INTRODUCTION TO GAUGE THEORY

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1. Bundles, connections and curvature

- M smooth manifold + G Lie group with Lie algebra $\mathfrak{g}$
- principal G–bundle P over M
  - smooth manifold P
  - free right G–action $R\colon P \times G \to P$
  - smooth projection to orbit space $\pi\colon P \to M = P/G$
  - local G-equivariant trivialisations $\pi^{-1}(U) \simeq U \times G$
- gauge group $\mathcal{G} = \mathrm{Aut}(P)$
- principal bundles and vector bundles
  - representation $\rho\colon G \to \mathrm{GL}(V) \rightsquigarrow$ vector bundle $E = P \times_\rho V$
    e.g. $\rho = \mathrm{Ad} \rightsquigarrow$ adjoint bundle $\mathrm{ad}\, P$
  - $G =$ stabiliser of $\phi_0 \in \bigotimes^r V \otimes \bigotimes^s V^*$, (E, ϕ) modelled on $(V, \phi_0) \rightsquigarrow$ principal G–bundle of ϕ–adapted frames
- Examples: Hopf circle bundle $S^3 \to S^1$, quaternionic Hopf bundle $S^7 \to S^4$, frame bundle

Exercise 5.1

- A connection on $\pi\colon P \to M$ is a G–invariant splitting $TP = \ker \pi_* \oplus H$ of
$$0 \to \ker \pi_* \to TP \to \pi^* TM \to 0$$
  H is called the horizontal subspace
- Since $\ker \pi_* \simeq P \times \mathfrak{g}$, a connection is a 1-form $A\colon TP \to \mathfrak{g}$ such that $R_g^* A = \mathrm{Ad}(g^{-1})A$
  - Product/trivial connection on $M \times G$ induced from identification $TG \simeq G \times \mathfrak{g}$
  - loc. trivialisation $P|_U \simeq U \times G$: $A =$ trivial $+\, a$ for a $\mathfrak{g}$-valued 1-form a on U
  - change trivialisation given by $u\colon U \to G \rightsquigarrow A =$ trivial $+\, u^{-1}du + u^{-1}au$
- infinite-dimensional affine space $\mathcal{A}$ of connections on P modelled on $\Omega^1(M; \mathrm{ad}\, P)$
  Note: action of gauge group $\mathcal{G}$ on $\mathcal{A}$ by pull-back

Exercise 5.2

- horizontal lift X^H of a vector field X on M: the unique G–invariant vector field on P such that $X^H \in H \subset TP$ at every point of P and $\pi_* X^H = X$
- parallel transport of A
  - hor. lift of path γ in M: path $\widetilde{\gamma}$ in P such that $\pi \circ \widetilde{\gamma} = \gamma$ and $\widetilde{\gamma}'(t) \in H_{\gamma(t)}$ $\forall t$
  - $\forall\, h \in \pi^{-1}(\gamma(0))$ $\exists!$ hor. lift $\rightsquigarrow$ parallel transport $\Pi_\gamma\colon \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1))$
  - γ loop $\rightsquigarrow \Pi_\gamma = R_g$ for some $g \in G$
- curvature $F_A \in \Omega^2(M; \mathrm{ad}\, P)$ of A
  - $\pi^* F_A(X, Y) = [X, Y]^H - [X^H, Y^H]$
  - $F_A = d_A \circ d_A$
  - $\nabla_A = d + a \rightsquigarrow F_A = da + a \wedge a$
- $A \in \mathcal{A}$, $a \in \Omega^1(M; \mathrm{ad}\, P)$: $F_{A+a} = F_A + d_A a + a \wedge a$
- Bianchi identity: $d_A F_A = 0$

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  - G–invariant polynomial $p\colon \mathfrak{g}\to\mathbb{R}\rightsquigarrow$ closed $(2\deg p)$–form $p(F_A)$ on M
  - characteristic class $[p(F_A)]\in H^{\deg p}_{dR}(M;\mathbb{R})$ independent of A
    e.g. $G=\mathrm{U}(k)$: $p_i(X)=\mathrm{Tr}(X^i)\rightsquigarrow$ Chern classes $c_i(E)$ of complex vector bundle E

M complex manifold, $E\to M$ complex vector bundle with Hermitian metric h

- Cauchy–Riemann operator on E: $\mathbb{C}$-linear map $\overline{\partial}_{\mathcal{E}}\colon\ \Omega^0(M;E)\to\Omega^{0,1}(M;E)$ such that
$$\overline{\partial}_{\mathcal{E}}(fs)=\overline{\partial}f\otimes s+f\,\overline{\partial}_{\mathcal{E}}s$$
  - space of Cauchy–Riemann operators: affine space modelled on $\Omega^{0,1}(M;\mathrm{End}\,E)$
- connection $\nabla\rightsquigarrow$ Cauchy–Riemann operator $\overline{\partial}_{\mathcal{E}}=\nabla^{0,1}$
- Chern connection: $h+\overline{\partial}_{\mathcal{E}}\implies\exists!$ unitary connection ∇ with $\overline{\partial}_{\mathcal{E}}=\nabla^{0,1}$
- $\mathcal{E}\to M$ holomorphic vector bundle with underlying smooth complex vector bundle E
  - Cauchy–Riemann operator $\overline{\partial}_{\mathcal{E}}$: in local holomorphic trivialisation $\overline{\partial}_{\mathcal{E}}=\overline{\partial}$
  - Cauchy–Riemann operator $\overline{\partial}_{\mathcal{E}}\rightsquigarrow$ sheaf of “holomorphic” sections $\mathcal{O}(\mathcal{E})=\ker\overline{\partial}_{\mathcal{E}}$
  - $\overline{\partial}_{\mathcal{E}}\circ\overline{\partial}_{\mathcal{E}}=0\iff E$ has the structure of a holomorphic bundle $\mathcal{E}$

