Let (D, g_D) be a Kähler Einstein metric with positive scalar curvature. The the total space K_D of the canonical line bundle over D carries a complete Calabi–Yau metric.*

Proof. We only prove the case where $\dim K_D = 3$. The general case is similar up to changing dimensional constants.

- $\pi\colon \Sigma^5 \to D^4$ unit circle bundle in K_D
- Kähler Einstein metric ω_1 on D, $\mathrm{Ric}(\omega_1) = \lambda\omega_1$, $\lambda > 0 \Longrightarrow [\omega_1] \propto c_1(K_D)$
- $2\pi^*\omega_1 = d\eta$ for connection 1-form $\eta\colon T\Sigma \to \mathbb{R} \rightsquigarrow$ horizontal subspace $\ker\eta \simeq \pi^*TD$
- tautological $(2,0)$-form $\omega_2 + i\omega_3$ with $d(\omega_2 + i\omega_3) = 3\eta \wedge (\omega_2 + i\omega_3)$

⇒ conical CY structure

$$\omega_{\mathrm{C}} = r dr \wedge \eta + r^2 \pi^*\omega_1 = d\left(\tfrac{1}{2} r^2 \eta\right), \qquad \Omega_{\mathrm{C}} = (dr + ir\eta) \wedge r^2(\omega_2 + i\omega_3)$$

on $\mathrm{C} = \mathbb{R}_+ \times \Sigma$ with induced metric $g_{\mathrm{C}} = dr^2 + r^2(\eta^2 + \pi^* g_D)$

- crepant resolution $p\colon K_D \to \mathrm{C} = K_D^\times$: $p^*\Omega_{\mathrm{C}}$ extends to holomorphic volume form Ω
- look for Kähler Ricci-flat metric $\omega = d\left(\tfrac{1}{2} u^2 \eta\right)$ for $u = u(r)$

$$2\omega^3 = 3 \operatorname{Re}\Omega \wedge \operatorname{Im}\Omega \qquad \Longleftrightarrow \qquad u^6 = r^6 + a^3, \quad a > 0$$

- $[\omega] = a[\omega_1]$ under $H^2(K_D) \simeq H^2(D)$

□

- $D = \mathbb{CP}^1$ is the Eguchi–Hanson metric, *i.e.* the metric of Theorem 3.4 with $\Gamma = \mathbb{Z}_2$
- $D = \mathbb{CP}^2 \Longrightarrow (\mathrm{C}, g_{\mathrm{C}}) = (\mathbb{C}^3/\mathbb{Z}_3, g_{flat})$ and $(K_D, g_{\omega,\Omega})$ is ALE
- $D = \mathbb{CP}^1 \times \mathbb{CP}^1 \Longrightarrow (\mathrm{C}, g_{\mathrm{C}})$ non-flat CY cone (the $\mathbb{Z}_2$–quotient of the conifold) and $(K_D, g_{\omega,\Omega})$ is asymptotically conical (AC)
- $D = \mathrm{Bl}_k\mathbb{CP}^2$ with $4 \leq k \leq 8 \Longrightarrow$ families of AC CY metrics (parametrised by variation of complex structure)

Exercise 35

- Note: ALE and AC asymptotic geometries don't exhaust possible asymptotic geometries of CY metrics with maximal volume growth

– Joyce (2000): QALE CY metrics on resolutions of $\mathbb{C}^n/\Gamma$ where $\Gamma \subset \mathrm{SU}(n)$ does not act freely on S^{2n-1}
– Li, Conlon–Rochon, Székelyhidi (2017): even on $\mathbb{C}^3$ there exists a complete CY metric with maximal volume growth asymptotic to $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$; qualitative description as a fibration of A_1 ALE spaces via

$$(z_1, z_2, z_3) \longmapsto z_1^2 + z_2^2 + z_3^2$$

3.4. **Kummer-type constructions.**

See §2 of my survey paper "Gravitational instatons and degenerations of Ricci-flat metrics on the K3 surface", available at `http://www.homepages.ucl.ac.uk/~ucahlfo/Publications_files/Survey_K3.pdf`.

3.5. **Exercises.**

Exercise 30. Fix $\kappa \in \mathbb{R}$. Consider separately the three cases $\kappa > 0$, $\kappa = 0$ and $\kappa < 0$.

