

BETHE-SOMMERFELD CONJECTURE FOR PERIODIC OPERATORS WITH STRONG PERTURBATIONS

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ABSTRACT. We consider a periodic self-adjoint pseudo-differential operator $H = (-\Delta)^m + B$, $m > 0$, in \mathbb{R}^d which satisfies the following conditions: (i) the symbol of B is smooth in \mathbf{x} , and (ii) the perturbation B has order less than $2m$. Under these assumptions, we prove that the spectrum of H contains a half-line. This, in particular implies the Bethe-Sommerfeld Conjecture for the Schrödinger operator with a periodic magnetic potential in all dimensions.

1. INTRODUCTION

Under very broad conditions, spectra of elliptic differential operators with periodic coefficients in $L^2(\mathbb{R}^d)$, $d \geq 1$, have a band structure, i.e. they represent a union of closed intervals (*bands*), possibly separated by spectrum-free intervals (*gaps*) (see [20] and [14]). Since the 30's it has been a general belief among the physicists that the number of gaps in the spectrum of the Schrödinger operator $H_V = -\Delta + V$ with a periodic electric potential (i.e. a real-valued function) V in dimension three must be finite. After the classical monograph [2] this belief is known as the Bethe-Sommerfeld conjecture. It is relatively straightforward to see that this conjecture holds for potentials which admit a partial separation of variables, as shown in [6], p.121. For general potentials this problem turned out to be quite difficult, and the first rigorous results appeared only in the beginning of the 80's. We do not intend to discuss these and more recent results in details, but refer to [29] for a more comprehensive survey and further references. Here we content ourselves with a very short description.

In the case of the Schrödinger operator H_V it is known that the number of gaps is generically infinite if $d = 1$ (see [20]). For $d \geq 2$ there has been a large number of publications proving the conjecture for H_V under various conditions on the potential and the periodicity lattice. First rigorous results for the Schrödinger operator relied on number-theoretic ideas, and they appeared in [19], [5] ($d = 2$) and [22]–[24] ($d \geq 2$). At that time it was found out that the complexity of the problem increases together with the dimension: the validity of the conjecture for dimensions $d \geq 4$ was established only for rational lattices, see [23]. Later the conjecture for arbitrary lattices was extended

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to $d = 4$ in [9]. The definitive result was obtained in the recent paper [16] where the Bethe-Sommerfeld conjecture was proved for the Schrödinger operator for any periodic lattice in all dimensions $d \geq 2$, with an arbitrary smooth potential V (see [31] and [32] for an alternative approach).

The study of the polyharmonic operator $(-\Delta)^m + V$, $m > 0$ in [23], [11] and [17], [18] revealed that large values of m facilitate the overlap of the spectral bands. Precisely, it was found that for $4m > d + 1$ the Bethe-Sommerfeld conjecture holds for arbitrary bounded self-adjoint perturbations V (see [17]), and if $8m > d + 3$, then it holds for arbitrary smooth potentials V (see [18]).

Returning to the case of the Schrödinger operator, we observe that the complexity of the problem increases dramatically when instead of the bounded potential perturbation one introduces in the Schrödinger operator a periodic magnetic potential $\mathbf{a} = (a_1, a_2, \dots, a_d)$: $(-i\nabla - \mathbf{a})^2 + V$. Until recently the Bethe-Sommerfeld conjecture for this operator was known to hold only for $d = 2$, see [15], [12]. A new step towards the study of higher order perturbations was made in [1], where the methods of [16] were extended to the operator

$$(1.1) \quad H = (-\Delta)^m + B, \quad m > 0,$$

with a symmetric pseudo-differential perturbation B of order $a < 2m - 1$ and arbitrary $d \geq 2$.

In the present paper we prove the Bethe-Sommerfeld conjecture for the operator (1.1) for arbitrary B of order $a < 2m$, see Theorems 2.1 and 2.2. In particular, our result covers the magnetic Schrödinger operator with a smooth periodic vector potential in any dimension $d \geq 2$.

Our proof is based on a subtle analysis of the Floquet eigenvalues of the operator H . It is known that the Floquet eigenvalues are divided in two groups: stable (or non-resonant) and unstable (or resonant). From the perturbation-theoretic point of view, the stable eigenvalues are generated by the isolated non-degenerate Floquet eigenvalues of the free operator $H_0 = (-\Delta)^m$, and hence they can be described using familiar methods of the theory. The unstable eigenvalues, on the contrary, are produced by the clusters of close (or even degenerate) eigenvalues of H_0 , and their detailed description is not that simple. Various asymptotic formulas for stable and/or unstable eigenvalues were obtained in [30], [31], [7], [8], [10], [12], [32], see also [11] for more references.

As papers [15], [12], [1] show, the increase of the order of the perturbation requires a more careful analysis of the resonant eigenvalues. The same observation can be made for higher dimensions, see [16], [32]. In the present paper we are forced to study the unstable eigenvalues in detail for arbitrary dimensions $d \geq 2$ and higher order perturbations. Associated with the partition of the eigenvalues into two groups, is a partition of the phase space into resonant and non-resonant zones (sets). In fact, the main focus of the present paper is the precise construction of these zones and understanding of the eigenvalues associated with them. Technically, our approach is a combination of methods of [16] and [28]. Our construction of the resonant zones is a simplified variant

of that suggested in [16]. However, in spite of the simplification, these are rather complicated geometrical objects, and the study of their properties is not straightforward. The reduction of the operator to the resonant and non-resonant parts is done using the “near-similarity” approach of [28]. It consists in finding a unitary operator U such that $A = U^* H U$ is “almost” an operator with constant coefficients. The operator U is sought in the form $e^{i\Psi}$ where Ψ is a self-adjoint periodic PDO, and hence we sometimes call this similarity transformation a “gauge transformation”. For $d = 1$ such a reduction to constant coefficients can be done (see [21], [27]), but for $d \geq 2$ only a partial reduction is possible. Namely, we explicitly describe the procedure of finding a pseudo-differential operator Ψ with symbol $\psi(\mathbf{x}, \boldsymbol{\xi})$ such that the operator $A = e^{-i\Psi} H e^{i\Psi}$ has constant coefficients in the non-resonant zone. Thus the Floquet eigenvalues in the non-resonant subspace can be found explicitly, which leads to relatively straightforward estimates for the band overlap. As far as the resonant zones are concerned, our construction ensures that on each of them the new operator A admits a partial separation of variables (see e.g. [8], [10] and [11], p. 143 for a similar observation for the Schrödinger operator in dimensions $d = 2, 3$). This fact enables us to show that the volume (more precisely, the angular measure) of the resonant sets is negligibly small compared to the non-resonant one. Having established this fact, we apply the combinatorial-geometric argument of [16], which allows us to deduce that the resonant zones do not destroy the band overlap obtained for the non-resonant one.

It is clear that the unitary transformation $U = e^{i\Psi}$ establishes a one-to-one correspondence between the Bloch functions for the operators H and A defined above. In particular, in the non-resonant zone the Bloch functions of A are just exponentials, and hence, applying the operator U one immediately obtains an explicit simple formula, involving the symbol $\psi(\mathbf{x}, \boldsymbol{\xi})$, for the Bloch functions of H . Thus it is not surprising that a similar representation appears in [12], where the spectral properties of the magnetic Schrödinger operator $(-i\nabla - \mathbf{a})^2 + V$ are studied in the case $d = 2$. The core of the approach of [12] is to construct a suitable high-energy approximation for the Bloch functions. In the non-resonant zone such an approximation is sought in the form $e^{i\mathbf{x}\cdot\boldsymbol{\xi} + iS(\mathbf{x}, \boldsymbol{\xi})}$ with a suitable phase $S(\mathbf{x}, \boldsymbol{\xi})$. Comparing formulas (30) from [12] and (4.12) in the present paper, we see that the leading terms of $S(\mathbf{x}, \boldsymbol{\xi})$ and $\psi(\mathbf{x}, \boldsymbol{\xi})$, as $\boldsymbol{\xi} \rightarrow \infty$, are in fact the same, as should be expected. In spite of this connection with the Bloch functions, in the present paper we prefer to use the operator language. We believe that it is precisely this choice, in combination with the use of PDO calculus, that made it possible to extend the previously known results on the Bethe-Sommerfeld Conjecture to the degree of generality described in this paper.

To conclude the introduction, we give a brief outline of the paper. In the next section, we introduce necessary notation, discuss the classes of pseudo-differential operators we will be using throughout and formulate the main result of the paper. In Section 3, we provide necessary information about the classes of pseudo-differential operators introduced in Section 2. The “gauge transformation” is studied in Section 4. In Section 5, we

describe the partition of the phase space into resonant and non-resonant zones. This section has a purely combinatorial-geometric character, and can be read separately from the rest of the paper. In Section 6, we construct the decomposition of A into an orthogonal sum over the resonant and non-resonant subspaces. On the basis of this decomposition we study the Floquet eigenvalues of A in Section 7. Sections 8, 9 are concerned with estimates for the volumes of resonant and non-resonant sets. These estimates become the central ingredient of the proof, completed in Section 10.

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2. PERIODIC PSEUDO-DIFFERENTIAL OPERATORS. MAIN RESULT

2.1. Classes of PDO's. Before we define the pseudo-differential operators (PDO's), we introduce the relevant classes of symbols. Let $\Gamma \in \mathbb{R}^d$ be a lattice. Denote by \mathcal{O} its fundamental domain. For example, for \mathcal{O} one can choose a parallelepiped spanned by a basis of Γ . The dual lattice and its fundamental domain are denoted by Γ^\dagger and \mathcal{O}^\dagger respectively. Sometimes we reflect the dependence on the lattice and write \mathcal{O}_Γ and $\mathcal{O}_\Gamma^\dagger$. In particular, in the case $\Gamma = (2\pi\mathbb{Z})^d$ one has $\Gamma^\dagger = \mathbb{Z}^d$ and it is natural to take $\mathcal{O} = [0, 2\pi)^d$, $\mathcal{O}^\dagger = [0, 1)^d$. For any measurable set $\mathcal{C} \subset \mathbb{R}^d$ we denote by $|\mathcal{C}|$ or $\text{vol}(\mathcal{C})$ its Lebesgue measure (volume). The volume of the fundamental domain does not depend on its choice, it is called the *determinant of the lattice* Γ and denoted $d(\Gamma) = |\mathcal{O}|$. By $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$ we denote the standard orthonormal basis in \mathbb{R}^d .

For any $u \in L^2(\mathcal{O})$ and $f \in L^2(\mathbb{R}^d)$ define the Fourier coefficients and Fourier transform respectively:

$$\hat{u}(\boldsymbol{\theta}) = \frac{1}{\sqrt{d(\Gamma)}} \int_{\mathcal{O}} e^{-i(\boldsymbol{\theta}, \mathbf{x})} u(\mathbf{x}) d\mathbf{x}, \quad \boldsymbol{\theta} \in \Gamma^\dagger, \quad (\mathcal{F}f)(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i(\boldsymbol{\xi}, \mathbf{x})} f(\mathbf{x}) d\mathbf{x}, \quad \boldsymbol{\xi} \in \mathbb{R}^d.$$

Let us now define the periodic symbols and PDO's associated with them. Let $b = b(\mathbf{x}, \boldsymbol{\xi})$, $\mathbf{x}, \boldsymbol{\xi} \in \mathbb{R}^d$, be a Γ -periodic complex-valued function, i.e.

$$b(\mathbf{x} + \boldsymbol{\gamma}, \boldsymbol{\xi}) = b(\mathbf{x}, \boldsymbol{\xi}), \quad \forall \boldsymbol{\gamma} \in \Gamma.$$

Let $w : \mathbb{R}^d \rightarrow \mathbb{R}$ be a locally bounded function such that $w(\boldsymbol{\xi}) \geq 1 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d$ and

$$(2.1) \quad w(\boldsymbol{\xi} + \boldsymbol{\eta}) \leq Cw(\boldsymbol{\xi})\langle \boldsymbol{\eta} \rangle^\kappa, \quad \forall \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d,$$

for some $\kappa \geq 0$. Here we have used the standard notation $\langle \mathbf{t} \rangle = \sqrt{1 + |\mathbf{t}|^2}$, $\forall \mathbf{t} \in \mathbb{R}^d$. We say that the symbol b belongs to the class $\mathbf{S}_\alpha = \mathbf{S}_\alpha(w) = \mathbf{S}_\alpha(w, \Gamma)$, $\alpha \in \mathbb{R}$, if for any $l \geq 0$ and any non-negative $s \in \mathbb{Z}$ the condition

$$(2.2) \quad |b|_{l,s}^{(\alpha)} := \max_{|\mathbf{s}| \leq s} \sup_{\boldsymbol{\xi}, \boldsymbol{\theta}} \langle \boldsymbol{\theta} \rangle^l w(\boldsymbol{\xi})^{-\alpha + |\mathbf{s}|} |\mathbf{D}_{\boldsymbol{\xi}}^{\mathbf{s}} \hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi})| < \infty, \quad |\mathbf{s}| = s_1 + s_2 + \dots + s_d,$$

is fulfilled. Here $\hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi})$ is the Fourier coefficient of the symbol $b(\cdot, \boldsymbol{\xi})$ with respect to the first variable. The quantities (2.2) define norms on the class \mathbf{S}_α . In the situations when it is not important for us to know the exact values of l, s , we denote the above norm by $|b|^{(\alpha)}$. In this case the inequality $A \leq C|b|^{(\alpha)}$ means that there exist values of l and s , and a constant $C > 0$, possibly depending on l, s , such that $A \leq C|b|_{l,s}^{(\alpha)}$. Similarly, when we write $|b|^{(\gamma)} \leq C|g|^{(\alpha)}$ for some symbols $b \in \mathbf{S}_\gamma, g \in \mathbf{S}_\alpha$, we mean that for any l and s the norm $|b|_{l,s}^{(\gamma)}$ is bounded by $|g|_{p,n}^{(\alpha)}$ with some p and n depending on l, s , and some constant $C = C_{l,s}$. In general, by C, c (with or without indices) we denote various positive constants, whose precise value is unimportant. Throughout the entire paper we adopt the following convention. An estimate (or an assertion) is said to be uniform in a symbol $b \in \mathbf{S}_\alpha$ if the constants in the estimate (or assertion) at hand depend only on the constants $C_{l,s}$ in the bounds $|b|_{l,s}^{(\alpha)} \leq C_{l,s}$. This is sometimes expressed by saying that an estimate (or assertion) is uniform in the symbol b satisfying $|b|^{(\alpha)} \leq C$.

We use the classes \mathbf{S}_α mainly with the weight $w(\boldsymbol{\xi}) = \langle \boldsymbol{\xi} \rangle^\beta, \beta \in (0, 1]$, which satisfies (2.1) for $\kappa = \beta$. Note that \mathbf{S}_α is an increasing function of α , i.e. $\mathbf{S}_\alpha \subset \mathbf{S}_\gamma$ for $\alpha < \gamma$. For later reference we write here the following convenient bounds that follow from definition (2.2) and property (2.1):

$$\text{decay:eq} \quad (2.3) \quad |\mathbf{D}_\xi^s \hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi})| \leq |b|_{l,s}^{(\alpha)} \langle \boldsymbol{\theta} \rangle^{-l} w(\boldsymbol{\xi})^{\alpha-s},$$

$$\text{differ:eq} \quad (2.4) \quad |\mathbf{D}_\xi^s \hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi} + \boldsymbol{\eta}) - \mathbf{D}_\xi^s \hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi})| \leq C |b|_{l,s+1}^{(\alpha)} \langle \boldsymbol{\theta} \rangle^{-l} w(\boldsymbol{\xi})^{\alpha-s-1} \langle \boldsymbol{\eta} \rangle^{\kappa|\alpha-s-1|} |\boldsymbol{\eta}|, \quad s = |s|,$$

with a constant C depending only on α, s . For a vector $\boldsymbol{\eta} \in \mathbb{R}^d$ introduce the symbol

$$\text{bboldeta:eq} \quad (2.5) \quad b_\eta(\mathbf{x}, \boldsymbol{\xi}) = b(\mathbf{x}, \boldsymbol{\xi} + \boldsymbol{\eta}), \quad \boldsymbol{\eta} \in \mathbb{R}^d,$$

so that $\hat{b}_\eta(\boldsymbol{\theta}, \boldsymbol{\xi}) = \hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi} + \boldsymbol{\eta})$. The bound (2.4) implies that for all $|\boldsymbol{\eta}| \leq C$ we have

$$\text{differ1:eq} \quad (2.6) \quad |b - b_\eta|_{l,s}^{(\alpha-1)} \leq C_s |b|_{l,s+1}^{(\alpha)} |\boldsymbol{\eta}|,$$

uniformly in $\boldsymbol{\eta}: |\boldsymbol{\eta}| \leq C$.

Now we define the PDO $\text{Op}(b)$ in the usual way:

$$\text{Op}(b)u(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int b(\mathbf{x}, \boldsymbol{\xi}) e^{i\langle \boldsymbol{\xi}, \mathbf{x} \rangle} (\mathcal{F}u)(\boldsymbol{\xi}) d\boldsymbol{\xi},$$

the integrals being over \mathbb{R}^d . Under the condition $b \in \mathbf{S}_\alpha$ the integral in the r.h.s. is clearly finite for any u from the Schwarz class $\mathcal{S}(\mathbb{R}^d)$. Moreover, the condition $b \in \mathbf{S}_0$ guarantees the boundedness of $\text{Op}(b)$ in $L^2(\mathbb{R}^d)$, see Proposition 3.1. Unless otherwise stated, from now on $\mathcal{S}(\mathbb{R}^d)$ is taken as a natural domain for all PDO's at hand. Observe that the operator $\text{Op}(b)$ is symmetric if its symbol satisfies the condition

$$\text{selfadj:eq} \quad (2.7) \quad \hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi}) = \overline{\hat{b}(-\boldsymbol{\theta}, \boldsymbol{\xi} + \boldsymbol{\theta})}.$$

We shall call such symbols *symmetric*.

Our aim is to study the spectrum of the operator

$$\boxed{\text{h:eq}} \quad (2.8) \quad \begin{cases} H = \text{Op}(h), \quad h(\mathbf{x}, \boldsymbol{\xi}) = h_0(\boldsymbol{\xi}) + b(\mathbf{x}, \boldsymbol{\xi}), \\ h_0(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{2m}, \quad m > 0, \\ b \in \mathbf{S}_\alpha(\langle \boldsymbol{\xi} \rangle^\beta), \quad \alpha\beta < 2m, \end{cases}$$

with a symmetric symbol b . The operator $\text{Op}(b)$ is infinitesimally H_0 -bounded, see Lemma 3.2, so that H is self-adjoint on the domain $D(H) = D(H_0) = \mathbf{H}^{2m}(\mathbb{R}^d)$. Due to the Γ -periodicity of the symbol b , the operator H commutes with the shifts along the lattice vectors, i.e.

$$H\mathcal{T}_\gamma = \mathcal{T}_\gamma H, \quad \gamma \in \Gamma.$$

with $(\mathcal{T}_\gamma u)(\mathbf{x}) = u(\mathbf{x} + \gamma)$. This allows us to use the *Floquet decomposition*.

quet:subject

2.2. Floquet decomposition. We identify the underlying Hilbert space $\mathcal{H} = \mathbf{L}^2(\mathbb{R}^d)$ with the direct integral

$$\mathfrak{G} = \int_{\mathcal{O}^\dagger} \mathfrak{H} d\mathbf{k}, \quad \mathfrak{H} = \mathbf{L}^2(\mathcal{O}).$$

This identification is implemented by the Gelfand transform

gelfand:eq

$$(2.9) \quad (Uu)(\mathbf{x}, \mathbf{k}) = \frac{1}{\sqrt{d(\Gamma^\dagger)}} e^{-i\langle \mathbf{k}, \mathbf{x} \rangle} \sum_{\gamma \in \Gamma} e^{-i\langle \mathbf{k}, \gamma \rangle} u(\mathbf{x} + \gamma), \quad \mathbf{k} \in \mathbb{R}^d,$$

which is initially defined on $u \in \mathcal{S}(\mathbb{R}^d)$ and extends by continuity to a unitary mapping from \mathcal{H} onto \mathfrak{G} . In terms of the Fourier transform the Gelfand transform is defined as follows: $(Uu)(\boldsymbol{\theta}, \mathbf{k}) = (\mathcal{F}u)(\boldsymbol{\theta} + \mathbf{k})$, $\boldsymbol{\theta} \in \Gamma^\dagger$. The unitary operator U reduces \mathcal{T}_γ to the diagonal form:

$$(U\mathcal{T}_\gamma U^{-1}f)(\cdot, \mathbf{k}) = e^{i\mathbf{k} \cdot \gamma} f(\cdot, \mathbf{k}), \quad \forall \gamma \in \Gamma.$$

Let us consider a self-adjoint operator A in \mathcal{H} which commutes with \mathcal{T}_γ for all $\gamma \in \Gamma$, i.e. $A\mathcal{T}_\gamma = \mathcal{T}_\gamma A$. We call such operators (Γ) -periodic. Then A is partially diagonalised by U (see [20]), that is, there exists a measurable family of self-adjoint operators (fibres) $A(\mathbf{k}), \mathbf{k} \in \mathcal{O}^\dagger$ acting in \mathfrak{H} , such that

direct:eq

$$(2.10) \quad UAU^* = \int_{\mathcal{O}^\dagger} A(\mathbf{k}) d\mathbf{k}.$$

It is easy to show that any periodic operator T , which is A -bounded with relative bound $\epsilon < 1$, can be also decomposed into a measurable set of fibers $T(\mathbf{k})$ in the sense that

$$(UTf)(\cdot, \mathbf{k}) = T(\mathbf{k})(Uf)(\cdot, \mathbf{k}), \quad \text{a.e. } \mathbf{k} \in \mathcal{O}^\dagger,$$

for all $f \in D(A)$. Moreover, the fibers $T(\mathbf{k})$ are $A(\mathbf{k})$ -bounded with the bound ϵ , and if T is symmetric, then the operator $A(\mathbf{k}) + T(\mathbf{k})$ is self-adjoint on $D(A(\mathbf{k}))$.

Suppose that the operator A (and hence $A(\mathbf{k})$) is bounded from below and that the spectrum of each $A(\mathbf{k})$ is discrete. Denote by $\lambda_j(A(\mathbf{k})), j = 1, 2, \dots$, the eigenvalues of $A(\mathbf{k})$ labeled in the ascending order. Define the counting function in the usual way:

$$N(\lambda, A(\mathbf{k})) = \#\{j : \lambda_j(A(\mathbf{k})) \leq \lambda\}, \lambda \in \mathbb{R}.$$

If $A = \text{Op}(a)$ with a real-valued symbol $a \in L_{\text{loc}}^\infty(\mathbb{R}^d)$ depending only on $\boldsymbol{\xi}$, then $A(\mathbf{k})$ is a self-adjoint PDO in \mathfrak{H} defined as follows:

$$A(\mathbf{k})u(\mathbf{x}) = \frac{1}{\sqrt{d(\Gamma)}} \sum_{\mathbf{m} \in \Gamma^\dagger} e^{i\mathbf{m}\cdot\mathbf{x}} a(\mathbf{m} + \mathbf{k}) \hat{u}(\mathbf{m}).$$

If $a(\boldsymbol{\xi}) \rightarrow \infty$ as $|\boldsymbol{\xi}| \rightarrow \infty$, then the spectrum of each $A(\mathbf{k})$ is purely discrete with eigenvalues given by $\lambda^{(\mathbf{m})}(\mathbf{k}) = a(\mathbf{m} + \mathbf{k}), \mathbf{m} \in \Gamma^\dagger$. Consequently, the number of eigenvalues below each $\lambda \in \mathbb{R}$ is essentially bounded from above uniformly in $\mathbf{k} \in \mathcal{O}^\dagger$. If T is a periodic symmetric operator which is A -bounded with a bound $\epsilon < 1$, then the spectrum of $A(\mathbf{k}) + T(\mathbf{k})$ is also purely discrete and the counting function is also bounded uniformly in \mathbf{k} . In particular, the above applies to the elliptic operator H defined in (2.8). In fact, applying the Gelfand transform (2.9) to $\text{Op}(b)$, one finds that, similarly to A considered above, the operator $H(\mathbf{k})$ is a PDO in \mathfrak{H} of the form

$$(2.11) \quad H(\mathbf{k})u(\mathbf{x}) = \frac{1}{\sqrt{d(\Gamma)}} \sum_{\mathbf{m} \in \Gamma^\dagger} e^{i\mathbf{m}\cdot\mathbf{x}} h(\mathbf{x}, \mathbf{m} + \mathbf{k}) \hat{u}(\mathbf{m}), \mathbf{k} \in \mathbb{R}^d.$$

The values $H(\mathbf{k})$ for $\mathbf{k} \in \mathcal{O}^\dagger$ determine $H(\mathbf{k})$ for all $\mathbf{k} \in \mathbb{R}^d$ due to the following unitary equivalence:

$$H(\mathbf{k} + \mathbf{m}) = e^{-i\mathbf{m}\cdot\mathbf{x}} H(\mathbf{k}) e^{i\mathbf{m}\cdot\mathbf{x}}, \mathbf{m} \in \Gamma^\dagger.$$

This implies, in particular, that

$$(2.12) \quad \lambda_j(H(\mathbf{k} + \mathbf{m})) = \lambda_j(H(\mathbf{k})), j = 1, 2, \dots,$$

for all $\mathbf{m} \in \Gamma^\dagger$. The images

$$\sigma_j = \bigcup_{\mathbf{k} \in \overline{\mathcal{O}^\dagger}} \lambda_j(H(\mathbf{k})),$$

are called *spectral bands of H* . The spectrum of H is the union

$$\sigma(H) = \bigcup_j \sigma_j.$$

Due to the mentioned boundedness of the counting function $N(\lambda, H(\mathbf{k}))$, each interval $(-\infty, \lambda]$ has non-empty intersection with finitely many spectral bands. When proving the Bethe-Sommerfeld conjecture we study the band overlap, which is characterized by the *the overlap function* $\zeta(\lambda), \lambda \in \mathbb{R}$, defined as the maximal number t such that the symmetric interval $[\lambda - t, \lambda + t]$ is entirely contained in one band, i.e.

$$(2.13) \quad \zeta(\lambda; H) = \begin{cases} \max_j \max\{t : [\lambda - t, \lambda + t] \subset \sigma_j\}, & \lambda \in \sigma(H); \\ 0, & \lambda \notin \sigma(H). \end{cases}$$

It is easy to see that ζ is continuous in λ . An equivalent definition of $\zeta(\lambda)$ is

$$\boxed{\text{zeta1:eq}} \quad (2.14) \quad \zeta(\lambda; H) = \sup\{t : \min_{\mathbf{k}} N(\lambda + t, H(\mathbf{k})) < \max_{\mathbf{k}} N(\lambda - t, H(\mathbf{k}))\}.$$

The function $\zeta(\lambda; H)$ was first introduced by M. Skriganov, see e.g. ^{Skr1}[23].