Exercise 5.3

2. Flat connections

- G compact Lie group, *e.g.* $G=\mathrm{U}(k)$
- $P\to M$ principal G–bundle
- moduli space of flat connections $\mathcal{M}=\{A\in\mathcal{A}\,|\,F_A=0\}/\mathcal{G}$
- as sets $\mathcal{M}=\mathrm{Hom}(\pi_1(M),G)/G$

Exercise 5.4

- fix $A\in\mathcal{M}$ and understand local structure of $\mathcal{M}$ near A
- introduce Banach spaces, *e.g.* Hölder spaces: $\mathcal{A}^{k,\alpha},\mathcal{G}^{k+1,\alpha}$
- deformation complex of a flat connection A
$$0\to\Omega^0(M;\mathrm{ad}\,P)\xrightarrow{d_A}\Omega^1(M;\mathrm{ad}\,P)\xrightarrow{d_A}\Omega^2(M;\mathrm{ad}\,P)\xrightarrow{d_A}\dots$$
  and cohomology groups H^0_A, H^1_A, H^2_A
- fix metric on $M\rightsquigarrow\Omega^1(M;\mathrm{ad}\,P)^{k,\alpha}=\mathrm{im}\,d_A\oplus\ker d_A^*$
  - $\mathrm{im}\,d_A$: tangent space to the orbit $\mathcal{G}\cdot A$
  - H^0_A: Lie algebra of stabiliser $\mathcal{G}^{2,\alpha}_A$ of A
    e.g. $G=\mathrm{U}(k),\mathrm{SU}(k)$: $\mathcal{G}^{k+1,\alpha}_A=\{1\}$ iff $H^0_A=0$
  - Slice Theorem $\rightsquigarrow$ local structure of $\mathcal{A}^{k,\alpha}/\mathcal{G}^{2,\alpha}$ near A
$$\mathcal{S}^{k,\alpha}_{A,\epsilon}/\mathcal{G}^{k+1,\alpha}_A=\{A+a\,|\,a\in\Omega^1(M;\mathrm{ad}\,P)^{k,\alpha},\|a\|_{C^{k,\alpha}}<\epsilon,d_A^*a=0\}/\mathcal{G}^{k+1,\alpha}_A$$
- $H^1_A=\{a\in\Omega^1(M;\mathrm{ad}\,P)^{k,\alpha}\,|\,d_Aa=0=d_A^*a\}$: gauge-fided infinitesimal deformations of A as a flat connection
- A is irreducible + $H^2_A=0$: $\mathcal{M}^{k+1,\alpha}$ is a smooth manifold near A with tangent space H^1_A (smooth structure independent of $\alpha\in(0,1),k\geq 1$)

Exercises 5.5, 5.6

3. The Yang–Mills functional

(M,g,vol) oriented Riemannian manifold, $P\to M$ principal G–bundle with G compact Lie group

- Yang–Mills functional on $\mathcal{A}$: $\mathcal{YM}(A)=\int_M|F_A|^2\,\mathrm{vol}$
- critical points: $d_A^*F_A=0$ (where $d_A^*=\pm*d_A*$)

Exercises 5.9, 5.10
Compactness:

- Strong Uhlenbeck Compactness: M^n closed, $\frac{n}{2} < p < \infty$ ($p \geq \frac{4}{3}$ if $n = 2$), A_i sequence of YM connections w/ uniformly bounded $\|F_{A_i}\|_{L^p} \Rightarrow$ after passing subsequence $\exists\ W^{2,p}$ gauge transformations g_i s.t. $g_i^* A_i \xrightarrow{C^\infty} A_\infty$ smooth YM connection
- Monotonicity Formula: $\exists\, c = c(M)$ s.t. $e^{cr^2} r^{4-n} \int_{B_r(po)} |F_A|^2$ is increasing in r
- $\rightsquigarrow$ smooth convergence of YM connections with unformly bounded energy away from a set of finite $(n-4)$–dimensional Hausdorff measure (Uhlenbeck)
  - in 4d convergence outside finitely many points

Exercise 5.11

4. Instantons

Exercise 5.12

- $(M^4, g) \rightsquigarrow \Lambda^2 T^*M = \Lambda^2_+ T^*M \oplus \Lambda^2_- T^*M$ (eingenspaces of $*$)
- A instanton/ASD connection if $F_A^+ = 0$
- $\mathcal{YM}(A) = -\int \langle F_A \wedge F_A \rangle_{\mathfrak{g}} + 2\int |F_A^+|^2 \Rightarrow$ instantons are absolute minima of $\mathcal{YM}$
- moduli space $\mathcal{M}$: $0 \to \Omega^0(M; \mathrm{ad}\, P) \xrightarrow{d_A} \Omega^1(M; \mathrm{ad}\, P) \xrightarrow{d_A^+} \Omega^2_+(M; \mathrm{ad}\, P) \to 0$

Exercises 5.13, 5.14

Donaldson's Diagonalisation Theorem:

- M^4 closed, smooth, oriented, simply connected w/ negative definite intersection form q
  $2m = \sharp\{\alpha \in H^2(M;\mathbb{Z}) \mid q(\alpha,\alpha) = 1\}$
- $P \to M^4$ SU(2)–bundle with $c_2(P) = 1$
- moduli space $\mathcal{M}$ of instantons on P
  - for generic metric g, $H_A^{2,+} = 0$ for all $A \in \mathcal{M}$ and $\mathcal{M}^{irr}$ smooth
  - if $\mathcal{M} \neq \emptyset$ then $\dim \mathcal{M} = 5$
  - reducibles $\rightsquigarrow$ singularities $p_1, \ldots, p_m \in \mathcal{M}$ modelled on cones $\overline{\mathbb{CP}^2}$ over
  - $\mathcal{M}^{irr}$ non-empty: $\forall p \in M$ and $\epsilon \ll 1\ \exists$ instanton $A_{p,\epsilon}$ "concentrated" in $B_\epsilon(p)$
- $\mathcal{M}^{irr}$ non-empty manifold with boundary $M \sqcup \coprod_m \overline{\mathbb{CP}^2}$
- M smoothly cobordant to $\coprod_m \overline{\mathbb{CP}^2} \Longrightarrow q = -\mathrm{id}$

Exercises 5.15, 5.16

5. Exercises

Exercise 5.1. In this exercise we study principal bundles on spheres.

