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PART I: Desingularizations of Calabi-Yau 3-folds w/ conical singul.

Main reference: Yatwin Chan, " , I & II, Q. J. Math (2006/2009)

§1. CY^3 in terms of torsion-free $SU(3)$-structures

Def An $SU(3)$-structure on a 6-mfld M^6 is a pair $(\omega, \Omega) \in \Omega^2(M;\mathbb{R}) \oplus \Omega^3(M,\mathbb{C})$ s.t. $\forall x \in M$ there exists $T_xM \simeq \mathbb{R}^6 \simeq \mathbb{C}^3$ w/

$$\omega_x = \frac{i}{2}(dz_j \wedge d\bar{z}_j) \qquad \Omega_x = dz_1 \wedge dz_2 \wedge dz_3$$

Equivalently, ω is a non-degenerate 2-form, $\mathrm{Re}\,\Omega$ is a stable 3-form in the sense of Hitchin, $\mathrm{Im}\,\Omega$ is the "dual" of $\mathrm{Re}\,\Omega$ and (i) $\omega \wedge \mathrm{Re}\,\Omega = 0$ (ii) $\frac{\omega^3}{3!} = \frac{1}{4}\mathrm{Re}\,\Omega \wedge \mathrm{Im}\,\Omega$

Rmks (a) If (ω, Ω) is an $SU(3)$-str. then $(\omega, e^{i\theta}\Omega)$ also is

(b) Ω defines an almost cplx str. J by $\theta \in \Omega^{1,0}(M,\mathbb{C})$ iff $\theta \wedge \Omega = 0$. Then ω is a $(1,1)$-form wrt J

Def (M^6, ω, Ω) is a Calabi-Yau 3-fold (CY^3) iff $d\omega = 0 = d\Omega$

Rmks: (a) The almost cplx structure is integrable iff $d\Omega = 0$

(b) $SU(3) \subset SO(6) \rightsquigarrow$ metric g w/ $\mathrm{Hol}(g) \subseteq SU(3)$ if (ω, Ω) is CY.

~~Conversely~~

§2. CY cones & Sasaki-Einstein mflds

$M^6 = C(\Sigma)$ $\quad (\omega, \Omega) \rightsquigarrow g = dr^2 + r^2 g_\Sigma$

$$\omega = r\,dr \wedge \eta + r^2 \omega_1 \qquad \Omega = r^2(\omega_2 + i\omega_3) \wedge (dr + ir\eta)$$

where $(\eta, \omega_1, \omega_2, \omega_3)$ is an $SU(2)$-structure on the 5-mfld Σ:

(i) η is a nowhere vanishing 1-form

(ii) $\omega_1, \omega_2, \omega_3$ are 2-forms s.t. $\omega_i \wedge \omega_j = \delta_{ij} v$ for a fixed 4-form s.t. $\eta \wedge v \neq 0$ at every pt.

Rmks: η defines at each pt a splitting $T_x\Sigma = \mathbb{R} \oplus \ker\eta_x$ ~~and~~
$\ker\eta_x$ is naturally an Euclidean vector space ($SU(2) \subset SO(4) \simeq SU(2)\cdot SU(2) \subset SO(5)$)
and $(\omega_1, \omega_2, \omega_3)$ gives an orthonormal triv. of $\Lambda^+(\ker\eta_x)^*$

Def $(\Sigma^5, \eta, \omega_1, \omega_2, \omega_3)$ is a Sasaki-Einstein 5-mfld iff $(C(\Sigma), \omega, \Omega)$ is a CY^3. Equivalently, $d\eta = 2\omega_1$, $d\omega_2 = -3\eta\wedge\omega_3$, $d\omega_3 = 3\eta\wedge\omega_2$

Def The Reeb vector field ξ is the ~~dual to~~ unique vector field in $(\ker\eta)^\perp$ w/ $\eta(\xi) = 1$.