The main result of the paper is the following Theorem:

main:thm **Theorem 2.1.** *Let $H = H_0 + \text{Op}(b)$ where $H_0 = (-\Delta)^m$ with some $m > 0$, and b is a symmetric symbol from $\mathbf{S}_\alpha(w)$, $w = \langle \boldsymbol{\xi} \rangle^\beta$, with some $\alpha \in \mathbb{R}$ and $\beta \in (0, 1)$ satisfying the condition*

$$\boxed{\text{alm1:eq}} \quad (2.15) \quad 2m - 2 > \beta(\alpha - 2).$$

Then the spectrum of the operator H contains a half-line, i.e. there exists a number $\lambda_0 \in \mathbb{R}$ such that $[\lambda_0, \infty) \subset \sigma(H)$. Moreover, there is a number $S \in \mathbb{R}$ and a constant $c > 0$ such that for each $\lambda \geq \lambda_0$ we have $\zeta(\lambda; H) \geq c\lambda^S$. The constant c and parameter λ_0 are uniform in b satisfying $|b|^{(\alpha)} \leq C$.

If one prefers stating the conditions on b in terms of the ‘‘standard’’ classes $\mathbf{S}_a(\langle \boldsymbol{\xi} \rangle)$, one can re-write Theorem ^{main:thm}2.1 as follows:

main1:thm **Theorem 2.2.** *Let $H = H_0 + \text{Op}(b)$ where $H_0 = (-\Delta)^m$ with some $m > 0$, and b is a symmetric symbol from $\mathbf{S}_a(w)$, $w = \langle \boldsymbol{\xi} \rangle$, with some $a < 2m$. Then the spectrum of the operator H contains a half-line, i.e. there exists a number $\lambda_0 \in \mathbb{R}$ such that $[\lambda_0, \infty) \subset \sigma(H)$. Moreover there is a number $S \in \mathbb{R}$ and a constant $c > 0$ such that for each $\lambda \geq \lambda_0$ we have $\zeta(\lambda; H) \geq c\lambda^S$. The constant c and parameter λ_0 are uniform in b satisfying $|b|^{(\alpha)} \leq C$.*

To deduce Theorem ^{main1:thm}2.2 from ^{main:thm}2.1 it suffices to note that $S_a(\langle \boldsymbol{\xi} \rangle) \subset S_a(\langle \boldsymbol{\xi} \rangle^\beta)$ for any $\beta \in (0, 1)$ and $\alpha = a\beta^{-1}$, and that for this α the condition ^{alm1:eq}(2.15) is equivalent to

$$\boxed{\text{alm2:eq}} \quad (2.16) \quad \beta > \frac{a}{2} - m + 1.$$

Remark 2.3. The magnetic Schrödinger operator $H = (-i\nabla - \mathbf{a})^2 + V$ with a smooth Γ -periodic vector-potential $\mathbf{a} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and electric potential $V : \mathbb{R}^d \rightarrow \mathbb{R}$, is a special case of the operator ^{h:eq}(2.8) with $h_0(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2$ and $b(\mathbf{x}, \boldsymbol{\xi}) = -2\mathbf{a}(\mathbf{x}) \cdot \boldsymbol{\xi} + i\nabla \cdot \mathbf{a}(\mathbf{x}) + |\mathbf{a}(\mathbf{x})|^2 + V(\mathbf{x})$. Thus defined symbols satisfy the conditions of Theorem ^{main1:thm}2.2 with $m = 1$ and $a = 1$. In this case any $\beta > 1/2$ satisfies ^{alm2:eq}(2.16).

Remark 2.4. In ^{BarPar}[1] the Bethe-Sommerfeld conjecture was proved for symmetric symbols b satisfying the conditions $|b|_{l,1}^{(a)} < \infty$ (here $w(\boldsymbol{\xi}) = \langle \boldsymbol{\xi} \rangle$) for some $a < 2m - 1$, and all $l \geq 1$. Although the restriction on the order of b is stronger than in Theorem ^{main1:thm}2.2, the paper ^{BarPar}[1] does not impose any conditions on derivatives w.r.t. $\boldsymbol{\xi}$ of order higher than one. In general, an interesting question is to find out how the smoothness of the perturbation in $\boldsymbol{\xi}$ affects the band overlap. We hope to address this issue in a further publication.

We conclude this Section by fixing some notations which will be used throughout the paper.

bre:subject

2.3. Some notational conventions. For any measurable set $\mathcal{C} \subset \mathbb{R}^d$ we denote by $\mathcal{P}(\mathcal{C})$ the operator $\text{Op}(\chi(\cdot; \mathcal{C}))$, where $\chi(\cdot; \mathcal{C})$ is the characteristic function of the set \mathcal{C} . We denote $\mathcal{H}(\mathcal{C}) = \mathcal{P}(\mathcal{C})\mathcal{H}$, $\mathcal{H} = \mathbf{L}^2(\mathbb{R}^d)$. Accordingly, the fibres $\mathcal{P}(\mathbf{k}, \mathcal{C})$, $\mathbf{k} \in \mathcal{O}^\dagger$, of $\mathcal{P}(\mathcal{C})$, which act in \mathfrak{H} , are PDO's with symbols $\sum_{\mathbf{m} \in \Gamma^\dagger} \chi(\mathbf{m} + \mathbf{k}; \mathcal{C})$. In other words, each $\mathcal{P}(\mathbf{k}; \mathcal{C})$ is a projection in \mathfrak{H} on the linear span of the exponentials

$$(2.17) \quad E_{\mathbf{m}}(\mathbf{x}) := \frac{1}{\sqrt{d(\Gamma)}} e^{i\mathbf{m} \cdot \mathbf{x}}, \quad \mathbf{m} \in \Gamma^\dagger : \mathbf{m} + \mathbf{k} \in \mathcal{C}.$$

The subspace $\mathcal{P}(\mathbf{k}; \mathcal{C})\mathfrak{H}$ of \mathfrak{H} is denoted by $\mathfrak{H}(\mathbf{k}; \mathcal{C})$.

Suppose that $\mathcal{H}(\mathcal{C})$ is an invariant subspace of the operator H defined in (2.8)^{h:eq}, that is $(H - iI)^{-1}\mathcal{H}(\mathcal{C}) \subset \mathcal{H}(\mathcal{C})$. Then the subspace $\mathfrak{H}(\mathbf{k}; \mathcal{C})$, $\mathbf{k} \in \mathcal{O}^\dagger$, is invariant for $H(\mathbf{k})$. We denote by $H(\mathbf{k}; \mathcal{C})$ the part of $H(\mathbf{k})$ in $\mathfrak{H}(\mathbf{k}; \mathcal{C})$, so that

$$H(\mathbf{k}) = H(\mathbf{k}; \mathcal{C}) \oplus H(\mathbf{k}; \mathbb{R}^d \setminus \mathcal{C}), \quad \mathbf{k} \in \mathcal{O}^\dagger,$$

where \oplus denotes the orthogonal sum. If $\mathcal{H}(\mathcal{C})$ is invariant for H , then we denote by $N(\lambda, H(\mathbf{k}; \mathcal{C}))$ the counting function of $H(\mathbf{k}; \mathcal{C})$ on the subspace $\mathfrak{H}(\mathbf{k}; \mathcal{C})$.

Each $\boldsymbol{\xi} \in \mathbb{R}^d$ can be uniquely represented as the sum $\boldsymbol{\xi} = \mathbf{m} + \mathbf{k}$, where $\mathbf{m} \in \Gamma^\dagger$ and $\mathbf{k} \in \mathcal{O}^\dagger$. We say that $\mathbf{m} =: [\boldsymbol{\xi}]$ is the integer part of $\boldsymbol{\xi}$ and $\mathbf{k} =: \{\boldsymbol{\xi}\}$ is the fractional part of $\boldsymbol{\xi}$.

The notation $B(\mathbf{x}_0, R)$ is used for the open ball in \mathbb{R}^d of radius $R > 0$, centered at $\mathbf{x}_0 \in \mathbb{R}^d$. We also write $B(R)$ for the open ball of radius R centered at 0.

For the reference convenience we copy here the conventions about symbol classes made earlier in this section. In the situations when it is not important for us to know the exact values of l, s in the norm $|b|_{l,s}^{(\alpha)}$, we denote the above norm by $|b|^{(\alpha)}$. In this case the inequality $A \leq C|b|^{(\alpha)}$ means that there exist values of l and s , and a constant $C > 0$, possibly depending on l, s , such that $A \leq C|b|_{l,s}^{(\alpha)}$. Similarly, when we write $|b|^{(\gamma)} \leq C|g|^{(\alpha)}$ for some symbols $b \in \mathbf{S}_\gamma, g \in \mathbf{S}_\alpha$, we mean that for any l and s the norm $|b|_{l,s}^{(\gamma)}$ is bounded by $|g|_{p,n}^{(\alpha)}$ with some p and n depending on l, s , and some constant $C = C_{l,s}$. In general, by C, c (with or without indices) we denote various positive constants, whose precise value is unimportant. Throughout the entire paper we adopt the following convention. An estimate (or an assertion) is said to be uniform in a symbol $b \in \mathbf{S}_\alpha$ if the constants in the estimate (or assertion) at hand depend only on the constants $C_{l,s}$ in the bounds $|b|_{l,s}^{(\alpha)} \leq C_{l,s}$. This is sometimes expressed by saying that an estimate (or assertion) is uniform in the symbol b satisfying $|b|^{(\alpha)} \leq C$.

We sometimes use notation $f \ll g$ or $g \gg f$ for two positive functions f, g , if there is a constant $C > 0$ independent of f, g such that $f \leq Cg$. If $f \ll g$ and $g \ll f$, then we write $f \asymp g$.

3. PROPERTIES OF PERIODIC PDO'S

In this section we collect various properties of periodic PDO's to be used in what follows.

3.1. Some basic results on the calculus of periodic PDO's. We begin by listing some elementary results for periodic PDO's, some of which can be found in [27].

Recall that $\mathfrak{S}(\mathbb{R}^d)$ is taken as a natural domain of $\text{Op}(b)$. Unless otherwise stated, all the symbols are supposed to belong to the class $\mathbf{S}_\alpha = \mathbf{S}_\alpha(w; \Gamma)$, $\alpha \in \mathbb{R}$, with an arbitrary function w satisfying (2.1) and a lattice Γ . The functions w and the lattice Γ are usually omitted from the notation.

Proposition 3.1. (See e.g. [27]) Suppose that $|b|_{l,0}^{(0)} < \infty$ with some $l > d$. Then $B = \text{Op}(b)$ is bounded in \mathcal{H} and $\|B\| \leq C|b|_{l,0}^{(0)}$, with a constant C independent of b .

Since $\text{Op}(b)u \in \mathfrak{S}(\mathbb{R}^d)$ for any $b \in \mathbf{S}_\alpha$ and $u \in \mathfrak{S}(\mathbb{R}^d)$, the product $\text{Op}(b)\text{Op}(g)$, $b \in \mathbf{S}_\alpha, g \in \mathbf{S}_\gamma$, is well defined on $\mathfrak{S}(\mathbb{R}^d)$. A straightforward calculation gives the following formula for the symbol $b \circ g$ of the product $\text{Op}(b)\text{Op}(g)$:

$$(b \circ g)(\mathbf{x}, \boldsymbol{\xi}) = \frac{1}{d(\Gamma)} \sum_{\boldsymbol{\theta}, \boldsymbol{\phi}} \hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi} + \boldsymbol{\phi}) \hat{g}(\boldsymbol{\phi}, \boldsymbol{\xi}) e^{i(\boldsymbol{\theta} + \boldsymbol{\phi})\mathbf{x}},$$

and hence

$$(3.1) \quad \widehat{(b \circ g)}(\boldsymbol{\chi}, \boldsymbol{\xi}) = \frac{1}{\sqrt{d(\Gamma)}} \sum_{\boldsymbol{\theta} + \boldsymbol{\phi} = \boldsymbol{\chi}} \hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi} + \boldsymbol{\phi}) \hat{g}(\boldsymbol{\phi}, \boldsymbol{\xi}), \quad \boldsymbol{\chi} \in \Gamma^\dagger, \boldsymbol{\xi} \in \mathbb{R}^d.$$

Here and below $\boldsymbol{\theta}, \boldsymbol{\phi} \in \Gamma^\dagger$. In particular, one sees that $\text{Op}(b)\text{Op}(w^\delta) = \text{Op}(bw^\delta)$ for any $\delta \in \mathbb{R}$. This observation leads to the following Lemma. We remind that the symbol $b_\boldsymbol{\eta}$ is defined in (2.5).

Lemma 3.2. Let $b \in \mathbf{S}_\alpha(w)$ with $w(\boldsymbol{\xi}) = \langle \boldsymbol{\xi} \rangle^\beta, \beta \in (0, 1]$. Then for any $u \in \mathfrak{S}(\mathbb{R}^d)$ and any $l > d$, we have

$$(3.2) \quad \|\text{Op}(b)u\| \leq C|b|_{l,0}^{(\alpha)} \|(H_0 + I)^{\tilde{\gamma}}u\|, \quad \tilde{\gamma} = \frac{\alpha\beta}{2m},$$

with a constant C independent of b, u . In particular, if $\alpha\beta < 2m$, then $\text{Op}(b)$ is H_0 -bounded with an arbitrarily small relative bound.

Moreover, for any $\boldsymbol{\eta} \in \mathbb{R}^d$ and any $l > d$,

$$(3.3) \quad \|(\text{Op}(b) - \text{Op}(b_\boldsymbol{\eta}))u\| \leq C|\boldsymbol{\eta}| |b|_{l,1}^{(\alpha)} \|(H_0 + I)^\gamma u\|, \quad \gamma = \frac{\beta(\alpha - 1)}{2m},$$

where the constant C does not depend on b, u , and is uniform in $\boldsymbol{\eta}$: $|\boldsymbol{\eta}| \leq \tilde{C}$.

Proof. Define $G = B\text{Op}(w^{-\alpha})$. As we have observed earlier, $G = \text{Op}(g)$ with $g = bw^{-\alpha}$, so that $g \in \mathbf{S}_0(w)$ and $|g|_{l,0}^{(0)} = |b|_{l,0}^{(\alpha)}$. Hence, by Lemma 3.1, $\|G\| \leq C|b|_{l,0}^{(\alpha)}$ and

$$(3.4) \quad \|\text{Op}(b)u\| = \|G\text{Op}(w^\alpha)u\| \leq C|b|_{l,0}^{(\alpha)} \|\text{Op}(w^\alpha)u\|.$$

As $\text{Op}(w^\alpha) \leq C(H_0 + I)^{\tilde{\gamma}}$, $\tilde{\gamma} = \alpha\beta(2m)^{-1}$, we get (3.2). formbound1:eq

The bound (3.3) follows from (3.2) when applied to the symbol $b - b_\eta$, and from the estimate (2.6). differ1:eq \square

The bound (3.2) allows one to give a proper meaning to the operator (2.8), since b is infinitesimally H_0 -bounded. The bound (3.3) will be useful in the study of the Floquet eigenvalues as functions of the quasi-momentum \mathbf{k} . h:eq

For general symbols b, g we have the following proposition (see e.g. [27]). Sob0

Proposition 3.3. *Let $b \in \mathbf{S}_\alpha$, $g \in \mathbf{S}_\gamma$. Then $b \circ g \in \mathbf{S}_{\alpha+\gamma}$ and*

$$|b \circ g|^{(\alpha+\gamma)} \leq C |b|^{(\alpha)} |g|^{(\gamma)},$$

with a constant C independent of b, g .

We are also interested in the estimates for symbols of commutators. For PDO's A, Ψ_l , $l = 1, 2, \dots, N$, denote

$$\begin{aligned} \text{ad}(A; \Psi_1, \Psi_2, \dots, \Psi_N) &= i[\text{ad}(A; \Psi_1, \Psi_2, \dots, \Psi_{N-1}), \Psi_N], \\ \text{ad}(A; \Psi) &= i[A, \Psi], \quad \text{ad}^N(A; \Psi) = \text{ad}(A; \Psi, \Psi, \dots, \Psi), \quad \text{ad}^0(A; \Psi) = A. \end{aligned}$$

For the sake of convenience we use the notation $\text{ad}(a; \psi_1, \psi_2, \dots, \psi_N)$ and $\text{ad}^N(a, \psi)$ for the symbols of multiple commutators. It follows from (3.1) that the Fourier coefficients of the symbol $\text{ad}(b, g)$ are given by prodsymb:eq

$$(3.5) \quad \widehat{\text{ad}(b, g)}(\boldsymbol{\chi}, \boldsymbol{\xi}) = \frac{i}{\sqrt{d(\Gamma)}} \sum_{\boldsymbol{\theta} + \boldsymbol{\phi} = \boldsymbol{\chi}} [\hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi} + \boldsymbol{\phi}) \hat{g}(\boldsymbol{\phi}, \boldsymbol{\xi}) - \hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi}) \hat{g}(\boldsymbol{\phi}, \boldsymbol{\xi} + \boldsymbol{\theta})],$$

$$\boldsymbol{\chi} \in \Gamma^\dagger, \quad \boldsymbol{\xi} \in \mathbb{R}^d.$$

Proposition 3.4. (See e.g. [27]) Sob0 *Let $b \in \mathbf{S}_\alpha$ and $g_j \in \mathbf{S}_{\gamma_j}$, $j = 1, 2, \dots, N$. Then $\text{ad}(b; g_1, \dots, g_N) \in \mathbf{S}_\gamma$ with*

$$\gamma = \alpha + \sum_{j=1}^N (\gamma_j - 1),$$

and

$$(3.6) \quad |\text{ad}(b; g_1, \dots, g_N)|^{(\gamma)} \leq C |b|^{(\alpha)} \prod_{j=1}^N |g_j|^{(\gamma_j)},$$

with a constant C independent of b, g_j .

3.2. Partition of the perturbation. From now on the weights in the definition of classes $\mathbf{S}_\alpha = \mathbf{S}_\alpha(w)$ are assumed to be $w(\boldsymbol{\xi}) = \langle \boldsymbol{\xi} \rangle^\beta$ with some $\beta \in (0, 1]$. Here we partition every symbol $b \in \mathbf{S}_\alpha$ into the sum of several symbols, restricted to different parts of the phase space. These symbols depend on the parameter $\rho \geq 1$, but this dependence is usually omitted from the notation. Later on, we will put $\rho = \lambda^{\frac{1}{2m}}$.

Let $\iota \in C^\infty(\mathbb{R})$ be a non-negative function such that

$$\boxed{\text{eta:eq}} \quad (3.7) \quad 0 \leq \iota \leq 1, \quad \iota(z) = \begin{cases} 1, & z \leq 1/4; \\ 0, & z \geq 1/2. \end{cases}$$

For $L \geq 1$ and $\boldsymbol{\theta} \in \Gamma^\dagger$, $\boldsymbol{\theta} \neq \mathbf{0}$, define the following C^∞ -cut-off functions:

$$\boxed{\text{el:eq}} \quad (3.8) \quad \begin{cases} e_{\boldsymbol{\theta}}(\boldsymbol{\xi}) = \iota\left(\left|\frac{|\boldsymbol{\xi} + \boldsymbol{\theta}/2|}{\rho} - 1\right|\right), \\ \ell_{\boldsymbol{\theta}}^>(\boldsymbol{\xi}) = 1 - \iota\left(\frac{|\boldsymbol{\xi} + \boldsymbol{\theta}/2|}{\rho} - 1\right), \\ \ell_{\boldsymbol{\theta}}^<(\boldsymbol{\xi}) = 1 - \iota\left(1 - \frac{|\boldsymbol{\xi} + \boldsymbol{\theta}/2|}{\rho}\right), \end{cases}$$

and

$$\boxed{\text{phizeta:eq}} \quad (3.9) \quad \begin{cases} \zeta_{\boldsymbol{\theta}}(\boldsymbol{\xi}; L) = \iota\left(\frac{|\boldsymbol{\theta}(\boldsymbol{\xi} + \boldsymbol{\theta}/2)|}{L|\boldsymbol{\theta}|}\right), \\ \varphi_{\boldsymbol{\theta}}(\boldsymbol{\xi}; L) = 1 - \zeta_{\boldsymbol{\theta}}(\boldsymbol{\xi}; L). \end{cases}$$

Note that $e_{\boldsymbol{\theta}} + \ell_{\boldsymbol{\theta}}^> + \ell_{\boldsymbol{\theta}}^< = 1$. The function $\ell_{\boldsymbol{\theta}}^>$ is supported on the set $|\boldsymbol{\xi} + \boldsymbol{\theta}/2| > 5\rho/4$, and $\ell_{\boldsymbol{\theta}}^<$ is supported on the set $|\boldsymbol{\xi} + \boldsymbol{\theta}/2| < 3\rho/4$. The function $e_{\boldsymbol{\theta}}$ is supported in the shell $\rho/2 \leq |\boldsymbol{\xi}| \leq 3\rho/2$. Omitting the parameter L and using the notation $\ell_{\boldsymbol{\theta}}$ for any of the functions $\ell_{\boldsymbol{\theta}}^>$ or $\ell_{\boldsymbol{\theta}}^<$, we point out that

$$\boxed{\text{symmetry:eq}} \quad (3.10) \quad \begin{cases} e_{\boldsymbol{\theta}}(\boldsymbol{\xi}) = e_{-\boldsymbol{\theta}}(\boldsymbol{\xi} + \boldsymbol{\theta}), \quad \ell_{\boldsymbol{\theta}}(\boldsymbol{\xi}) = \ell_{-\boldsymbol{\theta}}(\boldsymbol{\xi} + \boldsymbol{\theta}), \\ \varphi_{\boldsymbol{\theta}}(\boldsymbol{\xi}) = \varphi_{-\boldsymbol{\theta}}(\boldsymbol{\xi} + \boldsymbol{\theta}), \quad \zeta_{\boldsymbol{\theta}}(\boldsymbol{\xi}) = \zeta_{-\boldsymbol{\theta}}(\boldsymbol{\xi} + \boldsymbol{\theta}). \end{cases}$$

Note that the above functions satisfy the estimates

$$\boxed{\text{varphi:eq}} \quad (3.11) \quad \begin{cases} |\mathbf{D}_{\boldsymbol{\xi}}^s e_{\boldsymbol{\theta}}(\boldsymbol{\xi})| + |\mathbf{D}_{\boldsymbol{\xi}}^s \ell_{\boldsymbol{\theta}}(\boldsymbol{\xi})| \ll \rho^{-|s|}, \\ |\mathbf{D}_{\boldsymbol{\xi}}^s \varphi_{\boldsymbol{\theta}}(\boldsymbol{\xi}; L)| + |\mathbf{D}_{\boldsymbol{\xi}}^s \zeta_{\boldsymbol{\theta}}(\boldsymbol{\xi}; L)| \ll L^{-|s|}. \end{cases}$$

Let

$$\boxed{\text{T:eq}} \quad (3.12) \quad \Theta_r = \Theta_r(\Gamma) = \{\boldsymbol{\theta} \in \Gamma^\dagger : 0 < |\boldsymbol{\theta}| \leq r\}, \quad \Theta_r^0 = \Theta_r \cup \{\mathbf{0}\},$$

with some $r > 0$. We always assume that $1 \leq r \leq \rho^\varkappa$, where $\varkappa < \beta$ is a fixed (small) positive number the precise value of which will be chosen later. Using the above cut-off functions, for any symbol $b \in \mathbf{S}_\alpha(w)$ we introduce six new symbols $b^{\mathcal{L}\mathcal{F}}, b^{\mathcal{S}\mathcal{E}}, b^{\circ}, b^{\mathcal{L}\mathcal{E}}, b^{\mathcal{N}\mathcal{R}}, b^{\mathcal{R}}$

in the following way:

$$\boxed{\text{uparrow:eq}} \quad (3.13) \quad b^{\mathcal{LF}}(\mathbf{x}, \boldsymbol{\xi}; \rho) = \frac{1}{\sqrt{d(\Gamma)}} \sum_{\boldsymbol{\theta} \notin \Theta_r^0} \hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi}) e^{i\boldsymbol{\theta}\mathbf{x}},$$

$$\boxed{\text{sharp:eq}} \quad (3.14) \quad b^{\mathcal{LE}}(\mathbf{x}, \boldsymbol{\xi}; \rho) = \frac{1}{\sqrt{d(\Gamma)}} \sum_{\boldsymbol{\theta} \in \Theta_r} \hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi}) \ell_{\boldsymbol{\theta}}^>(\boldsymbol{\xi}) e^{i\boldsymbol{\theta}\mathbf{x}},$$

$$\boxed{\text{natural:eq}} \quad (3.15) \quad b^{\mathcal{NR}}(\mathbf{x}, \boldsymbol{\xi}; \rho) = \frac{1}{\sqrt{d(\Gamma)}} \sum_{\boldsymbol{\theta} \in \Theta_r} \hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi}) \varphi_{\boldsymbol{\theta}}(\boldsymbol{\xi}; \rho^\beta) e_{\boldsymbol{\theta}}(\boldsymbol{\xi}) e^{i\boldsymbol{\theta}\mathbf{x}},$$

$$\boxed{\text{flat:eq}} \quad (3.16) \quad b^{\mathcal{R}}(\mathbf{x}, \boldsymbol{\xi}; \rho) = \frac{1}{\sqrt{d(\Gamma)}} \sum_{\boldsymbol{\theta} \in \Theta_r} \hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi}) \zeta_{\boldsymbol{\theta}}(\boldsymbol{\xi}; \rho^\beta) e_{\boldsymbol{\theta}}(\boldsymbol{\xi}) e^{i\boldsymbol{\theta}\mathbf{x}},$$

$$\boxed{\text{downarrow:eq}} \quad (3.17) \quad b^{\mathcal{SE}}(\mathbf{x}, \boldsymbol{\xi}; \rho) = \frac{1}{\sqrt{d(\Gamma)}} \sum_{\boldsymbol{\theta} \in \Theta_r} \hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi}) \ell_{\boldsymbol{\theta}}^<(\boldsymbol{\xi}) e^{i\boldsymbol{\theta}\mathbf{x}},$$

$$\boxed{\text{o:eq}} \quad (3.18) \quad b^o(\mathbf{x}, \boldsymbol{\xi}; \rho) = b^o(\boldsymbol{\xi}; \rho) = \frac{1}{\sqrt{d(\Gamma)}} \hat{b}(0, \boldsymbol{\xi}).$$