(i) Show/convince yourself that isomorphism classes of principal G–bundles on the sphere S^n are in 1:1 correspondence with $\pi_{n-1}(G)$. (Hint: trivialise the bundle over two hemispheres and consider the gluing map over the equator.)

(ii) Describe principal U(1)–bundles on S^2.

(iii) Show that there are only two non-trivial SO(3)–bundles on S^2. Can you describe the non-trivial one? (Hint: think about rank 3 real vector bundles of the form $\underline{\mathbb{R}} \oplus L$, where $\underline{\mathbb{R}}$ is the trivial real line bundle and L is a complex line bundle.)

(iv) Show that every SU(2)–bundle on S^3 is trivial and that SU(2)–bundles on S^4 are classified by an integer.

(v) Show that there are only two principal SU(2)–bundles on S^5. Can you describe the non-trivial SU(2)–bundle on S^5? (Hint: use the homogeneous space presentation $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2)$.)

Exercise 5.2. Let $\pi\colon P \to M$ be a principal G–bundle with connection A and let $E = P \times_\rho V$ be an associated vector bundle of rank k.

(i) Show that there is a 1:1 correspondence between sections $s \in C^\infty(M;E)$ of E and G–equivariant V–valued functions $\widetilde{s} \in C^\infty(P;V)$ on P.

(ii) Show that the formula $s \mapsto d\tilde{s}|_H$ defines a covariant derivative on E, *i.e.* an $\mathbb{R}$–linear map $\nabla_A\colon\ C^\infty(M;E) \to C^\infty(M;T^*M \otimes E)$ satisfying the Leibniz rule $\nabla_A(fs) = df \otimes s + f\nabla_A s$ for all $f \in C^\infty(M)$ and $s \in C^\infty(M;E)$.

(iii) Work in a local trivialisation $P|_U \simeq U \times G$ where A is given in terms of a Lie algebra valued 1-form a on U. Note that since G is a subgroup of $\mathrm{GL}(k)$, a is simply a matrix of 1-forms. Similarly, a section s of $E|_U$ is simply a smooth map $s\colon\ U \to \mathbb{R}^k$. Show that $\nabla_A s = ds + as$, where d denotes the usual differential and as simply denotes the point-wise action of the matrix a on the vector s.

(iv) In the same way one extends the differential $d\colon\ C^\infty(M) \to \Omega^1(M)$ of functions to the exterior differential $d\colon\ \Omega^p(M) \to \Omega^{p+1}(M)$ of differential forms, show that the covariant derivative ∇_A extends to a covariant exterior differential operator $d_A\colon\ \Omega^p(M;E) \to \Omega^{p+1}(M;E)$ for any p. Here $\Omega^p(M;E)$ denotes the space of smooth section of the vector bundle $\Lambda^p T^*M \otimes E$.

Exercise 5.3. This exercise is about connections and Cauchy–Riemann operators on complex vector bundles over a complex manifold.

(i) Show the existence and uniqueness of the Chern connection. (Hint: in a unitary gauge, *i.e.* one where the Hermitian metric h is standard, write $\overline{\partial}_{\mathcal{E}} = \overline{\partial} + \alpha$ for some $k \times k$–matrix of $(0,1)$–forms α, and decompose α into is Hermitian and skew-Hermitian parts.)

(ii) Show that any choice of Cauchy–Riemann operator on a complex vector bundle E over a Riemann surface defines a holomorphic structure on E.

(iii) Let (E,h) be a Hermitian vector bundle over a complex manifold and let ∇ be a unitary connection on E. Decompose the bundle-valued 2-form F_∇ into (p,q)–types: $F_\nabla = F_\nabla^{2,0} + F_\nabla^{1,1} + F_\nabla^{0,2}$. Show that $\overline{\partial}_{\mathcal{E}} = \nabla^{0,1}$ satisfies $\overline{\partial}_{\mathcal{E}} \circ \overline{\partial}_{\mathcal{E}} = 0$ if and only if $F_\nabla = F_\nabla^{1,1}$.

Exercise 5.4. Let $\pi\colon\ P \to M$ be a principal G–bundle. Denote by $\mathcal{M}$ the moduli space of flat connections on P. You are going to show the existence of a bijection between $\mathcal{M}$ and $\mathrm{Hom}\left(\pi_1(M), G\right)/G$.

(i) Let γ_0 and γ_1 be smooth paths with same endpoints and suppose that there is a smooth homotopy $\varphi\colon\ [0,1]^2 \to M$ between γ_0 and γ_1 with fixed endpoints, *i.e.* $\{\phi(s,\cdot)\colon\ [0,1] \to M\}_s$ is a smooth family of smooth paths in M with fixed endpoints and $\phi(i,\cdot) = \gamma_i$ for $i = 0,1$. Show that the parallel transports Π_{γ_0} and Π_{γ_1} of a flat connection A coincide. (Hint: consider the commuting vector fields $X = \varphi_* \frac{\partial}{\partial s}$ and $Y = \varphi_* \frac{\partial}{\partial t}$ and their horizontal lifts, and use the fact that the flows of two commuting vector fields commute.)

