COMPLEX AND KÄHLER MANIFOLDS

1. Almost Hermitian geometry

Three subgroups of $\mathrm{GL}(2n,\mathbb{R})$:

- $\mathrm{GL}(n,\mathbb{C})$: preserves multiplication by i on $\mathbb{R}^{2n}\simeq\mathbb{C}^n$

$$J_0\colon \mathbb{R}^{2n}\to\mathbb{R}^{2n},\qquad J_0=\begin{pmatrix}0 & -1_n\\ 1_n & 0\end{pmatrix},\qquad J_0^2=-1_{2n}$$

- $\mathrm{Sp}(2n,\mathbb{R})$: preserves the non-degenerate skew-symmetric bilinear form

$$\omega_0=dx_1\wedge dy_1+\cdots+dx_n\wedge dy_n$$

  - ω_0 non-degenerate: $v\mapsto v\lrcorner\omega_0$ defines isomorphism $V\to V^*$
  - $\frac{1}{n!}\omega_0^n$ standard volume form on $\mathbb{R}^{2n}$

Recall: differential k-form $\alpha\in\Lambda^kV^*$ on a vector space V = k-multilinear alternating map

$$\alpha\colon V\times\cdots\times V\to\mathbb{R}.$$

Basis $\{e_1,\dots,e_m\}$ of V $\leftrightarrow$ basis $\{\varepsilon^1,\dots,\varepsilon^m\}$ of V^* $\leftrightarrow$ basis $\{\varepsilon^{i_1}\wedge\cdots\wedge\varepsilon^{i_k}\,|\,i_1<\cdots<i_k\}$ of Λ^kV^* defined in terms of minor determinants

- $\mathrm{U}(n)=\mathrm{GL}(n,\mathbb{C})\cap\mathrm{Sp}(2n,\mathbb{R})$: preserves compatible $(J_0,\omega_0)\rightsquigarrow$ positive definite symmetric bilinear form $g_0(\,\cdot\,,\,\cdot\,)=\omega_0(\,\cdot\,,J_0\,\cdot\,)$

Exercise 5.1

Definition 1.1. Let M be a smooth manifold of dimension $2n$.

(i) A Riemannian metric on M is a smooth section g of Sym^2T^*M which is positive definite at every point.
(ii) An almost complex structure on M is a smooth section J of $\mathrm{End}(TM)$ such that $J^2=-\mathrm{id}_{TM}$.
(iii) A non-degenerate 2-form on M is a smooth section of Λ^2T^*M that is non-degenerate at every point.

We say that (M^{2n},g,J,ω) is an almost Hermitian manifold if the Riemannian metric g, almost complex structure J and non-degenerate 2-form ω are related by $g(\,\cdot\,,\,\cdot\,)=\omega(\,\cdot\,,J\,\cdot\,)$.

2. Integrable almost complex structures and complex manifolds

M^{2n} with almost complex structure J

- decomposition $TM\otimes\mathbb{C}=T^{1,0}M\oplus T^{0,1}M$ into $\pm i$–eigenspaces of J
- $T^*M\otimes\mathbb{C}=\Lambda^{1,0}T^*M\oplus\Lambda^{0,1}T^*M\rightsquigarrow du=\partial u+\overline{\partial}u$ for every $u\in C^\infty(M;\mathbb{C})$
- Cauchy–Riemann operator: u J–holomorphic if $\overline{\partial}u=0$
- $\Lambda^2T^*M\otimes\mathbb{C}=\Lambda^{2,0}T^*M\oplus\Lambda^{1,1}T^*M\oplus\Lambda^{0,2}T^*M$
- $\alpha\in\Omega^{0,1}(M)\rightsquigarrow d\alpha=d^{2,-1}\alpha+\partial\alpha+\overline{\partial}\alpha$

Recall: exterior differential $d\colon\Omega^k(M)\to\Omega^{k+1}(M)$, the unique $\mathbb{R}$–linear map such that

(i) *for every smooth function f, $df\in\Omega^1(M)$ is the standard differential*
(ii) $d(\alpha\wedge\beta)=d\alpha\wedge\beta+(-1)^{\deg\alpha}\alpha\wedge d\beta$
(iii) $d\circ d=0$

- Nijenhuis tensor $N_J\colon\Lambda^{0,1}T^*M\to\Lambda^{2,0}T^*M$ such that $d^{2,-1}\alpha=-N_J(\alpha)$

Date: October 16, 2020. *E-mail*: l.foscolo@ucl.ac.uk.