The superscripts here are chosen to mean correspondingly: \mathcal{LF} = ‘large Fourier’ (coefficients), \mathcal{LE} = ‘large energy’, \mathcal{NR} = ‘non-resonance’, \mathcal{R} = ‘resonance’, \mathcal{SE} = ‘small energy’, $o=0$ -th Fourier coefficient. Sometimes the dependence of the introduced symbols on the parameter ρ is omitted from the notation. The corresponding operators are denoted by

$$B^{\mathcal{LF}} = \text{Op}(b^{\mathcal{LF}}), \quad B^{\mathcal{LE}} = \text{Op}(b^{\mathcal{LE}}), \quad B^{\mathcal{NR}} = \text{Op}(b^{\mathcal{NR}}), \\ B^{\mathcal{R}} = \text{Op}(b^{\mathcal{R}}), \quad B^{\mathcal{SE}} = \text{Op}(b^{\mathcal{SE}}), \quad B^o = \text{Op}(b^o).$$

By definition $\boxed{\text{eta:eq}}$ (3.7),

$$b = b^o + b^{\mathcal{SE}} + b^{\mathcal{R}} + b^{\mathcal{NR}} + b^{\mathcal{LE}} + b^{\mathcal{LF}}.$$

The role of each of these operator is easy to explain. The symbol $b^{\mathcal{LF}}$ contains only Fourier coefficients with $|\boldsymbol{\theta}| > r$, and the remaining symbols contain the Fourier coefficients with $|\boldsymbol{\theta}| \leq r$. Note that on the support of the functions $\hat{b}^{\mathcal{NR}}(\boldsymbol{\theta}, \cdot; \rho)$ and $\hat{b}^{\mathcal{R}}(\boldsymbol{\theta}, \cdot; \rho)$ we have

$$\boxed{\text{ds:eq}} \quad (3.19) \quad |\boldsymbol{\theta}| \leq \rho^\beta, \quad \frac{1}{2}\rho \leq |\boldsymbol{\xi} + \boldsymbol{\theta}/2| \leq \frac{3}{2}\rho, \quad \frac{1}{2}\rho - \frac{1}{2}\rho^\alpha \leq |\boldsymbol{\xi}| \leq \frac{3}{2}\rho + \frac{1}{2}\rho^\alpha.$$

On the support of $b^{\mathcal{SE}}(\boldsymbol{\theta}, \cdot; \rho)$ we have

$$\boxed{\text{portell<:eq}} \quad (3.20) \quad \left| \boldsymbol{\xi} + \frac{\boldsymbol{\theta}}{2} \right| \leq \frac{3}{4}\rho, \quad |\boldsymbol{\xi}| \leq \frac{3}{4}\rho + \frac{1}{2}\rho^\alpha.$$

On the support of $b^{\mathcal{LE}}(\mathbf{x}, \cdot; \rho)$ we have

$$\boxed{\text{portell>:eq}} \quad (3.21) \quad \left| \boldsymbol{\xi} + \frac{\boldsymbol{\theta}}{2} \right| \geq \frac{5}{4}\rho, \quad |\boldsymbol{\xi}| \geq \frac{5}{4}\rho - \frac{1}{2}\rho^\alpha.$$

The introduced symbols play a central role in the proof of the Main Theorem $\boxed{\text{main:thm}}$ 2.1. As we show in the course of the proof, due to $\boxed{\text{supportell<:eq}}$ (3.20) and $\boxed{\text{supportell>:eq}}$ (3.21) the symbols $b^{\mathcal{LF}}$, $b^{\mathcal{SE}}$ and

$b^{\mathcal{L}\mathcal{E}}$ make only a negligible contribution to the spectrum of the operator [\(2.8\)](#) near the point $\lambda = \rho^{2m}$. The only significant components of b are the symbols $b^{\mathcal{N}\mathcal{R}}, b^{\mathcal{R}}$ and b° . The symbol $b^{\mathcal{N}\mathcal{R}}$ will be transformed in the next Section into another symbol, independent of \mathbf{x} .

We will often combine $B^{\mathcal{R}}, B^{\mathcal{L}\mathcal{E}}$ and $B^{\mathcal{L}\mathcal{F}}, B^{\mathcal{S}\mathcal{E}}$: for instance $B^{\mathcal{R},\mathcal{L}\mathcal{E}} = B^{\mathcal{R}} + B^{\mathcal{L}\mathcal{E}}$, $B^{\mathcal{R},\mathcal{L}\mathcal{E},\mathcal{L}\mathcal{F}} = B^{\mathcal{R},\mathcal{L}\mathcal{E}} + B^{\mathcal{L}\mathcal{F}}$. A similar convention applies to the symbols. Under the condition $b \in \mathbf{S}_\alpha(w)$ the above symbols belong to the same class $\mathbf{S}_\alpha(w)$ and the following bounds hold:

$$\boxed{\text{subord:eq}} \quad (3.22) \quad |b^{\mathcal{R}}|_{l,s}^{(\alpha)} + |b^{\mathcal{N}\mathcal{R}}|_{l,s}^{(\alpha)} + |b^{\mathcal{L}\mathcal{E}}|_{l,s}^{(\alpha)} + |b^\circ|_{l,s}^{(\alpha)} + |b^{\mathcal{S}\mathcal{E}}|_{l,s}^{(\alpha)} + |b^{\mathcal{L}\mathcal{F}}|_{l,s}^{(\alpha)} \ll |b|_{l,s}^{(\alpha)}.$$

Indeed, let us check this for the symbol $b^{\mathcal{N}\mathcal{R}}$, for instance. According to [\(3.19\)](#) and [\(3.11\)](#), on the support of the function $\hat{b}^{\mathcal{N}\mathcal{R}}(\boldsymbol{\theta}, \cdot; \rho)$ we have

$$|\mathbf{D}^s \varphi_{\boldsymbol{\theta}}(\boldsymbol{\xi}, \rho^\beta)| \ll \rho^{-\beta|s|} \ll w^{-|s|},$$

$$|\mathbf{D}^s \ell_{\boldsymbol{\theta}}^>(\boldsymbol{\xi})| + |\mathbf{D}^s \ell_{\boldsymbol{\theta}}^<(\boldsymbol{\xi})| + |\mathbf{D}^s e_{\boldsymbol{\theta}}(\boldsymbol{\xi})| \ll \rho^{-|s|} \ll w^{-|s|}.$$

This immediately leads to the bound of the form [\(3.22\)](#) for the symbol $b^{\mathcal{N}\mathcal{R}}$.

The introduced operations also preserve symmetry. Precisely, calculate using [\(3.10\)](#):

$$\begin{aligned} \overline{\hat{b}^{\mathcal{R}}(-\boldsymbol{\theta}, \boldsymbol{\xi} + \boldsymbol{\theta})} &= \overline{\hat{b}(-\boldsymbol{\theta}, \boldsymbol{\xi} + \boldsymbol{\theta}) \zeta_{-\boldsymbol{\theta}}(\boldsymbol{\xi} + \boldsymbol{\theta}; \rho^\beta) e_{-\boldsymbol{\theta}}(\boldsymbol{\xi} + \boldsymbol{\theta})} \\ &= \hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi}) \zeta_{\boldsymbol{\theta}}(\boldsymbol{\xi}; \rho^\beta) e_{\boldsymbol{\theta}}(\boldsymbol{\xi}) = \hat{b}^{\mathcal{R}}(\boldsymbol{\theta}, \boldsymbol{\xi}). \end{aligned}$$

Therefore, by [\(2.7\)](#) the operator $B^{\mathcal{R}}$ is symmetric if so is B . The proof is similar for the rest of the operators introduced above.

Let us list some other elementary properties of the introduced operators. In the [Lemma](#) below we use the projection $\mathcal{P}(\mathcal{C})$, $\mathcal{C} \subset \mathbb{R}$ whose definition was given in [Subsection 2.3](#).

Lemma 3.5. *Let $b \in \mathbf{S}_\alpha(w)$, $w = \langle \boldsymbol{\xi} \rangle^\beta$, $\beta \in (0, 1]$, with some $\alpha \in \mathbb{R}$. Then the following hold:*

- (i) *The operator $\text{Op}(b^{\mathcal{S}\mathcal{E}})$ is bounded and*

$$\|\text{Op}(b^{\mathcal{S}\mathcal{E}})\| \ll |b|_{l,0}^{(\alpha)} \rho^{\beta \max(\alpha, 0)}.$$

Moreover,

$$(I - \mathcal{P}(B(7\rho/8))) \text{Op}(b^{\mathcal{S}\mathcal{E}}) = \text{Op}(b^{\mathcal{S}\mathcal{E}})(I - \mathcal{P}(B(7\rho/8))) = 0.$$

- (ii) *The operator $B^{\mathcal{R}}$ satisfies the following relations*

$$\mathcal{P}(B(3\rho/8))B^{\mathcal{R}} = B^{\mathcal{R}}\mathcal{P}(B(3\rho/8))$$

$$\boxed{\text{atorthog:eq}} \quad (3.23) \quad = (I - \mathcal{P}(B(13\rho/8)))B^{\mathcal{R}} = B^{\mathcal{R}}(I - \mathcal{P}(B(13\rho/8))) = 0,$$

and similar relations hold for the operator $B^{\mathcal{N}\mathcal{R}}$ as well.

Moreover, for any $\gamma \in \mathbb{R}$ one has $b^{\mathcal{N}\mathcal{R}}, b^{\mathcal{R}} \in \mathbf{S}_\gamma$ and

$$\boxed{\text{nat:eq}} \quad (3.24) \quad |b^{\mathcal{N}\mathcal{R}}|_{l,s}^{(\gamma)} + |b^{\mathcal{R}}|_{l,s}^{(\gamma)} \ll \rho^{\beta(\alpha-\gamma)} |b|_{l,s}^{(\alpha)},$$

for all l and s , with an implied constant independent of b and $\rho \geq 1$. In particular, the operators $B^{\mathcal{NR}}, B^{\mathcal{R}}$ are bounded and

$$\|B^{\mathcal{NR}}\| + \|B^{\mathcal{R}}\| \ll \rho^{\beta\alpha} |b|_{l,0}^{(\alpha)},$$

for any $l > d$.

(iii)

$$\mathcal{P}(B(9\rho/8))B^{\mathcal{L}\mathcal{E}} = B^{\mathcal{L}\mathcal{E}}\mathcal{P}(B(9\rho/8)) = 0.$$

(iv) If $R \leq 2\rho$, then

shar: eq

$$(3.25) \quad \|\mathcal{P}(B(R))B^{\mathcal{L}\mathcal{F}}\| + \|B^{\mathcal{L}\mathcal{F}}\mathcal{P}(B(R))\| \ll |b|_{l,0}^{(\alpha)} r^{p-l} \rho^{\beta \max(\alpha,0)},$$

for any $p > d$ and any $l \geq p$.

Proof. Proof of (i). By [\(2.3\)](#), [decay: eq](#)

$$|\hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi}; \rho)| \leq |b|_{l,0}^{(\alpha)} \langle \boldsymbol{\theta} \rangle^{-l} \langle \boldsymbol{\xi} \rangle^{\beta\alpha},$$

for any $l > 0$. It follows from [\(3.20\)](#) that [supportell<: eq](#)

latdecay: eq

$$(3.26) \quad |\hat{b}^{\mathcal{SE}}(\boldsymbol{\theta}, \boldsymbol{\xi}; \rho)| \ll |b|_{l,0}^{(\alpha)} \rho^{\beta \max(\alpha,0)} \langle \boldsymbol{\theta} \rangle^{-l}, \quad \forall l > 0.$$

By Proposition [3.1](#) [bound: prop](#) this implies the sought bound for the norm $\|\text{Op}(b^{\mathcal{SE}})\|$ [supportell<: eq](#) [downarrow: eq](#)

In view of [\(3.20\)](#), the second part of statement (i) follows from [\(3.17\)](#) by inspection. [bflatorthog: eq](#) [flat: eq](#) [natural: eq](#)

Proof of (ii). The relations [\(3.23\)](#) follow from the definitions [\(3.16\)](#) and [\(3.15\)](#) in view of [\(3.19\)](#). [ds: eq](#)

Furthermore, by [\(2.3\)](#) and [\(3.22\)](#), [decay: eq](#) [subord: eq](#)

$$|\mathbf{D}_{\boldsymbol{\xi}}^s \hat{b}^{\mathcal{NR}}(\boldsymbol{\theta}, \boldsymbol{\xi}; \rho)| + |\mathbf{D}_{\boldsymbol{\xi}}^s \hat{b}^{\mathcal{R}}(\boldsymbol{\theta}, \boldsymbol{\xi}; \rho)| \ll |b|_{l,s}^{(\alpha)} \langle \boldsymbol{\xi} \rangle^{\beta(\alpha-\gamma)} \langle \boldsymbol{\xi} \rangle^{\beta(\gamma-s)} \langle \boldsymbol{\theta} \rangle^{-l}.$$

Thus, using again [\(3.19\)](#), [ds: eq](#) we obtain:

$$|\mathbf{D}_{\boldsymbol{\xi}}^s \hat{b}^{\mathcal{NR}}(\boldsymbol{\theta}, \boldsymbol{\xi}; \rho)| + |\mathbf{D}_{\boldsymbol{\xi}}^s \hat{b}^{\mathcal{R}}(\boldsymbol{\theta}, \boldsymbol{\xi}; \rho)| \ll \rho^{\beta(\alpha-\gamma)} |b|_{l,s}^{(\alpha)} w^{\gamma-s} \langle \boldsymbol{\theta} \rangle^{-l}.$$

This means that $b^{\mathcal{NR}}, b^{\mathcal{R}} \in \mathbf{S}_{\gamma}$ for any $\gamma \in \mathbb{R}$ and [\(3.24\)](#) [nat: eq](#) holds. The bounds for the norms follow from [\(3.24\)](#) with $\gamma = 0$, and Proposition [3.1](#). [bound: prop](#)

Proof of (iii) is similar to (i). The required result follows from [\(3.21\)](#). [supportell>: eq](#)

Proof of (iv). By definition [\(3.12\)](#) [T: eq](#) the sum [\(3.13\)](#) [uparrow: eq](#) contains only those values of $\boldsymbol{\theta}$ for which $|\boldsymbol{\theta}| \geq r$. Thus, in view of [\(2.3\)](#) and [\(3.22\)](#), [decay: eq](#) [subord: eq](#) for any $l \geq p$ we have

$$|\hat{b}^{\mathcal{L}\mathcal{F}}(\boldsymbol{\theta}, \boldsymbol{\xi}; \rho)| \leq |b|_{l,0}^{(\alpha)} w^{\alpha} \langle \boldsymbol{\theta} \rangle^{-l} \ll r^{p-l} |b|_{l,0}^{(\alpha)} w^{\alpha} \langle \boldsymbol{\theta} \rangle^{-p},$$

Thus the symbol of $B^{\mathcal{L}\mathcal{F}}\mathcal{P}(B(R))$ is bounded by

$$Cr^{p-l} |b|_{l,0}^{(\alpha)} R^{\beta \max(\alpha,0)} \langle \boldsymbol{\theta} \rangle^{-p},$$

so that the sought estimate follows by Proposition [3.1](#). [bound: prop](#) The same argument leads to the same bound for $\mathcal{P}(B(R))B^{\mathcal{L}\mathcal{F}}$. \square

In what follows a central role is played by the operator of the form

$$\boxed{\text{newA}} \quad (3.27) \quad A := H_0 + B^o + B^{\mathcal{R}}$$

with some symmetric symbol $b \in \mathbf{S}_\alpha$. In the next Lemma we study the continuity of the Floquet eigenvalues $\lambda_j(A(\mathbf{k}))$, $j = 1, 2, \dots$, as functions of the quasi-momentum $\mathbf{k} \in \mathbb{R}^d$. Here, $A(\mathbf{k})$ are the fibers of the operator (3.27). To state the result, we introduce for any vector $\boldsymbol{\eta} \in \mathbb{R}^d$ the *distance on the torus*:

$$\boxed{\text{distorus:eq}} \quad (3.28) \quad |\boldsymbol{\eta}|_{\mathbb{T}} = \min_{\mathbf{m} \in \Gamma^\dagger} |\boldsymbol{\eta} - \mathbf{m}|.$$

$\boxed{\text{rturbed:thm}}$ **Theorem 3.6.** *Suppose that $\rho \geq 1$ and*

$$\boxed{\text{beta:eq}} \quad (3.29) \quad \beta(\alpha - 1) < 2m - 1.$$

If for some j

$$\boxed{\text{newA1}} \quad (3.30) \quad \lambda_j(A(\mathbf{k})) \asymp \rho^{2m},$$

then for any $l > d$ we have

$$\boxed{\text{rturbed:eq}} \quad (3.31) \quad |\lambda_j(A(\mathbf{k} + \boldsymbol{\eta})) - \lambda_j(A(\mathbf{k}))| \ll (1 + |b|_{l,1}^{(\alpha)}) |\boldsymbol{\eta}|_{\mathbb{T}} \rho^{2m-1}.$$

The implied constant in (3.31) depends on the constants in (3.30).

Proof. By Lemma $\boxed{\text{smallorthog:lem}}$ 3.5 (ii),

$$\mathcal{P}(\mathbf{k}, B(3\rho/8)) B^{\mathcal{R}}(\mathbf{k}) = B^{\mathcal{R}}(\mathbf{k}) \mathcal{P}(\mathbf{k}, B(3\rho/8)) = 0,$$

so that

$$\mathcal{P}(\mathbf{0}, B(3\rho/8 - |\mathbf{k}|)) B^{\mathcal{R}}(\mathbf{k}) = B^{\mathcal{R}}(\mathbf{k}) \mathcal{P}(\mathbf{0}, B(3\rho/8 - |\mathbf{k}|)) = 0.$$

Similarly, $\boxed{\text{flatorthog:eq}}$ (3.23) implies that

$$\left(I - \mathcal{P}(\mathbf{0}, B(13\rho/8 + |\mathbf{k}|)) \right) B^{\mathcal{R}}(\mathbf{k}) = B^{\mathcal{R}}(\mathbf{k}) \left(I - \mathcal{P}(\mathbf{0}, B(13\rho/8 + |\mathbf{k}|)) \right) = 0.$$

Thus, if one assumes that $|\mathbf{k}| \leq R$ with some R , then the operator $A(\mathbf{k})$ can be represented in the form

$$\boxed{\text{orthog:eq}} \quad (3.32) \quad A(\mathbf{k}) = A_-(\mathbf{k}) \oplus A_c(\mathbf{k}) \oplus A_+(\mathbf{k}),$$

with

$$A_{\pm}(\mathbf{k}) = \mathcal{P}_{\pm}(H_0(\mathbf{k}) + B^o(\mathbf{k})) \mathcal{P}_{\pm}, \quad A_c(\mathbf{k}) = \mathcal{P}_c A(\mathbf{k}) \mathcal{P}_c,$$

where by \mathcal{P}_{\pm} and \mathcal{P}_c we have denoted the following projections in \mathfrak{H} :

$$\mathcal{P}_- = \mathcal{P}(\mathbf{0}, B(3\rho/8 - R)),$$

$$\mathcal{P}_c = \mathcal{P}(\mathbf{0}, B(13\rho/8 + R)) - \mathcal{P}(\mathbf{0}, B(3\rho/8 - R)),$$

$$\mathcal{P}_+ = I - \mathcal{P}(\mathbf{0}, B(13\rho/8 + R)).$$

Due to [\(3.32\)](#) and a standard pigeonhole principle argument, in order to establish [\(3.31\)](#) it suffices to prove this inequality for eigenvalues (labeled in the standard ascending order) of each of the operators A_-, A_c, A_+ .

Suppose first that $|\boldsymbol{\eta}|_{\mathbb{T}} = |\boldsymbol{\eta}|$, so that

$$|\boldsymbol{\eta}| \leq R := \min_{\mathbf{0} \neq \mathbf{m} \in \Gamma^\dagger} |\mathbf{m}|.$$

The operator $A_0(\mathbf{k}) = H_0(\mathbf{k}) + B^o(\mathbf{k})$ has constant coefficients and its eigenvalues are found explicitly:

$$\mu_{\mathbf{m}}(\mathbf{k}) = (\mathbf{m} + \mathbf{k})^{2m} + b^o(\mathbf{m} + \mathbf{k}), \mathbf{m} \in \Gamma^\dagger.$$

Assuming that $|\mathbf{m} + \mathbf{k}| \geq 2R$, from [\(2.4\)](#) and [\(3.22\)](#) we obtain that

$$|\mu_{\mathbf{m}}(\mathbf{k} + \boldsymbol{\eta}) - \mu_{\mathbf{m}}(\mathbf{k})| \ll (|\mathbf{m} + \mathbf{k}|^{2m-1} + |b|_{0,1}^{(\alpha)} |\mathbf{m} + \mathbf{k}|^{\beta(\alpha-1)}) |\boldsymbol{\eta}|.$$

Due to conditions [\(3.29\)](#),

$$\boxed{\text{seig0:eq}} \quad (3.33) \quad |\mu_{\mathbf{m}}(\mathbf{k} + \boldsymbol{\eta}) - \mu_{\mathbf{m}}(\mathbf{k})| \ll (1 + |b|_{0,1}^{(\alpha)}) |\mathbf{m} + \mathbf{k}|^{2m-1} |\boldsymbol{\eta}|.$$

If we assume that the considered eigenvalue $\mu_{\mathbf{m}}(\mathbf{k})$ satisfies [\(3.30\)](#), then $|\mathbf{m} + \mathbf{k}| \asymp \rho$, and hence the inequality [\(3.33\)](#) implies that

$$\boxed{\text{seig1:eq}} \quad (3.34) \quad |\mu_{\mathbf{m}}(\mathbf{k} + \boldsymbol{\eta}) - \mu_{\mathbf{m}}(\mathbf{k})| \ll (1 + |b|_{0,1}^{(\alpha)}) \rho^{2m-1} |\boldsymbol{\eta}|.$$

In order to rewrite this bound for the eigenvalues $\lambda_j(A_{\pm}(\mathbf{k}))$ arranged in the ascending order, note that for $\mathbf{m} \in B(3\rho/8 - R)$ or $\mathbf{m} \notin B(13\rho/8 + R)$, the previous inequality is equivalent to the following one:

$$\begin{aligned} N(\lambda - C(1 + |b|_{0,1}^{(\alpha)}) \rho^{2m-1} |\boldsymbol{\eta}|; A_{\pm}(\mathbf{k})) &\leq N(\lambda; A_{\pm}(\mathbf{k} + \boldsymbol{\eta})) \leq \\ N(\lambda + C(1 + |b|_{0,1}^{(\alpha)}) \rho^{2m-1} |\boldsymbol{\eta}|; A_{\pm}(\mathbf{k})), \end{aligned}$$

which, in turn means that

$$\boxed{\text{seig:eq}} \quad (3.35) \quad |\lambda_j(A_{\pm}(\mathbf{k} + \boldsymbol{\eta})) - \lambda_j(A_{\pm}(\mathbf{k}))| \ll (1 + |b|_{0,1}^{(\alpha)}) \rho^{2m-1} |\boldsymbol{\eta}|$$

for all eigenvalues satisfying the condition $|\lambda_j(A_{\pm}(\mathbf{k}))| \asymp \rho^{2m}$.

Let us study the eigenvalues of A_c . This operator is bounded, and hence it suffices to find an upper bound for the sum of the norms

$$\|\mathcal{P}_c(A_0(\mathbf{k} + \boldsymbol{\eta}) - A_0(\mathbf{k}))\mathcal{P}_c\| + \|B^{\mathcal{R}}(\mathbf{k} + \boldsymbol{\eta}) - B^{\mathcal{R}}(\mathbf{k})\|.$$

Using [\(3.34\)](#) for $3\rho/8 - R \leq |\mathbf{m}| \leq 13\rho/8 + R$, we get

$$\boxed{\text{a0:eq}} \quad (3.36) \quad \|\mathcal{P}_c(A_0(\mathbf{k} + \boldsymbol{\eta}) - A_0(\mathbf{k}))\mathcal{P}_c\| \ll (1 + |b|_{0,1}^{(\alpha)}) \rho^{2m-1} |\boldsymbol{\eta}|.$$

(note that $\mathcal{P}_c A_0(\mathbf{k} + \boldsymbol{\eta}) \mathcal{P}_c$ and $\mathcal{P}_c A_0(\mathbf{k}) \mathcal{P}_c$ have the same eigenvectors). By Lemma [3.5\(ii\)](#), $b^{\mathcal{R}} \in \mathbf{S}_{\gamma}$ with any $\gamma \in \mathbb{R}$, so that [\(3.3\)](#) and [\(3.24\)](#) give:

$$\|B^{\mathcal{R}}(\mathbf{k} + \boldsymbol{\eta}) - B^{\mathcal{R}}(\mathbf{k})\| \ll |\boldsymbol{\eta}| |b^{\mathcal{R}}|_{l,1}^{(1)} \ll |\boldsymbol{\eta}| |b|_{l,1}^{(\alpha)} \rho^{\beta(\alpha-1)},$$

for any $l \geq d$. By $\beta(\alpha - 1) < 2m - 1$, so that the above estimate in combination with $\beta(\alpha - 1) < 2m - 1$, proves that

$$|\lambda_j(A_c(\mathbf{k} + \boldsymbol{\eta}) - \lambda_j(A_c(\mathbf{k})))| \ll |\boldsymbol{\eta}|(1 + |\mathbf{b}|_{l,1}^{(\alpha)})\rho^{2m-1},$$

uniformly in j . In its turn, this estimate together with $\beta(\alpha - 1) < 2m - 1$ leads to $\beta(\alpha - 1) < 2m - 1$.

