Böhm: Einstein metrics on low-dimensional spheres and other low-dimensional spaces

§. Introduction: Einstein metrics of spheres

S^n metric of constant sectional curvature

1973, Jensen: squashed metric on $S^{4m+3} = Sp(m+1)/Sp(m) \xrightarrow{Sp(1)} \mathbb{HP}^m = Sp(m+1)/Sp(1)\times Sp(m)$

1978, Bourguignon-Karcher: third Einstein metric on $S^{15} = Spin(9)/Spin(7)$

$$S^{15} \xrightarrow{S^7} S^8 \simeq \mathbb{OP}^1$$

1982, Ziller: these are all the homogeneous Einstein metric on spheres.

1998, Böhm: ~~Thm~~ There exists infinitely many Einstein metrics on S^5, S^6, S^7, S^8 and S^9.

2005, Boyer-Galicki-Kollár: continuous family of **Sasaki-Einstein** metrics on odd-dimensional spheres (also exotic spheres)

§1. Cohomogeneity 1 Einstein metrics

$(M^{n+1}, \bar g)$ $\quad \bar\nabla$ = LC connection of $\bar g$

family of equidistant hypersurfaces in M:

$\phi : I\times P^n \longrightarrow M \qquad \phi^* \bar g = dt^2 + g_t$

$N = \phi_* \frac{\partial}{\partial t}$ unit normal of the family

$L(t)$ = shape operatore of $\{t\}\times P$ = $\quad L(X) = \bar\nabla_X N \;\; \forall X \in TP$

(1) $g^{-1}\dot g = 2L$

(2) Riccati equation: $\dot L + L^2 + \bar R_N = 0$ where $\bar R_N(X) = \bar R(X,N)N$

(3) Gauss equations $\Rightarrow$ ~~$\overline{Ric}(X,Y) = Ric(X,Y) - (\operatorname{tr} L)\langle L(X),Y\rangle - 2\langle L(X),Y\rangle$~~

$$\overline{Ric}(X,Y) = Ric(X,Y) - H\langle L(X),Y\rangle + \langle L^2(X),Y\rangle + \langle \bar R_N(X),Y\rangle$$
$$= Ric(X,Y) - H\langle L(X),Y\rangle - \langle \dot L(X),Y\rangle \qquad H = \operatorname{tr} L$$

(4) Codazzi equation $\Rightarrow$ $\overline{Ric}(X,N) = \sum_i \langle (\nabla_{e_i}L)(X), e_i\rangle - \langle (\nabla_X L)e_i, e_i\rangle$
$$= -\operatorname{tr}(X \lrcorner d^\nabla L) \qquad (L\in\Omega^1(P;TP))$$

<u>Lemma</u> $\bar g$ is Einstein $\overline{Ric} = \Lambda \bar g$ iff

$$\begin{cases} g^{-1}\dot g = 2L \\ \dot L = r - HL - \Lambda\, id \\ \dot H + |L|^2 + \Lambda = 0 \quad (= \text{trace of Riccati equation}) \\ \operatorname{tr}(X\lrcorner d^\nabla L) = 0 \end{cases}$$

Rmk $Scal - (n-1)\Lambda = H^2 - |L|^2$

Lemma Given g_0, L_0 s.t. $tr(X \lrcorner d^\nabla L_0) = 0 \ \forall X \in TP$, $Scal(g_0) - (n-1)\Lambda = (trL_0)^2 - |L_0|^2$

let (g, L) sol. of

$$\begin{cases} g^{-1}g' = 2L \\ L' = R - HL - \Lambda id \end{cases}$$

w/ $(g(0), L(0)) = (g_0, L_0)$. If $Scal(\bar{g})$ is constant along $\{t\} \times P \ \forall t$, then $\bar{g}$ is Einstein w/ Einstein constant Λ.

proof.

$\frac{1}{2} d\overline{Scal} = div(\overline{Ric})$

$\frac{1}{2} d\overline{Scal}(X) = H\overline{Ric}(N,X) + \frac{d}{dt}\overline{Ric}(N,X) = 0$

$\Rightarrow \frac{d}{dt}\left(\overline{Ric}(N,X) V(t)\right) = 0$ because $\frac{\dot{V}}{V} = H$

$\frac{1}{2} d\overline{Scal}(N) = \frac{1}{2}\frac{d}{dt}\overline{Ric}(N,N) = H\Lambda - H\overline{Ric}(N,N)$

$\Rightarrow \frac{d}{dt}\left((\overline{Ric}(N,N) - \Lambda)V^2\right) = 0$

Of course, a situation where $d\overline{Scal}(X) = 0 \ \forall X \in TP$ is when g_t is a family of homogeneous metrics on P.