Rmk: (a) Regular, quasi-regular & irregular cones. / (b) Flow of the Reeb vector field: $\mathcal{L}_\xi \Omega = 3i\Omega$

Examples: $\mathbb{C}^3 = C(S^5) = \frac{1}{3} K^\times_{\mathbb{P}^2}$

conifold $\frac{1}{2} K^\times_{\mathbb{P}^1\times\mathbb{P}^1}$ $\quad \{z_1^2 + \dots + z_4^2 = 0\} \subset \mathbb{C}^4$

§3. CY^3 w/ conical singularities

Def Let $(\bar{M}_0, \omega_0, \Omega_0)$ be a CY^3 w/ only isolated singularities $\{x_1, \dots, x_n\}$
We say that $\bar{M}_0$ is a CY^3 w/ conical singularities ~~iff~~ w/ rate $\nu > 0$ modelled on the CY cone $(C_i, \omega_i, \Omega_i)$ ($C_i = C(\Sigma_i)$) at x_i if $\exists\, \epsilon > 0$, disjoint neighbourhoods U_i of x_i in M_0, and diffeos
$\Phi_i : \Sigma_i \times (0,\epsilon) \longrightarrow U_i \setminus \{x_i\}$ s.t.

$$|\nabla^k(\Phi_i^*\omega_0 - \omega_i)|_{g_{C_i}} = O(r^{\nu-k}) \quad |\nabla^k(\Phi_i^*\Omega_0 - \Omega_i)|_{g_{C_i}} = O(r^{\nu-k})$$

Assumption: CY^3 w/ conical singularities do exists
(For the moment known only if $C_i = \mathbb{C}^3/\Gamma_i$ is an orbifold)

Aim: desingularise M_0 by gluing in AC CY^3 Y_i asymptotic to C_i.

Def Let $(C=C(\Sigma), \omega_C, \Omega_C)$ be a CY cone. A CY mfld (Y, ω, Ω) is said to be asymptotically conical (AC) w/ asymptotic cone C w/ rate $\lambda<0$ if $\exists$ $K \overset{\text{cpct}}{\subseteq} Y$, $R>0$ & a diffeo $\Phi: \Sigma \times (R,\infty) \longrightarrow Y \setminus K$ s.t.

$$|\nabla^k(\Phi^*\omega - \omega_C)|_{g_C} = O(r^{\lambda-k}) \quad |\nabla^k(\Phi^*\Omega - \Omega_C)|_{g_C} = O(r^{\lambda-k})$$

Thm (Chan) Let $(M_0, \omega_0, \Omega_0)$ be a cpct CY^3 w/ conical singularities $\{x_1, \dots, x_n\}$ w/ rate $\nu_i>0$ each modelled on a CY cone $(C_i = C(\Sigma_i), \omega_{C_i}, \Omega_{C_i})$. $\forall i$ let $(Y_i, \omega_i, \Omega_i)$ be AC CY asymptotic to the cone C_i w/ rate $\lambda_i \leq -3$.

(i) If all $\lambda_i<-3$ then a desingularisation of M_0 w/ bubbles Y_i at x_i always exists.

(ii) If at least one $\lambda_i=3$ then there is a necessary & sufficient topological condition for a smoothing to exist. (More details later)

Examples:

(i) Crepant resolutions

$\pi: M \to C$ crepant resolution : π is an isom. outside of $E = Exc(\pi)$ & $\pi^*\Omega_C$ extends to a holomorphic vol. form Ω on M
($C = C(\Sigma)$)

$$H^1(\Sigma)=0 \to H^2_c(M) \to H^2(M) \to H^2(\Sigma) \to H^3_c(M) \simeq H_3(M) \simeq H_3(E) \to \dots$$

Thm (Goto) $\forall$ class in $[\omega] \in H^2(M)$ containing a Kähler metric, then there exists an AC CY metric in $[\omega]$ of rate $\lambda=-2$. If $[\omega] \in H^2_c(M)$ then the rate is $\lambda=-6$.

Concrete examples: Calabi's CY structure on K_D, D KE dPezzo

In particular, the orbifold singularity $\mathbb{C}^3/\mathbb{Z}_3 = K^\times_{\mathbb{P}^2}$ can always be resolved.