(ii) Deduce that there is a map $\Psi\colon\ \mathcal{M} \to \mathrm{Hom}\left(\pi_1(M), G\right)/G$.

(iii) Show that Ψ is a bijection. (Hint: for the surjectivity, let $p\colon\ \widetilde{M} \to M$ be the universal cover of M and consider the standard action of $\pi_1(M)$ on $\widetilde{M}$; for a homomorphism $\rho\colon\ \pi_1(M) \to G$ consider $(\widetilde{M} \times G)/\pi_1(M)$, where $\pi_1(M)$ acts on G via ρ.)

(iv) Set $G = \mathrm{U}(1)$ and let M be a torus $\mathbb{T} = V/\Lambda$ for Λ a lattice in a finite-dimensional vector space V. Let P be the trivial principal $\mathrm{U}(1)$–bundle on $\mathbb{T}$. By making explicit the abstract result of part (iii), prove that flat connections on P are parametrised by the dual torus $\mathbb{T}^\vee = V^\vee/\Lambda^\vee$, where $V^\vee$ is the dual vector space and $\Lambda^\vee = \{\alpha \in V^\vee \,|\, \alpha(v) \in \mathbb{Z},\ \forall\, v \in \Lambda\}$.

Exercise 5.5. Let $(\Sigma, g_\Sigma, \omega_\Sigma, j_\Sigma)$ be a Riemann surface with Riemannian metric g_Σ, area form ω_Σ and complex structure j_Σ related by $\omega_\Sigma(\,\cdot\,,\,\cdot\,) = g_\Sigma(j_\Sigma\,\cdot\,,\,\cdot\,)$. Let G be a compact Lie group (say, $G = \mathrm{SU}(2)$) and let P be a principal G–bundle on Σ. You are going to show that the moduli space $\mathcal{M}^{irr}$ of irreducible flat connections on P is a Kähler manifold, as explained in a seminal paper by Atiyah–Bott (Philos. Trans. Roy. Soc. London Ser. A, 1983).

(i) Show that if A is an irreducible flat connection then $H^2_A = 0$, so that the moduli space $\mathcal{M}^{irr}$ is a smooth manifold. (Hint: use Poincaré Duality.)

(ii) Let $\langle \cdot\,, \cdot \rangle_{\mathfrak{g}}$ be an Ad–invariant positive definite inner product on $\mathfrak{g}$ (it exists by compactness of G). For all $a, b \in \Omega^1(\Sigma; \operatorname{ad} P)$ define

$$\omega(a,b) = \int_\Sigma \langle a \wedge b \rangle_{\mathfrak{g}}.$$

Let A an irreducible flat connection. Show that restring ω to elements of $H^1_A = T_{[A]}\mathcal{M}^{irr}$ induces a symplectic form on $\mathcal{M}^{irr}$. (Hint: formally, the point is that $\mathcal{M}^{irr}$ is the infinite dimensional symplectic quotient of $(\mathcal{A}^{irr}, \omega)$ by the action of $\mathcal{G}$; indeed, the moment map for this action is $\mu\colon\ \mathcal{A}^{irr} \to \Omega^2(\Sigma; \operatorname{ad} P) \simeq \Omega^0(\Sigma; \operatorname{ad} P)^*$, $\mu(A) = F_A$.)

(iii) Show that $*_\Sigma : \Omega^1(\Sigma; \operatorname{ad} P) \to \Omega^1(\Sigma; \operatorname{ad} P)$ preserves H^1_A and defines a complex structure on $\mathcal{M}^{irr}$ compatible with the symplectic form ω of part (i).

When G is a unitary group, a famous theorem of Narasimhan–Seshadri (*cf.* Donaldson, J. Differential Geom., 1983) shows that the map $\nabla_A \mapsto \nabla^{0,1}_A$ identifies the complex manifold $\mathcal{M}^{irr}$ with a moduli space of stable holomorphic vector bundles on Σ.

Exercise 5.6. Let (M, g_M, vol_M) be an oriented Riemannian manifold, let G be a compact Lie group and $\pi\colon\ P \to M$ a principal G–bundle. Consider the complexification G^c of G with Lie algebra $\mathfrak{g}^c = \mathfrak{g} \oplus i\mathfrak{g}$. For example, if $G = \mathrm{SU}(2)$ then $G^c = \mathrm{SL}(2, \mathbb{C})$. Let $P^c = P \times_G G^c$, where G acts on G^c by left multiplication, and observe that P^c is a principal G^c–bundle. We consider flat connections A^c on P^c. The condition $F_{A^c} = 0$ is now preserved by the complex gauge group $\mathcal{G}^c$. As the theory of quotients in complex geometry indicates, we can only hope to define a moduli space of complex flat connections $\mathcal{M}^c$ if we impose a stability condition: $\mathcal{M}^c = \{A^c \in \mathcal{A}^c \,|\, F_{A^c} = 0, A^c \text{ “stable”}\}/\mathcal{G}^c$.

(i) The construction of $\mathcal{M}^c$ was given by Corlette (J. Differential Geom., 1988) exploiting an infinite dimensional Kähler quotient construction.

(a) Observe that any connection A^c on P^c can be written as $A^c = A + i\phi$ for a connection A on P and $\phi \in \Omega^1(\Sigma; \operatorname{ad} P^c)$. Show that A^c is flat if and only if

$$F_A - \phi \wedge \phi = 0 = d_A\phi.$$

(b) Let $\mathcal{A}^c$ be the infinite dimensional complex affine space of connections on P^c. Use an Ad–invariant inner product $\langle \cdot\,, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ and the complex structure on $\mathfrak{g}^c = \mathfrak{g} \oplus i\mathfrak{g}$ to endow $\mathcal{A}^c$ with a Kähler structure with Kähler form

$$\omega(a,b) = \int_M \mathrm{Re}\, \langle ia \wedge *_M \overline{b} \rangle_{\mathfrak{g}}.$$

(Here for $x = x_1 + ix_2 \in \mathfrak{g}^c$ we define $ix = -x_2 + ix_1$ and $\overline{x} = x_1 - ix_2$.)