For a general $\boldsymbol{\eta}$, note that according to $\beta(\alpha - 1) < 2m - 1$, $\lambda_j(A(\mathbf{k} + \boldsymbol{\eta})) = \lambda_j(A(\mathbf{k} + \mathbf{m} + \boldsymbol{\eta}))$ for any $\mathbf{m} \in \Gamma^\dagger$. Choose \mathbf{m} in such a way that

$$|\boldsymbol{\eta}|_{\mathbb{T}} = |\boldsymbol{\eta} + \mathbf{m}|.$$

Denote $\boldsymbol{\eta}_1 = \boldsymbol{\eta} + \mathbf{m}$ and use the first part of the proof for $\boldsymbol{\eta}_1$. \square

4. A “GAUGE TRANSFORMATION”

In this and all the subsequent sections we assume that $\mathbf{S}_\alpha = \mathbf{S}_\alpha(w)$ with $w(\boldsymbol{\xi}) = \langle \boldsymbol{\xi} \rangle^\beta$, $\beta \in (0, 1]$. Recall that we study spectral properties of the operator H defined in $\beta \in (0, 1]$. Our ultimate goal is to prove that each sufficiently large λ belongs to the spectrum of H . We are going to use the notation from the previous section with the parameter $\rho = \lambda^{\frac{1}{2m}} \geq 1$.

4.1. Preparation. Our strategy is to find a unitary operator which reduces $H = H_0 + \text{Op}(b)$ to another PDO, whose symbol, up to some controllable small errors, depends only on $\boldsymbol{\xi}$. The sought unitary operator is constructed in the form $U = e^{i\Psi}$ with a suitable bounded self-adjoint Γ -periodic PDO Ψ . This is why we sometimes call it a “gauge transformation”. It is useful to consider $e^{i\Psi}$ as an element of the group

$$U(t) = \exp\{i\Psi t\}, \quad \forall t \in \mathbb{R}.$$

We assume that the operator $\text{ad}(H_0, \Psi)$ is bounded, so that $U(t)D(H_0) = D(H_0)$. This assumption will be justified later on. Let us express the operator

$$A_t := U(-t)HU(t)$$

via its (weak) derivative with respect to t :

$$A_t = H + \int_0^t U(-t') \text{ad}(H; \Psi)U(t') dt'.$$

By induction it is easy to show that

$$(4.1) \quad A_1 = H + \sum_{j=1}^M \frac{1}{j!} \text{ad}^j(H; \Psi) + R_{M+1}^{(1)},$$

$$R_{M+1}^{(1)} := \int_0^1 dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_M} U(-t_{M+1}) \text{ad}^{M+1}(H; \Psi)U(t_{M+1}) dt_{M+1}.$$

The operator Ψ is sought in the form

$$\boxed{\text{psik:eq}} \quad (4.2) \quad \Psi = \sum_{k=1}^M \Psi_k, \quad \Psi_k = \text{Op}(\psi_k),$$

with symbols ψ_k from some suitable classes \mathbf{S}_σ , $\sigma = \sigma_k$ to be specified later on. Substitute this formula in (4.1) and rewrite, regrouping the terms:

$$\boxed{\text{rtilde:eq}} \quad (4.3) \quad \begin{aligned} A_1 &= H_0 + B + \sum_{j=1}^M \frac{1}{j!} \sum_{l=j}^M \sum_{k_1+k_2+\dots+k_j=l} \text{ad}(H; \Psi_{k_1}, \Psi_{k_2}, \dots, \Psi_{k_j}) \\ &\quad + R_{M+1}^{(1)} + R_{M+1}^{(2)}, \\ R_{M+1}^{(2)} &:= \sum_{j=1}^M \frac{1}{j!} \sum_{k_1+k_2+\dots+k_j \geq M+1} \text{ad}(H; \Psi_{k_1}, \Psi_{k_2}, \dots, \Psi_{k_j}). \end{aligned}$$

Changing this expression yet again produces

$$\begin{aligned} A_1 &= H_0 + B + \sum_{l=1}^M \text{ad}(H_0; \Psi_l) + \sum_{j=2}^M \frac{1}{j!} \sum_{l=j}^M \sum_{k_1+k_2+\dots+k_j=l} \text{ad}(H_0; \Psi_{k_1}, \Psi_{k_2}, \dots, \Psi_{k_j}) \\ &\quad + \sum_{j=1}^M \frac{1}{j!} \sum_{l=j}^M \sum_{k_1+k_2+\dots+k_j=l} \text{ad}(B; \Psi_{k_1}, \Psi_{k_2}, \dots, \Psi_{k_j}) + R_{M+1}^{(1)} + R_{M+1}^{(2)}. \end{aligned}$$

Next, we switch the summation signs and decrease l by one in the second summation:

$$\begin{aligned} A_1 &= H_0 + B + \sum_{l=1}^M \text{ad}(H_0; \Psi_l) + \sum_{l=2}^M \sum_{j=2}^l \frac{1}{j!} \sum_{k_1+k_2+\dots+k_j=l} \text{ad}(H_0; \Psi_{k_1}, \Psi_{k_2}, \dots, \Psi_{k_j}) \\ &\quad + \sum_{l=2}^{M+1} \sum_{j=1}^{l-1} \frac{1}{j!} \sum_{k_1+k_2+\dots+k_j=l-1} \text{ad}(B; \Psi_{k_1}, \Psi_{k_2}, \dots, \Psi_{k_j}) + R_{M+1}^{(1)} + R_{M+1}^{(2)}. \end{aligned}$$

Now we introduce the notation

$$B_1 := B,$$

$$\boxed{\text{bl:eq}} \quad (4.4) \quad B_l := \sum_{j=1}^{l-1} \frac{1}{j!} \sum_{k_1+k_2+\dots+k_j=l-1} \text{ad}(B; \Psi_{k_1}, \Psi_{k_2}, \dots, \Psi_{k_j}), \quad l \geq 2,$$

$$\boxed{\text{tl:eq}} \quad (4.5) \quad T_l := \sum_{j=2}^l \frac{1}{j!} \sum_{k_1+k_2+\dots+k_j=l} \text{ad}(H_0; \Psi_{k_1}, \Psi_{k_2}, \dots, \Psi_{k_j}), \quad l \geq 2.$$

We emphasise that the operators B_l and T_l depend only on $\Psi_1, \Psi_2, \dots, \Psi_{l-1}$. Let us make one more rearrangement:

$$A_1 = H_0 + B + \sum_{l=1}^M \text{ad}(H_0, \Psi_l) + \sum_{l=2}^M B_l + \sum_{l=2}^M T_l + R_{M+1},$$

$$\boxed{\text{r:eq}} \quad (4.6) \quad R_{M+1} = B_{M+1} + R_{M+1}^{(1)} + R_{M+1}^{(2)}.$$

Now we can specify our algorithm for finding Ψ_k 's. The symbols ψ_k will be found from the following system of commutator equations:

$$\boxed{\text{psil:eq}} \quad (4.7) \quad \text{ad}(H_0; \Psi_1) + B_1^{\text{NR}} = 0,$$

$$\boxed{\text{psil:eq}} \quad (4.8) \quad \text{ad}(H_0; \Psi_l) + B_l^{\text{NR}} + T_l^{\text{NR}} = 0, \quad l \geq 2,$$

and hence

$$\boxed{\text{lm:eq}} \quad (4.9) \quad \begin{cases} A_1 = A_0 + X_M^{\mathcal{R}} + X_M^{\text{SE}, \mathcal{L}\mathcal{E}, \mathcal{L}\mathcal{F}} + R_{M+1}, \\ X_M = \sum_{l=1}^M B_l + \sum_{l=2}^M T_l, \\ A_0 = H_0 + X_M^{(o)}. \end{cases}$$

Below we denote by x_M the symbol of the PDO X_M . Recall that by Lemma [3.5\(ii\)](#), [smallorthog:lem](#), the operators $B_l^{\text{NR}}, T_l^{\text{NR}}$ are bounded, and therefore, in view of [\(4.7\)](#), [\(4.8\)](#), [psil:eq](#), so is the commutator $\text{ad}(H_0; \Psi)$. This justifies the assumption made in the beginning of the formal calculations in this Section.

4.2. Commutator equations. Recall that $h_0(\xi) = |\xi|^{2m}$ with $m > 0$. Before proceeding to the study of the commutator equations [\(4.7\)](#), [\(4.8\)](#) note that for ξ in the support of the function $\hat{b}^{\text{NR}}(\theta, \cdot; \rho)$ the symbol

$$\boxed{\text{taut:eq}} \quad (4.10) \quad \tau_\theta(\xi) = h_0(\xi + \theta) - h_0(\xi) = (|\xi|^2 + 2\theta \cdot (\xi + \theta/2))^m - |\xi|^{2m}$$

satisfies the bound

$$|\tau_\theta(\xi)| \asymp \rho^{2m-2} |\theta \cdot (\xi + \theta/2)|,$$

which easily follows from [\(3.19\)](#). Using [\(3.19\)](#) again, we conclude that

$$|\theta| \rho^{2m-2+\beta} \ll \tau_\theta(\xi) \ll |\theta| \rho^{2m-1}.$$

Note also that

$$|\mathbf{D}_\xi^s \tau_\theta(\xi)| \ll |\theta| \rho^{2m-1-s}, \quad |\mathbf{s}| = s.$$

Therefore,

$$\boxed{\text{tau:eq}} \quad (4.11) \quad |\mathbf{D}_\xi^s \tau_\theta^{-1}| \ll |\theta|^{-1} \rho^{-2m+2-\beta(1+s)} \ll |\theta|^{-1} w^{-(2m-2)\beta-1-1-s}, \quad \theta \neq 0,$$

for all ξ in the support of the function $\hat{b}^{\text{NR}}(\theta, \cdot; \rho)$. This estimate will come in handy in the next lemma.

commut:lem

Lemma 4.1. *Let $A = \text{Op}(a)$ be a symmetric PDO with $a \in \mathbf{S}_\omega$. Then the PDO Ψ with the Fourier coefficients of the symbol $\psi(\mathbf{x}, \boldsymbol{\xi}; \rho)$ given by*

$$(4.12) \quad \begin{cases} \hat{\psi}(\boldsymbol{\theta}, \boldsymbol{\xi}; \rho) = i \frac{\hat{a}^{\text{NR}}(\boldsymbol{\theta}, \boldsymbol{\xi}; \rho)}{\tau_{\boldsymbol{\theta}}(\boldsymbol{\xi})}, & \boldsymbol{\theta} \neq 0, \\ \hat{\psi}(\mathbf{0}, \boldsymbol{\xi}; \rho) = 0, \end{cases}$$

solves the equation

adb:eq

$$(4.13) \quad \text{ad}(H_0; \Psi) + \text{Op}(a^{\text{NR}}) = 0.$$

Moreover, the operator Ψ is bounded and self-adjoint, its symbol ψ belongs to \mathbf{S}_γ with any $\gamma \in \mathbb{R}$ and the following bound holds:

$$(4.14) \quad |\psi|_{l,s}^{(\gamma)} \ll \rho^{\beta(\sigma-\gamma)} |a|_{l-1,s}^{(\omega)},$$

where

sigma:eq

$$(4.15) \quad \sigma = \omega - (2m - 2)\beta^{-1} - 1.$$

Proof. For brevity we omit ρ from the notation. Let t be the symbol of $\text{ad}(H_0; \Psi)$. The Fourier transform $\hat{t}(\boldsymbol{\theta}, \boldsymbol{\xi})$ is easy to find using (3.1):

$$\hat{t}(\boldsymbol{\theta}, \boldsymbol{\xi}) = i(h_0(\boldsymbol{\xi} + \boldsymbol{\theta}) - h_0(\boldsymbol{\xi}))\hat{\psi}(\boldsymbol{\theta}, \boldsymbol{\xi}) = i\tau_{\boldsymbol{\theta}}(\boldsymbol{\xi})\hat{\psi}(\boldsymbol{\theta}, \boldsymbol{\xi}).$$

Therefore, by definition (3.15), the equation (4.13) amounts to

$$i\tau_{\boldsymbol{\theta}}(\boldsymbol{\xi})\hat{\psi}(\boldsymbol{\theta}, \boldsymbol{\xi}) = -\hat{a}^{\text{NR}}(\boldsymbol{\theta}, \boldsymbol{\xi}; \rho) = -\hat{a}(\boldsymbol{\theta}, \boldsymbol{\xi}; \rho)\varphi_{\boldsymbol{\theta}}(\boldsymbol{\xi}; \rho^\beta)e_{\boldsymbol{\theta}}(\boldsymbol{\xi}), \quad |\boldsymbol{\theta}| \leq r.$$

By definition of the functions $\varphi_{\boldsymbol{\theta}}, e_{\boldsymbol{\theta}}$, the function $\hat{\psi}$ given by (4.12) is defined for all $\boldsymbol{\xi}$. Moreover, the symbol $\hat{\psi}$ satisfies the condition (2.7), so that Ψ is a symmetric operator.

In order to prove that $\psi \in \mathbf{S}_\gamma$ for all $\gamma \in \mathbb{R}$, note that according to (3.24) and (2.3),

$$|\mathbf{D}_\xi^s \hat{a}^{\text{NR}}(\boldsymbol{\theta}, \boldsymbol{\xi}; \rho)| \ll \rho^{\beta(\omega-\gamma)} |a|_{l,s}^{(\omega)} w^{\gamma-s} |\boldsymbol{\theta}|^{-l}.$$

Together with (4.11) this implies that

$$|\mathbf{D}_\xi^s \hat{\psi}(\boldsymbol{\theta}, \boldsymbol{\xi}; \rho)| \ll \rho^{-\beta\gamma} |a|_{l,s}^{(\omega)} w^{\sigma+\gamma-s} |\boldsymbol{\theta}|^{-l-1},$$

so that $\psi \in \mathbf{S}_\gamma$ and it satisfies (4.14).

The estimate (4.14) with $\gamma = 0, s = 0$, and Proposition 3.1 ensure the boundedness of Ψ . \square

Let us apply Lemma 4.1 to equations (4.7) and (4.8).

Lemma 4.2. *Let $b \in \mathbf{S}_\alpha$ be a symmetric symbol, $\rho \geq 1$, and let*

$$(4.16) \quad \begin{cases} \sigma = & \alpha - (2m - 2)\beta^{-1} - 1, \\ \sigma_j = & j(\sigma - 1) + 1, \\ \epsilon_j = & j(\sigma - 1) + (2m - 2)\beta^{-1} + 2, \end{cases}$$

$j = 1, 2, \dots$. Then there exists a sequence of self-adjoint bounded PDO's Ψ_j , $j = 1, 2, \dots$ with the symbols ψ_j such that $\psi_j \in \mathbf{S}_\gamma$ for any $\gamma \in \mathbb{R}$, [\(4.7\)](#) and [\(4.8\)](#) hold, and

$$\boxed{\text{psik1:eq}} \quad (4.17) \quad |\psi_j|^{(\gamma)} \ll \rho^{\beta(\sigma_j - \gamma)} (|b|^{(\alpha)})^j, \quad j \geq 1.$$

The symbols b_j , t_j of the corresponding operators B_j , T_j belong to \mathbf{S}_γ for any $\gamma \in \mathbb{R}$ and

$$\boxed{\text{bt:eq}} \quad (4.18) \quad |b_j|^{(\gamma)} + |t_j|^{(\gamma)} \ll \rho^{\beta(\epsilon_j - \gamma)} (|b|^{(\alpha)})^j, \quad j \geq 2.$$

If $\rho^{\beta(\sigma-1)} |b|^{(\alpha)} \ll 1$, then for any M and $\psi = \sum_{j=1}^M \psi_j$ the following bounds hold:

$$\boxed{\text{xm:eq}} \quad (4.19) \quad |\psi|^{(\gamma)} \ll \rho^{\beta(\sigma - \gamma)} |b|^{(\alpha)}, \quad \forall \gamma \in \mathbb{R}; \quad |x_M|^{(\alpha)} \ll |b|^{(\alpha)},$$

$$\boxed{\text{rm:eq}} \quad (4.20) \quad \|R_{M+1}\| \ll (|b|^{(\alpha)})^{M+1} \rho^{\beta\epsilon_{M+1}};$$

uniformly in b satisfying $\rho^{\beta(\sigma-1)} |b|^{(\alpha)} \ll 1$.

Proof. The existence of $\psi_1 \in \mathbf{S}_\gamma$ with required properties follows from Lemma [4.1](#). Further proof is by induction. [commut:lem](#)

Suppose that ψ_k with $k = 1, 2, \dots, K-1$ satisfy [\(4.17\)](#). In order to conclude that ψ_K also satisfies [\(4.17\)](#), first we need to check that b_K and t_K satisfy [\(4.18\)](#). [psik1:eq](#) [bt:eq](#)

Step I. Estimates for b_j . Note that

$$\boxed{\text{gmatoeeps:eq}} \quad (4.21) \quad \begin{aligned} \epsilon_j &= \sigma_{j-1} + \alpha - 1 \\ &= (j-1)\sigma + \alpha - (j-1), \quad j \geq 2. \end{aligned}$$

To begin with, we prove that all the symbols b_j with $j \leq K$, satisfy the estimate [\(4.18\)](#). We first obtain a bound for $\text{ad}(b; \psi_{k_1}, \psi_{k_2}, \dots, \psi_{k_q})$ with $k_1 + k_2 + \dots + k_q = j-1$. To this end we use [\(4.17\)](#) with $\gamma = (\omega - \alpha)q^{-1} + 1$ for each ψ_{k_n} and Proposition [3.4](#) to conclude that [commut0:prop](#)

$$\boxed{\text{b:eq}} \quad (4.22) \quad |\text{ad}(b; \psi_{k_1}, \psi_{k_2}, \dots, \psi_{k_q})|^{(\omega)} \ll |b|^{(\alpha)} \prod_{n=1}^q |\psi_{k_n}|^{(\gamma)} \ll (|b|^{(\alpha)})^j \prod_{n=1}^q \rho^{\beta(\sigma_{k_n} - \gamma)},$$

for any $\omega \in \mathbb{R}$. Obviously, we have:

$$\begin{aligned} \sum_{n=1}^q \beta(\sigma_{k_n} - \gamma) &= \beta(q(1 - \gamma) + \sum_{n=1}^q k_n(\sigma - 1)) = \beta((j-1)(\sigma - 1) - \omega + \alpha) \\ &= \beta(\sigma_{j-1} + \alpha - 1 - \omega) = \beta(\epsilon_j - \omega). \end{aligned}$$

Now, [\(4.4\)](#) implies [bl:eq](#)

$$|b_j|^{(\omega)} \ll \rho^{\beta(\epsilon_j - \omega)} (|b|^{(\alpha)})^j,$$

for all $\omega \in \mathbb{R}$, i.e. b_j satisfies [\(4.18\)](#) for all $j \leq K$. [bt:eq](#)

Step II. Estimates for t_j . For the symbols t_j the proof is by induction. First of all, note that

$$\text{ad}(h_0; \psi_1, \psi_1) = -\text{ad}(b^{\mathcal{NR}}, \psi_1),$$

so that, using [\(3.22\)](#), [Proposition 3.4](#), and [\(4.17\)](#) with $\gamma = \omega - \alpha + 1$, we obtain

$$|\mathrm{ad}(h_0; \psi_1, \psi_1)|^{(\omega)} \ll |b|^{(\alpha)} |\psi_1|^{(\gamma)} \ll \rho^{\beta(\sigma_1 - \gamma)} (|b|^{(\alpha)})^2.$$

By [\(4.16\)](#), we have $\sigma_1 - \gamma = \sigma_1 + \alpha - 1 - \omega = \epsilon_2 - \omega$, and hence t_2 satisfies [\(4.18\)](#). Suppose that all t_k with $k \leq j-1 \leq K-1$ satisfy [\(4.18\)](#). As we have already established that all $b_k, k \leq K$ satisfy [\(4.18\)](#), by definition [\(4.8\)](#) and [\(3.22\)](#) all $\mathrm{ad}(h_0; \psi_k), k \leq j-1$, satisfy the same bound, i.e.

$$|\mathrm{ad}(h_0, \psi_k)|^{(\gamma)} \ll \rho^{\beta(\epsilon_k - \gamma)} (|b|^{(\alpha)})^k, k \leq j-1.$$

Using the bound [\(4.17\)](#) for $\psi_k, k \leq j-1$, with $\gamma = (\omega - 1)q^{-1} + 1$, and applying [Proposition 3.4](#), we obtain for $k_1 + k_2 + \dots + k_q = j, q \geq 2$ the following estimate:

$$|\mathrm{ad}(h_0; \psi_{k_1}, \psi_{k_2}, \dots, \psi_{k_q})|^{(\omega)} = |\mathrm{ad}(\mathrm{ad}(h_0; \psi_{k_1}); \psi_{k_2}, \dots, \psi_{k_q})|^{(\omega)}$$

$$\boxed{\text{h0:eq}} \quad (4.23) \quad \ll |\mathrm{ad}(h_0; \psi_{k_1})|^{(\gamma)} \prod_{n=2}^q |\psi_{k_n}|^{(\gamma)} \ll (|b|^{(\alpha)})^j \rho^{\beta(\epsilon_{k_1} - \gamma)} \prod_{n=2}^q \rho^{\beta(\sigma_{k_n} - \gamma)},$$

for any $\omega \in \mathbb{R}$. Obviously, we have:

$$\begin{aligned} \beta(\epsilon_{k_1} - \gamma) + \sum_{n=2}^q \beta(\sigma_{k_n} - \gamma) &= \beta((\sigma - 1) \sum_{n=1}^q k_n + 2 + (2m - 2)\beta^{-1} + q - 1 - q\gamma) \\ &= \beta(j(\sigma - 1) + 2 + (2m - 2)\beta^{-1} + q(1 - \gamma) - 1) \\ &= \beta(\epsilon_j - \omega). \end{aligned}$$

This leads to [\(4.18\)](#) for all $t_j, j \leq K$.

Step III. In order to handle Ψ_K , we use the solution Ψ of the equation [\(4.13\)](#) constructed in [Lemma 4.1](#). Then from definition [\(4.8\)](#) and [steps I, II](#) we immediately conclude that $\psi_K \in \mathbf{S}_\gamma$ for any $\gamma \in \mathbb{R}$. Moreover, estimate [\(4.14\)](#) with $\omega = \gamma$ implies

$$\begin{aligned} |\psi_K|^{(\gamma)} &\ll \rho^{\beta(-(2m-2)\beta^{-1}-1)} (|b_K|^{(\gamma)} + |t_K|^{(\gamma)}) \\ &\ll \rho^{\beta(\epsilon_K - \gamma - (2m-2)\beta^{-1}-1)} (|b|^{(\alpha)})^K. \end{aligned}$$

Since $\epsilon_K - (2m - 2)\beta^{-1} - 1 = \sigma_K$, the required result follows.

Step IV. Proof of [\(4.20\)](#) and [\(4.19\)](#). We assume that $\rho^{\beta(\sigma-1)} |b|^{(\alpha)} \ll 1$ throughout. Before treating the remainder R_{M+1} (see [\(4.6\)](#) for its definition), let us prove estimates [\(4.19\)](#) for the symbols $\psi = \sum_{j=1}^M \psi_j$ and $x_M = b + \sum_{j=2}^M (b_j + t_j)$. Using [\(4.17\)](#), we get

$$|\psi|^{(\gamma)} \ll \sum_{j=1}^M \rho^{\beta(\sigma_j - \gamma)} (|b|^{(\alpha)})^j = \rho^{\beta(1 - \gamma)} \sum_{j=1}^M (\rho^{\beta(\sigma-1)} |b|^{(\alpha)})^j,$$

which implies that

$$\boxed{\text{globalpsi:eq}} \quad (4.24) \quad |\psi|^{(\gamma)} \ll \rho^{\beta(\sigma - \gamma)} |b|^{(\alpha)} \left(1 + (\rho^{\beta(\sigma-1)} |b|^{(\alpha)})^{M-1} \right) \ll \rho^{\beta(\sigma - \gamma)} |b|^{(\alpha)}$$

with an arbitrary $\gamma \in \mathbb{R}$. Similarly, in view of [\(4.18\)](#) and [\(4.21\)](#),

$$\begin{aligned} |x_M|^{(\alpha)} &\ll |b|^{(\alpha)} + \sum_{j=2}^M \rho^{\beta(\epsilon_j - \alpha)} (|b|^{(\alpha)})^j = |b|^{(\alpha)} + \sum_{j=2}^M \rho^{\beta(j-1)(\sigma-1)} (|b|^{(\alpha)})^j \\ &\ll |b|^{(\alpha)} \left(1 + (\rho^{\beta(\sigma-1)} |b|^{(\alpha)})^{M-1} \right) \ll |b|^{(\alpha)}. \end{aligned} \tag{4.25}$$

Now we apply these estimates to find upper bounds for R_{M+1} .

The remainder R_{M+1} consists of three components. To estimate the first one, B_{M+1} , note that in view of [\(4.18\)](#), $b_{M+1} \in \mathbf{S}_0$ and $|b_{M+1}|^{(0)} \ll \rho^{\beta\epsilon_{M+1}} (|b|^{(\alpha)})^{M+1}$. By [Proposition 3.1](#), we conclude that the norm of B_{M+1} is bounded by $(|b|^{(\alpha)})^{M+1} \rho^{\beta\epsilon_{M+1}}$ as required.