(ii) Affine smoothings of complete intersection singularities

(van Coevering): every CY cone can be realised as an algebraic variety in $\mathbb{C}^N$ in such a way that the natural $\mathbb{C}^*$ on C is the restriction of a diagonal $\mathbb{C}^*$-action on $\mathbb{C}^N$ w/ weights $w_1, \ldots, w_N > 0$

Suppose $C \subset \mathbb{C}^N$ is a complete intersection & it is regular

($C = \frac{1}{k} K_D^\times$ where $D = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, d\mathbb{P}_4, d\mathbb{P}_5$, $Bl_6 \mathbb{P}^2, Bl_5 \mathbb{P}^2$)

M affine smoothing of $C = C(\Sigma) = \frac{1}{k} K_D^\times$

Prop: (Conlon-Hein)
(i) $M = X \setminus D$ where X Fano of index ≥ 2 & $-K_X = (k+1)[D]$ (
(ii) $H^2_c(M) = 0$ & $b^2(M) = b^2(X) - 1$
(iii) Existence of AC CY metric

Thm (Chi Li) The rate of convergence $\lambda = \max\{-6, -3\frac{l}{k}\}$ where $l = \min(d_i - e_i) \geq 1$. (For generic deformations in the examples $\lambda = -3$)

Explicit examples:

(a) conifold $\{z_1^2 + z_2^2 + z_4^2 + z_5^2 = \epsilon\} = M_\epsilon \simeq T^*S^3$ $\qquad C = \frac{1}{2} K_{\mathbb{P}^1 \times \mathbb{P}^1}^\times$

cohomogeneity one AC CY str. (Candelas - de la Ossa, Stenzel)

$\lambda = -3$

(b) cubic cone $z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0$

Any affine smoothing is $\sum_i z_i^3 + \sum_{1 \leq i < j \leq 4} t_{ij} z_i z_j + \sum_{i=1}^4 t_i z_i = \epsilon$

$\lambda = -6$ if all $t_{ij} = 0$.

Rmk. Corresponding to Chan's Theorem in this case we have theorems in algebraic geometry

Friedman (1991): $\exists$ topological obstruction to smooth ODP

(the ~~obstruction~~ obstruction coincides w/ Chan's one, cf. later)

Gross (1997): any ~~other~~ complete intersection singularity different from a node can be smoothed out.

(iii) Small resolutions

Case not covered by Chan's Thm is when $\lambda = -2$ for a crepant res. In particular, the conifold has two ~~explicit~~ small resolutions:

$Bl_0 \mathbb{C}^3 \supset \hat{C}$ exceptional div. $\simeq \mathbb{P}^1 \times \mathbb{P}^1$ & can contract ~~one of the~~ either of the two-rulings. On each of this $\exists$ an explicit AC CY metric w/ rate $\lambda = -2$ $\left(H^2_c = 0 \text{ so } H^2 \simeq H^2(\Sigma) \right)$ which is cohomogeneity one. Now, if M_0 is a cpct nodal CY then the question is when there exists a Kähler ($\Leftrightarrow$ projective) small resolution.

§4. Aspects of the proof of Chan's Theorem

(A) The unobstructed case: gluing $SU(3)$-structures

Prop. (Darboux Thm)

(i) Let $(M_0, \omega_0, \Omega_0)$ be a CY w/ conical sing. $x_1, \ldots, x_n$ modelled on CY cones $(C_i, \omega_{C_i}, \Omega_{C_i})$ w/ rate $\nu > 0$. Then $\exists\ \epsilon > 0$, $x_i \in U_i \subset M_0$ and diffeo $\Phi_i : \Sigma_i \times (0, \epsilon) \longrightarrow U_i^*$ s.t. $\Phi_i^* \omega_0 = \omega_{C_i}$ & $|\nabla^k(\Phi_i^* \Omega_0 - \Omega_{C_i})|_{g_{C_i}} = O(r^{\nu - k})$

(ii) Let (Y, ω_Y, Ω_Y) be an AC CY^3 ~~w/~~ asymptotic to (C, ω_Y, Ω_Y) w/ rate $\lambda < -2$. Then $\exists\ R > 0$, $K \overset{\text{cpct}}{\subseteq} M$ & diffeo $\Phi : \Sigma \times (R, \infty) \longrightarrow Y \setminus K$ s.t.