(c) Show that $\mathcal{G}$ acts on $\mathcal{A}^c$ preserving ω and with moment map $\mu(A^c) = d^*_A\phi$, where $A^c = A + i\phi$.

When G is a unitary group and we work with a complex vector bundle E, a complex connection $A^c \in \mathcal{A}^c$ is said “stable” if there is no complex sub-bundle of E preserved by ∇_{A^c} (in general stability is defined in terms of A^c–parallel reductions of structure group of P^c to a strict parabolic subgroup of G^c). Corlette has shown that for every stable flat connection A^c there exists $g \in \mathcal{G}^c$, unique up to an element of $\mathcal{G}$, such that $g^*A^c = A + i\phi$ with

$$F_A - \phi \wedge \phi = 0, \qquad d_A\phi = 0, \qquad d^*_A\phi = 0. \tag{5.7}$$

We can therefore construct $\mathcal{M}^c$ as the space of (irreducible) solutions to (5.7) modulo the action of the gauge group $\mathcal{G}$.

(ii) Consider the special case where $(M, g_M, \mathrm{vol}_M) = (\Sigma, g_\Sigma, \omega_\Sigma)$ is a Riemann surface.

(a) Using the complex structure on Σ, write $\phi = \Phi + \overline{\Phi}$ with $\Phi \in \Omega^{1,0}(\Sigma; \operatorname{ad} P^c)$ and $\overline{\Phi}$ its complex conjugate (using the complex structure on Σ and $\mathfrak{g}^c = \mathfrak{g} \oplus i\mathfrak{g}$). Show that (5.7) are then equivalent to Hitchin equations

$$F_A - [\Phi, \overline{\Phi}] = 0, \qquad \overline{\partial}_A\Phi = 0. \tag{5.8}$$

(b) Show that the complex structure of Σ induces a different complex structure on $\mathcal{M}^c$. (Hint: see part (ii) of Exercise 5.5.)

In fact $\mathcal{M}^c$ is a hyperkähler manifold, first studied by Hitchin (Proc. London Math. Soc., 1987). In the complex structure of part (ii).(b) it is known as the moduli space of Higgs bundles.

Exercise 5.9. Show that an irreducible Yang–Mills connection on a 2-dimensional manifold must be flat. (Hint: if A is a Yang–Mills connection on a surface, $*F_A$ is a parallel section of the adjoint bundle.)

Exercise 5.10. Let A be a Yang–Mills connection on a principal G–bundle P over a closed manifold (M^n, g). We say that A is stable if for every non-zero infinitesimal variation $a \in \Omega^1(M; \operatorname{ad} P)$ satisfying the gauge fixing condition $d_A^* a = 0$ one has

$$\frac{d^2}{dt^2}|_{t=0}\mathcal{YM}(A_t) \geq 0,$$

where A_t is any smooth 1-parameter family of connections on P with $A_0 = A$ and $\frac{d}{dt}|_{t=0}A_t = a$.

(i) Show that

$$\frac{d^2}{dt^2}|_{t=0}\mathcal{YM}(A_t) = 2\int_M \langle \mathcal{J}a, a\rangle_{T^*M\otimes\mathfrak{g}} \operatorname{vol}_M,$$

where $\mathcal{J}$ is the operator $\mathcal{J}a = \triangle_A a + F_A * a$. Here $\triangle_A = d_A d_A^* + d_A^* d_A$ is the covariant Hodge Laplacian and $F_A * a$ denotes the adjoint-valued 1-form given by $\sum_i [F_A(e_i, \cdot\,), a(e_i)]$ in any orthonormal frame $\{e_1, \dots, e_n\}$ on M. (Hint: for any Ad–invariant inner product on $\mathfrak{g}$ one has $\langle X, [Y, Z]\rangle = \langle [X, Y], Z\rangle$.)

(ii) Suppose that $\omega \in \Omega^2(M; \operatorname{ad} P)$ satisfies $d_A\omega = 0 = d_A^*\omega$, let X be a vector field on M and consider the adjoint-valued 1-form $a = X \lrcorner \omega$. Show that if $X = \nabla f$ is of gradient type then

$$d_A^* a = 0,$$

and that if X is a gradient conformal Killing field, *i.e.* X is of gradient type and there is a function λ such that $\nabla X = \lambda \operatorname{id}$, then

$$\mathcal{J}a = (\nabla^*\nabla X)\lrcorner\omega + X\lrcorner(\nabla_A^*\nabla_A\omega) + a \circ \operatorname{Ric} + 2F_A * a.$$

(Hint: you can use the Weitzenböck formula $\triangle_A a = \nabla_A^*\nabla_A a + a \circ \operatorname{Ric} + F_A * a$. Also, as usual in Riemannian geometry it is convenient to work in an orthonormal frame in which Christoffel symbols vanish at a given point in M.)

(iii) If (M^n, g) is the standard round sphere $\mathbb{S}^n$ of unit radius in $\mathbb{R}^{n+1}$, any ω as in part (ii) satisfies

$$\nabla_A^*\nabla_A\omega + 2(n-2)\omega + F_A * \omega = 0,$$

where $(F_A * \omega)(u, v) = \sum_i [F_A(e_i, u), \omega(e_i, v)] - [F_A(e_i, v), \omega(e_i, u)]$. Also note that the restriction X of a constant vector field on $\mathbb{R}^{n+1}$ to $\mathbb{S}^n$ is a gradient conformal Killing field. Apply part (ii) to $\omega = F_A$ and any such X: show that

$$\mathcal{J}(X\lrcorner F_A) = -(n-4)\, X\lrcorner F_A$$

and conclude that there are no stable Yang–Mills connections on any round sphere of dimension $n \geq 5$.