G cpt Lie gp acting by isometries on $(M^{n+1}, \bar{g})$ w/ cohomogeneity 1

$M/G = \mathbb{R}, [0, +\infty), S^1, [0,T]$

$t \in (0,T)$ $P = G/K$ principal orbit

$t = 0, T$ $Q_1 = G/H_1$, $Q_2 = G/H_2$ $H_i > K$ $H_i/K = S^{d-1}$ collapsing

$q \in Q \subset M$ $T_qM/T_qQ = V \supset H$ w/ cohomogeneity 1

neighbhd of $Q \subset M$ is equiv. diffeo to $G \times_H V$ and $P = G \times_H S^{d_s}$

Assumption: The space of G-invariant metrics on P is 2-dimensional

$$\bar{g} = dt^2 + f^2(t) g_S + h^2(t) g_Q$$

where g_S = std metric on S^{d_s}

g_Q = G-inv. metric on Q (necessarily Einstein)

Example: $G = SO(p+1) \times SO(q+1) < SO(p+2) \times SO(q+1) \curvearrowright S^{p+1} \times S^q$

$G = SO(p+1) \times SO(q+1) < SO(p+q+2) \curvearrowright S^{p+q+1}$

Want to write equations for (f,h). We need to look at Besse Chapter 9 for a formula for Ric:

$P \to Q$ Riemannian submersion w/ totally geodesic fibers

$\mathcal{H}$ = horiz. distr. $\qquad$ $\mathcal{V}$ = vertical space

$A_X Y = {}^{\mathcal{V}}\nabla_X Y = \frac{1}{2}[X,Y] \qquad X, Y$ horizontal

$A_X U = \mathcal{H}\nabla_X U$

$Ric^P(U,V) = Ric^S(U,V) + \langle AU, AV\rangle$

$Ric^P(X,Y) = Ric^Q(X,Y) - 2\langle A_X, A_Y\rangle$

$Ric^P(X,U) = 0$ if the connection is Yang-Mills connection $d_\theta^*\theta = 0$.

$\langle AU, AV\rangle = \sum_{j=1}^{d_Q} \langle A_{X_i}U, A_{X_i}V\rangle \qquad \langle A_X, A_Y\rangle = \sum_{j=1}^{d_Q} \langle A_X X_j, A_Y X_j\rangle$

Set $|A|^2 = \frac{1}{d_Q}\sum_{i,j=1}^{d_Q} |A_{X_i}X_j|^2$ wrt to the metric $g_Q + g_S$ on P

Then scaling + G-invariance $\Rightarrow$

$$\begin{cases} Ric^P(U,V) = \dfrac{d_S - 1}{f^2}\langle U,V\rangle + \dfrac{d_Q}{d_S}\dfrac{f^2}{h^4}|A|^2 \\ Ric^P(X,Y) = \dfrac{Ric^Q}{h^2} - 2\dfrac{f^2}{h^4}|A|^2 \end{cases}$$

This gives the equations:

$$(\text{O.D.E.}) \begin{cases} \left(\dfrac{\dot f}{f}\right)' + \left(d_Q\dfrac{\dot h}{h} + d_S\dfrac{\dot f}{f}\right)\dfrac{\dot f}{f} - \dfrac{d_S - 1}{f^2} - \dfrac{d_Q}{d_S}\dfrac{f^2}{h^4}|A|^2 + \Lambda = 0 \\ \left(\dfrac{\dot h}{h}\right)' + \left(d_Q\dfrac{\dot h}{h} + d_S\dfrac{\dot f}{f}\right)\dfrac{\dot h}{h} - \dfrac{Ric^Q}{h^2} - 2\dfrac{f^2}{h^4}|A|^2 + \Lambda = 0 \end{cases}$$

Together w/ constraints:

$$\begin{cases} d_Q\dfrac{\ddot h}{h} + d_S\dfrac{\ddot f}{f} + \Lambda = 0 \\ \overline{Ric}(X,N) = 0 \quad \forall X \in TP \end{cases}$$

Next we investigate existence of solutions that extends smoothly over Q:

Conditions for smoothness: $\begin{cases} f(t) = t + O(t^3) \\ h(t) = \bar{h}_0 + O(t^2) \end{cases}$ for some $\bar{h}_0 > 0$.

