NON-COMPACT HYPERKÄHLER MANIFOLDS

1. References

General references are [2], [11] and [16]. They all contain chapters on hyperkähler manifolds and of course a lot more. A general reference on hyperkähler manifolds which contains most of what I talked about is the survey paper [10].

At the beginning of the course I talked about Riemannian holonomy groups: you can find more about this in [2, Chapter 10], [11, Chapters 2 and 3] and the survey paper [3].

The first examples of hyperkähler manifolds due to Calabi are described in the seminal paper [4] and Beauville's compact examples (together with results about the structure of compact Kähler Ricci-flat manifolds) are in [1]. (Unfortunately both these papers are in French!)

For basics on Einstein 4–manifolds and self-duality, read [2, Section 6.D and Chapter 13].

Twistor spaces of hyperkähler manifolds and the hyperkähler quotient construction are very nicely explained in [8].

Kronheimer's construction and classification of ALE spaces is explained in [10]. I couldn't find a nice description of the Taub–NUT metric and the Gibbons–Hawking construction except maybe for [7, Construction 2.3 in §2] (this paper discusses a limit of the Ricci-flat Kähler metric on a K3 different from the orbifold limit of the Kummer construction).

Finally, [6] gives the Kummer construction of the Kähler Ricci-flat metric on K3.

2. Exercises

Exercise 2.1. Let P be the Lie group $SU(2)$ and G the subgroup $U(1)$ embedded in $SU(2)$ as the diagonal matrices, $e^{it} \mapsto M_t = \operatorname{diag}(e^{it}, e^{-it})$. Then $P/G \simeq \mathbb{S}^2$ and $SU(2) \to \mathbb{S}^2$ is a principal $U(1)$–bundle.

(i) Identify $SU(2)$ with $\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}$ via $g\colon\ (z_1, z_2) \longmapsto \begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix}$.
A basis $\{\eta_1, \eta_2, \eta_3\}$ of left invariant 1–forms on $SU(2)$ is defined by $g^{-1}dg = \eta_1\sigma_1 + \eta_2\sigma_2 + \eta_3\sigma_3$, where

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \sigma_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

is a basis of $\mathfrak{su}_2$. Calculate η_1, η_2, η_3. (If you prefer, instead of complex coordinates (z_1, z_2) you can using Euler angles

$$(z_1, z_2) = \left(e^{\frac{i}{2}(\theta+\psi)} \cos\left(\tfrac{\phi}{2}\right), e^{\frac{i}{2}(\theta-\psi)} \sin\left(\tfrac{\phi}{2}\right) \right),$$

$\theta \in [0, 2\pi)$, $\phi \in [0, \pi)$ and $\psi \in [0, 4\pi)$.)

(ii) Prove that $\eta_3\colon\ T\mathbb{S}^3 \to \mathbb{R}$ defines a connection on the principal bundle $\mathbb{S}^3 \to \mathbb{S}^2$.

(iii) Let $\lambda\colon\ U(1) \times \mathbb{C} \to C$ be the representation of $U(1)$ of weight $m \in \mathbb{Z}$, $\lambda(e^{it}, z) = e^{imt}z$. Define the complex line bundle $\mathcal{O}(m) \to \mathbb{S}^2$ as $\mathcal{O}(m) = SU(2) \times_\lambda \mathbb{C} := (P \times \mathbb{C})/\sim_\lambda$, where $(A, z) \sim_\lambda (A', z')$ iff $(A', z') = (AM_t, e^{imt}z)$. Describe the space of sections of $\mathcal{O}(m)$. (Hint: interpret a section f of $\mathcal{O}(m)$ as a function $f\colon\ \mathbb{S}^3 \to \mathbb{C}$ satisfying a certain $U(1)$–equivariance property.)

Exercise 2.2. Prove that the holonomy group of the round sphere $(\mathbb{S}^n, g_{\text{round}})$ is $SO(n)$.

Exercise 2.3. Construct a metric g on $\mathbb{R}^4$ with the following properties: g is flat outside the union of two disjoint balls; each point $x \in \mathbb{R}^4$ has a neighbourhood U_x such that the holonomy of g restricted to U_x is contained in $U(2)$; however, globally $\mathrm{Hol}(g) = SO(4)$.