- J integrable if $N_J \equiv 0$

Exercise 5.2

Theorem 2.1 (Newlander–Nirenberg, 1957)**.** *The almost complex structure J is induced by a holomorphic atlas on M iff it is integrable.*

Exercises 5.3 and 5.4

3. Symplectic and Kähler manifolds

M^{2n} with non-degenerate 2-form ω

- ω symplectic if $d\omega = 0$
- Darboux Theorem: existence of coordinate charts $\varphi\colon U \to \mathbb{R}^{2n}$ such that $(\varphi^{-1})^*\omega = \omega_0$
- Moser Stability Theorem: M closed, $\{\omega_t\}_{t\in[0,1]}$ family of cohomologous symplectic forms $\Rightarrow \exists$ family $\{\psi_t\}_{t\in[0,1]}$ of diffeomorphisms with $\psi_0 = \mathrm{id}$ and $\psi_t^*\omega_t = \omega_0$.

Exercise 5.5
Recall: $d \circ d = 0 \rightsquigarrow$ deRham complex of M^n

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0$$

and deRham cohomology $H^k_{dR}(M) = \{\alpha \in \Omega^k(M) \,|\, d\alpha = 0\}/d\,\Omega^{k-1}(M)$.

Definition 3.1. An almost Hermitian manifold (M^{2n}, g, J, ω) is Kähler iff J is integrable and ω is symplectic. In this case ω is often referred to as the Kähler form.

Exercises 5.6, 5.7, 5.8 and 5.9

- (M^{2n}, J) complex: $N_J \equiv 0 \Rightarrow \overline{\partial} \circ \overline{\partial} = 0 \rightsquigarrow$
- Dolbeault complex

$$0 \to \Omega^{p,0}(M) \xrightarrow{\overline{\partial}} \Omega^{p,1}(M) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{p,n}(M) \to 0$$

  and Dolbeault cohomology $H^{p,q}_{\overline{\partial}}(M) = \{\alpha \in \Omega^{p,q}(M) \,|\, \overline{\partial}\alpha = 0\}/\overline{\partial}\,\Omega^{p,q-1}(M)$
- Hodge theory for closed Kähler manifolds: $H^{p,q}_{\overline{\partial}}(M) = \overline{H^{q,p}_{\overline{\partial}}(M)}$ and $H^k_{dR}(M;\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(M)$

  Proof. Some global analysis on manifolds needed for this:
  - M closed: $H^k_{dR}(M) \xleftrightarrow{1:1} \mathcal{H}^k(M) = \{\alpha \in \Omega^k(M) \,|\, \triangle\alpha = (dd^* + d^*d)\alpha = 0\}$
  - (M, g, J, ω) Kähler $\Rightarrow$ J parallell with respect to the Levi-Civita connection of g $\Rightarrow$ $\triangle$ preserves type (p,q)-decomposition □