(i.e. both h^2, f^2 have to be even, $f(0) = 0$)

Prop. $\forall\, h_0 \in (0,+\infty)$, $\exists!$ analytic fct (h,f) s.t. $(f(0), f'(0), h(0), h'(0)) = (0, 1, h_0, 0)$ sol. to O.D.E. Moreover, $dt^2 + f^2 g_S + h^2 g_Q$ is an Einstein metric on $[0,T)\times P$ that depends continuosly on $h_0 > 0$.

proof.

~~ye~~ $\frac{ds}{dt} = f \sim 1$ for t small

$f \equiv s$

$y = (y_1, y_2, y_3)$ solves $\dot{y}(s) = A(y(s)) + \frac{1}{s} B(y(s))$. say $|y - y_0| < 2\varepsilon$

where A, B bounded Lipschitz fct in a neighbd of $y_0 = (1, h_0, 0)$, $h_0 > 0$

Moreover, one can check that: $B(y_0) = 0$ and

$(m+1)\,\mathrm{id} - d_{y_0} B$ is uniformly invertible for every $m \geq 0$

$\Rightarrow \exists$ formal power series solutions $y_m(s)$ $\forall$ $m \geq 0$.

$$\Phi(\xi) = y_0 - y_m(s) + \int_0^s A(y_m(\tau) + \xi) + \frac{1}{\tau} B(y_m(\tau) + \xi)\, d\tau$$

$\Phi : \mathcal{O}_{m+1} \longrightarrow \mathcal{O}_{m+1}$ $\qquad \mathcal{O}_{m+1} := \{ \xi : [0,T] \to \mathbb{R}^3 \mid \|\xi\|_{m+1} = \sup_{s \in [0,T]} \frac{|\xi|}{s^{m+1}} < +\infty \}$

Choose $T = T(m)$ s.t. $\begin{cases} |y_m - y_0| < \varepsilon & \forall\ s \in [0,T] \\ \|\Phi(0)\|_{m+1} < \frac{\varepsilon}{2} \end{cases}$

$\|\Phi(\xi_1) - \Phi(\xi_2)\|_{m+1} \leq \frac{C}{m+1} \|\xi_1 - \xi_2\|_{m+1}$. Choose m large enough so that

$\Phi : B_\varepsilon(0) \subset \mathcal{O}_{m+1} \longrightarrow \mathcal{O}_{m+1}$ is a contraction.

As for the last statement observe that $V(t)\, \overline{Ric}(X, N)$ and $V^2(t)(\overline{Ric}(N,N) - \Lambda)$ are certainly 0 at $t=0$ b/c $V(0) = 0$ and $\bar{g}$ is smooth. ∎

§2. The sine-cone

$dt^2 + \sin^2(t)(\alpha_1 g_S + \alpha_2 g_Q)$ w/ α_1, α_2 constant

singularity at $t = 0, \pi$

is Einstein $\Lambda = d = \dim P$ iff $\alpha_1 g_S + \alpha_2 g_Q$ is Einstein w/ Ric constant $= d-1$

The canonical variation:

$g_Q + w^2 g_S$

$$S(w) = w^{2d_S/d} \left[\frac{d_S(d_S - 1)}{w^2} + d_Q Ric^Q - w^2 d_Q |A|^2 \right]$$

$$S'(w) = 0 \iff \frac{d_Q + 2d_S}{d_S} |A|^2 w^2 - Ric^Q + \frac{d_S - 1}{w^2} = 0$$

Case I: $|A| = 0$, $\exists!\ \bar{w} := \sqrt{\frac{Ric^Q}{d_S - 1}}$ if $Ric^Q > 0$, $d_S > 1$

Case II: $|A| > 0$, $\exists$ w_1, w_2 s.t. $S'(w) = 0$, $w_1, w_2 > 0$ iff $\Delta = (Ric^Q)^2 - 4|A|^2 \frac{(d_Q + 2d_S)(d_S - 1)}{d_S} > 0$

or $\bar{w}$ if $\Delta = 0$

In each case, we let $\bar{w}$ be the point where $S(w)$ attains a minimum.

Next set $b^2 = \frac{S(\bar{w})}{\bar{w}^{2d_S/d}\, d(d-1)} = \frac{Ric^Q - 2|A|^2 \bar{w}^2}{d-1}$ so that

$Ric\left(h^2(g_Q + w^2 g_S)\right) = (d-1)$.

§3. Critical points of $w = f/h$.