Remark. The exercise above shows that in general the holonomy group of a Riemannian metric cannot be calculated locally. However, if the metric g is real analytic (for example if it is Einstein), then the holonomy is completely determined by looking at a neighbourhood of a point.

Exercise 2.4. Show that an $SU(m)$ structure (g, ω, Ω) on manifold M^{2m} is torsion-free iff $d\omega = 0 = d\Omega$.

Exercise 2.5. Show that $Sp(m)$ is a compact, connected and simply connected Lie group. Calculate its dimension. (Hint: use the fact that $Sp(1) = SU(2)$ and $\mathbb{S}^{4m-1} = Sp(m)/Sp(m-1)$.)

Exercise 2.6. Work out all the possibilities for the holonomy group of a simply connected Kähler Ricci flat manifold of real dimension 8. (Hint: the only Ricci flat symmetric space is $\mathbb{R}^n$.)

Exercise 2.7. Let X be a complex manifold of complex dimension $m+1$, $m \geq 2$, such that the anticanonical bundle K_X^{-1} is ample. Consider a smooth anticanonical divisor $M \in |-K_X|$.

(i) Show that X has finite fundamental group. (Hint: use Calabi Conjecture to prove that X admits a Kähler metric with positive Ricci curvature.)

(ii) Since K_X^{-1} is ample, Kodaira Vanishing Theorem says that $h^{p,0}(X) = 0$ for all $p > 1$ and therefore the holomorphic Euler characteristic $\chi(X, \mathcal{O}_X) := \sum_{p=0}^{m+1} (-1)^p h^{0,p}(X) = 1$. Deduce that X is simply connected. (Hint: look at how the holomorphic characteristic behaves under finite coverings.)

(iii) Use Lefschetz Hyperplane Theorem to deduce that M is also simply connected.

(iv) Use the Adjunction Formula to show that K_M is trivial.

(v) Use the exact sequence
$$0 \to K_X \to \mathcal{O}_X \to \mathcal{O}_M \to 0$$
and the fact that $H^i(X, K_X) = 0$ for all $i \leq m$ (by Nakano Vanishing Theorem) to show that $h^{p,0}(M) = 0$ for all $0 < p < m$.

(vi) Deduce that M admits a metric g with $\mathrm{Hol}(g) = SU(m)$.

(vii) Justify the fact that a hypersurface of degree $m+2$ in $\mathbb{CP}^{m+1}$ admits a metric with holonomy equal to $SU(m)$.

Exercise 2.8. Let V be a 4–dimensional vector space endowed with a positive definite inner product and a volume form $\mathrm{dv} \in \Lambda^4 V^*$.

(i) Using dv and the wedge product define a non-degenerate pairing q on $\Lambda^2 V^*$. Show that q has signature $(3,3)$. Let $\Lambda^{\pm} V^*$ be maximal positive/negative subspaces of $(\Lambda^2 V^*, q)$.

(ii) Show that the induced action of $SL(V) \simeq SL(4, \mathbb{R})$ (*i.e.* the matrices that preserve dv) on $\Lambda^2 V^*$ defines a double cover $SL(4,\mathbb{R}) \to SO(3,3)$. Restricting to compact subgroups, we see that $SO(4) \to SO(3)^+ \times SO(3)^-$ is a double-cover; here $SO(3)^{\pm}$ is the induced action of $SO(4)$ on $\Lambda^{\pm} V^*$.

(iii) Identify V with $\mathbb{H}$ and $SU(2)$ with the unit sphere $\mathbb{S}^3 \subset \mathbb{H}$. Define a map $SU(2) \times SU(2) \times \mathbb{H}$, by $(q_1, q_2, x) \mapsto q_1 x \overline{q_2}$ (using quaternionic multiplication). Show that this defines a double cover $SU(2)^+ \times SU(2)^- \to SO(4)$.

(iv) Show that this induces a double cover of $U(1) \times SU(2)^- \to U(2)$, where $U(1) \subset SU(2)^+$ are the diagonal matrices.

(v) Show that $U(2)$ acts on $\Lambda^- V^*$ as $SO(3)^-$ and on $\Lambda^+ V^*$ as the subgroup $SO(2) \subset SO(3)^+$ preserving the standard Kähler form ω_1 on $\mathbb{H} \simeq \mathbb{C}^2$.