4. Holomorphic bundles and Chern connections

(M, J) complex manifold + $E \to M$ smooth complex vector bundle of rank k

- connection: $\mathbb{C}$-linear map $\nabla\colon C^\infty(M;E) \to \Omega^1(M;E) = C^\infty(M; T^*M \otimes E)$ such that

$$\nabla(fs) = df \otimes s + f\nabla s$$

  for every $s \in C^\infty(M;E)$ and $f \in C^\infty(E;\mathbb{C})$
- in a local trivialisation (or gauge) $E|_{\mathcal{U}} \simeq \mathcal{U} \times \mathbb{C}^k$: $\nabla s = ds + As$ for $A \in \Omega^1(\mathcal{U};\mathbb{C}) \otimes \mathfrak{gl}(k,\mathbb{C})$
- change of gauge: $g\colon \mathcal{U} \to \mathrm{GL}(k,\mathbb{C}) \rightsquigarrow g^*\nabla = d + gAg^{-1} - (dg)g^{-1}$, *i.e.* $g^*\nabla(s) = g\nabla(g^{-1}s)$
- (E, h) Hermitian bundle: ∇ unitary if $\nabla h = 0 \iff$ in a local unitary gauge $\nabla = d + A$ with $A \in \Omega^1(M) \otimes \mathfrak{u}(k)$
- Cauchy–Riemann operator on E: $\mathbb{C}$-linear map $\overline{\partial}_{\mathcal{E}}\colon C^\infty(M;E) \to \Omega^{0,1}(M;E)$ such that

$$\overline{\partial}_{\mathcal{E}}(fs) = \overline{\partial}f \otimes s + f\,\overline{\partial}_{\mathcal{E}}s$$

  for every $s \in C^\infty(M;E)$ and $f \in C^\infty(E;\mathbb{C})$.
- connection $\nabla \rightsquigarrow$ Cauchy–Riemann operator $\overline{\partial}_{\mathcal{E}} = \nabla^{0,1}$

- Chern connection: Hermitian metric h + Cauchy-Riemann operator $\overline{\partial}_{\mathcal{E}}$ on $E \Longrightarrow \exists!$ unitary connection ∇ with $\overline{\partial}_{\mathcal{E}} = \nabla^{0,1}$

  Proof. In a local unitary gauge $(E,h)|_{\mathcal{U}} \simeq \mathcal{U} \times \mathbb{C}^k$ with standard Hermitian metric on $\mathbb{C}^k$
  - $\overline{\partial}_{\mathcal{E}} = \overline{\partial} + \alpha$ for some $\alpha \in \Omega^{0,1}(\mathcal{U};\mathbb{C}) \otimes \mathfrak{gl}(k,\mathbb{C})$
  - set $\nabla = d + A$ where $A = \alpha - \alpha^*$ □

Recall: curvature $F_\nabla \in \Omega^2(M; \operatorname{End} E)$ of connection ∇

(i) *locally $F_\nabla = dA + A \wedge A$ if $\nabla = d + A$*
(ii) *$F_\nabla = d_\nabla \circ d_\nabla$ where $d_\nabla \colon \Omega^k(M;E) \to \Omega^{k+1}(M;E)$ is the exterior covariant differential*
(iii) *curvature as obstruction to integrability: $F_\nabla = 0 \Rightarrow$ locally $E|_{\mathcal{U}} \simeq \mathcal{U} \times \mathbb{C}^k$ with $\nabla = d$*

$\mathcal{E} \to M$ holomorphic vector bundle with underlying smooth complex vector bundle E

- Cauchy–Riemann operator $\overline{\partial}_{\mathcal{E}}$: in local holomorphic trivialisation $\overline{\partial}_{\mathcal{E}} = \overline{\partial}$
- Cauchy–Riemann operator $\overline{\partial}_{\mathcal{E}} \rightsquigarrow$ sheaf of "holomorphic" sections $\mathcal{O}(\mathcal{E}) = \ker \overline{\partial}_{\mathcal{E}}$
- $\overline{\partial}_{\mathcal{E}} \circ \overline{\partial}_{\mathcal{E}} = 0 \Longleftrightarrow E$ has the structure of a holomorphic bundle $\mathcal{E}$

  Proof. See §2.2.2 in Donaldson–Kronheimer. □

Exercise 5.10

5. Exercises

Exercise 5.1 (More linear algebra)**.** Let V be a finite-dimensional vector space endowed with a positive definite symmetric bilinear form g_0.