Consider now $R_{M+1}^{(1)}$ defined in [\(4.1\)](#). Using [\(4.24\)](#) with $\gamma = -(M+1)^{-1}\alpha + 1$ and applying [Proposition 3.4](#), we conclude that

$$|\mathrm{ad}^{M+1}(b; \psi)|^{(0)} \ll |b|^{(\alpha)} (|\psi|^{(\gamma)})^{M+1} \ll \rho^{\beta(M+1)(\sigma-\gamma)} (|b|^{(\alpha)})^{M+2}.$$

In view of [\(4.21\)](#),

$$(M+1)(\sigma-\gamma) = M\sigma + \alpha - M + \sigma - 1 = \epsilon_{M+1} + \sigma - 1,$$

so that [Proposition 3.1](#) yields:

$$\|\mathrm{ad}^{M+1}(B; \Psi)\| \ll \rho^{\beta\epsilon_{M+1}} (|b|^{(\alpha)})^{M+1} \rho^{\beta(\sigma-1)} |b|^{(\alpha)} \ll \rho^{\beta\epsilon_{M+1}} (|b|^{(\alpha)})^{M+1}. \tag{4.26}$$

To estimate the norm of $\mathrm{ad}^{M+1}(H_0; \Psi)$, note that by [\(4.25\)](#) and [\(3.24\)](#),

$$|x_M^{\mathrm{NR}}|^{(\gamma)} \ll \rho^{\beta(\alpha-\gamma)} |b|^{(\alpha)}$$

for any $\gamma \in \mathbb{R}$. It follows from [\(4.7\)](#) and [\(4.8\)](#) that $\mathrm{ad}(h_0, \psi) + x_M^{\mathrm{NR}} = 0$. Thus, using the above bound and [\(4.24\)](#) with $\gamma = M(1+M)^{-1}$, we obtain with the help of [Proposition 3.4](#) that

$$\begin{aligned} |\mathrm{ad}^{M+1}(h_0; \psi)|^{(0)} &= |\mathrm{ad}^M(x_M^{\mathrm{NR}}; \psi)|^{(0)} \ll |x_M^{\mathrm{NR}}|^{(\gamma)} (|\psi|^{(\gamma)})^M \\ &\ll \rho^{\beta M(\sigma-\gamma) + \beta(\alpha-\gamma)} (|b|^{(\alpha)})^{M+1}. \end{aligned}$$

According to [\(4.21\)](#),

$$M(\sigma-\gamma) + \alpha - \gamma = M\sigma + \alpha - M - (M+1)\gamma + M = \epsilon_{M+1},$$

so that by [Proposition 3.1](#),

$$\|\mathrm{ad}^{M+1}(H_0; \Psi)\| \ll \rho^{\beta\epsilon_{M+1}} (|b|^{(\alpha)})^{M+1}.$$

Together with [\(4.26\)](#) this leads to the estimate of the form [\(4.20\)](#) for $R_{M+1}^{(1)}$.

Following the same strategy, one easily obtains the sought estimate for the norm of the $R_{M+1}^{(2)}$ defined in [\(4.3\)](#). This completes the proof of [\(4.20\)](#). \square

Let us now summarize the results of this section in the following Theorem: the implications of the above Lemma for the operator $H = H_0 + \mathrm{Op}(b)$, defined in [\(2.8\)](#).

induction:thm

Theorem 4.3. *Let $b \in \mathbf{S}_\alpha(w)$, $w(\boldsymbol{\xi}) = \langle \boldsymbol{\xi} \rangle^\beta$, $\beta \in (0, 1]$, $\alpha \in \mathbb{R}$ be a symmetric symbol, and let H be the operator defined in (2.8). Suppose that the condition (2.15) is satisfied. Then for any positive integer M there exist symmetric symbols $\psi = \psi_M$, $x = x_M$, and a self-adjoint bounded operator R_{M+1} satisfying the following properties:*

- (1) $\psi \in \mathbf{S}_\gamma$ for all $\gamma \in \mathbb{R}$, $x \in \mathbf{S}_\alpha$, and

$$|\psi|^{(\gamma)} \ll \rho^{\beta(\sigma-\gamma)} |b|^{(\alpha)},$$

$$|x|^{(\alpha)} \ll |b|^{(\alpha)},$$

$$\|R_{M+1}\| \ll (|b|^{(\alpha)})^{M+1} \rho^{\beta\epsilon_{M+1}},$$

uniformly in b satisfying $|b|^{(\alpha)} \ll 1$;

- (2) The operator $A_1 = e^{-i\Psi} H e^{i\Psi}$, $\Psi = \text{Op}(\psi)$, has the form

a1:eq

$$(4.27) \quad A_1 = A_0 + X^{\mathcal{R}} + X^{\text{SE}, \mathcal{L}\mathcal{E}, \mathcal{L}\mathcal{F}} + R_{M+1}, \quad A_0 = H_0 + X^o.$$

Proof. Note that the condition (2.15) is equivalent to $\sigma < 1$. Thus the existence of symbols ψ , x and the operator R_{M+1} with required properties follows from Lemma 4.2. In particular, the claimed upper bounds are direct consequences of (4.19) and (4.20). \square

ezhiki:sect

5. GEOMETRY OF CONGRUENT POINTS: RESONANT SETS

In the course of the proof we need a substantial number of certain lattice-geometric constructions. They are discussed in this section. First, we fix the notation.

For any vector $\boldsymbol{\xi} \in \mathbb{R}^d$, $\boldsymbol{\xi} \neq 0$, we denote $\mathbf{e}(\boldsymbol{\xi}) = \boldsymbol{\xi}|\boldsymbol{\xi}|^{-1}$. For any $\boldsymbol{\eta} \in \mathbb{R}^d$ the vector $\boldsymbol{\xi}_\eta$ is the projection of $\boldsymbol{\xi}$ onto the one-dimensional subspace spanned by $\boldsymbol{\eta}$, i.e. $\boldsymbol{\xi}_\eta = (\boldsymbol{\xi} \cdot \mathbf{e}(\boldsymbol{\eta}))\mathbf{e}(\boldsymbol{\eta})$. Let $\Theta_r \subset \Gamma^\dagger$ and $\Theta_r^0 \subset \Gamma^\dagger$ be the sets defined in (3.12). We always assume that

$$r \geq r_0, \quad \text{where } r_0 \text{ is such that } \Theta_r^0 \text{ contains } d \text{ linearly independent lattice vectors.}$$

We say that a subspace $\mathfrak{V} \subset \mathbb{R}^d$ is a *lattice r -subspace* if \mathfrak{V} is spanned by some linearly independent lattice vectors from the set Θ_r^0 . A one dimensional lattice subspace spanned by a vector $\boldsymbol{\theta} \in \Theta_r$ is denoted by $\mathfrak{V}(\boldsymbol{\theta})$. The set of all lattice r -subspaces of dimension n is denoted by $\mathcal{V}(n) = \mathcal{V}(r, n)$, $n = 0, 1, \dots, d$, and $\mathcal{W} = \mathcal{W}(r) = \cup_{n=0}^d \mathcal{V}(r, n)$. We have included in this list the zero dimensional subspace $\mathfrak{X} = \{0\}$. For any vector $\boldsymbol{\xi} \in \mathbb{R}^d$ we denote by $\boldsymbol{\xi}_{\mathfrak{V}}$ its orthogonal projection on $\mathfrak{V} \in \mathcal{V}(r, n)$. In particular, $\boldsymbol{\xi}_{\mathfrak{X}} = \mathbf{0}$.

The notation w_n is used for the volume of the unit ball in \mathbb{R}^n .

5.1. Elementary geometrical estimates. We begin with estimates for distances between lattice subspaces and lattice points.

cassels:lem

Lemma 5.1. (1) *For any $\mathfrak{V} \in \mathcal{V}(r, d-1)$ there exists a vector $\boldsymbol{\gamma} \in \Gamma$ such that $\boldsymbol{\gamma}_{\mathfrak{V}} = \mathbf{0}$, and $|\boldsymbol{\gamma}| \leq 2d(\Gamma) w_{d-1}^{-1} \pi^{1-d} r^{d-1}$.*

- (2) *For any $\mathfrak{W} \in \mathcal{V}(r, n)$, $n \leq d-1$, and any $\boldsymbol{\nu} \notin \mathfrak{W}$, $\boldsymbol{\nu} \in \Gamma^\dagger$, one has*

$$\text{dist}(\boldsymbol{\nu}, \mathfrak{W}) \geq d(\Gamma)^{-1} w_{d-1} \pi^d r^{1-d}.$$

Proof. Denote by $\mathbf{e} \in \mathbb{R}^d$ a unit vector, orthogonal to \mathfrak{V} . For $t \geq 1$, let $\mathcal{A}_t \subset \mathbb{R}^d$ be the cylindrical set

$$\mathcal{A}_t = \{\boldsymbol{\xi} \in \mathbb{R}^d : |\boldsymbol{\xi}_{\mathfrak{V}}| < 2\pi r^{-1}, |\boldsymbol{\xi}_{\mathbf{e}}| \leq t\},$$

which is obviously convex and symmetric about the origin. Moreover,

$$\text{vol}(\mathcal{A}_t) = t(2\pi)^{d-1}r^{-d+1}w_{d-1}.$$

By [\[4\]](#), §III.2.2 Theorem II, under the condition $\text{vol}(\mathcal{A}_t) \geq d(\Gamma)2^d$ the set \mathcal{A}_t contains at least two points $\pm\boldsymbol{\gamma} \in \Gamma$. The above condition is satisfied if $t = d(\Gamma)w_{d-1}^{-1}2^d(2\pi)^{1-d}r^{d-1}$.

By definition of \mathcal{A}_t , for any $\mathbf{m} \in \Theta_r \cap \mathfrak{V}$ we have $|\boldsymbol{\gamma} \cdot \mathbf{e}(\mathbf{m})| < 2\pi r^{-1}$, and hence $|\boldsymbol{\gamma} \cdot \mathbf{m}| < 2\pi r^{-1}|\mathbf{m}| \leq 2\pi$. Since $\boldsymbol{\gamma} \in \Gamma$ and $\mathbf{m} \in \Gamma^\dagger$, the latter inequality implies that $\boldsymbol{\gamma} \cdot \mathbf{m} = 0$, so $\boldsymbol{\gamma}_{\mathfrak{V}} = 0$. It is also clear that $|\boldsymbol{\gamma}| \leq t = 2d(\Gamma)w_{d-1}^{-1}\pi^{1-d}r^{d-1}$.

To prove the second statement, we find a subspace $\mathfrak{W} \in \mathcal{V}(r, d-1)$ such that $\boldsymbol{\nu} \notin \mathfrak{W}$ and $\mathfrak{W} \subset \mathfrak{V}$, and denote by $\boldsymbol{\gamma} \in \Gamma$ the vector orthogonal to \mathfrak{W} , found at the first step of the proof. Since

$$\text{dist}(\boldsymbol{\nu}, \mathfrak{W}) \geq \text{dist}(\boldsymbol{\nu}, \mathfrak{V}) = \frac{1}{|\boldsymbol{\gamma}|}|\boldsymbol{\nu} \cdot \boldsymbol{\gamma}| \geq \frac{2\pi}{|\boldsymbol{\gamma}|},$$

the required inequality follows from the first part of the Lemma. \square

actions:lem

Lemma 5.2. *Let $\mathfrak{V} \in \mathcal{V}(r, m)$, $m \leq d-1$.*

(1) *If $\boldsymbol{\theta} \in \Theta_r$ and $\boldsymbol{\theta} \notin \mathfrak{V}$, then for any $\boldsymbol{\xi} \in \mathfrak{V} + \mathfrak{V}(\boldsymbol{\theta})$ we have*

$$|\boldsymbol{\xi}| \ll r^d(|\boldsymbol{\xi}_{\boldsymbol{\theta}}| + |\boldsymbol{\xi}_{\mathfrak{V}}|).$$

(2) *For any subspace $\mathfrak{W} \in \mathcal{V}(r, n)$, $n \leq d-1$,*

$$|\boldsymbol{\xi}_{\mathfrak{W}+\mathfrak{W}}| \ll r^{nd}(|\boldsymbol{\xi}_{\mathfrak{W}}| + |\boldsymbol{\xi}_{\mathfrak{W}}|).$$

Proof. Let $\boldsymbol{\theta} \notin \mathfrak{V}$, and let $\boldsymbol{\xi} \in \mathfrak{V} + \mathfrak{V}(\boldsymbol{\theta})$. By Lemma [5.1](#), [cassels:lem](#)

$$(5.1) \quad |\boldsymbol{\theta}_{\mathbf{e}}| = \text{dist}(\boldsymbol{\theta}, \mathfrak{V}) \gg r^{1-d},$$

where $\mathbf{e} \in \mathfrak{V} + \mathfrak{V}(\boldsymbol{\theta})$ is a unit vector orthogonal to \mathfrak{V} . We have:

$$(5.2) \quad \boldsymbol{\xi} = \boldsymbol{\xi}_{\mathbf{e}} + \boldsymbol{\xi}_{\mathfrak{V}},$$

so that

$$\boldsymbol{\xi}_{\boldsymbol{\theta}} \cdot \boldsymbol{\theta} = \boldsymbol{\xi} \cdot \boldsymbol{\theta} = \boldsymbol{\xi}_{\mathbf{e}} \cdot \boldsymbol{\theta}_{\mathbf{e}} + \boldsymbol{\xi}_{\mathfrak{V}} \cdot \boldsymbol{\theta}_{\mathfrak{V}}.$$

Since $|\boldsymbol{\xi}_{\mathbf{e}} \cdot \boldsymbol{\theta}_{\mathbf{e}}| = |\boldsymbol{\xi}_{\mathbf{e}}| |\boldsymbol{\theta}_{\mathbf{e}}|$, using [\(5.1\)](#) we obtain

$$|\boldsymbol{\xi}_{\mathbf{e}}| \leq \frac{1}{|\boldsymbol{\theta}_{\mathbf{e}}|} (|\boldsymbol{\xi}_{\mathfrak{V}}| + |\boldsymbol{\xi}_{\boldsymbol{\theta}}|) |\boldsymbol{\theta}| \ll r^d (|\boldsymbol{\xi}_{\mathfrak{V}}| + |\boldsymbol{\xi}_{\boldsymbol{\theta}}|).$$

Together with [\(5.2\)](#) this implies the required bound.

To prove the second statement we may assume that $\boldsymbol{\xi} \in \mathfrak{V} + \mathfrak{W}$. Let $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \dots, \boldsymbol{\theta}_n \in \Theta_r \cap \mathfrak{W}$ be a linearly independent set of lattice vectors. Denote by $\mathfrak{W}^{(j)}$, $j = 1, 2, \dots, n$

the subspace spanned by $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \dots, \boldsymbol{\theta}_j$. Applying the first part of the Lemma repeatedly, we get

$$\begin{aligned} |\boldsymbol{\xi}| &\ll r^d(|\boldsymbol{\xi}_{\boldsymbol{\theta}_n}| + |\boldsymbol{\xi}_{\mathfrak{W}^{(n-1)+\mathfrak{Y}}}|) \\ &\ll r^d(|\boldsymbol{\xi}_{\boldsymbol{\theta}_n}| + r^d(|\boldsymbol{\xi}_{\boldsymbol{\theta}_{n-1}}| + |\boldsymbol{\xi}_{\mathfrak{W}^{(n-2)+\mathfrak{Y}}}|)) \ll \dots \ll r^{nd} \left(\sum_{j=1}^n |\boldsymbol{\xi}_{\boldsymbol{\theta}_j}| + |\boldsymbol{\xi}_{\mathfrak{Y}}| \right). \end{aligned}$$

Noticing that $|\boldsymbol{\xi}_{\boldsymbol{\theta}}| \leq |\boldsymbol{\xi}_{\mathfrak{W}}|$ for any $\boldsymbol{\theta} \in \mathfrak{W}$, we get the proclaimed estimate. \square

5.2. Congruent vectors. For a non-zero vector $\boldsymbol{\theta}$ we define *the resonant layer* corresponding to $\boldsymbol{\theta}$ by

$$\text{layer:eq} \quad (5.3) \quad \Lambda(\boldsymbol{\theta}) := \{\boldsymbol{\xi} \in \mathbb{R}^d, |\boldsymbol{\xi} \cdot \boldsymbol{\theta}| < \rho^{\alpha_1} |\boldsymbol{\theta}|\}.$$

Here and below, $\alpha_1 \in (0, 1)$ is a fixed number which will be specified later, and $\rho \geq 1$. For the sake of uniformity of the notation, in case $\boldsymbol{\theta} = \mathbf{0}$ we denote $\Lambda(\mathbf{0}) = \mathbb{R}^d$.

Definition 5.3. Let $\boldsymbol{\theta}, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \dots, \boldsymbol{\theta}_m$ be some vectors from Θ_r^0 , which are not necessarily distinct.

- 1 (1) We say that two vectors $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d$ are *$\boldsymbol{\theta}$ -resonant congruent* if both $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are inside $\Lambda(\boldsymbol{\theta})$ and $(\boldsymbol{\xi} - \boldsymbol{\eta}) = l\boldsymbol{\theta}$ with $l \in \mathbb{Z}$. In this case we write $\boldsymbol{\xi} \leftrightarrow \boldsymbol{\eta} \pmod{\boldsymbol{\theta}}$. In particular, each $\boldsymbol{\xi} \in \mathbb{R}^d$ is $\mathbf{0}$ -resonant congruent to itself.
- 2 (2) For each $\boldsymbol{\xi} \in \mathbb{R}^d$ we denote by $\Upsilon_{\boldsymbol{\theta}}(\boldsymbol{\xi})$ the set of all points which are $\boldsymbol{\theta}$ -resonant congruent to $\boldsymbol{\xi}$. For $\boldsymbol{\theta} \neq \mathbf{0}$ we say that $\Upsilon_{\boldsymbol{\theta}}(\boldsymbol{\xi}) = \emptyset$ if $\boldsymbol{\xi} \notin \Lambda(\boldsymbol{\theta})$.
- 3 (3) We say that $\boldsymbol{\eta}$ is *$\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \dots, \boldsymbol{\theta}_m$ -resonant congruent* to $\boldsymbol{\xi}$, if there exists a sequence $\boldsymbol{\xi}_j \in \mathbb{R}^d, j = 0, 1, \dots, m$ such that $\boldsymbol{\xi}_0 = \boldsymbol{\xi}, \boldsymbol{\xi}_m = \boldsymbol{\eta}$, and $\boldsymbol{\xi}_j \in \Upsilon_{\boldsymbol{\theta}_j}(\boldsymbol{\xi}_{j-1})$ for $j = 1, 2, \dots, m$.
- (4) We say that $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ are *resonant congruent*, if $\boldsymbol{\eta}$ is $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \dots, \boldsymbol{\theta}_m$ -resonant congruent to $\boldsymbol{\xi}$ with some $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \dots, \boldsymbol{\theta}_m \in \Theta_r$. The set of all points, resonant congruent to $\boldsymbol{\xi}$, is denoted by $\Upsilon(\boldsymbol{\xi})$. We use the notation $\mathbf{N}(\boldsymbol{\xi}) = \{\mathbf{m} \in \Gamma^\dagger : \boldsymbol{\xi} + \mathbf{m} \in \Upsilon(\boldsymbol{\xi})\}$, i.e. $\mathbf{N}(\boldsymbol{\xi}) = \Upsilon(\boldsymbol{\xi}) - \boldsymbol{\xi} \subset \Gamma^\dagger$. For points $\boldsymbol{\eta} \in \Upsilon(\boldsymbol{\xi})$ we write $\boldsymbol{\eta} \leftrightarrow \boldsymbol{\xi}$.

Remark 5.4. (1) Note that according to the above definition every point in \mathbb{R}^d is resonant congruent to some point. In particular, every vector $\boldsymbol{\xi}$ is $\mathbf{0}$ -resonant congruent to itself, i.e. $\Upsilon_{\mathbf{0}}(\boldsymbol{\xi}) = \{\boldsymbol{\xi}\}$.

(2) It is clear that the resonant congruence defines an equivalence relation, so that if $\boldsymbol{\eta} \leftrightarrow \boldsymbol{\xi}$ then we have $\Upsilon(\boldsymbol{\xi}) = \Upsilon(\boldsymbol{\eta})$.

(3) Note also that if $\boldsymbol{\eta}$ is $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \dots, \boldsymbol{\theta}_m$ -resonant congruent to $\boldsymbol{\xi}$, then $\boldsymbol{\xi}$ is $\boldsymbol{\theta}_m, \boldsymbol{\theta}_{m-1}, \dots, \boldsymbol{\theta}_1$ -resonant congruent to $\boldsymbol{\eta}$, and $\boldsymbol{\xi}$ is $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \dots, \boldsymbol{\theta}_{m-1}, \boldsymbol{\theta}_m, \boldsymbol{\theta}_{m-1}, \dots, \boldsymbol{\theta}_1$ -resonant congruent to itself.

Let us establish some immediate properties of congruent points:

Lemma 5.5. For any vectors $\boldsymbol{\xi} \in \mathbb{R}^d, \boldsymbol{\theta} \in \Theta_r$, and any $\boldsymbol{\nu} \perp \boldsymbol{\theta}$ we have $\Upsilon_{\boldsymbol{\theta}}(\boldsymbol{\xi}) + \boldsymbol{\nu} \subset \Upsilon_{\boldsymbol{\theta}}(\boldsymbol{\xi} + \boldsymbol{\nu})$.

As a consequence, if $\mathbf{N}(\boldsymbol{\xi}) \subset \mathfrak{V}$, then for any vector $\boldsymbol{\nu} \in \mathfrak{V}^\perp$ we have $\Upsilon(\boldsymbol{\xi}) + \boldsymbol{\nu} \subset \Upsilon(\boldsymbol{\xi} + \boldsymbol{\nu})$ and $\mathbf{N}(\boldsymbol{\xi}) \subset \mathbf{N}(\boldsymbol{\xi} + \boldsymbol{\nu})$.

Proof. If $\Upsilon_\theta(\boldsymbol{\xi})$ is empty, the result is obvious. For a non-empty $\Upsilon_\theta(\boldsymbol{\xi})$ the first statement is an immediate consequence of Definition [5.3](#) in view of the orthogonality of $\boldsymbol{\theta}$ and $\boldsymbol{\nu}$. Under the conditions $\mathbf{N}(\boldsymbol{\xi}) \subset \mathfrak{V}$, $\boldsymbol{\nu} \in \mathfrak{V}^\perp$, this leads to the inclusion $\mathbf{N}(\boldsymbol{\xi}) \subset \mathbf{N}(\boldsymbol{\xi} + \boldsymbol{\nu})$. \square

The next Lemma is the first of many results, establishing some inequalities for congruent points and/or lattice subspaces. To avoid unnecessary repetitions in their formulations, we adopt the following convention.

Convention 5.6. (1) Each inequality (e.g. \ll , \gg or \asymp), involving points of the Euclidean space and/or lattice subspace(s), is assumed to be uniform in those objects.
 (2) The constants in the inequalities are allowed to depend on the dimension d and the lattice Γ^\dagger only.
 (3) We say that a certain statement, involving points of the Euclidean space and/or lattice subspace(s), holds for sufficiently large ρ , if there is a number $\rho_0 > 0$, independent of the points and/or subspace(s) at hand, such that the statement holds for all $\rho \geq \rho_0$.

Lemma 5.7. *If $\boldsymbol{\eta} \in \Upsilon(\boldsymbol{\xi})$, then*

$$(5.4) \quad |\boldsymbol{\xi} - \boldsymbol{\eta}| \ll \rho^{\alpha_1} r^{(d-1)d}.$$

In particular,

$$(5.5) \quad \max_{\mathbf{N}(\boldsymbol{\xi})} |\mathbf{m}| \ll \rho^{\alpha_1} r^{d(d-1)}, \quad N(\boldsymbol{\xi}) := \text{card } \mathbf{N}(\boldsymbol{\xi}) \ll \rho^{d\alpha_1} r^{d^2(d-1)},$$

uniformly in $\boldsymbol{\xi}$.

The proof of this Lemma relies on the following result:

Lemma 5.8. (i) *Let $\boldsymbol{\eta} \in \Upsilon(\boldsymbol{\xi})$ be $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \dots, \boldsymbol{\theta}_m$ -resonant congruent to $\boldsymbol{\xi}$, and let*

$$\text{span}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \dots, \boldsymbol{\theta}_m) =: \mathfrak{V} \subset \mathcal{V}(n)$$

with some $n \leq d$. Then

$$(5.6) \quad |\boldsymbol{\xi}_{\mathfrak{V}}| \ll \rho^{\alpha_1} r^{(n-1)d}.$$

(ii) Suppose that

$$\text{span } \mathbf{N}(\boldsymbol{\xi}) = \mathfrak{V} \in \mathcal{V}(n),$$

with some $n \leq d$. Then

$$(5.7) \quad |\boldsymbol{\xi}_{\mathfrak{V}}| \ll \rho^{\alpha_1} r^{(n-1)d}.$$

Proof of Lemma 5.7. As the relation $\xi \leftrightarrow \eta$ is symmetric, Lemma 5.8 implies that $|\xi_{\mathfrak{W}}| + |\eta_{\mathfrak{W}}| \ll \rho^{\alpha_1 r^{(d-1)d}}$. As $\xi - \eta \in \mathfrak{W}$, we have

$$|\xi - \eta| = |(\xi - \eta)_{\mathfrak{W}}| \leq |\xi_{\mathfrak{W}}| + |\eta_{\mathfrak{W}}| \ll \rho^{\alpha_1 r^{(d-1)d}},$$

which is (5.4). The first estimate in (5.5) follows from the inequality

$$\max_{\mathbf{m} \in \mathcal{N}(\xi)} |\mathbf{m}| \leq \sup_{\eta \in \Upsilon(\xi)} |\xi - \eta|$$

and from (5.4). The bound for $N(\xi)$ is simply an estimate for the number of lattice points in the ball of radius $\ll \rho^{\alpha_1 r^{(d-1)d}}$. \square

Proof of Lemma 5.8. The proof of the first part is by induction. For $n = 1$ the statement is an immediate consequence of the definitions. Assume that (5.6) holds for all $n \leq k$, $k \geq 1$, and let us prove (5.6) for $n = k + 1$.