This last result is due to Simons, and the proof you have given is due to Bourguignon–Lawson (Comm. Math. Phys., 1981).

Exercise 5.11. The exercise is about the monotonicity formula for Yang–Mills connections on $\mathbb{R}^n$.

(i) Fix a connection A on a principal G–bundle $\pi\colon P \to M$ over a manifold M. Let X be a vector field on M and denote by φ_t its flow. Consider the horizontal lift X^H on P and its flow φ_t^H, which is a G–equivariant diffeomorphism of P such that $\pi \circ \varphi_t^H = \varphi_t$. Consider the 1-parameter family of connections on P defined by $A_t = (\varphi_t^H)^* A$. Here we think of connections as G–equivariant 1-forms on P with values in $\mathfrak{g}$. Show that $\frac{d}{dt}|_{t=0} A_t = X \lrcorner F_A$. (Hint: use the interpretation of curvature in terms of Lie bracket of horizontal vector fields.)

(ii) In the same notation of part (i), let now $\Omega \subset M$ be a domain with smooth boundary and assume that $X|_{\partial\Omega}$ is a unit outward normal for $\partial\Omega$. Show that if A (and therefore A_t) is a Yang–Mills connection then

$$\frac{d}{dt}|_{t=0} \int_\Omega |F_{A_t}|^2 = \int_{\partial\Omega} |X \lrcorner F_A|^2.$$

(Hint: apply the first variation of the Yang–Mills energy and calculate explicitly the boundary term. For the latter calculation it is convenient to fix a boundary defining function ρ for $\partial\Omega$ such that $X = \partial_\rho$. At points of $\partial\Omega$ we can then write $F_A = F_A|_{\partial\Omega} + d\rho \wedge X \lrcorner F_A$, and observe that $*F_A|_{\partial\Omega}$ vanishes along $\partial\Omega$.)

(iii) Use part (ii) to conclude that if A is a Yang–Mills connection on $\mathbb{R}^n$ and $B_r = B_r(p)$ is a ball of radius r centred at a point p, then

$$\frac{d}{dr}\left(r^{4-n} \int_{B_r} |F_A|^2 \right) = 2r^{4-n} \int_{\partial B_r} |\partial_\rho \lrcorner F_A|^2$$

where ρ is the radial coordinate on $\mathbb{R}^n$ centred at p. (Hint: use scaling, *i.e.* the flow of the Euler vector field, to replace the fixed connection A on the varying domains B_r with a family of Yang–Mills connections A_r on the fix domain B_1.)

(iv) Deduce from part (iii) that for $n \geq 5$ there is no non-flat Yang–Mills connection with finite Yang–Mills energy on $\mathbb{R}^n$. (Hint: what happens as $r \to \infty$?)

(v) Suppose that A is a Yang–Mills connection on $\mathbb{R}^n \backslash \{0\}$ such that the energy ratio $r^{4-n} \int_{B_r(0)} |F_A|^2$ is constant in r. Show that A is the radial extension of a Yang–Mills connection on $\mathbb{S}^{n-1}$. (Hint: first observe that the principal bundle on $\mathbb{R}^n \setminus \{0\}$ must be the radial extension of a fixed bundle on $\mathbb{S}^{n-1}$ and show that there exists a gauge such that A has no $d\rho$–component.)

Exercise 5.12. Let V be a 4-dimensional vector space endowed with a positive definite inner product and a volume form $\mathrm{dv} \in \Lambda^4 V^*$.

(i) Using dv and the wedge product define a non-degenerate pairing q on $\Lambda^2 V^*$. Show that q has signature $(3,3)$. Let $\Lambda^\pm V^*$ be maximal positive/negative subspaces of $(\Lambda^2 V^*, q)$.

(ii) Show that the induced action of $\mathrm{SL}(V) \simeq \mathrm{SL}(4,\mathbb{R})$ (*i.e.* the matrices that preserve dv) on $\Lambda^2 V^*$ defines a double cover $\mathrm{SL}(4,\mathbb{R}) \to \mathrm{SO}(3,3)$. Restricting to compact subgroups, we see that $\mathrm{SO}(4) \to \mathrm{SO}(3)^+ \times \mathrm{SO}(3)^-$ is a double-cover; here $\mathrm{SO}(3)^\pm$ is the induced action of $\mathrm{SO}(4)$ on $\Lambda^\pm V^*$.

(iii) Identify V with the quaternions $\mathbb{H}$ and $\mathrm{SU}(2)$ with the unit sphere $\mathbb{S}^3 \subset \mathbb{H}$. Define a map $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbb{H} \to \mathbb{H}$ by $(q_1, q_2, x) \mapsto q_1 x \overline{q_2}$. Show that this defines a double cover $\mathrm{SU}(2)^+ \times \mathrm{SU}(2)^- \to \mathrm{SO}(4)$.

(iv) Show that this induces a double cover of $\mathrm{U}(1) \times \mathrm{SU}(2)^- \to \mathrm{U}(2)$, where $\mathrm{U}(1) \subset \mathrm{SU}(2)^+$ is the subgroup of diagonal matrices.

(v) Show that $\mathrm{U}(2)$ acts on $\Lambda^- V^*$ as $\mathrm{SO}(3)^-$ and on $\Lambda^+ V^*$ as the subgroup $\mathrm{SO}(2) \subset \mathrm{SO}(3)^+$ preserving the standard Kähler form ω on $\mathbb{H} \simeq \mathbb{C}^2$.