(i) Show that $A \mapsto g_0(A\,\cdot\,,\,\cdot\,)$ defines an isomorphism between the vector space $\mathfrak{so}(V,g_0)$ of skew-symmetric endomorphisms of (V,g_0) (*i.e.* endomorphisms $A\colon V \to V$ such that $g_0(Au,v) = -g_0(u,Av)$ for all $u,v \in V$) and Λ^2V^*.
(ii) Show that ω_0 is the image of J_0 under the isomorphism defined above.
(iii) Deduce that any two of g_0, J_0, ω_0 determine the third one and that $\mathrm{U}(n) = \mathrm{SO}(2n) \cap \mathrm{GL}(n,\mathbb{C}) = \mathrm{SO}(2n) \cap \mathrm{Sp}(2n,\mathbb{R})$.

Exercise 5.2 (The Nijenhuis tensor)**.** Let J be an almost complex structure on M.

(i) Verify that $d^{2,-1}$ is tensorial, *i.e.* $d^{2,-1}(f\alpha) = f\,d^{2,-1}\alpha$ for any function f and $(0,1)$-form α. Deduce the existence of the Nijenhuis tensor N_J, *i.e.* the existence of a section N_J of $\mathrm{Hom}(\Lambda^{0,1}T^*M, \Lambda^{2,0}T^*M)$ such that $d^{2,-1}\alpha = -N_J(\alpha)$ for every $(0,1)$-form α.
(ii) Under the isomorphism
$$\mathrm{Hom}(\Lambda^{0,1}T^*M, \Lambda^{2,0}T^*M) \simeq \left(\Lambda^{0,1}T^*M\right)^* \otimes \Lambda^{2,0}T^*M \simeq T^{0,1}M \otimes \Lambda^{2,0}T^*M$$
identify N_J with the skew-symmetric map $N_J\colon T^{1,0}M \times T^{1,0}M \to T^{0,1}M$ defined by $N_J(X,Y) = [X,Y]^{0,1}$.
(Hint: use the fact that $d\alpha(X,Y) = X\cdot\alpha(Y) - Y\cdot\alpha(X) - \alpha([X,Y])$ for every 1-form α and vector fields X, Y.)

Exercise 5.3 (The Newlander–Nirenberg Theorem)**.** In this exercise you discuss some easy aspects of the proof of the Newlander–Nirenberg Theorem 2.1.

(i) Show the easy implication of the Newlander–Nirenber Theorem: if J is induced by a holomorphic atlas then J must be integrable.
(ii) Show that the converse "hard" implication amounts to showing that, assuming J is integrable, for every point $p \in M$ there exists an open set $\mathcal{U} \subset M$ containing p and J–holomorphic functions $z_1,\dots,z_n\colon \mathcal{U} \to \mathbb{C}$ such that $(z_1,\dots,z_n)\colon \mathcal{U} \to \mathbb{C}^n$ is a diffeomorphism onto its image.

Exercise 5.4 (Almost complex structures on spheres)**.** In this exercise you construct almost complex structures on $\mathbb{S}^2$ and $\mathbb{S}^6$. The latter is non-integrable and deciding whether $\mathbb{S}^6$ carries an integrable almost complex structure is a famous open problem. It can be shown (Borel–Serre, 1953) that spheres of dimensions $\neq 2, 6$ cannot carry almost complex structures.

(i) Let $\mathbb{S}^2$ denote the unit sphere $\mathbb{S}^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ in $\mathbb{R}^3$. Given $x \in \mathbb{S}^2$ identify $T_x\mathbb{S}^2$ with the plane $x^\perp \subset \mathbb{R}^3$. Show that the formula

$$J_x(u) = x \times u,$$

where $\times$ denotes the cross product on $\mathbb{R}^3$, defines an almost complex structure on $\mathbb{S}^2$. Show that this almost complex structure is integrable.
(Hint: you can answer the last question without doing any computation.)