Lemma

Let U_{h_0} one of the sol. to O.D.E. closing smoothly on Q. Then critical pts of $w = \frac{f}{h}$ are non-degenerate.

proof. At a critical pt. of w, $\ddot{w} = \underset{\neq 0}{C_w} S'(w)$ ■

Rmk Reinterpretation of critical pts of w:

$$L = \frac{1}{2} g^{-1}\dot{g} = \begin{pmatrix} \dot{f}/f & \\ & \dot{h}/h \end{pmatrix} \qquad H = \operatorname{tr} L = d_S \frac{\dot{f}}{f} + d_Q \frac{\dot{h}}{h}$$

$$\overset{\circ}{L} = \begin{pmatrix} \frac{d_Q}{d_S + d_Q} \frac{\dot{w}}{w} & \\ & -\frac{d_S}{d_S + d_Q} \frac{\dot{w}}{w} \end{pmatrix} \Rightarrow \operatorname{tr}\left((\overset{\circ}{L})^*(\overset{\circ}{L})\right) = |\overset{\circ}{L}|^2 = \left(\frac{d_S d_Q}{d_S + d_Q}\right)\left(\frac{\dot{w}}{w}\right)^2$$

- Orbital volume fct:

$$\frac{\dot{V}}{V} = H \qquad \begin{cases} \dot{H} + |L|^2 + d = 0 \iff \dot{H} + \frac{1}{d} H^2 + |\overset{\circ}{L}|^2 + d = 0 \\ Scal - d(d-1) = H^2 - |L|^2 = \frac{d-1}{d} H^2 - |\overset{\circ}{L}|^2 \end{cases}$$

$$Scal = \frac{d_S(d_S - 1)}{f^2} + \frac{d_Q Ric^Q}{h^2} - \frac{f^2}{h^4} d_Q |A|^2$$

Lemma If $|A| = 0$, the $\exists!$ $t_0 \in [0,T)$ s.t. $H(t_0) = 0$.

proof.

i) Riccati comparison: $H \leq d\cot(t)$ so $T \leq \pi$

$\Rightarrow \forall \bar{h} > 0$, $\exists C > 0$ s.t. $V(t) \leq C_{\bar{h}}$ $\forall t \in [0,T)$, $\forall U_{h_0}$ w/ $h_0 \geq \bar{h}$

ii) Since $V = f^{d_S} h^{d_Q}$, $f \to 0 \iff h \to +\infty$ & $h \to 0 \iff f \to +\infty$

but $Scal \leq d(d-1) + d\cot(T)$ so, $h, f \leq C$

iii) In particular, $Scal \geq C$ and therefore $|L|$ cannot blow-up. ■

Def A sol. U_{h_0} is called symmetric if $\exists\, t_0 \in (0,T)$ s.t. $\dot{w}(t_0) = 0 = \dot{V}(t_0)$, i.e. $|L(t_0)| = 0$.

Rmk: If U_{h_0} is symmetric then can double across max. vol. orbit to get Einstein metric on the doubling of the disc bdle $G \times_H D^{q+1} \to Q$.

Example: $S^p \times S^{q+1}$

~~Lemma~~ **Def** If U_{h_0} is not symmetric, $\# C_w(h_0) = \#$ crit. pts of w before max vol. orb.

~~Lemma~~ **Thm** (Intermidiate Value Thm)

$0 < h_1 < h_2$, U_{h_1}, U_{h_2} not symmetric, then $\exists$ at least $|\# C_w(h_1) - \# C_w(h_2)|$ $h_0 \in (h_1, h_2)$ s.t. U_{h_0} is symmetric.

proof.

(i) $\nexists$ $h_0 \in (h_1, h_2)$ w/ U_{h_0} symmetric $\Longrightarrow \# C_w(h_1) = \# C_w(h_2)$

(ii) $\exists !$ $h_0 \in (h_1, h_2)$ w/ U_{h_0} symmetric $\Longrightarrow |\# C_w(h_1) - \# C_w(h_2)| \leq 1$ ∎