$\Phi^* \omega_Y = \omega_C$ & $|\nabla^k(\Phi^* \Omega_Y - \Omega_C)|_{g_C} = O(r^{\lambda - k})$.

Suppose (Y, ω_Y, Ω_Y) is AC w/ rate $\lambda < -3$. Then $[\Omega_Y - \Omega_C] = 0 \in H^3(\Sigma, \mathbb{C})$

~~The gluing occurs~~ Fix $t > 0$ sufficiently small and rescale $(Y, t^2\omega_Y, t^3\Omega_Y)$

Gluing occurs in the region $t^\alpha < r_i < 2t^\alpha$, $\alpha \in (0,1)$, using diffeos given in the previous proposition.

$t^2\omega_Y = \omega_{C_i} = \omega_0$

$t^3\Omega_Y = \Omega_{C_i} + dA_i$. $\Omega_0 = \Omega_{C_i} + dB_i \rightsquigarrow \Omega_t = \Omega_{C_i} + d\left(\chi_i A_i + (1-\chi_i)B_i\right)$

The upshot is: • a smooth mfld M_t

• a closed non-degenerate 2-form ω_t

• two closed stable 3-form $\mathrm{Re}\,\Omega_t$, $\mathrm{Im}\,\Omega_t$

w/ $\omega_t \wedge \mathrm{Re}\,\Omega_t$, $2\omega_t^3 - 3\,\mathrm{Re}\,\Omega_t \wedge \widehat{\mathrm{Re}\,\Omega_t}$, $\mathrm{Im}\,\Omega_t - \widehat{\mathrm{Re}\,\Omega_t}$ small

In particular, $(\omega_t, \mathrm{Re}\,\Omega_t + i\,\mathrm{Im}\,\Omega_t)$ is close to define an $SU(3)$ structure.

but it satisfies the PDE!

Q: How to get an $SU(3)$-structure that almost satisfies the PDE? Rmk: No problem in the crepant resolution case.

There are many different ways to do this. Chan's solution: pass to G_2!

On $S^1_s \times M^6_t$ the 3-form $\varphi_t = ds \wedge \omega_t + \mathrm{Re}\,\Omega_t$

(i) is closed

(ii) defines a G_2-structure

In fact this corresponds to consider the $SU(3)$ structure $(\tilde{\omega}_t, \tilde{\Omega}_t)$ on M_t defined as follows:

$\omega_t \wedge \mathrm{Re}\,\Omega_t = \gamma_t \wedge \omega_t^2 \rightsquigarrow \mathrm{Re}\,\tilde{\Omega}_t = \mathrm{Re}\,\Omega_t - \gamma_t \wedge \omega_t \rightsquigarrow \mathrm{Im}\,\tilde{\Omega}_t$

$3\,\mathrm{Re}\,\tilde{\Omega}_t \wedge \mathrm{Im}\,\tilde{\Omega}_t = 2 f_t^6\, \omega_t^3 \rightsquigarrow \tilde{\omega}_t = f_t^2\, \omega_t$

Then applying Joyce to deform φ_t into a torsion-free G_2-structure corresponds to make $(\tilde{\omega}_t, \tilde{\Omega}_t)$ into a torsion-free $SU(3)$-structure.