(ii) The existence of the octonions $\mathbb{O}$ implies the existence of a cross product on $\mathbb{R}^7 = \operatorname{Im}\mathbb{O}$, a bilinear alternating map $\times\colon \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ with the properties that $u \times v$ is orthogonal to u and v and has norm $|u \times v|^2 = |u|^2|v|^2 - (u \cdot v)^2$. One simply sets $u \times v = \operatorname{Im}(uv)$, where uv is octonionic multiplication. Show that the formula of part (i) defines an almost complex structure J on the 6-sphere $\mathbb{S}^6$ and that the non-associativity of the 7-dimensional cross product makes J non-integrable.

Exercise 5.5 (Moser's trick)**.** In this exercise you prove the Moser Stability Theorem.

(i) We begin with some observations about time-dependent vector fields. Let $\psi\colon [0,1] \times M \to M$ be a smooth map such that $\psi_t = \psi(t, \cdot)\colon M \to M$ is a diffeomorphism for every t. Define a 1-parameter family of vector fields $\{X_t\}_{t\in[0,1]}$ on M by $X_t = d_{(t,\cdot)}\psi\left(\frac{d}{dt}\right)$. Conversely, if M is closed (or if the vector fields are compactly supported) a family of vector fields $\{X_t\}_{t\in[0,1]}$ generates a family $\{\psi_t\}_{t\in[0,1]}$ of diffeomorphisms with $\frac{d}{dt}\psi_t = X_t \circ \psi_t$ and $\psi_0 = \mathrm{id}$. Let $\{\alpha_t\}_{t\in[0,1]}$ be a smooth 1-parameter family of k-forms on M. Show that

$$\frac{d}{dt}\psi_t^*\alpha_t = \psi_t^*\left(\tfrac{d}{dt}\alpha_t + \mathcal{L}_{X_t}\alpha_t\right).$$

(ii) Let $\{\omega_t\}_{t\in[0,1]}$ be a smooth family of symplectic forms and suppose that $\frac{d}{dt}\omega_t = d\mu_t$ for a smooth family $\{\mu_t\}_{t\in[0,1]}$ of 1-forms. Construct a family $\{X_t\}_{t\in[0,1]}$ of vector fields such that $\mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t = 0$.
(Hint: recall Cartan's Magic Formula $\mathcal{L}_X\alpha = d(X \lrcorner \alpha) + X \lrcorner d\alpha$.)

(iii) Deduce Moser's Stability Theorem from parts (i) and (ii).

(iv) Let (M, J) be an almost complex manifold and suppose that ω_0 and ω_1 are two symplectic forms on M compatible with J. Show that $\omega_t = (1-t)\omega_0 + t\omega_1$ is a family of symplectic forms compatible with J.
(Hint: consider the corresponding family of Riemannian metrics using the fact that the set of positive definite bilinear forms on $\mathbb{R}^n$ is convex.)

(v) Let (M, g_0, J_0, ω_0) be an almost Hermitian manifold with ω symplectic (such manifolds are sometimes called almost Kähler). Consider the sets

$$\mathcal{J}(\omega_0) = \{\omega_0\text{–compatible almost complex structure } J\},$$

$$\mathcal{H}(J_0, [\omega_0]) = \{J_0\text{–compatible symplectic form } \omega \text{ with } [\omega] = [\omega_0] \in H^2_{dR}(M)\}$$

and the subset $\mathcal{Y}(J_0, \omega_0) \subset \mathrm{Diff}_0(M) \times \mathcal{H}(J_0, [\omega_0])$ consisting of pairs (f, ω) where f is a diffeomorphism isotopic to the identity such that $f^*\omega = \omega_0$. Use parts (iii) and (iv) to construct a map $\Phi\colon \mathcal{Y}(J_0, \omega_0) \to \mathcal{J}(\omega_0)$.