(vi) Deduce that on a Kähler surface (M, ω), $\Lambda^+ M = [\![\Lambda^{2,0} M]\!] \oplus \mathbb{R}\omega$ and $\Lambda^- M = [\![\Lambda_0^{1,1} M]\!]$, where $\Lambda_0^{1,1} M$ is the space of $(1,1)$-forms orthogonal to ω. Here for a complex vector space W we denote by $[\![W]\!]$ the real vector space such that $W \oplus \overline{W} = [\![W]\!] \otimes_{\mathbb{R}} \mathbb{C}$.

Exercise 5.13. Let (M^4, g) be a closed, smooth, oriented, simply connected manifold and let $E \to M$ be a rank 2 Hermitian complex vector bundle with $c_1(E) = 0$ (so E has structure group $\mathrm{SU}(2)$) and $c_2(E) = 1$. For any instanton A on E the Atiyah–Singer Index Theorem yields

$$\dim H^0_A - \dim H^1_A + \dim H^{2,+}_A = -5.$$

Here $H^\bullet_A$ are the comohomology groups of the instanton deformation complex. We also assume that $H^{2,+}_A = 0$ for any instanton A on E.

(i) Show that topological splittings $E = L \oplus L^{-1}$ for a line bundle L are in $1:1$ correspondence with half the classes $\alpha \in H^2(M; L)$ such that $q(\alpha, \alpha) = -1$. (Hint: take $\pm\alpha = c_1(L)$ and calculate $c_2(L \oplus L^{-1})$.)

(ii) Show that for any class α as in part (i) there exists a unique $\mathrm{U}(1)$ instantons A_α on the line bundle L_α with $c_1(L_\alpha) = \alpha$ (and therefore a unique reducible instanton $A_\alpha \oplus A_{-\alpha}$ on the split bundle $E = L_\alpha \oplus L_\alpha^{-1}$).

(iii) Show that the stabiliser $\mathcal{G}_A$ of $A = A_\alpha \oplus A_{-\alpha}$ in $\mathcal{G}$ is the subgroup $\mathrm{U}(1)$ of constant gauge transformations that are diagonal in the decomposition $E = L_\alpha \oplus L_\alpha^{-1}$. Deduce that $H^0_A = \mathbb{R}$.

(iv) Show that $\mathfrak{su}(E) = \underline{\mathbb{R}} \oplus L^2_\alpha$ and correspondingly $H^1_A = H^1(\underline{\mathbb{R}}) \oplus H^1(L^2_\alpha)$.

(v) Show that $H^1(\underline{\mathbb{R}}) = 0$ and $H^1_A = H^1(L^2_\alpha)$ is 6-dimensional.

(vi) Show that the action of $\mathcal{G}_A$ on $H^1(L^2_\alpha)$ is the diagonal action of $\mathrm{U}(1)$ on $\mathbb{C}^3$ with weight 2, *i.e.* $e^{i\theta} \cdot (z_1, z_2, z_3) = (e^{2i\theta} z_1, e^{2i\theta} z_2, e^{2i\theta} z_3)$. Deduce that a neighbourhood of the reducible connection $[A]$ in the moduli space of instantons on E is modelled on $\mathbb{C}^3/\mathrm{U}(1)$, *i.e.* , up to a choice of orientation, a cone over $\overline{\mathbb{CP}^2}$.

Exercise 5.14. Identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$ via $\mathbf{x} = x_0 + x_1 i + x_2 j + x_3 k$ and $\mathfrak{su}_2$ with $\operatorname{Im}\mathbb{H}$ endowed with the Lie bracket $[\mathbf{x}, \mathbf{y}] = \mathbf{xy} - \mathbf{yx}$.

(i) Show that $\frac{1}{2} d\overline{\mathbf{x}} \wedge d\mathbf{x} = \omega_1 i + \omega_2 j + \omega_3 k$, where, for any cyclic permutation (ijk) of (123), $\omega_i = dx_0 \wedge dx_i - dx_j \wedge dx_k$ is an anti-self-dual 2-form.

(ii) Show that

$$A = d + \tfrac{1}{2} \frac{\overline{\mathbf{x}}\, d\mathbf{x} - d\overline{\mathbf{x}}\, \mathbf{x}}{1 + \mathbf{x}\overline{\mathbf{x}}}$$

defines an $\mathrm{SU}(2)$ instanton on the trivial bundle over $\mathbb{R}^4$.

(iii) Show that for $|\mathbf{x}| \gg 0$, we have $A \sim d + u^{-1} du$ with $u(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|}$.

(iv) By a change of variable $\mathbf{x} = \mathbf{y}^{-1}$ near $\mathbf{x} = \infty$ and composing with the gauge transformation u^{-1} from part (iii), show that A extends to an instanton on a bundle E on the conformal compactification $\mathbb{S}^4 = \mathbb{R}^4 \cup \{\infty\}$ of $\mathbb{R}^4$.

(v) Calculate $\mathcal{YM}(A)$ and deduce that $c_2(E) = 1$.

(vi) By pulling back A by scalings, for every $\epsilon > 0$ construct an instanton A_ϵ on the same bundle with curvature

$$F_\epsilon = \frac{\epsilon^2\, d\overline{\mathbf{x}} \wedge d\mathbf{x}}{(\epsilon^2 + |\mathbf{x}|^2)^2}.$$

(vii) Study the behaviour of A_ϵ as $\epsilon \to 0$: show that A_ϵ converges smoothly to the trivial connection on compact subsets of $\mathbb{H} \setminus \{0\}$, while $|F_\epsilon|^2\, \mathrm{dv}_{\mathbb{R}^4}$ converges to a multiple of the Dirac delta at the origin in the sense of currents.