(vi) Deduce that on a Kähler surface (M, ω), $\Lambda^+ M = [\![\Lambda^{2,0} M]\!] \oplus \mathbb{R}\omega$ and $\Lambda^- M = [\![\Lambda_0^{1,1} M]\!]$, where $\Lambda_0^{1,1} M$ are the $(1,1)$-forms on $T_x M$ orthogonal to ω. (Hint: use the identifications $\Lambda^2 V^* \simeq \mathfrak{so}(4)$, $[\![\Lambda^{1,1} V]\!] \simeq \mathfrak{u}(2)$.)

Exercise 2.9. Let M be a hypersurface of degree d in $\mathbb{CP}^3$. Denote with $h \in H^2(\mathbb{CP}^3, \mathbb{Z})$ the class dual to a hyperplane, *i.e.* $h = c_1(H)$, where H is the line bundle $\mathcal{O}(1)$ over $\mathbb{CP}^3$. The total Chern class $c = 1 + c_1 + c_2 + c_3$ of $\mathbb{CP}^3$ is $c(\mathbb{CP}^3) = (1+h)^4$ (where of course $h^n = 0$ for all $n \geq 4$). Use the exact sequence

$$0 \to TM \to T\mathbb{CP}^3|_M \to \mathcal{O}(d)|_M \to 0$$

to calculate the total Chern class of M (here TX is the holomorphic tangent bundle of the complex manifold X).

Exercise 2.10. Let M be a quartic surface in $\mathbb{CP}^3$.

(i) Using the previous exercise, find the top Chern class $c_2(M)$.
(ii) Show that M has Euler characteristic $\chi(M) = 24$. (Hint: the Euler class $e(M)$ is equal to $c_2(M)$.)
(iii) Show that $b_2(M) = 22$. (Hint: M is simply connected by the Lefschetz Hyperplane Theorem.)
(iv) Use the fact that M has holonomy $SU(2)$ to calculate $h^{2,0}(M) = h^{0,2}(M)$ and $h^{1,1}(M)$.
(v) Show that $b^+(M) = 3$ and $b^-(M) = 19$. (Hint: use Exercise 8.)

Exercise 2.11. An oriented Riemannian 4–manifold is called *anti-self-dual* (ASD) if $W^+ = 0$. Let M be a compact ASD manifold with $\mathrm{Scal} = 0$.

(i) Use the Weitzenböck formula $\triangle\alpha = \nabla^*\nabla\alpha - 2W^+(\alpha) + \frac{\mathrm{Scal}}{3}\alpha$ for $\alpha \in \Omega^+(M)$, to show that every harmonic self-dual form on M is parallel.
(ii) Deduce that either $b^+(M) = 0$, or $b^+(M) = 1$ and M is Kähler, or $b^+(M) = 3$ and M is hyperkähler. (Hint: you can use the fact that $U(2)$ is precisely the subgroup of $SO(4)$ that fixes a self-dual form and the group that fixes a plane in $\Lambda^+(\mathbb{R}^4)^*$ is precisely $SU(2)$.)

Exercise 2.12. In the lectures we saw that the twistor space of $\mathbb{S}^4$ is $\mathbb{CP}^3$. Deduce from this that the twistor space of $\mathbb{R}^4$ is the total space of $\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1$. (Hint: write $\mathbb{R}^4 = \mathbb{S}^4 \setminus \{\infty\}$ and use the conformal invariance of the twistor space.)

Exercise 2.13. Fix a positive number $\lambda \neq 1$. Think of $M = \mathbb{S}^3 \times \mathbb{S}^1$ as $(\mathbb{R}^4 \setminus \{0\})/\mathbb{Z}$, where $\mathbb{Z}$ acts via $n \cdot x = \lambda^n x$.