Let $\eta \in \Upsilon(\xi)$ be $\theta_1, \theta_2, \dots, \theta_m$ -resonant congruent to ξ , and let $\xi_0, \xi_1, \dots, \xi_m$ be the vectors from Definition 5.3(3). Let $l \geq k, l < m$, be a number, uniquely defined by the following conditions: $\text{span}(\theta_1, \dots, \theta_{l+1}) = \mathfrak{W} \subset \mathcal{V}(k+1)$, and $\mathfrak{W} := \text{span}(\theta_1, \dots, \theta_l) \neq \mathfrak{W}$. Clearly, $\dim \mathfrak{W} = k$. Since $\xi_{l+1} \leftrightarrow \xi_l \pmod{\theta_{l+1}}$, by Definition 5.3(II) we have $|(\xi_l)_{\theta_{l+1}}| < \rho^{\alpha_1}$. At the same time, since $\xi_l \leftrightarrow \xi$, we also have $|(\xi_l)_{\mathfrak{W}}| \ll \rho^{\alpha_1 r^{(k-1)d}}$ by the induction assumption, as $\mathfrak{W} \in \mathcal{V}(k)$. Thus, according to Lemma 5.2(I),

$$(5.8) \quad |(\xi_l)_{\mathfrak{W}}| \ll r^d (|(\xi_l)_{\mathfrak{W}}| + |(\xi_l)_{\theta_{l+1}}|) \ll \rho^{\alpha_1 r^{kd}}.$$

Noticing that $\xi - \xi_l \in \mathfrak{W}$, we infer from the induction assumption again that

$$|\xi - \xi_l| = |(\xi - \xi_l)_{\mathfrak{W}}| \leq |\xi_{\mathfrak{W}}| + |(\xi_l)_{\mathfrak{W}}| \ll \rho^{\alpha_1 r^{(k-1)d}},$$

Together with (5.8) this leads to $|\xi_{\mathfrak{W}}| \ll \rho^{\alpha_1 r^{kd}}$. Thus by induction, (5.6) is proved.

In order to prove part (ii), we use the observation made in Remark 5.4(3) to conclude that under the assumption $\text{span } \mathcal{N}(\xi) = \mathfrak{W}$ there are vectors $\theta_1, \dots, \theta_m$ such that ξ is $\theta_1, \dots, \theta_m$ -resonant congruent to itself and $\text{span}(\theta_1, \dots, \theta_m) = \mathfrak{W}$. It remains to use the first part of the Lemma. \square

One important conclusion of Lemma 5.7 is that the numbers $\text{card } \Upsilon(\xi)$ are bounded uniformly in $\xi \in \mathbb{R}^d$.

We need further notions related to congruency:

Definition 5.9. We say that a point $\xi \in \mathbb{R}^d$ is θ -non-critical, if for a sufficiently small $\epsilon > 0$ we have $\Upsilon_{\theta}(\xi + \mu) = \Upsilon_{\theta}(\xi) + \mu$ for all $\mu : |\mu| < \epsilon$. Otherwise we call ξ θ -critical.

If ξ is θ -non-critical for all $\theta \in \Theta_r$, we call it non-critical.

We say that the set $\Upsilon(\xi)$ is non-critical, if it consists of non-critical points. In other words, for a sufficiently small $\epsilon > 0$, we have $\Upsilon(\xi + \mu) = \Upsilon(\xi) + \mu$ if $|\mu| < \epsilon$. Otherwise $\Upsilon(\xi)$ is said to be critical.

If $\Upsilon(\xi)$ is non-critical, then the set $\mathcal{N}(\cdot)$ remains constant in a neighbourhood of ξ .

It is clear that if $\theta \neq 0$, then the set

$$\{\xi \in \mathbb{R}^d : \xi \cdot \mathbf{e}(\theta) \not\equiv \pm \rho^{\alpha_1} \pmod{|\theta|}\},$$

consists of θ -non-critical points. Each $\xi \in \mathbb{R}^d$ is obviously $\mathbf{0}$ -non-critical. Thus, the set of $\xi \in \mathbb{R}^d$ for which $\Upsilon(\xi)$ is non-critical, is open and of full measure.

ant:subject

5.3. Resonant sets and their properties. Our aim is to construct a collection of sets $\Xi(\mathfrak{W}) \subset \mathbb{R}^d$ parametrised by subspaces $\mathfrak{W} \in \mathcal{V}(r, n)$ for $n = 0, 1, \dots, d$, and depending on the parameter $\rho > 0$, satisfying the following properties:

exhaust:eq

$$(5.9) \quad \mathbb{R}^d = \bigcup_{\mathfrak{W} \in \mathcal{W}(r)} \Xi(\mathfrak{W}),$$

disjoint:eq

$$(5.10) \quad \Xi(\mathfrak{W}_1) \cap \Xi(\mathfrak{W}_2) = \emptyset \quad \text{for} \quad \mathfrak{W}_1 \neq \mathfrak{W}_2,$$

span:eq

$$(5.11) \quad \text{For each } \xi \in \Xi(\mathfrak{W}), \text{ one has } \Upsilon(\xi) \subset \Xi(\mathfrak{W}) \quad \text{and} \quad \mathbf{N}(\xi) \subset \mathfrak{W}.$$

The required sets will depend on the arbitrarily chosen real parameters $\alpha_0, \alpha_1, \dots, \alpha_d$ and \varkappa , satisfying the conditions $0 = \alpha_0 < \alpha_1 < \alpha_2 < \dots < \alpha_d < 1$ and

condition1

$$(5.12) \quad \begin{cases} r \leq \rho^\varkappa, \\ \alpha_{n+1} > \alpha_n + 2\varkappa d^2, \quad n = 0, 1, \dots, d-1. \end{cases}$$

From now on these conditions are always assumed to hold. Under these conditions the inequalities (5.5) give for sufficiently large ρ :

card:eq

$$(5.13) \quad \begin{cases} |\mathbf{m}| \leq \rho^{\frac{\alpha_1 + \alpha_2}{2}}, \quad \mathbf{m} \in \mathbf{N}(\xi); \\ \text{card } \Upsilon(\xi) \leq \rho^{\frac{\alpha_1 + \alpha_2}{2} d}, \end{cases}$$

uniformly in ξ . The next Lemma is a straightforward consequence of the above inequalities:

bups:lem

Lemma 5.10. *Let $|\xi| \geq \rho/2$. Then*

$$\max_{\Upsilon(\xi)} |\eta| \asymp \min_{\Upsilon(\xi)} |\eta| \asymp |\xi|,$$

uniformly in ξ .

For each lattice subspace $\mathfrak{W} \in \mathcal{V}(n)$, $n = 0, 1, \dots, d$ we introduce the (open) sets

xi1:eq

$$(5.14) \quad \Xi_1(\mathfrak{W}) := \{\xi \in \mathbb{R}^d, |\xi_{\mathfrak{W}}| < \rho^{\alpha_n}\}, \quad \Xi_2(\mathfrak{W}) := \bigcup_{\xi \in \Xi_1(\mathfrak{W})} \Upsilon(\xi),$$

and

xi:eq

$$(5.15) \quad \Xi(\mathfrak{W}) := \begin{cases} \Xi_2(\mathfrak{W}) \setminus \bigcup_{m=n+1}^d \bigcup_{\mathfrak{W} \in \mathcal{V}(m)} \Xi_2(\mathfrak{W}), & n < d; \\ \Xi_2(\mathfrak{W}), & n = d. \end{cases}$$

The set $\Xi(\mathfrak{W})$, $\mathfrak{W} \neq \mathfrak{X}$, is referred to as the *resonant set*, associated with the lattice subspace \mathfrak{W} . By definition of $\Xi_2(\mathfrak{W})$, we have $\xi \in \Xi_2(\mathfrak{W})$ if and only if $\Upsilon(\xi) \subset \Xi_2(\mathfrak{W})$, so that

$$\boxed{\text{xi2:eq}} \quad (5.16) \quad \xi \in \Xi(\mathfrak{W}) \text{ if and only if } \Upsilon(\xi) \subset \Xi(\mathfrak{W}).$$

This observation immediately leads to the natural representation of $\Xi(\mathfrak{W})$ as a union of non-intersecting equivalence classes:

$$\boxed{\text{equiv:eq}} \quad (5.17) \quad \Xi(\mathfrak{W}) = \bigcup_{\xi \in \Xi(\mathfrak{W})/\leftrightarrow} \Upsilon(\xi).$$

Note that $\Xi_1(\mathfrak{W}(\theta)) = \Lambda(\theta)$ for each $\theta \in \Theta_r$. For $\mathfrak{W} = \mathfrak{X}$ we have $\Xi_1(\mathfrak{X}) = \Xi_2(\mathfrak{X}) = \mathbb{R}^d$. The sets

$$\boxed{\text{resonant:eq}} \quad (5.18) \quad \mathcal{B} := \Xi(\mathfrak{X}) = \mathbb{R}^d \setminus \bigcup_{m \geq 1} \bigcup_{\mathfrak{W} \in \mathcal{V}(m)} \Xi_2(\mathfrak{W}) \quad \text{and} \quad \mathcal{D} = \mathbb{R}^d \setminus \mathcal{B}$$

are called the *non-resonant set* and *resonant set* of \mathbb{R}^d respectively. The sets introduced above obviously depend on the parameter ρ . Whenever necessary, the dependence on ρ is reflected in the notation, e.g. $\Xi(\mathfrak{W}; \rho)$, $\mathcal{B}(\rho)$, $\mathcal{D}(\rho)$.

Remark 5.11. To illustrate the difference between the sets $\Xi_1(\mathfrak{W})$ and $\Xi_2(\mathfrak{W})$ assume for simplicity that the set Θ_r^0 contains two non-zero vectors only, which we denote θ, η . Away from the intersection with $\Lambda(\eta)$, the set $\Xi_2(\theta)$ coincides with $\Xi_1(\theta) = \Lambda(\theta)$. However, near $\Lambda(\theta) \cap \Lambda(\eta)$ the set $\Xi_2(\theta)$ acquires some extra points. Namely, along with each point $\xi \in \Lambda(\theta)$ the set $\Xi_2(\theta)$ contains the points from $\Lambda(\eta)$ obtained by translating ξ by $\pm\eta, \pm 2\eta, \dots$ etc.

Note an immediate consequence of the definition $\boxed{\text{xi:eq}}$ and Lemma $\boxed{\text{lem:reachability1}}$ 5.7:

$\boxed{\text{properties3}}$ **Lemma 5.12.** Assume $\boxed{\text{eq:condition1}}$ (5.12). Let $\mathfrak{W} \in \mathcal{V}(n)$, $n = 1, 2, \dots, d$, and $\xi \in \Xi_2(\mathfrak{W})$. Then for sufficiently large ρ we have:

$$\boxed{\text{proj:eq}} \quad (5.19) \quad |\xi_{\mathfrak{W}}| \leq \begin{cases} 2\rho^{\frac{\alpha_1 + \alpha_2}{2}}, & n = 1, \\ 2\rho^{\alpha_n}, & n = 2, \dots, d. \end{cases}$$

Proof. By definition $\boxed{\text{xi1:eq}}$ (5.14), $\xi \in \Upsilon(\eta)$ for some $\eta \in \Xi_1(\mathfrak{W})$. Thus, by $\boxed{\text{card:eq}}$ (5.13),

$$|\xi_{\mathfrak{W}}| \leq |\eta_{\mathfrak{W}}| + \max_{\mathbf{m} \in \mathcal{N}(\eta)} |\mathbf{m}| < \rho^{\alpha_n} + \rho^{\frac{\alpha_1 + \alpha_2}{2}}.$$

In view of monotonicity of α_j 's, this proves $\boxed{\text{proj:eq}}$ (5.19). \square

The previous Lemma shows that the resonant sets $\Xi(\mathfrak{W})$, $\mathfrak{W} \neq \mathfrak{X}$, are “small” relative to the non-resonant set $\mathcal{B} = \Xi(\mathfrak{X})$. More precisely, we show that the resonant set \mathcal{D} has a small angular measure. To this end for each $\theta \in \Theta_r$ define

$$\tilde{\Lambda}(\theta) = \{\xi \in \mathbb{R}^d : |\xi \cdot \mathbf{e}(\theta)| < 2\rho^{\alpha_{d-1}}\},$$

cf. (5.3). By Lemma 5.12, for any $\mathfrak{W} \in \mathcal{V}(n), n \leq d-1$ we have

$$\Xi(\mathfrak{W}) \subset \bigcup_{\boldsymbol{\theta} \in \mathfrak{W} \cap \Theta_r} \tilde{\Lambda}(\boldsymbol{\theta})$$

so that

$$\mathcal{D} \setminus B(\rho/8) \subset \bigcup_{\boldsymbol{\theta} \in \Theta_r} \tilde{\Lambda}(\boldsymbol{\theta}) \setminus B(\rho/8).$$

An elementary calculation shows that

$$\tilde{\Lambda}(\boldsymbol{\theta}) \setminus B(\rho/8) \subset S(\boldsymbol{\theta}; \rho) \times [\rho/8, \infty), \quad S(\boldsymbol{\theta}; \rho) := \{\boldsymbol{\Omega} \in \mathbb{S}^{d-1} : |\boldsymbol{\Omega} \cdot \mathbf{e}(\boldsymbol{\theta})| < 16\rho^{\alpha_{d-1}-1}\},$$

for all sufficiently large ρ . Let

$$(5.20) \quad S(\rho) = \bigcup_{\boldsymbol{\theta} \in \Theta_r} S(\boldsymbol{\theta}; \rho), \quad T(\rho) = \mathbb{S}^{d-1} \setminus S(\rho).$$

Lemma 5.13. *Let the sets $S(\rho) \subset \mathbb{S}^{d-1}, T(\rho) \subset \mathbb{S}^{d-1}$ be as defined above. Then*

$$(5.21) \quad \text{vol}_{\mathbb{S}^{d-1}} S(\rho) \ll \rho^{\alpha_{d-1}}, \quad \text{vol}_{\mathbb{S}^{d-1}} T(\rho) \asymp 1,$$

for sufficiently large ρ .

Proof. The elementary bound

$$\text{vol}_{\mathbb{S}^{d-1}} S(\boldsymbol{\theta}; \rho) = \int_{\mathbb{S}^{d-2}} \int_{|\cos \omega| \leq 16\rho^{\alpha_{d-1}-1}} \sin^{d-2} \omega d\omega d\hat{\boldsymbol{\Omega}} \ll \rho^{\alpha_{d-1}-1},$$

together with the observation that the number of the sets $\tilde{\Lambda}(\boldsymbol{\theta})$ is bounded above by $\text{card } \Theta_r \ll r^d$, gives the estimate

$$\text{vol}_{\mathbb{S}^{d-1}} S(\rho) \ll r^d \rho^{\alpha_{d-1}-1} \ll \rho^{\alpha_{d-1}}.$$

Here we have used (5.12).

The second bound in (5.21) immediately follows from the first one by definition (5.20). \square

In what follows, apart from the non-resonant set $\mathcal{B}(\rho)$, the set

$$(5.22) \quad \tilde{\mathcal{B}}(\rho) = T(\rho) \times [\rho/8, \infty)$$

will play an important role. Since $(\mathcal{D} \setminus B(\rho/8)) \subset S(\rho) \times [\rho/8, \infty)$, and $\mathcal{B} = \mathbb{R}^d \setminus \mathcal{D}$ (see (5.18)), we have

$$\tilde{\mathcal{B}}(\rho) \subset \mathcal{B}(\rho).$$

Let us now proceed with our study of the resonant sets $\Xi(\mathfrak{W})$. Introduce a new notion:

Definition 5.14. Two subspaces $\mathfrak{V}_1, \mathfrak{V}_2$ are said to be *strongly distinct*, if they are distinct, and neither of $\mathfrak{V}_1, \mathfrak{V}_2$ is a subspace of the other.

An equivalent definition of this property is that $\dim(\mathfrak{V}_1 + \mathfrak{V}_2) > \max(\dim \mathfrak{V}_1, \dim \mathfrak{V}_2)$.

Here is a simple relation between the sets Ξ_1, Ξ_2 :

xi12:lem

Lemma 5.15. *Assume [\(5.12\)](#). Then for any strongly distinct lattice subspaces \mathfrak{V} and \mathfrak{W} we have*

$$(5.23) \quad \Xi_2(\mathfrak{V}) \cap \Xi_2(\mathfrak{W}) \subset \Xi_1(\mathfrak{V} + \mathfrak{W}),$$

for sufficiently large ρ .

Proof. Let $\mathfrak{V} \in \mathcal{V}(n)$, $\mathfrak{W} \in \mathcal{V}(m)$, and assume without loss of generality that $m \geq n$.

Suppose first that $m \geq 2$. By Lemma [5.12](#), for any $\xi \in \Xi_2(\mathfrak{V}) \cap \Xi_2(\mathfrak{W})$ we have

$$|\xi_{\mathfrak{V}}| < 2\rho^{\alpha_m}, \quad |\xi_{\mathfrak{W}}| < 2\rho^{\alpha_m}.$$

Therefore, by Lemma [5.2](#), for the projection onto the subspace $\mathfrak{A} = \mathfrak{V} + \mathfrak{W}$ we have

$$|\xi_{\mathfrak{A}}| \ll \rho^{\alpha_m} r^{d^2}.$$

Since \mathfrak{V} and \mathfrak{W} are strongly distinct, we have $p := \dim \mathfrak{A} > m$, and hence, by [\(5.12\)](#), for sufficiently large ρ the right hand side is bounded above by ρ^{α_p} , which implies the proclaimed inclusion in view of the definition [\(5.14\)](#).

Suppose that $n = m = 1$, so that by Lemma [5.12](#)

$$|\xi_{\mathfrak{V}}|, |\xi_{\mathfrak{W}}| < 2\rho^{\frac{\alpha_1 + \alpha_2}{2}}.$$

Lemma [5.2](#) gives again that $|\xi_{\mathfrak{A}}| \ll \rho^{\frac{\alpha_1 + \alpha_2}{2}} r^{d^2}$. Since $p := \dim \mathfrak{A} = 2$, by [\(5.12\)](#), for sufficiently large ρ the right hand side is bounded above by ρ^{α_2} , which implies [\(5.23\)](#) again. \square

Lemma [5.15](#) has a number of very useful consequences. First of all note that in the definition [\(5.15\)](#) we could have written $\mathfrak{V} \subset \mathfrak{W}$ without changing the set $\Xi(\mathfrak{V})$. Precisely, we have the following lemma.

Lemma 5.16. *Let*

$$(5.24) \quad \Xi'(\mathfrak{V}) := \Xi_2(\mathfrak{V}) \setminus \bigcup_{m > n} \bigcup_{\mathfrak{W} \in \mathcal{V}(m), \mathfrak{V} \subset \mathfrak{W}} \Xi_2(\mathfrak{W}).$$

Then for sufficiently large ρ , we have $\Xi(\mathfrak{V}) = \Xi'(\mathfrak{V})$ for all $\mathfrak{V} \in \mathcal{V}(n)$, $n \leq d - 1$.

Proof. Let $\mathfrak{V} \in \mathcal{V}(r, n)$. Clearly, $\Xi(\mathfrak{V}) \subset \Xi'(\mathfrak{V})$. Let us prove the opposite inclusion. On the r.h.s. of the formula [\(5.15\)](#) one can replace $\Xi_2(\mathfrak{W})$ by $\Xi_2(\mathfrak{W}) \cap \Xi_2(\mathfrak{V})$. For strongly distinct \mathfrak{W} and \mathfrak{V} , from Lemma [5.15](#) we obtain that

$$\Xi_2(\mathfrak{W}) \cap \Xi_2(\mathfrak{V}) \subset \Xi_2(\mathfrak{W} + \mathfrak{V}).$$

For $m > n$ this entails the inclusion

$$\begin{aligned} \bigcup_{\mathfrak{W} \in \mathcal{V}(m)} \Xi_2(\mathfrak{W}) \cap \Xi_2(\mathfrak{V}) &\subset \left(\bigcup_{\mathfrak{W} \in \mathcal{V}(m), \mathfrak{V} \not\subset \mathfrak{W}} \Xi_2(\mathfrak{W} + \mathfrak{V}) \right) \cup \left(\bigcup_{\mathfrak{W} \in \mathcal{V}(m), \mathfrak{V} \subset \mathfrak{W}} \Xi_2(\mathfrak{W}) \right) \\ &\subset \bigcup_{p \geq m} \bigcup_{\mathfrak{W} \in \mathcal{V}(p), \mathfrak{V} \subset \mathfrak{W}} \Xi_2(\mathfrak{W}), \end{aligned}$$

so that $\Xi(\mathfrak{V}) \supset \Xi'(\mathfrak{V})$, as required. \square

equiv:lem

xi11:eq

The next lemma shows that all resonant sets $\Xi(\mathfrak{V})$ are non-empty. It is given here for the sake of completeness, and it will not be used in the subsequent argument.

Lemma 5.17. *Let $\mathfrak{V} \in \mathcal{V}(n)$, $n \leq d$. For any $R \gg \rho$ and sufficiently large ρ , we have*

$$(5.25) \quad \text{vol}(\Xi(\mathfrak{V}) \cap B(R)) \gg \rho^{n\alpha_n} R^{d-n}.$$

Proof. Let us fix a subspace $\mathfrak{V} \in \mathcal{V}(n)$. By definition [\(5.14\)](#),

$$V_0(\rho, R) := \text{vol}(\Xi_2(\mathfrak{V}) \cap B(R)) \geq \text{vol}(\Xi_1(\mathfrak{V}) \cap B(R)) \gg \rho^{n\alpha_n} R^{d-n}.$$

For $n = d$ we have $\mathfrak{V} = \mathbb{R}^d$ and $\Xi_2(\mathfrak{V}) = \Xi(\mathfrak{V})$, so the above lower bound yields [\(5.25\)](#).

Suppose that $n \leq d - 1$. By [Lemma 5.12](#), for any $\mathfrak{V} \in \mathcal{V}(n)$ and any $\mathfrak{W} \in \mathcal{V}(m)$, $m = n + 1, \dots, d$, such that $\mathfrak{V} \subset \mathfrak{W}$, we have

$$\text{vol}(\Xi_2(\mathfrak{W}) \cap \Xi_2(\mathfrak{V}) \cap B(0, R)) \ll \rho^{n\alpha_n} \rho^{(m-n)\alpha_m} R^{d-m}.$$

If $n = d - 1$, then definition [\(5.15\)](#) ensures that

$$\begin{aligned} \text{vol}(\Xi(\mathfrak{V}) \cap B(R)) &\gg V_0(\rho, R) - C\rho^{(d-1)\alpha_{d-1} + \alpha_d} \\ &\gg \rho^{(d-1)\alpha_{d-1}} R(1 - C\rho^{\alpha_{d-1}}) \gg \rho^{(d-1)\alpha_{d-1}} R. \end{aligned}$$

For $n \leq d - 2$ we recall that the number of lattice subspaces $\mathfrak{W} \in \mathcal{V}(m)$, $m \leq d - 1$, is less than Cr^{d^2} with a universal constant C . This leads to the estimate

$$\begin{aligned} V(\rho, R) &:= \text{vol} \bigcup_{m \geq n+1} \bigcup_{\substack{\mathfrak{W} \in \mathcal{V}(m) \\ \mathfrak{V} \subset \mathfrak{W}}} (\Xi_2(\mathfrak{W}) \cap \Xi_2(\mathfrak{V}) \cap B(R)) \\ &\ll r^{d^2} \sum_{m=n+1}^{d-1} \rho^{n\alpha_n} \rho^{(m-n)\alpha_m} R^{d-m} + \rho^{n\alpha_n} \rho^{(d-n)\alpha_d} \\ &\ll \rho^{n\alpha_n} R^{d-n} \left[\sum_{m=n+1}^{d-1} r^{d^2} \rho^{(m-n)\alpha_m} R^{n-m} + \rho^{(d-n)\alpha_d} R^{n-d} \right]. \end{aligned}$$

Therefore, under the condition $R \gg \rho$, using [\(5.12\)](#), we obtain that

$$\frac{V(\rho, R)}{V_0(\rho, R)} \ll \sum_{m=n+1}^d \rho^{(m-n)(\alpha_d-1)} \rightarrow 0, \rho \rightarrow \infty.$$

Now the required estimate [\(5.25\)](#) follows from [Lemma 5.16](#). \square

Lemma 5.18. *Let $\mathfrak{V} \in \mathcal{V}(n)$ and $\mathfrak{W} \in \mathcal{V}(m)$ be two strongly distinct lattice subspaces. Then for sufficiently large ρ we have $\Xi(\mathfrak{V}) \cap \Xi_2(\mathfrak{W}) = \emptyset$. In particular, if $\theta \notin \mathfrak{V}$, then $\Xi(\mathfrak{V}) \cap \Lambda(\theta) = \emptyset$.*

Proof. By [Lemma 5.15](#), for sufficiently large ρ ,

$$\Xi(\mathfrak{V}) \cap \Xi_2(\mathfrak{W}) \subset \Xi_2(\mathfrak{V}) \cap \Xi_2(\mathfrak{W}) \subset \Xi_2(\mathfrak{U}), \quad \mathfrak{U} = \mathfrak{V} + \mathfrak{W}.$$

Since $\dim \mathfrak{U} > n$, by definition (5.15), the left hand side is empty, as required.

As $\Lambda(\theta) = \Xi_1(\mathfrak{V}(\theta)) \subset \Xi_2(\mathfrak{V}(\theta))$, the second statement follows immediately. \square

Corollary 5.19. *Let $\mathfrak{V} \in \mathcal{V}(n)$. If $\xi \in \Xi(\mathfrak{V})$, then $\Upsilon(\xi) \subset \Xi(\mathfrak{V})$, and for sufficiently large ρ also $\mathbf{N}(\xi) \subset \mathfrak{V}$.*

Proof. The inclusion $\Upsilon(\xi) \subset \Xi(\mathfrak{V})$ immediately follows from (5.16).