Exercise 5.6 (Projective manifolds)**.** In this exercise you show that complex projective space $\mathbb{CP}^n$ and its holomorphic submanifolds are Kähler.

(i) Show that $\omega_{\mathrm{FS}} = i\partial\overline{\partial}\log\left(1 + |z|^2\right)$ is a Kähler form on $\mathbb{C}^n$. Here $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$.

(ii) Show that if $\varphi(z_1, \ldots, z_n) = z_1^{-1}(1, z_2, \ldots, z_n)$ then $\varphi^*\omega_{\mathrm{FS}} = \omega_{\mathrm{FS}}$.

(iii) Show that the charts $(\mathcal{U}_i, \varphi_i)$ defined by

$$\varphi_i \colon \mathcal{U}_i = \{[z_0 : \cdots : z_n] \,|\, z_i \neq 0\} \to \mathbb{C}^n, \qquad \varphi_i([z_0 : \cdots : z_n]) = z_i^{-1}(z_0, \ldots, \check{z}_i, \ldots, z_n),$$

where $\check{z}_i$ means that we drop the ith coordinates, form a holomorphic atlas on $\mathbb{CP}^n$.

(iv) Show that the formula $\omega = \varphi_i^* \omega_{\mathrm{FS}}$ over $\mathcal{U}_i$ defines a Kähler form on $\mathbb{CP}^n$, called the Fubini–Study Kähler form.

(v) Let M be a complex submanifold of $\mathbb{CP}^n$. Show that M is Kähler. *(Hint: consider the restriction of the Fubini–Study Kähler form to M.)*

Exercise 5.7 (Hopf surface)**.** In this exercise you show that there are complex manifolds that cannot be Kähler.

(i) Let (M, ω) be a closed symplectic manifold. Show that the deRham cohomology class $[\omega] \in H^2_{dR}(M)$ cannot vanish. *(Hint: use Stokes' Theorem and the fact that ω^n is a volume form.)*

(ii) Fix a real number $s \neq 0$ and let $\mathbb{Z}$ act on $\mathbb{C}^2 \setminus \{0\}$ by $n \cdot (z_1, z_2) = (e^{ns} z_1, e^{ns} z_2)$. Define $M^4 = \mathbb{C}^2 \setminus \{0\} / \mathbb{Z}$. Show that M is a complex manifold.

(iii) Show that M is diffeomorphic to $S^1 \times S^3$ and conclude that M cannot be Kähler. *(Hint: work in spherical coordinates on $\mathbb{C}^2 \setminus \{0\} \simeq \mathbb{R}^+ \times \mathbb{S}^3$.)*

Exercise 5.8 (Kodaira–Thurston manifold)**.** In this exercise you show that there are symplectic manifolds that cannot be Kähler.

(i) Show that if (M, g, J, ω) is a closed Kähler manifold then $H^1(M; \mathbb{R})$ is even dimensional.

(ii) Let $(n, m) \in \mathbb{Z}^2$ act on $\mathbb{R}^2$ by translation $(x, y) \mapsto (x + n, y + m)$ and on $\mathbb{T}^2$ via the matrix $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$. Let M be the 4-manifold $M = (\mathbb{R}^2 \times \mathbb{T}^2)/\mathbb{Z}^2$. Show that the standard symplectic form on $\mathbb{R}^2 \times \mathbb{T}^2$ descends to M and defines a symplectic form on M.

