Bathe-Sommerfeld Conjecture

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We consider periodic pseudo-differential operators

$$H = h(x, D),$$

where $x \in \mathbb{R}^d$ and h is periodic in x, i.e. $h(x+\gamma,\xi) = h(x,\xi)$ for all $\gamma \in \Gamma$, and $\Gamma \subset \mathbb{R}^d$ is a lattice of the full rank.
We assume H to be elliptic; the standard examples are: periodic Schrödinger operator

$$H = -\Delta + V$$

with smooth periodic potential $V = V(x)$, $x \in \mathbb{R}^d$ and periodic magnetic Schrödinger operator

$$H = (i\nabla + a)^2 + V$$

with smooth periodic scalar potential $V = V(x)$ and smooth vector potential $a = a(x)$.

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If $d = 1$, then the number of gaps is almost always infinite.

Bethe-Sommerfeld conjecture

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If $d \geq 2$, the number of gaps of periodic Schrödinger operator $H = -\Delta + V$ is always finite.

Proved:
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$d = 3$; arbitrary d with rational Γ: M.Skriganov (1985)
The lattice Γ is rational, if $\forall \gamma_1, \gamma_2, \gamma_3 \in \Gamma$ we have $\frac{(\gamma_1, \gamma_2)}{|\gamma_3|^2} \in \mathbb{Q}$.

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Also, papers by Yu.Karpeshina ($d = 3$, singular potential),
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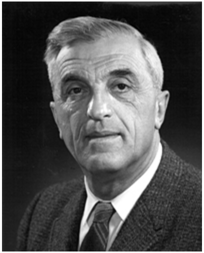
For the magnetic Schrödinger operator $H = (i\nabla + a)^2 + V$ the Bethe-Sommerfeld conjecture was proved only for $d = 2$ (A. Mohamed, 1997).

Important tool when working with periodic problems: Floquet-Bloch decomposition.



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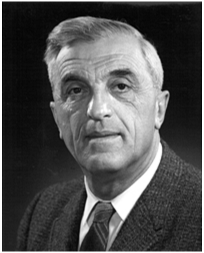

$$H = \int_{\oplus} H(k) dk,$$

where $H(k) = h(x, \xi + k)$ ($H(k) := (i\nabla + k)^2 + V$ in the Schrödinger case) acts in $L^2(\mathbb{R}^d/\Gamma)$, $k \in \mathbb{R}^d/\Gamma'$, and Γ' is the (analytical) dual to Γ. This means that

$$\sigma(H) = \cup_{k \in \mathbb{R}^d/\Gamma'} \sigma(H(k)).$$

The spectrum of $H(k)$ consists of eigenvalues:

$$\sigma(H(k)) = \{\lambda_1(k) \leq \lambda_2(k) \leq \dots\}.$$

Now we can define

$$\ell_j := \cup_{k \in \mathbb{R}^d/\Gamma'} \lambda_j(k)$$

as the n-th spectral band, so that $\sigma(H) = \cup_j \ell_j$. Then for each λ we can define two functions:

$$m(\lambda) = \#\{j : \lambda \in \ell_j\}$$

(the multiplicity of overlapping) and

$$\zeta(\lambda) = \zeta(\lambda; H) = \max_j \max\{t : [\lambda - t, \lambda + t] \subset \ell_j\}.$$

(the overlapping function)

Important property:

$$\zeta(\lambda; A + B) \geq \zeta(\lambda; A) - ||B||.$$

Therefore, if we define

$$\tilde{\zeta}(\lambda; H) := \inf\{||A||, \lambda \notin \sigma(H + A)\},$$

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The equality here holds if H has constant coefficients, but not in general.

Theorem. (A.Sobolev,LP, 2001)

Let $d = 2, 3, 4$. Then for sufficiently large λ we have:

Dimension	$m(\lambda) \gg$	$\zeta(\lambda) \gg$
2	$\lambda^{\frac{1}{4}}$	$\lambda^{\frac{1}{4}}$
3	$\lambda^{\frac{1}{2}}$	1
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Unfortunately, the method does not work for $d \geq 5$!

Suppose, we want to prove just the conjecture (not the bounds on m or ζ). Then we can use the following strategy (the approach of Skriganov). Denote $N = N_\lambda(k) = \#\{\lambda_j(k) < \lambda\}$. Then the following statements are equivalent:

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Thus, our aim is to obtain a non-trivial lower bound for $||N_\lambda - \langle N_\lambda \rangle||$.

Denote by $N^0 = N^0_\lambda$ the unperturbed counting function. It is equal to the number of points Γ' inside a ball with center k and radius $\rho := \sqrt{\lambda}$. We have:

$$||N_\lambda - \langle N_\lambda \rangle|| \geq ||N^0_\lambda - \langle N^0_\lambda \rangle|| - ||N_\lambda - N^0_\lambda|| - ||\langle N_\lambda \rangle - \langle N^0_\lambda \rangle||$$

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If Γ is rational, we can choose L_∞ norm in our estimates to prove the conjecture.
If Γ is irrational, the best known lower bounds for $||N^0_\lambda - \langle N^0_\lambda \rangle||_\infty$ and $||N^0_\lambda - \langle N^0_\lambda \rangle||_1$ are essentially the same!

Theorem. (D.Kendall;M.Skriganov;A.Sobolev,LP)

For sufficiently large λ the following estimates hold:

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Assume that $\int_{\mathbb{R}^d/\Gamma} V(x)dx = 0$. Then we have:

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Therefore, for the method to work, we need $\frac{d-1}{4} > \frac{d-3}{2}$, i.e. $d < 5$!

If we want to prove the conjecture for all d and all lattices, we need to study the eigenvalues of $H(k)$. There are two types of eigenvalues of these operators: stable (corresponding to perturbations of simple eigenvalues, lying not too close to other eigenvalues) and unstable (corresponding to perturbations of a cluster of eigenvalues lying close together). It is relatively straightforward to compute stable eigenvalues with high precision. Unstable eigenvalues cause the main problem.

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Let $d \geq 2$*. Then the Bethe-Sommerfeld conjecture holds for operators* $H = (-\Delta)^m + q$ *with periodic pseudo-differential operators* q *of order smaller than* $2m - 1$*.*

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Let $d \geq 2$*. Then the Bethe-Sommerfeld conjecture holds for operators* $H = (-\Delta)^m + q$ *with periodic pseudo-differential operators* q *of order smaller than* $2m$*. In particular, this conjecture holds for periodic magnetic Schrödinger operators.*