(i) Find the solution $\mathrm{sn}_\kappa(r)$ of the IVP

$$f'' = -f, \qquad f(0) = 0, \qquad f'(0) = 1.$$

(ii) Show that the unique solution of

$$\lambda' + \lambda^2 + \kappa = 0$$

with $\lambda \sim \frac{1}{r}$ as $r \to 0$ is $\mathrm{ct}_\kappa(r) = \frac{\mathrm{sn}'_\kappa(r)}{\mathrm{sn}_\kappa(r)}$.

(iii) Identify the complete Riemannian manifold (M_κ, g_κ) which contains $(0, R_\kappa) \times S^{n-1}$ endowed with the metric $dr^2 + \mathrm{sn}^2_\kappa(r) g_{S^{n-1}}$ as a dense open set. Here $R_\kappa = \frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$ and $R_\kappa = \infty$ otherwise.

Exercise 31. Let (M^n, g) be a complete Riemannian manifold with non-negative Ricci curvature. Show that if there exists $p \in M$ such that

$$\lim_{r\to\infty} \frac{\mathrm{Vol}_g\left(B\left(p, r\right)\right)}{\omega_n r^n} = 1$$

then (M^n, g) is isometric to flat space $(\mathbb{R}^n, g_{Eucl})$. (Hint: what can you say about S in the proof of Theorem 3.1?)

Exercise 32. Let (M^n, g) be an ALE manifold asymptotic to $\mathbb{R}^n/\Gamma$ with $\mathrm{Ric}(g) \geq 0$. You are going to show that M has finite fundamental group.

(i) Show that either M is isometric to $\mathbb{R}^n$ or it does not carry parallel 1-forms. (Hint: use Exercise 31 to show that either M is isometric to $\mathbb{R}^n$ or $\Gamma \neq \{1\}$; in the latter case use the fact that Γ cannot preserve any vector in $\mathbb{R}^n$.)
(ii) Observe that $\pi_1(M \setminus K)$ is finite for every large enough compact set $K \subset M$.
(iii) Show that $\pi_1(M \setminus K) \to \pi_1(M)$ is surjective. (Hint: if not construct a finite cover of M with at least two ALE ends and use Theorem 2.4 and part (a).)

Exercise 33. In this exercise we construct the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_2$ and compare it with its smoothing.

(i) Identify $\mathbb{C}^2/\mathbb{Z}_2$ with the hypersurface X of equation $xy + z^2 = 0$ in $\mathbb{C}^3$. (Hint: consider invariant polynomials.)
(ii) Define $\widetilde{X}$ as the strict transform of X in the blow-up of $\mathbb{C}^3$ at the origin. Show that $\widetilde{X}$ is smooth and that the exceptional locus of the resolution $\pi\colon \widetilde{X} \to X$ is a rational curve Σ.
(iii) Let $\mathrm{Bl}_0\mathbb{C}^2$ denote the blow-up of $\mathbb{C}^2$ at the origin. Show that $\mathrm{Bl}_0\mathbb{C}^2 \to \widetilde{X}$ is a double cover branched along the exceptional curve Σ.
(iv) Deduce that $\Sigma \cdot \Sigma = -2$ and therefore that $\widetilde{X} = T^*\mathbb{CP}^1$. (Hint: $\mathrm{Bl}_0\mathbb{C}^2$ and $T^*\mathbb{CP}^1$ are the holomorphic line bundles $\mathcal{O}(-1)$ and, respectively, $\mathcal{O}(-2)$ over $\mathbb{CP}^1$.)

(v) A different way to desingularise X is to consider the smooth hypersurface X_t for $t \in \mathbb{C}^*$ defined by the equation $xy + z^2 = t$ in $\mathbb{C}^3$. You are going to show that X_1 is diffeomorphic to $\widetilde{X}$.
  - Show that there exists a linear change of variables in $\mathbb{C}^3$ such that X_1 is defined by the equation $z_1^2 + z_2^2 + z_3^2 = 1$.
  - Writing $z_a = x_a + iy_a$, $a = 1, 2, 3$, identify X_1 with TS^2, the tangent bundle of the unit 2-sphere in $\mathbb{R}^3$.
  - Deduce that X_1 is diffeomorphic to $\widetilde{X}$.