To prove the second statement, it suffices to show that if $\eta \in \Upsilon_\theta(\xi)$ with some $\theta \in \Theta_r$, then $\theta \in \mathfrak{V}$. The assumption $\eta \in \Upsilon_\theta(\xi)$ means, in particular, that $\xi \in \Lambda(\theta)$. By Lemma 5.18 the latter inclusion is possible only if $\theta \in \mathfrak{V}$. \square

Now we are in position to prove that the constructed sets $\Xi(\mathfrak{V})$ satisfy the conditions (5.9), (5.10) and (5.11):

Theorem 5.20. *Let $\alpha_0, \alpha_1, \dots, \alpha_d$ and \varkappa be some numbers satisfying (5.12). Then for sufficiently large ρ the collection of sets $\Xi(\mathfrak{V}) \subset \mathbb{R}^d$, $\mathfrak{V} \in \mathcal{W}(r)$, defined by the equalities (5.14), (5.15), satisfies the properties (5.9), (5.10) and (5.11).*

Proof. Proof of (5.9). Let us prove that

$$(5.26) \quad \Xi^{(n)} := \bigcup_{m \geq n} \bigcup_{\mathfrak{V} \in \mathcal{V}(m)} \Xi(\mathfrak{V}) = \bigcup_{m \geq n} \bigcup_{\mathfrak{V} \in \mathcal{V}(m)} \Xi_2(\mathfrak{V}) =: \Xi_2^{(n)},$$

for all $n = 0, 1, 2, \dots, d$. If $n = d$, then $\Xi^{(d)} = \Xi(\mathbb{R}^d) = \Xi_2(\mathbb{R}^d) = \Xi_2^{(d)}$. Suppose that $0 \leq n \leq d - 1$. By definition (5.15) and backward induction,

$$\Xi_2^{(n)} = \bigcup_{\mathfrak{V} \in \mathcal{V}(n)} \Xi_2(\mathfrak{V}) \bigcup \Xi_2^{(n+1)} = \bigcup_{\mathfrak{V} \in \mathcal{V}(n)} \Xi(\mathfrak{V}) \bigcup \Xi_2^{(n+1)} = \bigcup_{\mathfrak{V} \in \mathcal{V}(n)} \Xi(\mathfrak{V}) \bigcup \Xi^{(n+1)} = \Xi^{(n)}.$$

Therefore $\mathbb{R}^d = \Xi_2(\mathbb{R}^d) \subset \Xi_2^{(0)} = \Xi^{(0)}$, as claimed.

Proof of (5.10). Let \mathfrak{V}_1 and \mathfrak{V}_2 be distinct lattice subspaces, $\dim \mathfrak{V}_j = p_j$, $j = 1, 2$. If $\mathfrak{V}_1 \subset \mathfrak{V}_2$, then $p_1 < p_2$, and it follows from Definition (5.15) that

$$\Xi(\mathfrak{V}_1) \subset \Xi_2(\mathfrak{V}_1) \setminus \Xi(\mathfrak{V}_2),$$

so that $\Xi(\mathfrak{V}_1) \cap \Xi(\mathfrak{V}_2) = \emptyset$. If \mathfrak{V}_1 and \mathfrak{V}_2 are strongly distinct, then the required result follows from Lemma 5.18.

Proof of (5.11). See Corollary 5.19. \square

5.4. Scaling properties of the sets $\Upsilon(\xi)$.

Lemma 5.21. *Suppose that $\xi \in \Xi(\mathfrak{V})$ and $\xi + \nu \in \Xi(\mathfrak{V})$ with some $\nu \in \mathfrak{V}^\perp$. Then for sufficiently large ρ ,*

- (1) $\mathbf{N}(\xi) = \mathbf{N}(\xi + \nu)$,
- (2) For $\xi \in \Xi(\mathfrak{V})$ the set $\mathbf{N}(\xi)$ depends only on the vector $\xi_{\mathfrak{V}}$.

Proof. By Corollary 5.19 $\mathbf{N}(\xi), \mathbf{N}(\xi + \nu) \subset \mathfrak{V}$, so Part II follows from Lemma 5.5. Statement I is a rephrased statement I. \square

Lemma 5.22. *Suppose that $\xi \in \Xi(\mathfrak{V})$ with some $\mathfrak{V} \in \mathcal{V}(n)$, $n \leq d$, and $\xi_t = \xi + t\nu$, $\nu = \xi_{\mathfrak{V}^\perp}$ with $t \geq 0$. Then for sufficiently large ρ the vector $\xi_t \in \Xi(\mathfrak{V})$ for all $t \geq 0$.*

We precede the proof with another Lemma:

Lemma 5.23. *Suppose that $\xi \in \Xi(\mathfrak{V})$ with some $\mathfrak{V} \in \mathcal{V}(n)$, $n \leq d$, and $\xi_t = \xi + t\nu$, $\nu = \xi_{\mathfrak{V}^\perp}$ with $t \geq 0$. Then for sufficiently large ρ we have $N(\xi) = N(\xi_t)$ for all $t \geq 0$, and, in particular, $N(\xi_t) \subset \mathfrak{V}$.*

Proof. For $n = d$ the result is trivial, so we assume that $n \leq d - 1$. We consider the case $n > 0$; the case $n = 0$ is similar, and we leave it to the reader.

Let us fix a $t > 0$. By Corollary 5.19, $N(\xi) \subset \mathfrak{V}$, so that $N(\xi) \subset N(\xi_t)$ by Lemma 5.5.

In order to prove the opposite inclusion, it suffices to prove that $\Upsilon_\theta(\xi_t) - t\nu \subset \Upsilon_\theta(\xi)$ for each $\theta \in \Theta_r^0$ such that $\Upsilon_\theta(\xi_t) \neq \emptyset$. If $\theta = \mathbf{0}$, then the above assertion is obvious. Suppose that $\theta \neq \mathbf{0}$. Let $\mu_t \in \Upsilon_\theta(\xi_t)$, so we need to show that $\mu := \mu_t - t\nu \in \Upsilon_\theta(\xi)$.

First, we prove that $\theta \in \mathfrak{V}$. Suppose the converse, i.e. that $\theta \notin \mathfrak{V}$, and define $\mathfrak{W} = \mathfrak{V} + \mathfrak{V}(\theta) \in \mathcal{V}(n+1)$. We may assume that $|\xi_{\mathfrak{W}}| \geq \rho^{\alpha_{n+1}}$. Indeed, otherwise we would have $|\xi_{\mathfrak{W}}| < \rho^{\alpha_{n+1}}$, which would imply that $\xi \in \Xi_1(\mathfrak{W}) \subset \Xi_2(\mathfrak{W})$, but the latter is impossible in view of the condition $\xi \in \Xi(\mathfrak{V})$. Since

$$(5.27) \quad |\xi_{\mathfrak{W}}| \leq 2\rho^{\frac{\alpha_n + \alpha_{n+1}}{2}},$$

see Lemma 5.12, we can claim by virtue of Lemma 5.2 and conditions (5.12) that

$$|\xi_\theta| \geq r^{-d}|\xi_{\mathfrak{W}}| - |\xi_{\mathfrak{W}}| \gg \rho^{\alpha_{n+1}}r^{-d}.$$

Consequently,

$$|\nu_\theta| = |\xi_{\mathfrak{V}^\perp} \cdot \mathbf{e}(\theta)| \geq |\xi_\theta| - |\xi_{\mathfrak{W}}| \gg \rho^{\alpha_{n+1}}r^{-d}.$$

Therefore,

$$\begin{aligned} |(\xi_t)_\theta| &= |\xi_\theta + t\nu_\theta| = |(\xi_{\mathfrak{V}})_\theta + (t+1)\nu_\theta| \\ &\geq (t+1)|\nu_\theta| - |\xi_{\mathfrak{V}}| \geq \rho^{\alpha_{n+1}}r^{-d} \geq \rho^{\alpha_1}, \quad t \geq 0, \end{aligned}$$

so that $\xi_t \notin \Lambda(\theta)$, $t \geq 0$, which contradicts the fact that $\xi_t \leftrightarrow \mu_t \pmod{\theta}$. Thus, $\theta \in \mathfrak{V}$.

As $\nu \in \mathfrak{V}^\perp$, the inclusion $\mu_t \in \Upsilon_\theta(\xi_t)$ implies $\mu \in \Upsilon_\theta(\xi)$ by Lemma 5.5. Therefore, $\Upsilon_\theta(\xi_t) = \Upsilon_\theta(\xi) + t\nu$, which in turn implies $\Upsilon(\xi_t) = \Upsilon(\xi) + t\nu$ and $N(\xi_t) = N(\xi)$. The inclusion $N(\xi_t) \subset \mathfrak{V}$ follows from $N(\xi) \subset \mathfrak{V}$. \square

Proof of Lemma 5.22. As in the previous proof assume that $n \leq d - 1$. Let us fix a $t > 0$. Since $\xi \in \Xi(\mathfrak{V}) \subset \Xi_2(\mathfrak{V})$, one can find a vector $\eta \in \Xi_1(\mathfrak{V}) \cap \Upsilon(\xi)$. As $\nu \in \mathfrak{V}^\perp$, this implies that $\eta_t := \eta + t\nu \in \Xi_1(\mathfrak{V})$ and by Lemma 5.5, $\xi_t \in \Upsilon(\eta_t) \subset \Xi_2(\mathfrak{V})$. Thus, in view of Lemma 5.16 it remains to prove that $\xi_t \notin \Xi_2(\mathfrak{W})$ for any $\mathfrak{W} \supsetneq \mathfrak{V}$. Suppose the contrary, i.e. for some $\mathfrak{W} \in \mathcal{V}(m)$, $m > n$, $\mathfrak{W} \supset \mathfrak{V}$, we have $\xi_t \in \Xi_2(\mathfrak{W})$. We show that under this condition we have $\xi \in \Xi_2(\mathfrak{W})$, which would contradict the assumption $\xi \in \Xi(\mathfrak{V})$.

Denote $\mathfrak{U} := \mathfrak{W} \ominus \mathfrak{V}$ (\mathfrak{U} does not have to be a lattice r -subspace). By Definition (xi1:eq) (5.14) there is a vector $\boldsymbol{\mu}_t \in \Xi_1(\mathfrak{W})$, resonant congruent to $\boldsymbol{\xi}_t$. By Lemma (coupl:lem) (5.23), $\mathbf{N}(\boldsymbol{\xi}_t) = \mathbf{N}(\boldsymbol{\xi}) \subset \mathfrak{V}$, so that $(\boldsymbol{\mu}_t)_{\mathfrak{U}} = (\boldsymbol{\xi}_t)_{\mathfrak{U}}$. Therefore, for the vector $\boldsymbol{\mu} = \boldsymbol{\mu}_t - t\boldsymbol{\nu}$ we also have $\boldsymbol{\mu}_{\mathfrak{U}} = \boldsymbol{\xi}_{\mathfrak{U}}$. This implies that

$$\begin{aligned} |(\boldsymbol{\mu}_t)_{\mathfrak{W}}|^2 &= |\boldsymbol{\mu}_{\mathfrak{W}} + t\boldsymbol{\nu}_{\mathfrak{W}}|^2 = |\boldsymbol{\mu}_{\mathfrak{W}} + t\boldsymbol{\xi}_{\mathfrak{U}}|^2 = |\boldsymbol{\mu}_{\mathfrak{W}} + t\boldsymbol{\mu}_{\mathfrak{U}}|^2 \\ &= |\boldsymbol{\mu}_{\mathfrak{W}} + (t+1)\boldsymbol{\mu}_{\mathfrak{U}}|^2 = |\boldsymbol{\mu}_{\mathfrak{W}}|^2 + (t+1)^2|\boldsymbol{\mu}_{\mathfrak{U}}|^2 \geq |\boldsymbol{\mu}_{\mathfrak{W}}|^2, t \geq 0. \end{aligned}$$

By definition (xi1:eq) (5.14), $|(\boldsymbol{\mu}_t)_{\mathfrak{W}}| < \rho^{\alpha_m}$ and hence $|\boldsymbol{\mu}_{\mathfrak{W}}| < \rho^{\alpha_m}$, i.e. $\boldsymbol{\mu} \in \Xi_1(\mathfrak{W})$. On the other hand, since $\boldsymbol{\mu}_t \in \Upsilon(\boldsymbol{\xi}_t)$, Lemma (coupl:lem) (5.23) entails that $\boldsymbol{\mu} \in \Upsilon(\boldsymbol{\xi})$, and thus $\boldsymbol{\mu} \in \Xi(\mathfrak{V})$. This contradicts the inclusion $\boldsymbol{\mu} \in \Xi_1(\mathfrak{W})$, by definition of $\Xi(\mathfrak{V})$. Consequently, $\boldsymbol{\xi}_t \notin \Xi_2(\mathfrak{W})$ for any $\mathfrak{W} \supset \mathfrak{V}$, and the proof of the Lemma is complete. \square

6. INVARIANT SUBSPACES FOR THE ‘‘GAUGED’’ OPERATOR

The resonant sets $\Xi(\mathfrak{V})$ are designed to describe the invariant subspaces of the periodic PDO's having the form

$$\boxed{\text{model:eq}} \quad (6.1) \quad A = H_0 + B^o + B^{\mathfrak{R}}$$

with the symbols $h_0(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{2m}$ and $b \in \mathbf{S}_{\alpha}(w)$, where $\alpha \in \mathbb{R}$, $w(\boldsymbol{\xi}) = \langle \boldsymbol{\xi} \rangle^{\beta}$, $\beta \in (0, 1)$. By (decay:eq) (2.3) and (subord:eq) (3.22),

$$\boxed{\text{bm:eq}} \quad (6.2) \quad |\mathbf{D}_{\boldsymbol{\xi}}^{\mathfrak{s}} \hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi})| + |\mathbf{D}_{\boldsymbol{\xi}}^{\mathfrak{s}} \hat{b}^o(\boldsymbol{\theta}, \boldsymbol{\xi})| + |\mathbf{D}_{\boldsymbol{\xi}}^{\mathfrak{s}} \hat{b}^{\mathfrak{R}}(\boldsymbol{\theta}, \boldsymbol{\xi})| \ll |b|_{l,s}^{(\alpha)} \langle \boldsymbol{\xi} \rangle^{(\alpha-|\mathfrak{s}|\beta)} \langle \boldsymbol{\theta} \rangle^{-l},$$

for all \mathfrak{s} . We always assume that (alm1:eq) (2.15) is satisfied, so that

$$\boxed{\text{alm:eq}} \quad (6.3) \quad 2m > \alpha\beta, \quad 2m - 1 > \beta(\alpha - 1), \quad 2m - 2 > \beta(\alpha - 2).$$

This guarantees that the symbol b and its first two derivatives grow slower than the principal symbol h_0 and its corresponding derivatives respectively.

In order to use the resonant sets $\Xi(\mathfrak{V})$ constructed previously, set

$$\alpha_1 = \beta,$$

and assume that the condition (eq:condition1) (5.12) is satisfied. In addition to the symbol (flat:eq) (3.16), for any lattice subspace $\mathfrak{V} \in \mathcal{V}(n)$, $n = 1, 2, \dots, d$ we define

$$\boxed{\text{bflatv:eq}} \quad (6.4) \quad b_{\mathfrak{V}}^{\mathfrak{R}}(\mathbf{x}, \boldsymbol{\xi}; \rho) = \frac{1}{\sqrt{d(\Gamma)}} \sum_{\boldsymbol{\theta} \in \Theta_r \cap \mathfrak{V}} \hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi}; \rho) \zeta_{\boldsymbol{\theta}}(\boldsymbol{\xi}; \rho^{\beta}) e_{\boldsymbol{\theta}}(\boldsymbol{\xi}) e^{i\boldsymbol{\theta} \cdot \mathbf{x}}.$$

It is clear that the above symbol retains from $b^{\mathfrak{R}}$ only the Fourier coefficients with $\boldsymbol{\theta} \in \mathfrak{V}$. Introduce also the notation for the appropriate reduced version of the model operator (6.1):

$$A_{\mathfrak{V}} = H_0 + B^o + B_{\mathfrak{V}}^{\mathfrak{R}}, \quad B_{\mathfrak{V}}^{\mathfrak{R}} = \text{Op}(b_{\mathfrak{V}}^{\mathfrak{R}}).$$

Recall (see Subject (fibre:subject) (2.3)) that for any set $\mathcal{C} \subset \mathbb{R}^d$ we denote by $\mathcal{P}(\mathcal{C})$ the operator $\chi(\mathbf{D}; \mathcal{C})$, where $\chi(\cdot; \mathcal{C})$ is the characteristic function of the set \mathcal{C} . Accordingly, for the operators in the Floquet decomposition acting on the torus, we define $\mathcal{P}(\mathbf{k}; \mathcal{C})$ to be $\chi(\mathbf{D} + \mathbf{k}; \mathcal{C})$.

In what follows we still apply Convention [uniform:conv](#) 5.6, and add to it one more rule: the estimates which we obtain are also uniform in the symbol b , satisfying the condition $|b|^{(\alpha)} \ll 1$.

Lemma 6.1. *Let b be as above. Then for sufficiently large ρ , and any $\mathfrak{V} \in \mathcal{V}(n)$ we have*

$$\boxed{\text{inv:eq}} \quad (6.5) \quad B^{\mathfrak{R}}\mathcal{P}(\Xi(\mathfrak{V})) = B_{\mathfrak{V}}^{\mathfrak{R}}\mathcal{P}(\Xi(\mathfrak{V})) = \mathcal{P}(\Xi(\mathfrak{V}))B_{\mathfrak{V}}^{\mathfrak{R}}\mathcal{P}(\Xi(\mathfrak{V})),$$

and

$$\boxed{\text{inv0:eq}} \quad (6.6) \quad B^{\mathfrak{R}}(\mathbf{k})\mathcal{P}(\mathbf{k}; \Upsilon(\boldsymbol{\mu})) = \mathcal{P}(\mathbf{k}; \Upsilon(\boldsymbol{\mu}))B_{\mathfrak{V}}^{\mathfrak{R}}(\mathbf{k})\mathcal{P}(\mathbf{k}; \Upsilon(\boldsymbol{\mu})),$$

for any $\mathbf{k} \in \mathcal{O}^\dagger$ and any $\boldsymbol{\mu} \in \Xi(\mathfrak{V})$ with $\{\boldsymbol{\mu}\} = \mathbf{k}$.

Proof. Assume without loss of generality that $b^{\mathfrak{R}}$ has only one non-zero Fourier coefficient, i.e.

$$\boxed{\text{odin:eq}} \quad (6.7) \quad b^{\mathfrak{R}}(\mathbf{x}, \boldsymbol{\xi}; \rho) = \frac{1}{\sqrt{d(\Gamma)}} \hat{b}(\boldsymbol{\theta}, \boldsymbol{\xi}; \rho) \zeta_{\boldsymbol{\theta}}(\boldsymbol{\xi}; \rho^\beta) e_{\boldsymbol{\theta}}(\boldsymbol{\xi}) e^{i\boldsymbol{\theta}\mathbf{x}},$$

so that $b_{\mathfrak{V}}^{\mathfrak{R}} = 0$ if $\boldsymbol{\theta} \notin \mathfrak{V}$. According to [floquet:eq](#) (2.11),

$$(B^{\mathfrak{R}}(\mathbf{k})u)(\mathbf{x}) = \frac{1}{d(\Gamma)} \sum_{\mathbf{m} \in \Gamma^\dagger} \hat{b}^{\mathfrak{R}}(\boldsymbol{\theta}, \mathbf{m} + \mathbf{k}) e^{i(\mathbf{m} + \boldsymbol{\theta})\mathbf{x}} \hat{u}(\mathbf{m}),$$

$$\boxed{\text{foquetups:eq}} \quad (6.8) \quad (B^{\mathfrak{R}}(\mathbf{k})\mathcal{P}(\mathbf{k}; \Upsilon(\boldsymbol{\mu}))u)(\mathbf{x}) = \frac{1}{d(\Gamma)} \sum_{\mathbf{m}: \mathbf{m} + \mathbf{k} \in \Upsilon(\boldsymbol{\mu})} \hat{b}^{\mathfrak{R}}(\boldsymbol{\theta}, \mathbf{m} + \mathbf{k}) e^{i(\mathbf{m} + \boldsymbol{\theta})\mathbf{x}} \hat{u}(\mathbf{m}),$$

for any $u \in \mathbf{L}^2(\mathbb{T}^d)$. Observe that by virtue of [phizeta:eq](#) (3.9) for any $\boldsymbol{\xi} := \mathbf{m} + \mathbf{k} \in \text{supp } \zeta_{\boldsymbol{\theta}}(\cdot; \rho^\beta)$ we have

$$|\boldsymbol{\xi} \cdot \boldsymbol{\theta}| \leq |\boldsymbol{\theta}(\boldsymbol{\xi} + \boldsymbol{\theta}/2)| + |\boldsymbol{\theta}|^2/2 \leq (\rho^\beta/2 + r/2)|\boldsymbol{\theta}| < \rho^\beta|\boldsymbol{\theta}|,$$

that is, $\boldsymbol{\xi} \in \Lambda(\boldsymbol{\theta})$, and a similar calculation shows that $\boldsymbol{\xi} + \boldsymbol{\theta} \in \Lambda(\boldsymbol{\theta})$ as well. By Lemma [properties2:lem](#) 5.18, $\Upsilon(\boldsymbol{\mu}) \cap \Lambda(\boldsymbol{\theta}) = \emptyset$, if $\boldsymbol{\theta} \notin \mathfrak{V}$, so it follows from [foquetups:eq](#) (6.8) that

$$B^{\mathfrak{R}}(\mathbf{k})\mathcal{P}(\mathbf{k}; \Upsilon(\boldsymbol{\mu})) = B_{\mathfrak{V}}^{\mathfrak{R}}(\mathbf{k})\mathcal{P}(\mathbf{k}; \Upsilon(\boldsymbol{\mu})) = 0, \quad \text{if } \boldsymbol{\theta} \notin \mathfrak{V}.$$

In the case $\boldsymbol{\theta} \in \mathfrak{V}$, by Definition [reachability:defn](#) 5.3, the points $\boldsymbol{\xi} := \mathbf{m} + \mathbf{k} \in \Upsilon(\boldsymbol{\mu})$ and $\boldsymbol{\xi} + \boldsymbol{\theta}$ are $\boldsymbol{\theta}$ -resonant congruent, so that $\boldsymbol{\xi} + \boldsymbol{\theta} \in \Upsilon(\boldsymbol{\mu})$. This completes the proof of [inv0:eq](#) (6.6).

Using [equiv:eq](#) (5.17) we get from [inv0:eq](#) (6.6):

$$B^{\mathfrak{R}}(\mathbf{k})\mathcal{P}(\mathbf{k}; \Xi(\mathfrak{V})) = \mathcal{P}(\mathbf{k}; \Xi(\mathfrak{V}))B_{\mathfrak{V}}^{\mathfrak{R}}(\mathbf{k})\mathcal{P}(\mathbf{k}; \Xi(\mathfrak{V})).$$

Taking the direct integral in \mathbf{k} yields [inv:eq](#) (6.5). □

6.1. **Operator A in the invariant subspaces.** Due to the properties (5.9) and (5.10), the formulas (6.5) and (6.6) imply the following orthogonal decomposition for the Floquet fibres $A(\mathbf{k})$:

$$(6.9) \quad A(\mathbf{k}) = \bigoplus_{\mathfrak{Y} \in \mathcal{W}(r)} A(\mathbf{k}; \Xi(\mathfrak{Y})) = \bigoplus_{\mathfrak{Y} \in \mathcal{W}(r)} \bigoplus_{\substack{\mu \in \Xi(\mathfrak{Y}) / \leftrightarrow \\ \{\mu\} = \mathbf{k}}} A_{\mathfrak{Y}}(\mathbf{k}; \Upsilon(\mu)).$$

Since $\text{card } \Upsilon(\mu) < \infty$, see (5.13), for each $\mu \in \Xi(\mathfrak{Y})$ the operator $A_{\mathfrak{Y}}(\mathbf{k}; \Upsilon(\mu))$ is finite dimensional. In the basis

$$E_{[\mu] + \mathbf{m}}(\mathbf{x}), \quad \mathbf{m} \in \mathbf{N}(\mu)$$

(see definition (2.17)), of the subspace $\mathfrak{H}(\mathbf{k}; \Upsilon(\mu))$, the operator $A(\mathbf{k}; \Upsilon(\mu))$ reduces to the matrix $\mathcal{A}(\mu)$ with the entries

$$(6.10) \quad \mathcal{A}_{\mathbf{m}, \mathbf{n}}(\mu) = \frac{1}{\sqrt{d(\Gamma)}} \hat{a}(\mathbf{m} - \mathbf{n}, \mu + \mathbf{n}; \rho), \quad \mathbf{m}, \mathbf{n} \in \mathbf{N}(\mu).$$

Denote by $\lambda_j(\mathcal{A}(\mu))$, $j = 1, 2, \dots, N(\mu) = \text{card } \mathbf{N}(\mu)$ the eigenvalues of the matrix $\mathcal{A}(\mu)$, arranged in descending order. It is easy to check that the matrices $\mathcal{A}(\mu)$ and $\mathcal{A}(\mu')$ with $\mu' \in \Upsilon(\mu)$, are unitarily equivalent, so that the eigenvalues do not depend on the choice of μ , but only on the set $\Upsilon(\mu)$.

Lemma 6.2. *Let $\lambda_j(\mathcal{A}(\mu))$ be the eigenvalues introduced above. Then for sufficiently large ρ , for all $|\mu| \gg \rho$, and for all $j = 1, 2, \dots, N(\mu)$ one has*

$$\lambda_j(\mathcal{A}(\mu)) \asymp \min_{\eta \in \Upsilon(\mu)} |\eta|^{2m} \asymp \max_{\eta \in \Upsilon(\mu)} |\eta|^{2m} \asymp |\mu|^{2m},$$

uniformly in μ .

