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Harmonic Maps from surfaces to reductive homogeneous spaces

(after Burstall–Pedit)

References:

Burstall, Pedit, "Harmonic maps via AKS theory, §§ 3.1, 3.2 (up to Thm 3.2)

Burstall, Rawnsley, "Twistor theory for Riemannian symmetric spaces", Chapter 1

§. Harmonic Maps

$u : (\Sigma, g_\Sigma) \longrightarrow (N, g)$ smooth map

$du \in \Omega^1(\Sigma; u^*TN) \qquad E(u) = \frac{1}{2}\int_\Sigma |du|^2_{g_\Sigma \otimes g}\, dv_\Sigma$

u harmonic map $\iff$ u is a critical pt of E

$X \in \Omega^0(\Sigma; u^*TN) \rightsquigarrow u_t(x) = \exp^N_{u(x)}(tX(x))$

$\frac{d}{dt}\Big|_{t=0} E(u_t) = \int_\Sigma \langle d_{u^*\nabla} X, du \rangle_{g_N \otimes g}\, dv_\Sigma \qquad \nabla =$ LC connection on N

Euler-Lagrange eq.: $d^*_{u^*\nabla} du = 0$.

§. Geometry of homogeneous spaces

$G := \mathrm{Isom}(N, g)$ $G \circlearrowright N$ transitively $\qquad x_0 \in N \quad K := \mathrm{Stab}_{x_0} G \rightsquigarrow N \simeq G/K$

$\mathfrak{g} = \mathrm{Lie}(G) \qquad \mathfrak{k} = \mathrm{Lie}(K)$

Assume: G/K is reductive: $\exists$ Ad_K-inv. complement $\mathfrak{m}$ of $\mathfrak{k}$ in $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$

$[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$

Metric g on $N \longleftrightarrow \mathrm{Ad}_K$-inv. scalar product on $\mathfrak{m}$

Example (k-symmetric spaces)

$\tau \in \mathrm{Aut}(G) \quad \tau^k = \mathrm{id} \qquad (G^\tau)_0 \subset K \subset G^\tau$

$\rightsquigarrow \tau \in \mathrm{Aut}(\mathfrak{g})$

$\mathfrak{g} \otimes \mathbb{C} = \sum_{i \in \mathbb{Z}_k} \mathfrak{g}_i \qquad \mathfrak{g}_0 = \mathfrak{k} \otimes \mathbb{C} \qquad \mathfrak{m} \otimes \mathbb{C} = \sum_{\substack{i \in \mathbb{Z}_k \\ i \neq 0}} \mathfrak{g}_i$

$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \quad \overline{\mathfrak{g}_i} = \mathfrak{g}_{-i}$

$k = 2 \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$, G/K Riemannian symmetric space

One class of examples: $\underbrace{G \times \dots \times G}_{k\text{-times}} / \Delta G$

(a) The tangent bdle of a reductive homogeneous space

$G \xrightarrow{\pi} G/K = N$ principal K-bdle

V is a K-rep. $\rightsquigarrow$ associated bdle $G \times_K V \longrightarrow N$

$G \times_K V = (G \times V)/K$ where $k.(g, v) = (gk, k^{-1} \cdot v)$

$$\begin{array}{ccc} G\times\mathfrak{g} & \longrightarrow & TN \\ (g,\xi) & \longmapsto & \left(\pi(g),\ \frac{d}{dt}\Big|_{t=0} \exp\left(t\,Ad_g(X)\right)\cancel{g}\pi(g)\right)\end{array} \qquad \begin{array}{ccc} N\times\mathfrak{g} & & TN \\ \cong & & \cong \\ P: G\times_K\mathfrak{g} & \longrightarrow & G\times_K\mathfrak{m}\end{array}$$

descends to a map $G\times_K\mathfrak{g}\longrightarrow TN$ which induces an isomorphism $TN \simeq G\times_K\mathfrak{m}$

Rmk: $G\times_K\mathfrak{g}\simeq N\times\mathfrak{g}$. More in general, if V is a G-rep. then by restriction it is also a K-rep. $\rightsquigarrow$ $G\times_K V$.
But in this case $G\times_K V\simeq N\times V$ via $(g,v)\longmapsto(\pi(g), Ad_g(v))$

$$\beta: TN \xrightarrow{\sim} G\times_K\mathfrak{m} \hookrightarrow G\times_K\mathfrak{g} \xrightarrow{\sim} G\times\mathfrak{g} \xrightarrow{pr} \mathfrak{g}$$

The "Maurer-Cartan form" of N

θ = Maurer-Cartan form of G $\quad \theta: TG \xrightarrow{\sim} G\times\mathfrak{g}\xrightarrow{pr}\mathfrak{g}$
$\quad (R_g^*\theta = Ad_g^{-1}\theta)$

Maurer-Cartan eq.: $d\theta + \frac{1}{2}[\theta,\theta]=0$, i.e. $d+\theta$ is a flat connection on $G\times\mathfrak{g}\longrightarrow G$.