Exercise 5.15. Let (M^3, g) be a closed oriented Riemannian 3-manifold, fix a principal G–bundle P and consider the spaces $\mathcal{A}$ and $\mathcal{G}$ of connections on P and gauge transformations. We consider instantons on $\mathbb{R}_t \times M$ on the pull-back of P.

(i) Show that up to gauge equivalence any connection $\mathbb{A}$ on $\mathbb{R} \times M$ can be put in "temporal gauge" $\mathbb{A} = A_t$ for a 1-parameter family of connections on M depending smoothly on t. Deduce that gauge equivalence classes of connections on $\mathbb{R} \times M$ can be interpreted as curves in $\mathcal{A}/\mathcal{G}$.

(ii) Show that the curvature of $\mathbb{A} = A_t$ is $F_{\mathbb{A}} = F_{A_t} + dt \wedge \partial_t A_t$ (for a suitable interpretation of the last term).

(iii) Show that $\mathbb{A} = A_t$ is an instanton if and only if the curve $t \mapsto A_t \in \mathcal{A}$ satisfies the "flow" equation $\partial_t A_t = - * F_{A_t}$.

(iv) Consider the 1-form γ on $\mathcal{A}$ defined by $\gamma_A(a) = \int_M \langle a \wedge F_A \rangle_{\mathfrak{g}}$. Show that γ is closed. (Hint: for any $a, b \in \Omega^1(M; \operatorname{ad} P)$, thought of as constant vector fields on $\mathcal{A}$, one has $(d\gamma)_A(a,b) = a \cdot (\gamma_A(b)) - b \cdot (\gamma_A(a))$.)

(v) Fix a based point $A_0 \in \mathcal{A}$ and define the function (Chern–Simons functional) on $\mathcal{A}$

$$\mathrm{CS}_{A_0}(A) = \int_M \langle d_{A_0} a \wedge a + \tfrac{2}{3} a \wedge a \wedge a \rangle_{\mathfrak{g}},$$

where $a = A - A_0$. Show that $d\mathrm{CS}_{A_0} = \gamma$.

(vi) Introduce the Riemannian metric $(a, b) \mapsto \int_M \langle a \wedge *b \rangle_{\mathfrak{g}}$ on $\mathcal{A}$. Show that $\gamma^\flat_A = - * F_A$ and therefore the flow equations in (iii) can be interpreted as the gradient flow of the Chern–Simons functional.

Exercise 5.16. Identify $\mathbb{C}^3 \setminus \{0\}$ with $(0, \infty) \times S^5$ with coordinate r on the first factor. Regard S^5 as a principal U(1)–bundle $\pi\colon S^5 \to \mathbb{CP}^2$ and fix the connection $\eta\colon TS^5 \to \mathfrak{u}(1) = i\mathbb{R}$ with curvature $d\eta = 2i\pi^*\omega_{\mathrm{FS}}$ a multiple of the Fubini–Study Kähler form ω_{FS} on $\mathbb{CP}^2$. Consider the Levi-Civita connection A' of $(\mathbb{CP}^2, g_{\mathrm{FS}})$ on $E' = T\mathbb{CP}^2$: since the Fubini–Study metric is Kähler, the structure group of A' is U(2). Since the Fubini–Study metric has constant scalar curvature, identities for the curvature of constant scalar curvature Kähler metric imply that $F_{A'} = 2i\lambda \operatorname{id}_E \otimes \omega_{\mathrm{FS}} + F^-_{A'}$ for a constant $\lambda \in \mathbb{R}$. Thus A' is not an instanton, but its self-dual part is constrained to be a constant multiple of $\operatorname{id}_E \otimes \omega_{\mathrm{FS}}$; such connections on Kähler surfaces, and generalisations to higher dimensional Kähler manifolds, are called Hermitian Yang–Mills connections.

(i) Show that $E = \pi^*E' \to S^5$ is the unique non-trivial rank 2 complex vector bundle. (Hint: you can think about the principal bundle $P' = \mathrm{SU}(3) \to \mathrm{SU}(3)/\mathrm{U}(2) = \mathbb{CP}^2$ of unitary frames of E and its lift to $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2)$.)

(ii) Fix the connection $A := \pi^*A' - \lambda \operatorname{id}_E \otimes \eta$ on E. By abuse of notation, denote by (E, A) the radial extension of E and A to $\mathbb{C}^3 \setminus \{0\}$. You are going to show that A is a Yang–Mills connection on $\mathbb{C}^3 \setminus \{0\}$.

   (a) Show that $F_A = F_A^{1,1}$. (Hint: use the fact that $F_{A'}$ and ω_{FS} are both of type $(1,1)$ on $\mathbb{CP}^2$ and that $\mathbb{C}^3 \setminus \{0\} \to \mathbb{CP}^2 = \mathbb{C}^3 \setminus \{0\}/\mathbb{C}^*$ is a holomorphic projection.)

   (b) Show that $F_A \wedge \omega_0^2 = 0$, where $\omega_0 = d\left(\frac{1}{2} r^2 \eta\right)$ is the standard Kähler form on $\mathbb{C}^3$. (Connections satisfying the conditions of part (a) and (c) are sometimes called Calabi–Yau instantons.)

   (c) It is know that if κ is a $(1,1)$-form on $\mathbb{C}^3$ satisfying $\kappa \wedge \omega_0^2 = 0$ then $*\kappa = -\kappa \wedge \omega_0$. Use this fact to show that A is a Yang–Mills connection.

Note that A yields an example of a Yang-Mills connection with a singularity of codimension 6 ($|F_A| = O(r^{-2})$ and the bundle E does not even extend topologically to $0 \in \mathbb{C}^3$), showing that in general there are higher codimension singularities beyond the codimension-4 "bubbling" set.