(iii) Calculate the first deRham cohomology group of M and conclude that M cannot carry any Kähler structure. *(Hint: you can either study $\mathbb{Z}^2$–invariant closed 1-forms on $\mathbb{R}^2 \times \mathbb{T}^2$ or calculate the fundamental group of M, thus its first homology, and then use the fact that deRham cohomology is dual to homology with real coefficients.)*

Exercise 5.9 (Hyperkähler 4-manifolds)**.** A hyperkähler triple on a 4-manifold is a triple $(\omega_1, \omega_2, \omega_3)$ of symplectic forms satisfying

$$\omega_i \wedge \omega_j = \delta_{ij} \tfrac{1}{3} \left(\omega_1^2 + \omega_2^2 + \omega_3^2 \right).$$

(i) Work on $\mathbb{R}^4$ with coordinates (x_0, x_1, x_2, x_3). Show that the forms

$$\omega_i = dx_0 \wedge dx_i + dx_j \wedge dx_k,$$

where (ijk) is a cyclic permutation of (123), define a hyperkähler triple. Show also that you can identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$ so that the almost complex structures corresponding to ω_1, ω_2 and ω_3 using the standard inner product on $\mathbb{R}^4$ as in Exercise 5.1 are, respectively, left multiplications by i, j and k.

(ii) Let $(M^4, \omega_1, \omega_2, \omega_3)$ be a manifold endowed with a hyperkähler triple. Write $\omega = \omega_1$ and $\omega_c = \omega_2 + i\omega_3$. Show that $\omega_c \wedge \omega_c = 0$ and deduce the existence of an almost complex structure J for which a 1-form α is of type $(1, 0)$ if and only if $\alpha \wedge \omega_c = 0$. *(Hint: you can use the fact that $\omega_c \wedge \omega_c = 0$ if and only if ω_c is decomposable,* i.e. *it can locally be written as $\omega_c = \theta_1 \wedge \theta_2$ for linearly independent complex 1-forms θ_1, θ_2.)*

(iii) Show that J is integrable. *(Hint: observe that $d\overline{\omega_c} = 0$ and differentiate the relation $\alpha \wedge \overline{\omega_c} = 0$ valid for any $(0, 1)$-form α to deduce that $d^{2,-1}\alpha = 0$.)*

(iv) Show that (ω, J) are compatible and that, denoting by g the resulting metric, (M, g, J, ω) is Kähler.
(Hint: show that $\omega \wedge \omega_c = 0$ and deduce that ω is of type $(1,1)$ with respect to J. Positivity of the resulting metric can be shown by arguing that on each tangent space $\omega_1, \omega_2, \omega_3$ must be linearly equivalent to the hyperkähler triple of part (i).)

Exercise 5.10 (Bundles and connections)**.** This problem is a collection of three separate questions about holomorphic bundles and connections.

(i) Show that any choice of Cauchy–Riemann operator on a complex vector bundle E over a Riemann surface defines a holomorphic structure on E.

(ii) Let (E, h) be a Hermitian vector bundle over a complex manifold and let ∇ be a unitary connection on E. Decompose the bundle-valued 2-form F_∇ into (p, q)–types: $F_\nabla = F_\nabla^{2,0} + F_\nabla^{1,1} + F_\nabla^{0,2}$. Show that $\overline{\partial}_{\mathcal{E}} = \nabla^{0,1}$ satisfies $\overline{\partial}_{\mathcal{E}} \circ \overline{\partial}_{\mathcal{E}} = 0$ if and only if $F_\nabla = F_\nabla^{1,1}$.

(iii) Set $\mathcal{O}_{\mathbb{CP}^n}(-1) = \{([z], v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} \,|\, v \in \mathbb{C}z\}$.
  (a) Show that the projection onto the first factor induces on $\mathcal{O}_{\mathbb{CP}^n}(-1)$ the structure of a holomorphic line bundle over $\mathbb{CP}^n$.
  (b) Endow $\mathbb{C}^{n+1}$ with its standard Hermitian metric and denote by h the induced Hermitian metric on $\mathcal{O}_{\mathbb{CP}^n}(-1)$. Calculate the curvature of its Chern connection.

6. Some references

- J.-P. Demailly, Complex Analytic and Differential Geometry, Sections V.1–7, VIII.8
- A. Cannas da Silva, Lectures on symplectic geometry, Chapters 6–8 and 12–17
- S. Donaldson and P. Kronheimer, The geometry of 4-manifolds, Sections 2.1.5 and 2.2