Lemma 1 $\pi^*\beta = Ad_g(\theta_{\mathfrak{m}})$

proof:

$\xi\in\mathfrak{g}$ $\qquad X := \frac{d}{dt}\Big|_{t=0}\exp\left(t\,Ad_g(\xi)\right)\cdot\pi(g)$

$$\tilde{X}_g = (L_g)_*\xi = \frac{d}{dt}\Big|_{t=0} g\exp(t\xi) = \frac{d}{dt}\Big|_{t=0}\exp(t\,Ad_g\xi)\,g$$

By definition: $\quad \theta_g(\tilde{X}) = \xi \qquad \beta_{\pi(g)}(X) = Ad_g(\xi_{\mathfrak{m}})$ ■

(b) The Levi-Civita connection of a reductive homogeneous space

On the bdle $G\times_K\mathfrak{g}\simeq N\times\mathfrak{g}$ there are 2 natural connections:

d = trivial connection

$\overline{\nabla}$ = canonical connection

$\theta_{\mathfrak{k}}: TG\xrightarrow{\theta}\mathfrak{g}\longrightarrow\mathfrak{k}$ connection 1-form on the principal K-bdle $G\xrightarrow{\pi}N$

$$\left(\begin{array}{l} \cdot\ X = \frac{d}{dt}\Big|_{t=0} g\exp(t\xi) = \frac{d}{dt}\Big|_{t=0}\exp(t\,Ad_g\xi) \Rightarrow \theta(X)=\xi \\ \cdot\ Ad_g(R_g^*\theta_{\mathfrak{k}}) = Ad_g(R_g^*\theta) - Ad_g(R_g^*\theta_{\mathfrak{m}}) = \theta-\theta_{\mathfrak{m}} = \theta_{\mathfrak{k}} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow Ad_g(R_g^*\theta)=\theta\ +\ \mathfrak{m}\text{ is }Ad_K\text{-inv.}\end{array}\right)$$

$\overline{\nabla}$ = associated connection on $G\times_K\mathfrak{g}$

Rmk: G-inv. section $\Longleftrightarrow$ $\overline{\nabla}$-parallel sections of whatever associated bundle to π.
In particular, $\overline{\nabla}$ is a metric connection.

Lemma 2 (i) $\forall s \in C^\infty(N;\mathfrak{g})$, $\overline{\nabla}_X s = X\cdot s - [\beta(X), s]$

(ii) $dP = [\mathrm{ad}_\beta, P]$

(iii) $d\beta = [\beta\wedge\beta] - \frac{1}{2}P[\beta,\beta]$

proof.

(i) $s: C^\infty(N;\mathfrak{g}) = \Gamma(N; N\times\mathfrak{g}) = \Gamma(N; G\times_K \mathfrak{g}) \rightsquigarrow \tilde{s}: G \longrightarrow \mathfrak{g}$ s.t.

$\tilde{s}(g) = \mathrm{Ad}_g^{-1} s(\pi(g))$

$d\tilde{s} = \mathrm{Ad}_g^{-1}\left(-[\mathrm{Ad}_g\theta, s] + ds\right) \Rightarrow d\tilde{s} + [\theta_{\mathfrak{k}}, \tilde{s}] = \mathrm{Ad}_g^{-1}\left(-[\mathrm{Ad}_g(\theta-\theta_{\mathfrak{k}}), s] + ds\right)$

(ii) P is G-inv. $\Rightarrow$ it is $\overline{\nabla}$-parallel: $dP + \mathrm{ad}_\beta\circ P + P\circ\mathrm{ad}_\beta = 0$

(iii) By Lemma 1 $\pi^*\beta = \mathrm{Ad}_g\theta_{\mathfrak{m}} \Rightarrow$

$\pi^* d\beta = \mathrm{Ad}_g\left([\theta\wedge\theta_{\mathfrak{m}}] + d\theta_{\mathfrak{m}}\right)$