(B) The obstructed case: constructed an approximate solution

On the AC CY (Y, ω_Y, Ω_Y) we have $[\Omega_Y - \Omega_C] \neq 0 \in H^3(\Sigma, \mathbb{C})$

Assume that $\exists$ closed primitive $(1,2)$-form ξ on C s.t.

$$|\nabla^k(\Omega_Y - \Omega_C - \xi)|_{g_C} = O(r^{-3-\delta-k}), \quad \delta > 0$$

In fact
$$\left.\begin{aligned} \mathrm{Re}\,\xi &= \tfrac{1}{r}\, dr \wedge \tau_1 - \eta \wedge \tau_2 \\ \mathrm{Im}\,\xi &= \tfrac{1}{r}\, dr \wedge \tau_2 - \eta \wedge \tau_1 \end{aligned}\right\} \text{ for } \tau_1, \tau_2 \text{ harmonic 2-forms on the SE } \Sigma^5$$

The idea is to look for a closed primitive $(1,2)$-form χ on $M_0 \setminus \{x_1, \dots, x_n\}$ s.t. $\chi = \xi_i + O(r^{-3+\delta})$ close to x_i (i.e. we infinitesimally deform the cplx str on M_0 so that it matches at higher order w/ that of Y_i)

To find χ we use weighted Sobolev spaces à la Lockart-McOwen.

Def $u \in L^p_\beta$ iff $\int (r^{-\beta} u)^p \, r^{-6} \, dv < +\infty$

Now, extend ξ_i to some 3-form χ' on M_0 w/ $|\nabla^k(\chi' - \xi_i)| = O(r^{-3+\delta})$

Look for η s.t. ($\chi = \chi' + \eta$ is closed) $\begin{cases} d\eta = -d\chi' \in L^p_{-4+\delta} \\ d^*\eta = 0 \end{cases}$

$\eta \in W^{1,q}_{-3+\delta}$. By duality coker $\simeq \ker(d + d^*)|_{L^2_{-2-\delta}}$ $\exists\, \alpha \in \Omega^4$

$$\alpha = r\, dr \wedge \eta \wedge \tau_i + O(r^{-2+\delta}) \text{ for some harmonic 2-form } \tau_i \text{ on } \Sigma_i$$

Then

$$\int d\chi' \wedge * \alpha = \sum_{i=1}^n \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(x_i)} \chi' \wedge * \alpha = \pm \sum_{i=1}^n \int \xi_i \wedge \tau_i$$

$$= \sum_i [\xi_i] \cup [\tau_i] \quad \tau_i \in \operatorname{im} H^2(M_0) \to \bigoplus_{i=1}^n H^2(\Sigma_i)$$

One can show (Stokes' Thm) that the annihilator of $\operatorname{im} H^2(M_0) \to \bigoplus_i H^2(\Sigma_i)$ is $\operatorname{im} H^3(M_0) \to \bigoplus_{i=1}^n H^3(\Sigma_i)$

So the necessary & sufficient condition for χ to exists is

$$\bigoplus_{i=1}^n [\xi_i] \cancel{\in \operatorname{im}} \, H^3(M_0).$$

Rmk: Friedman's result is stated in terms of a small resolution $\hat{M}$ of a nodal CY M_0. Let C_i be exp. curve over x_i.

Exact sequence for $(\hat{M}, \hat{M} \setminus \bigcup_{i=1}^{n} C_i)$:

$$
\begin{array}{ccccccc}
\cdots \to & H^3(\hat{M} \setminus \bigcup_i C_i) & \longrightarrow & H^4(\hat{M}, \hat{M} \setminus \bigcup_{i=1}^{n} C_i) & \to & H^4(\hat{M}) & \to \cdots \\
& \cong & & \cong \text{ Thom isom} & & \cong & \\
& H^3(M_0) & & \bigoplus_{i=1}^{n} H^0(C_i) & & H_2(\hat{M}) & \\
& & & \cong & & & \\
& & & \bigoplus_{i=1}^{n} H_2(C_i) & & & \\
& & & \cong & & & \\
& & & \bigoplus_{i=1}^{n} H^3(\Sigma_i) & & &
\end{array}
$$

Friedman's condition: $\sum_i \lambda_i [C_i] = 0 \in H_2(\hat{M})$, $\lambda_i \neq 0 \; \forall i$