Exercise 34. Let $\mathbb{H}^2$ be endowed with the flat hyperkähler structure $\boldsymbol{\omega}$ and consider the action of $\mathrm{U}(1) \subset \mathrm{Sp}(2)$ on $\mathbb{H}$ defined by $e^{i\theta} \cdot (q_1, q_2) = (e^{i\theta} q_1, e^{i\theta} q_2)$. Denote by ξ the vector field generating this $\mathrm{U}(1)$–action and define $\boldsymbol{\mu}\colon \mathbb{H}^2 \to \operatorname{Im}\mathbb{H}$ by $\boldsymbol{\mu}(q_1, q_2) = \overline{q}_1 i q_1 + \overline{q}_2 i q_2$.

(i) Show that $d\boldsymbol{\mu} = \boldsymbol{\omega}(\xi, \cdot)$.

(ii) Fix a direction, say i, in $\operatorname{Im}\mathbb{H}$ and split $\boldsymbol{\mu} = \mu_{\mathbb{R}} \oplus \mu_c$ according to the decomposition $\operatorname{Im}\mathbb{H} = \mathbb{R} \oplus \mathbb{C}$. Show that for every $(\zeta_{\mathbb{R}}, \zeta_c) \in \mathbb{R} \oplus \mathbb{C}$ there is a bijection $\mu_c^{-1}(\zeta_c)/\mathbb{C}^* \simeq \mu_{\mathbb{R}}^{-1}(\zeta_{\mathbb{R}}) \cap \mu_c^{-1}(\zeta_c)/\mathrm{U}(1)$. Here the action of $\mathbb{C}^*$ on $\mathbb{H}^2$ extends in the obvious way the action of $\mathrm{U}(1)$, *i.e.* if we identify $\mathbb{H}^2$ with $\mathbb{C}^4$ by writing $q_a = z_a + \overline{w}_a j$ for $a = 1, 2$ then $t \cdot (z_1, w_1, z_2, w_2) = (tz_1, t^{-1} w_1, tz_2, t^{-1} w_2)$.

(iii) Use part (b) to identify $\boldsymbol{\mu}^{-1}(0, \zeta_c)/\mathrm{U}(1)$ with the affine variety of equation $xy + z(z - \zeta_c) = 0$ in $\mathbb{C}^3$.

(iv) Identify $\boldsymbol{\mu}^{-1}(\zeta, 0)/\mathrm{U}(1)$ with $T^*\mathbb{CP}^1$ and relate the parameter ζ to the area of the zero-section.

Exercise 35. In this exercise we work with the notation introduced in the proof of Theorem 3.6. Let $D = \mathbb{CP}^1 \times \mathbb{CP}^1$ be endowed with its standard Kähler Einstein metric ω_1, *i.e.* ω_1 is the sum of (appropriate multiples) of the area forms of the two factors. Observe that D carries a closed primitive $(1, 1)$–form ω_0, the difference between the area forms of the two factors. Normalise ω_0 so that $\omega_0^2 = -\omega_1^2$.

(i) Show that $\pi^*\omega_0 \wedge \pi^*\omega_1 = 0 = \pi^*\omega_0 \wedge (\omega_2 + i\omega_3)$.

(ii) Show that for every $-a < b < a$ there exists a complete CY metric $\omega_{a,b}$ on K_D with $[\omega] = a[\omega_1] + b[\omega_0]$ under the isomorphism $H^2(K_D) \simeq H^2(D)$. (Hint: try $\omega_{a,b} = a\pi^*\omega_1 + b\pi^*\omega_0 + d\left(\frac{1}{2}u^2\eta\right)$ with $u = u(r)$ satisfying $u(0) = 0$.)

(iii) Show that $\omega_{a,b}$ is AC asymptotic to C with rate 6 if $b = 0$ and rate 2 otherwise. Here we say that (M, g) is asymptotic to $(\mathrm{C}, g_{\mathrm{C}})$ with rate $\tau > 0$ if there exists a compact set $K \subset M$, $R > 0$ and a diffeomorphism $f\colon (R, \infty) \times \Sigma \to M$ such that

$$|\nabla^k (g_{\mathrm{C}} - f^*g)|_{g_{\mathrm{C}}} = O(r^{-\tau-k}) \text{ for all } k \geq 0.$$

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