(i) Show that the twistor space $\tilde{Z}$ of $\mathbb{R}^4 \setminus \{0\}$ is $\mathbb{CP}^3 \setminus \{\ell_0, \ell_\infty\}$ where $\ell_0 = \{z_0 = 0 = z_1\}$ and $\ell_\infty = \{z_2 = 0 = z_3\}$ in homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$. Furthermore, show that $\tilde{Z}$ admits a holomorphic projection onto $\mathbb{CP}^1$. Why does such a projection have to exist?
(ii) Consider the matrix $A = \mathrm{diag}\,(1, 1, \lambda, \lambda)$. Show that the action of $\mathbb{Z}$ on $\mathbb{CP}^3$ given by $n \cdot [z] = [A^n z]$ restricts to $\tilde{Z}$ to give a covering of the action of $\mathbb{Z}$ on $\mathbb{R}^4 \setminus \{0\}$. Deduce that the twistor space of $\mathbb{S}^3 \times \mathbb{S}^1$ is $\tilde{Z}/\mathbb{Z}$.
(iii) From part (i) there exists a holomorphic projection $\pi\colon Z \to \mathbb{CP}^1$ whose fibres are complex manifolds $(\mathbb{S}^3 \times \mathbb{S}^1, J)$ (Hopf surfaces). Show that these are never Kähler manifolds.

Exercise 2.14. Let $M = T^*\mathbb{R}^3$ be endowed with the canonical symplectic form $\omega = \sum_{i=1}^3 dx_i \wedge dy_i$, where (x_1, x_2, x_3) are coordinates on $\mathbb{R}^3$ and y is the 1–form $y_1 dx_1 + y_2 dx_2 + y_3 dx_3$.

(i) Show that the moment map for the action of $G = \mathbb{R}^3$ by translations on $\mathbb{R}^3$ is $\mu(x, y) = y$.
(ii) Show that the moment map for the action of $G = SO(3)$ by rotations on M is $\mu(x, y) = x \times y$ after identifying $\mathfrak{so}(3)$ with $\mathbb{R}^3$.

Exercise 2.15. Suppose that M is a (complete non-compact) hyperkähler manifold and that $U(1)$ acts freely on M by isometries preserving ω_1 and rotating ω_2, ω_3. In other words, if X denotes the vector field on M generating the $U(1)$–action, then $\mathcal{L}_X g = 0 = \mathcal{L}_X \omega_1$, $\mathcal{L}_X \omega_2 = \omega_3$ and $\mathcal{L}_X \omega_3 = -\omega_2$. Furthermore, assume that there exists a moment map $\mu\colon M \to \mathfrak{u}(1)^* \simeq \mathbb{R}$ for the $U(1)$–action on (M, ω_1).

(i) Show that μ is a global Kähler potential for the Kähler forms ω_2 and ω_3.

(ii) Can you apply this to Calabi's metric on $M = T^*\mathbb{CP}^n$? Let J_1 be the standard complex structure on M (the one that identifies it with the complex manifold $T^*\mathbb{CP}^n$); can (M, J_1) be biholomorphic to (M, J_2) or (M, J_3)?

Exercise 2.16. Show that the Taub–NUT metric can be obtained as the hyperkähler quotient of $\mathbb{C}^2 \times \mathbb{C}^* \times \mathbb{C}$ with respect to the $U(1)$–action $e^{i\theta} \cdot (z_1, z_2, w_1, w_2) = (e^{i\theta} z_1, e^{-i\theta} z_2, e^{i\theta} w_1, w_2)$. Here $\mathbb{C}^2 \times \mathbb{C}^* \times \mathbb{C}$ is endowed with the hyperkähler structure

$$\omega_1 = \frac{i}{2}\left(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2 + \frac{dw_1}{w_1} \wedge \frac{d\overline{w}_1}{w_1} + dw_2 \wedge d\overline{w}_2 \right),$$

$$\omega_2 + i\omega_3 = dz_1 \wedge dz_2 + \frac{dw_1}{w_1} \wedge dw_2.$$

Exercise 2.17. Consider the multi-Taub-NUT manifold (M, g) obtained by the Gibbons–Hawking construction with the harmonic function $h = \sum_{i=1}^n \frac{1}{2|x - p_i|}$. Suppose for simplicity that $p_i = (a_i, 0, 0)$ with $a_i < a_{i+1}$.

(i) Show that M is simply connected and its second homology is generated by $(n-1)$ spheres of self-intersection -2 and which mutually intersect according to the Dynkin diagram A_{n-1}.

(ii) Show that there exists a complex structure on M compatible with the metric such that all these spheres are holomorphic (and in particular minimal submanifolds).

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