θ flat: $\begin{cases} d\theta_{\mathfrak{k}} + \frac{1}{2}[\theta_{\mathfrak{k}},\theta_{\mathfrak{k}}] + \frac{1}{2}[\theta_{\mathfrak{m}},\theta_{\mathfrak{m}}]_{\mathfrak{k}} = 0 \\ d\theta_{\mathfrak{m}} + [\theta_{\mathfrak{k}},\theta_{\mathfrak{m}}] + \frac{1}{2}[\theta_{\mathfrak{m}},\theta_{\mathfrak{m}}]_{\mathfrak{m}} = 0 \end{cases}$

$\Rightarrow \quad \pi^* d\beta = \mathrm{Ad}_g\left([\theta_{\mathfrak{m}},\theta_{\mathfrak{m}}] - \frac{1}{2}[\theta_{\mathfrak{m}},\theta_{\mathfrak{m}}]_{\mathfrak{m}}\right)$ ∎

Using the projection P we obtain connections $\nabla^0 = P\circ d$ and $\nabla^1 = P\circ\overline{\nabla}$ on $TN \simeq G\times_K\mathfrak{m}$

Lemma 3 $s\in\Omega^0(N; N\times\mathfrak{g})$ s.t. $s(x)\in G\times_K\mathfrak{m}$ $\forall x\in N$

(i) $\nabla^0 s = ds - [\beta, s] + P[\beta, s]$

(ii) $\nabla^1 s = ds - [\beta, s]$

proof.

(i) $\nabla^0 s = P(ds) = ds - (dP)s \overset{\text{Lemma 2 (ii)}}{=} ds + [\beta, (Ps)] - P[\beta, s]$

$(d(Ps) = ds = (dP)s + P\,ds)$ ∎

(ii) $\nabla^1 = P\circ\overline{\nabla} = P\circ d + P\circ(\underbrace{\overline{\nabla} - d}_{=\,\mathrm{ad}_\beta})$

Lemma 4 Let T denote the torsion of ∇^1, then $\beta\circ T^1 = -\frac{1}{2}P[\beta,\beta]$

proof. $\beta(\nabla^1_X Y) = X\cdot\beta(Y) - [\beta(X),\beta(Y)] \Rightarrow \beta\circ T^1 = d\beta - [\beta,\beta]$ ∎

The Levi-Civita connection of N is $\frac{1}{2}(\nabla^0+\nabla^1) = \nabla^1 - \frac{1}{2}T^1 = d - [\beta,\cdot] - \frac{1}{2}P[\beta,\cdot]$

when we think $TN \subset N \times \mathfrak{g}$

§. Harmonic maps to homogeneous spaces

$u: \Sigma \longrightarrow N = G/K$

Euler-Lagrange eq.: $u^*\beta \in \Omega^1(\Sigma; \mathfrak{g})$ $\qquad P: \Sigma\times\mathfrak{g} \longrightarrow u^*(G\times_K \mathfrak{m})$

$$d * u^*\beta - [u^*\beta, *u^*\beta] - \frac{1}{2}P[u^*\beta, *u^*\beta] = 0$$

(Note: $[u^*\beta, *u^*\beta] = 0 \quad \forall$ $\mathfrak{g}$-valued 1-form)

Now assume u^*G is a trivial principal K-bdle (always true loc.)

Then we can lift u to $\tilde{u}: \Sigma \longrightarrow G$

$\alpha := \tilde{u}^*\theta \qquad u^*\beta = Ad_{\tilde{u}}\,\alpha_{\mathfrak{m}}$

$\alpha = \alpha_{\mathfrak{m}} + \alpha_{\mathfrak{k}}$

Euler-Lagrange eq.:

$$d*\alpha_{\mathfrak{m}} + [\alpha, *\alpha_{\mathfrak{m}}] - [\alpha_{\mathfrak{m}}, *\alpha_{\mathfrak{m}}] - \frac{1}{2}\underbrace{[\alpha_{\mathfrak{m}}, *\alpha_{\mathfrak{m}}]_{\mathfrak{m}}}_{\text{actually this is}} = 0$$