~~Corollary~~ **§4. Limit of U_{h_0} as $h_0 \to 0$**

. Bubbles

Rescaling U_{h_0} $h_0 \to 0$ $\rightsquigarrow \tilde{U}_{h_0}$ $\quad h_0^{-2} \bar{g} = h_0^{-2} dt^2 + f^2(t) g_S + h^2(s) g_Q$

$$\begin{cases} \left(\frac{\dot f}{f}\right)' + \left(d_Q \frac{\dot h}{h} + d_S \frac{\dot f}{f}\right) \frac{\dot f}{f} - \frac{d_S - 1}{f^2} + d h_0^2 = 0 \\ \left(\frac{\dot h}{h}\right)' + \left(d_Q \frac{\dot h}{h} + d_S \frac{\dot f}{f}\right) \frac{\dot h}{h} - \frac{d_Q - 1}{h^2} + d h_0^2 = 0 \\ d_Q \frac{\ddot h}{h} + d_S \frac{\ddot f}{f} + d h_0^2 = 0 \\ \overline{Ric}(X, N) = 0 \end{cases}$$

Continuous family ~~of~~ on $[0, +\infty)$. $h_0 = 0$: Ricci-flat metric

. A Lyapunov fct

$$(g, h) \in Sym^+(V \oplus p)^K \oplus Sym(V \oplus p)^K = TSym^+(V \oplus p)$$

$$\mathcal{E}_\Lambda := \{ (g, h) \mid Scal(g) - \Lambda(d-1) = [tr(g^{-1}h)]^2 - |h|_g^2 \} \qquad |h|_g^2 = tr(g^{-1}h)$$

<u>Lemma</u> If P is not a torus or $\Lambda \neq 0$, then $\mathcal{E}_\Lambda$ is a smooth hypersurface in $TSym^+(V \oplus p)^K$ invariant under the "Einstein flow."

Def $\mathcal{K} : \mathcal{E}_\Lambda \to \mathbb{R}$, $\quad \mathcal{K}(g, h) = V^{\frac{2}{d}} \left(|\mathring{h}|_g^2 + Scal(g) \right)$

$$= V^{\frac{2}{d}} \left(\frac{d-1}{d} H^2 + \Lambda(d-1) \right) \qquad H = tr(g^{-1}h)$$

<u>Lemma</u> $\frac{d\mathcal{K}}{dt} = -\frac{2}{d} \frac{d-1}{d} V^{2/d} H |\mathring{L}|^2$

<u>Lemma</u> The ~~critical~~ critical pts of $\mathcal{K}$ ~~on~~ $\mathcal{E}_\Lambda$ are cones, $\mathcal{K} \equiv d(d-1)$

~~Prop~~: The cones w/ $w = \bar{w}$ is a local attractor for the "Einstein flow" on $\mathcal{E}_\Lambda$

Proof. g_P non-degenerate minimum of $V^{\frac{2}{d}}\,\mathrm{Scal}(g) \Longrightarrow g_P$ non-degenerate minimum of $\varkappa$ on $\mathcal{E}_\Lambda \Longrightarrow$ level sets of $\varkappa$ are tubes around the cone ■

- The Ricci-flat case

$$g = V^{2/d}\hat{g} \qquad \overset{\circ}{L} = \tfrac{1}{2}\hat{g}^{-1}\hat{g}$$

$$\begin{cases} (\overset{\circ}{L})' + \dfrac{\dot V}{V}\overset{\circ}{L} - V^{-2/d}\,\overset{\circ}{\mathrm{ric}}(\hat g) = 0 \\ \left(\dfrac{\dot V}{V}\right)' + \left(\dfrac{\dot V}{V}\right)^2 - V^{-2/d}\,\mathrm{Scal}(\hat g) = 0 \\ \left(\dfrac{\dot V}{V}\right)' + \dfrac{1}{d}\left(\dfrac{\dot V}{V}\right)^2 + |\overset{\circ}{L}|^2 = 0 \end{cases}$$

$$\hat g = \varrho^{2/d_S} g_S + \varrho^{-2/d_Q} g_Q$$

$$\mathrm{Scal}(\varrho) = d_S(d_S-1)\varrho^{-2/d_S} + d_Q(d_Q-1)\varrho^{2/d_Q}$$

$$\mathrm{Scal}'(\varrho) = -\frac{2}{\varrho}\left[(d_S-1)\varrho^{-2/d_S} - (d_Q-1)\varrho^{2/d_Q}\right]$$

$$\overset{\circ}{L} = \begin{pmatrix} \frac{1}{d_S}\frac{\dot\varrho}{\varrho} & \\ & -\frac{1}{d_Q}\frac{\dot\varrho}{\varrho} \end{pmatrix} \qquad |\overset{\circ}{L}|^2 = \left(\frac{1}{d_S} + \frac{1}{d_Q}\right)\left(\frac{\dot\varrho}{\varrho}\right)^2$$