+ Maurer Cartan

$$\begin{cases} d*\alpha_{\mathfrak{m}} + [\alpha_{\mathfrak{k}}, *\alpha_{\mathfrak{m}}] - \frac{1}{2}[\alpha_{\mathfrak{m}}, *\alpha_{\mathfrak{m}}]_{\mathfrak{m}} = 0 \\ d\alpha_{\mathfrak{k}} + \frac{1}{2}[\alpha_{\mathfrak{k}}, \alpha_{\mathfrak{k}}] + \frac{1}{2}[\alpha_{\mathfrak{m}}, \alpha_{\mathfrak{m}}]_{\mathfrak{k}} = 0 \\ d\alpha_{\mathfrak{m}} + [\alpha_{\mathfrak{k}}, \alpha_{\mathfrak{m}}] + \frac{1}{2}[\alpha_{\mathfrak{m}}, \alpha_{\mathfrak{m}}]_{\mathfrak{m}} = 0 \end{cases}$$

Assume $\dim \Sigma = 2$ & Σ is a Riemann surface (conformal invariance of the energy fct.)

$\Sigma \times \mathfrak{g} \longrightarrow \Sigma \qquad A = d + \alpha_{\mathfrak{k}} \qquad \Phi \in \Omega^{1,0}(\Sigma; \mathfrak{m}\otimes\mathbb{C})$

$\alpha_{\mathfrak{m}} = \Phi + \overline{\Phi} \qquad *\alpha_{\mathfrak{m}} = -i\Phi + i\overline{\Phi}$

Rewrite the equations:

$$\begin{cases} (d\Phi + [\alpha_{\mathfrak{k}}, \Phi]) - (d\overline{\Phi} + [\alpha_{\mathfrak{k}}, \overline{\Phi}]) = 0 \\ d\alpha_{\mathfrak{k}} + \frac{1}{2}[\alpha_{\mathfrak{k}}, \alpha_{\mathfrak{k}}] + [\Phi, \overline{\Phi}]_{\mathfrak{k}} = 0 \\ (d\Phi + [\alpha_{\mathfrak{k}}, \Phi]) + (d\overline{\Phi} + [\alpha_{\mathfrak{k}}, \overline{\Phi}]) + [\Phi, \overline{\Phi}]_{\mathfrak{m}} = 0 \end{cases}$$

Assume: $[\Phi, \overline{\Phi}]_{\mathfrak{m}} = 0$ (e.g. N is a symmetric space, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$)

$F_A + [\Phi, \overline{\Phi}] = 0 \qquad \overline{\partial}_A \Phi = 0$

__Lemma 5__ $\forall\, \zeta \in S^1 \subset \mathbb{C}^*$ the connection $A_\zeta = A + \zeta \Phi + \bar{\zeta} \bar{\Phi}$ on $\Sigma \times \mathfrak{g}$ is flat.

proof. $F_{A_\zeta} = F_A + \zeta \bar{\partial}_A \Phi + \bar{\zeta}\, \partial_A \bar{\Phi} + [\Phi, \bar{\Phi}]$ ■

So to the harmonic map $u: \Sigma \longrightarrow N$ s.t. $u^* G$ is a trivial K-bdle & $[\tilde{u}^* \theta_{\mathfrak{m}}, \tilde{u}^* \theta_{\mathfrak{m}}]_{\mathfrak{m}} = 0$ for a lift of u to $\tilde{u}: \Sigma \longrightarrow G$ we associated a loop of flat connections A_ζ on the trivial bdle $\Sigma \times \mathfrak{g}$.

Conversely, if $A_\zeta = A + \zeta \Phi + \bar{\zeta} \bar{\Phi}$

A: connection on $\Sigma \times \mathfrak{k}$ $\qquad \Phi \in \Omega^{1,0}(\Sigma; \mathfrak{m})$

is a loop of flat connections s.t. $[\Phi, \bar{\Phi}]_{\mathfrak{m}} = 0$ & Σ is simply connected then we obtain a loop of harmonic maps $u_\zeta : \Sigma \longrightarrow G/K$:

indeed $\pi_1(\Sigma) = 1 \Rightarrow A_\zeta = \tilde{u}_\zeta^* \theta$ for some $\tilde{u}_\zeta : \Sigma \longrightarrow G$.

(sketch proof. Lift any curve γ in Σ to a curve in $\Sigma \times G$ parallel w/ respect to A_ζ by solving an O.D.E. Since A_ζ is flat one can check that this lift is independent of the homotopy class of γ. Since $\pi_1(\Sigma) = 1$ we see that we can define $\tilde{u}_\zeta$ by solving ODEs along curves)