Change variable: $\dfrac{d\tau}{dt} = V^{-1/d} \longleftrightarrow V^{\frac{1}{d}}\dfrac{d}{dt} = \dfrac{d}{d\tau}$

$$\begin{cases} \dfrac{\ddot V}{V} + \dfrac{d}{d_S d_Q}\left(\dfrac{\dot\varrho}{\varrho}\right)^2 = 0 \\ \dfrac{\ddot V}{V} + \dfrac{d-1}{d}\left(\dfrac{\dot V}{V}\right)^2 - \mathrm{Scal}(\varrho) = 0 \\ \left(\dfrac{\dot\varrho}{\varrho}\right)' + \dfrac{d-1}{d}\dfrac{\dot V}{V}\left(\dfrac{\dot\varrho}{\varrho}\right) - \dfrac{d_S d_Q}{d}\varrho\,\mathrm{Scal}'(\varrho) = 0 \end{cases}$$

Using the first 2, this give a decoupled eq. on ϱ:

$$\left(\frac{\dot\varrho}{\varrho}\right)' + \sqrt{\frac{d-1}{d}}\sqrt{\varkappa(\varrho,\dot\varrho)} - \frac{d_S d_Q}{d}\varrho\,\mathrm{Scal}'(\varrho) = 0$$

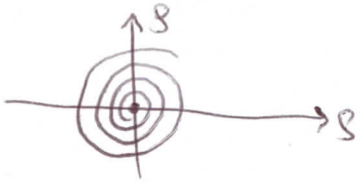
Rmk: More in general $\mathrm{Sym}^+(V\oplus\mathfrak{p}) = GL(d+1,\mathbb{R})/O(d+1) = SL(d+1,\mathbb{R})/SO(d+1) \times \mathbb{R}_+$

$$\frac{D^2\hat g}{dt^2} + \sqrt{\frac{d-1}{d}}\sqrt{\varkappa(\hat g,\hat g')}\,\hat g + 2\nabla S(\hat g) = 0$$

and $\mathrm{Sym}^+(V\oplus\mathfrak{p})^K$ is totally geodesic.

• Properties of this:

$\varrho = 0$ unique zero and if $d \leq 9$ it is a stable focus



$\Longrightarrow h_0^{-1} U_{h_0} \longrightarrow U_0$ asymptotically conical Ricci-flat w/ infinitely many totally umbilic hypersurfaces.

Prop. $U_{h_0} \longrightarrow$ sine-cone and $\#C_W(h_0) \longrightarrow +\infty$ as $h_0 \to 0$.

proof. Use $h_0^{-1} U_{h_0} \to U_0$ and Lyapunov fct $\mathcal{K}$. ■

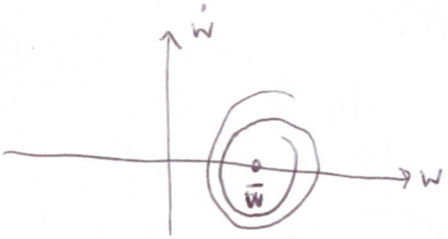
§5. Results

Rmk $h = \cos t$, $f = \sin t$ (round metric on a sphere) has $\#C_W = 0$.

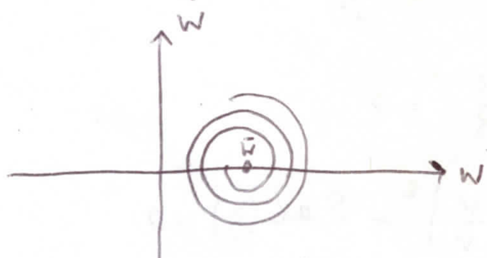
Thm $\exists$ infinitely many Einstein metrics on $S^p \times S^q$

Space of max vol. orbits:

$$\begin{cases} \frac{1}{h^2}\left[\frac{d_S(d_S-1)}{w^2} + d_Q(d_Q-1) \right] = d(d-1) - \frac{d_S d_Q}{d}\left(\frac{\dot w}{w}\right)^2 & \Leftrightarrow \ \mathrm{Scal} - d(d-1) = -|\mathring{L}|^2 \\ d_Q \frac{\dot h}{h} + d_S \frac{\dot w}{w} = 0 & \Leftrightarrow \ H = 0 \end{cases}$$



starting at Q_1



starting at Q_2

Thm $\exists$ infinitely many Einstein metrics on S^5, S^6, S^7, S^8, S^9

Rmk 1: All these metrics (up to a finite number) are not homogeneous

Rmk 2: All these metrics are non-isometric.