Proof. The operator A has the form $H_0 + B^{o, \mathcal{R}}$, and since $\alpha\beta < 2m$ (see (6.3)), by Lemma 3.2, the perturbation $B^{o, \mathcal{R}}$ is infinitesimally H_0 -bounded, so that $cH_0 - \tilde{C} \leq A \leq CH_0 + \tilde{C}$ with some positive constants C, c, \tilde{C} . Therefore, the same bounds hold for the fibers $H_0(\mathbf{k})$ and $A(\mathbf{k})$. As a consequence, the restriction of both operators to the subspace $\mathfrak{H}(\mathbf{k}, \Upsilon(\mu))$ satisfy the same inequalities:

$$cH_0(\mathbf{k}; \Upsilon(\mu)) - \tilde{C} \leq A(\mathbf{k}; \Upsilon(\mu)) \leq CH_0(\mathbf{k}; \Upsilon(\mu)) + \tilde{C}.$$

Now the claimed inequalities follows from Lemma 5.10. \square

If $\Upsilon(\mu)$ is non-critical (see Definition 5.9), the set $\mathbf{N}(\mu)$ remains constant in a neighbourhood of μ . Since the entries of the matrix \mathcal{A} depend continuously on μ , we conclude that the eigenvalues $\lambda_j(\mathcal{A}(\cdot))$ are continuous in a neighbourhood of such a point μ . Moreover, by virtue of Lemma 5.21, for any $\mu \in \Xi(\mathfrak{Y})$ the set $\mathbf{N}(\mu)$ remains constant if $\mu_{\mathfrak{Y}}$ is kept constant, and hence it makes sense to study the eigenvalues as functions of the component $\nu = \mu_{\mathfrak{Y}^\perp}$. Define the matrix

$$\tilde{A}(t) = \mathcal{A}(\mu_{\mathfrak{Y}} + t\mathbf{e}(\nu)), \quad \mathbf{e}(\nu) = \frac{\nu}{|\nu|},$$

with a real-valued parameter $t \geq t_0 := |\boldsymbol{\nu}|$. By (6.10) the entries of this matrix are

$$(6.11) \quad \tilde{\mathcal{A}}_{\mathbf{m},\mathbf{n}}(t) = \frac{1}{\sqrt{d(\Gamma)}} \hat{a}(\mathbf{m} - \mathbf{n}, \boldsymbol{\mu}_{\mathfrak{V}} + \mathbf{n} + t\mathbf{e}(\boldsymbol{\nu}); \rho), \quad \mathbf{m}, \mathbf{n} \in \mathbf{N}(\boldsymbol{\mu}).$$

By Lemma 5.22 the matrix $\tilde{\mathcal{A}}$ is well-defined on the interval $[t_0, \infty)$.

Lemma 6.3. *Let (6.3) be satisfied. Suppose that $\boldsymbol{\mu} \in \Xi(\mathfrak{V})$ and $|\boldsymbol{\mu}| \asymp \rho$. Then*

$$(6.12) \quad \lambda_j(\tilde{\mathcal{A}}(t_2)) - \lambda_j(\tilde{\mathcal{A}}(t_1)) \asymp \rho^{2m-1}(t_2 - t_1),$$

for any $t_1, t_2 \asymp t_0$, $t_0 \leq t_1 < t_2$, uniformly in $j = 1, 2, \dots, \mathbf{N}(\boldsymbol{\mu})$, $\boldsymbol{\mu}$ and \mathfrak{V} .

Proof. By Lemma 5.12, $|\boldsymbol{\mu}_{\mathfrak{V}}| \leq 2\rho^{\alpha_d}$, so that $t_0 = |\boldsymbol{\nu}| \asymp |\boldsymbol{\mu}| \asymp \rho$. By the elementary perturbation theory, it would suffice to establish for the matrix

$$\tilde{\mathcal{A}}(t_1, t_2) = \tilde{\mathcal{A}}(t_2) - \tilde{\mathcal{A}}(t_1)$$

the relation

$$(\tilde{\mathcal{A}}(t_1, t_2)u, u) \asymp \rho^{2m-1}(t_2 - t_1)\|u\|^2, \quad t_1, t_2 \asymp \rho, t_2 > t_1 \geq t_0,$$

for all $u \in \mathfrak{H}$. The entries of this matrix are

$$\int_{t_1}^{t_2} \mathfrak{Y}_{\mathbf{m},\mathbf{n}}(t) dt, \quad \mathfrak{Y}_{\mathbf{m},\mathbf{n}}(t) = \frac{d}{dt} \tilde{\mathcal{A}}_{\mathbf{m},\mathbf{n}}(t).$$

We show that the matrix $\mathfrak{Y}(t)$ satisfies

$$(6.13) \quad (\mathfrak{Y}(t)u, u) \asymp \rho^{2m-1}\|u\|^2, \quad t \asymp \rho,$$

for all $u \in \mathfrak{H}$, uniformly in $\boldsymbol{\mu}$, \mathfrak{V} , and the symbol b . Denote

$$\boldsymbol{\mu}_t = \boldsymbol{\mu}_{\mathfrak{V}} + t\mathbf{e}(\boldsymbol{\nu}), \quad \boldsymbol{\nu} = \boldsymbol{\mu}_{\mathfrak{V}^\perp}.$$

Corollary 5.19 implies that $\mathbf{N}(\boldsymbol{\mu}) \subset \mathfrak{V}$. Therefore, $\mathfrak{Y}(t)$ is the sum of the matrix with diagonal entries

$$\frac{d}{dt} h_0(\boldsymbol{\mu}_t + \mathbf{m}) = 2m|\boldsymbol{\mu}_t + \mathbf{m}|^{2m-2}t, \quad \mathbf{m} \in \mathbf{N}(\boldsymbol{\mu}),$$

and the matrix $\mathcal{Z}(t)$ with the entries

$$\mathcal{Z}_{\mathbf{m},\mathbf{n}}(t) = \frac{1}{\sqrt{d(\Gamma)}} \nabla_{\boldsymbol{\xi}} \hat{b}^{\rho, \mathcal{R}}(\mathbf{m} - \mathbf{n}, \boldsymbol{\xi}; \rho) \cdot \mathbf{e}(\boldsymbol{\nu}) \Big|_{\boldsymbol{\xi} = \boldsymbol{\mu}_t + \mathbf{n}}.$$

By (5.13), $|\mathbf{m}| \leq \rho^{\alpha_d}$, and hence

$$(6.14) \quad \rho \ll t \leq |\boldsymbol{\mu}_t + \mathbf{m}| \ll \rho, \quad \mathbf{m} \in \mathbf{N}(\boldsymbol{\mu}).$$

Thus

$$(6.15) \quad \frac{d}{dt} h_0(\boldsymbol{\mu}_t + \mathbf{m}) \asymp \rho^{2m-1}.$$

Also, by (6.2),

$$|\mathcal{Z}_{\mathbf{m},\mathbf{n}}(t)| \ll \langle \mathbf{m} - \mathbf{n} \rangle^{-l} |\boldsymbol{\mu}_t + \mathbf{n}|^{(\alpha-1)\beta} \ll \langle \mathbf{m} - \mathbf{n} \rangle^{-l} \rho^{(\alpha-1)\beta},$$

for any $l > 0$. Assuming that $l > d$, from here we get:

$$\begin{aligned} \|\mathcal{Z}(t)\| &\leq \max_{\mathbf{n}} \sum_{\mathbf{m}} |\mathcal{Z}_{\mathbf{m},\mathbf{n}}(t)| \\ &\leq C_l \rho^{(\alpha-1)\beta} \sup_{\mathbf{n}} \max_{\mathbf{m}} \langle \mathbf{m} - \mathbf{n} \rangle^{-l} \ll C \rho^{(\alpha-1)\beta}. \end{aligned}$$

This, together with (6.3) and (6.15), leads to (6.13), which implies (6.12), as required. \square

7. GLOBAL DESCRIPTION OF THE EIGENVALUES OF OPERATOR $A(\mathbf{k})$

global:sect

In this section we continue the study of the discrete spectrum of the fibres $A(\mathbf{k})$. Our aim is to construct a function $g : \mathbb{R}^d \rightarrow \mathbb{R}$, which establishes a one-to-one correspondence between the points of \mathbb{R}^d and the eigenvalues of $A(\mathbf{k})$. More precisely, we seek a function g such that

- (1) for every $\boldsymbol{\xi} \in \mathbb{R}^d$ the value $g(\boldsymbol{\xi})$ is an eigenvalue of the operator $A(\mathbf{k})$, $\mathbf{k} = \{\boldsymbol{\xi}\}$, and
- (2) for every $j \in \mathbb{N}$ there exists a uniquely defined point $\boldsymbol{\xi}$ with $\{\boldsymbol{\xi}\} = \mathbf{k}$ such that $g(\boldsymbol{\xi}) = \lambda_j(A(\mathbf{k}))$.

In other words, we intend to label the eigenvalues of $A(\mathbf{k})$ by the points of the lattice Γ^\dagger , shifted by \mathbf{k} . The construction of the convenient function g is conducted using the decomposition (6.9), individually in the invariant subspaces generated by the sets $\Xi(\mathfrak{X})$.

We begin with the non-resonant set $\mathcal{B} = \Xi(\mathfrak{X})$, $\mathfrak{X} = \{0\}$. On the subspace $\mathcal{H}(\Xi(\mathfrak{X}))$ the symbol of the operator A is \mathbf{x} -independent, and it is $a^\circ(\boldsymbol{\xi}) = h_0(\boldsymbol{\xi}) + b^\circ(\boldsymbol{\xi})$. Therefore the eigenvalues of the operator $A(\mathbf{k})$ are given by $a^\circ(\boldsymbol{\mu} + \mathbf{k})$, $\boldsymbol{\mu} \in \Gamma^\dagger$, $\boldsymbol{\mu} + \mathbf{k} \in \Xi(\mathfrak{X})$. Clearly, it is natural to label the eigenvalues by lattice points. Let us define

$$g(\boldsymbol{\xi}) = a^\circ(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in \Xi(\mathfrak{X}) = \mathcal{B}.$$

According to (6.2),

ginb:eq

$$(7.1) \quad g(\boldsymbol{\xi}) = h_0(\boldsymbol{\xi}) + b^\circ(\boldsymbol{\xi}), \quad \left| \frac{\partial}{\partial |\boldsymbol{\xi}|} b^\circ(\boldsymbol{\xi}) \right| \leq C \langle \boldsymbol{\xi} \rangle^{(\alpha-1)\beta}, \quad \left| \frac{\partial^2}{\partial |\boldsymbol{\xi}|^2} b^\circ(\boldsymbol{\xi}) \right| \leq C \langle \boldsymbol{\xi} \rangle^{(\alpha-2)\beta}$$

for all $\boldsymbol{\xi} \in \mathcal{B}$.

Suppose now that $\boldsymbol{\xi} \in \Xi(\mathfrak{W})$ with some non-trivial lattice subspace \mathfrak{W} . In view of (6.9), it suffices to define the function g on the sets $\Upsilon(\boldsymbol{\xi})$. Let us label all numbers $\boldsymbol{\eta} \in \Upsilon(\boldsymbol{\xi})$, $\boldsymbol{\xi} \in \Xi(\mathfrak{W})$ in the increasing order of their length $|\boldsymbol{\eta}|$ by natural numbers from the set $\{1, 2, \dots, N(\boldsymbol{\xi})\}$; if there are two different vectors $\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}} \in \Upsilon(\boldsymbol{\xi})$ with $|\boldsymbol{\eta}| = |\tilde{\boldsymbol{\eta}}|$, we label them in the lexicographic order of their coordinates, i.e. we put $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_d)$ before $\tilde{\boldsymbol{\eta}} = (\tilde{\eta}_1, \tilde{\eta}_2, \dots, \tilde{\eta}_d)$ if either $\eta_1 < \tilde{\eta}_1$, or $\eta_1 = \tilde{\eta}_1$ and $\eta_2 < \tilde{\eta}_2$, etc. Such a labeling associates in a natural way with each point $\boldsymbol{\eta} \in \Upsilon(\boldsymbol{\xi})$ a positive integer $\ell = \ell(\boldsymbol{\eta}) \leq N(\boldsymbol{\xi})$. Clearly, this number does not depend on the choice of the point $\boldsymbol{\xi}$ as long as $\boldsymbol{\xi}$ remains within the same \leftrightarrow -equivalence class. In particular,

newindices

$$(7.2) \quad |\boldsymbol{\eta}|^{2m} = \lambda_{\ell(\boldsymbol{\eta})}(H_0(\mathbf{k}; \Upsilon(\boldsymbol{\eta}))).$$

Now for every $\boldsymbol{\eta} \in \mathbb{R}^d$ we define

$$g(\boldsymbol{\eta}) := \lambda_{\ell(\boldsymbol{\eta})}(\mathcal{A}(\boldsymbol{\eta})),$$

where $\mathcal{A}(\cdot)$ is the matrix defined in (6.10). Note that in view of Lemma 6.2

$$(7.3) \quad g(\boldsymbol{\eta}) \asymp |\boldsymbol{\eta}|^{2m}, \quad |\boldsymbol{\eta}| \gg \rho,$$

for sufficiently large ρ .

In order to analyse the continuity of $g(\cdot)$, we assume that $\boldsymbol{\eta}$ is a non-critical point, i.e. the set $\mathbf{N}(\cdot)$ remains constant in a neighbourhood of $\boldsymbol{\eta}$, see Definition 5.9. Furthermore, in the non-critical set, $\ell(\boldsymbol{\eta})$ remains constant, if the point $\mathbf{0}$ stays away from the Voronoi hyper-planes associated with pairs of points from the set $\boldsymbol{\Upsilon}(\boldsymbol{\eta})$. Recall that the Voronoi hyper-plane for a pair $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in \mathbb{R}^d$ is the set of all points $\mathbf{z} \in \mathbb{R}^d$ such that $|\boldsymbol{\eta}_1 - \mathbf{z}| = |\boldsymbol{\eta}_2 - \mathbf{z}|$. Thus, the function $g(\cdot)$ is continuous on an open set of full measure in \mathbb{R}^d .

In each set $\Xi(\mathfrak{V})$ the labeling function ℓ possesses the following important property.

lem:new1

Lemma 7.1. *Let $\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}} \in \Xi(\mathfrak{V})$ satisfy $\boldsymbol{\nu} := \tilde{\boldsymbol{\eta}} - \boldsymbol{\eta} \perp \mathfrak{V}$. Then $\ell(\tilde{\boldsymbol{\eta}}) = \ell(\boldsymbol{\eta})$.*

Proof. Recall that by Lemma 5.21, $\boldsymbol{\Upsilon}(\boldsymbol{\eta}) + \boldsymbol{\nu} = \boldsymbol{\Upsilon}(\tilde{\boldsymbol{\eta}})$. Let $\boldsymbol{\eta} \leftrightarrow \boldsymbol{\mu}$, so that $(\boldsymbol{\eta} + \boldsymbol{\nu}) \leftrightarrow (\boldsymbol{\mu} + \boldsymbol{\nu})$. By Corollary 5.19, $\boldsymbol{\mu}_{\mathfrak{V}^\perp} = \boldsymbol{\eta}_{\mathfrak{V}^\perp}$, and hence the inequality $|\boldsymbol{\mu}| \leq |\boldsymbol{\eta}|$ holds or does not hold simultaneously with $|\boldsymbol{\mu} + \boldsymbol{\nu}| \leq |\boldsymbol{\eta} + \boldsymbol{\nu}|$. The same property is true for the coordinates of the vectors involved. Therefore, $\ell(\boldsymbol{\eta}) = \ell(\boldsymbol{\eta} + \boldsymbol{\nu})$. \square

The next lemma allows us to establish smoothness of the function g with respect to the variable $\boldsymbol{\eta}_{\mathfrak{V}^\perp}$.

Lemma 7.2. *Let $\mathfrak{V} \in \mathcal{V}(n), 1 \leq n \leq d-1$ and let (6.3) be satisfied. Suppose that $\boldsymbol{\eta} \in \Xi(\mathfrak{V})$ and $|\boldsymbol{\eta}| \asymp \rho$, and let $\boldsymbol{\nu} = \boldsymbol{\eta}_{\mathfrak{V}^\perp}$. Then for sufficiently large ρ , on the interval $[t_0, \infty)$, $t_0 := |\boldsymbol{\nu}|$, the function $\tilde{g}(t; \boldsymbol{\eta}) := g(\boldsymbol{\eta}_{\mathfrak{V}} + t\mathbf{e}(\boldsymbol{\nu}))$ satisfies*

$$(7.4) \quad \tilde{g}(t_2, \boldsymbol{\eta}) - \tilde{g}(t_1, \boldsymbol{\eta}) \asymp \rho^{2m-1}(t_2 - t_1),$$

for any $t_1, t_2 \in [t_0, \infty)$ such that $t_1 < t_2$ and $t_1, t_2 \asymp t_0$, uniformly in $\boldsymbol{\eta} \in \Xi(\mathfrak{V})$ and \mathfrak{V} .

Proof. Recall that in view of Lemma 5.22, $\boldsymbol{\eta}_{\mathfrak{V}} + t\mathbf{e}(\boldsymbol{\nu}) \in \Xi(\mathfrak{V})$ for all $t > t_0$. Thus by Lemma 7.1, $\ell(\boldsymbol{\eta}) = \ell(\boldsymbol{\eta}_{\mathfrak{V}} + t\mathbf{e}(\boldsymbol{\nu})) =: \ell$ for $t \in [t_0, \infty)$. Therefore,

$$\tilde{g}(t; \boldsymbol{\eta}) = \lambda_{\ell}(\mathcal{A}(\boldsymbol{\eta}_{\mathfrak{V}} + t\mathbf{e}(\boldsymbol{\nu}))).$$

It remains to apply Lemma 6.3. \square

In order to study the global properties of g , note that by the above construction there is a bijection $J : \Gamma^\dagger + \mathbf{k} \rightarrow \mathbb{N}$ such that

$$g(\boldsymbol{\eta}) = \lambda_{J(\boldsymbol{\eta})}(A(\mathbf{k})), \quad \mathbf{k} = \{\boldsymbol{\eta}\},$$

for all $\boldsymbol{\eta} \in \mathbb{R}^d$. For the following Lemma recall that the distance on the torus $|\cdot|_{\mathbb{T}}$ is defined in (3.28).

partialofs2

Lemma 7.3. *Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ be such that $|\mathbf{a}| \asymp \rho$. Then there exists a vector $\mathbf{n} \in \Gamma^\dagger$ such that*

lip:eq

$$(7.5) \quad |g(\mathbf{b} + \mathbf{n}) - g(\mathbf{a})| \ll \rho^{2m-1} |\mathbf{b} - \mathbf{a}|_{\mathbb{T}},$$

for sufficiently large ρ .

Suppose in addition, that $\mathbf{m} \in \Gamma^\dagger, \mathbf{m} \neq \mathbf{0}$, is a vector such that $|\mathbf{a} + \mathbf{m}| \asymp \rho$. Then there exists a $\tilde{\mathbf{n}} \in \Gamma^\dagger$, such that $\mathbf{n} \neq \tilde{\mathbf{n}}$ and

lip1:eq

$$(7.6) \quad |g(\mathbf{b} + \tilde{\mathbf{n}}) - g(\mathbf{a} + \mathbf{m})| \ll \rho^{2m-1} |\mathbf{b} - \mathbf{a}|_{\mathbb{T}},$$

for sufficiently large ρ .

Proof. As $\lambda_{J(\mathbf{a})}(A(\mathbf{k})) = g(\mathbf{a})$, by [\(7.3\)](#), we have

$$|\lambda_{J(\mathbf{a})}(A(\mathbf{k}))| \asymp \rho^{2m}.$$

Denote $\mathbf{k} = \{\mathbf{a}\}$, $\mathbf{k}_1 = \{\mathbf{b}\}$. Recall that the condition [\(3.29\)](#) is satisfied due to [\(6.3\)](#), so by [Theorem 3.6](#)

lip2:eq

$$(7.7) \quad |\lambda_{J(\mathbf{a})}(A(\mathbf{k})) - \lambda_{J(\mathbf{a})}(A(\mathbf{k}_1))| \ll \rho^{2m-1} |\mathbf{k} - \mathbf{k}_1|_{\mathbb{T}} = \rho^{2m-1} |\mathbf{b} - \mathbf{a}|_{\mathbb{T}}.$$

Let $\mathbf{p} \in \mathbb{R}^d$ be a vector such that $\{\mathbf{p}\} = \mathbf{k}_1$ and $g(\mathbf{p}) = \lambda_{J(\mathbf{a})}(A(\mathbf{k}_1))$. Now [\(7.7\)](#) implies [\(7.5\)](#) with $\mathbf{n} = \mathbf{p} - \mathbf{b}$.

In order to prove [\(7.6\)](#), we use [\(7.7\)](#) with $\mathbf{a} + \mathbf{m}$ instead of \mathbf{a} . Then, as above, one can find a vector $\tilde{\mathbf{p}}$ such that $\{\tilde{\mathbf{p}}\} = \mathbf{k}_1$ and $g(\tilde{\mathbf{p}}) = \lambda_{J(\mathbf{a}+\mathbf{m})}(A(\mathbf{k}_1))$. Since J is one-to-one, we have $J(\mathbf{a} + \mathbf{m}) \neq J(\mathbf{a})$, and hence $\mathbf{p} \neq \tilde{\mathbf{p}}$. As a consequence, $\tilde{\mathbf{n}} = \tilde{\mathbf{p}} - \mathbf{b} \neq \mathbf{n}$, as required, and [\(7.7\)](#) again leads to [\(7.6\)](#). \square

8. ESTIMATES OF VOLUMES

In this section we continue the investigation of the operator of the form [\(6.1\)](#), with a symbol $b \in \mathbf{S}_\alpha(w)$, $w(\boldsymbol{\xi}) = \langle \boldsymbol{\xi} \rangle^\beta$, with parameters α, β , satisfying the conditions [\(6.3\)](#). Let $g : \mathbb{R}^d \rightarrow \mathbb{R}$ be the function defined in the previous section, and let $\mathcal{B}(\rho), \mathcal{D}(\rho)$ and $\tilde{\mathcal{B}}(\rho)$ be the sets introduced in [\(5.18\)](#) and [\(5.22\)](#) respectively.

Let $\delta \in (0, \lambda/4]$, $\lambda = \rho^{2m}$, and let

ABD:eq

$$(8.1) \quad \begin{cases} \mathcal{A}(\rho, \delta) = \mathcal{A}(g; \rho, \delta) := g^{-1}([\lambda - \delta, \lambda + \delta]), \\ \mathcal{B}(\rho, \delta) = \mathcal{B}(g; \rho, \delta) := \mathcal{A}(\rho, \delta) \cap \mathcal{B}(\rho), \\ \mathcal{D}(\rho, \delta) = \mathcal{D}(g; \rho, \delta) := \mathcal{A}(\rho, \delta) \cap \mathcal{D}(\rho), \\ \tilde{\mathcal{B}}(\rho, \delta) = \tilde{\mathcal{B}}(g; \rho, \delta) := \mathcal{A}(\rho, \delta) \cap \tilde{\mathcal{B}}(\rho). \end{cases}$$

The estimates for the volumes of the above sets are very important for our argument.

em:volumeCA

Lemma 8.1. *Let A be the operator [\(6.1\)](#), and let α, β satisfy the conditions [\(6.3\)](#). Then for any $\delta \in (0, \rho^{2m}/4]$ and for all sufficiently large ρ , the following estimates hold*

_tildeCB:eq

$$(8.2) \quad \text{vol } \tilde{\mathcal{B}}(\rho, \delta) \asymp \delta \rho^{d-2m},$$

and

$$\text{eq:volumeCD} \quad (8.3) \quad \text{vol}(\mathcal{D}(\rho, \delta)) \ll \delta \rho^{d-1-2m+\alpha_d}.$$

Here $\alpha_d \in (0, 1)$ is the number defined together with $\alpha_1, \alpha_2, \dots, \alpha_{d-1}$ at the beginning of Subsection 5.3 (see (5.12)).

Before proving the above lemma we find a convenient representation of the set $\tilde{\mathcal{B}}(\rho, \delta)$. Since $\tilde{\mathcal{B}}(\rho, \delta) \subset \mathcal{B}(\rho)$, for all $\boldsymbol{\xi} \in \tilde{\mathcal{B}}(\rho, \delta)$ the function g is defined by the formula (7.1), and in particular, it is continuous. For all $\boldsymbol{\Omega} \in T(\rho)$ (see (5.20) for definition), we introduce the subsets of the real line defined as follows:

$$\text{iomd:eq} \quad (8.4) \quad I(\boldsymbol{\Omega}; \rho, \delta) = \{t > 0 : \rho^{2m} - \delta \leq g(t\boldsymbol{\Omega}) \leq \rho^{2m} + \delta\}.$$

By (7.1), for $t \in I(\boldsymbol{\Omega}; \rho, \delta)$ we have $\rho/2 < t < 2\rho$, and hence $t\boldsymbol{\Omega} \in \tilde{\mathcal{B}}(\rho, \delta)$. If ρ is sufficiently large, by virtue of (7.1), for these values of t the function $g(t\boldsymbol{\Omega})$ is strictly increasing, and hence $I(\boldsymbol{\Omega}; \rho, \delta)$ is a closed interval. Moreover, (7.1) implies the relation

$$\text{iomd1:eq} \quad (8.5) \quad |I(\boldsymbol{\Omega}; \rho, \delta)| \asymp \delta \rho^{1-2m}$$

for its length, uniformly in $\boldsymbol{\Omega}$. By construction,

$$\text{tildecb:eq} \quad (8.6) \quad \tilde{\mathcal{B}}(\rho, \delta) = \bigcup_{\boldsymbol{\Omega} \in T(\rho)} I(\boldsymbol{\Omega}; \rho, \delta)\boldsymbol{\Omega}.$$

Proof of Lemma 8.1. In view of Lemma 5.13 and of the bound (8.5), we obtain from (8.6):

$$\text{volume_tildeCB:eq} \quad \text{vol } \tilde{\mathcal{B}}(\rho, \delta) = \int_{T(\rho)} \int_{I(\boldsymbol{\Omega}; \rho, \delta)} t^{d-1} dt d\boldsymbol{\Omega} \asymp \delta \rho^{d-2m}.$$

This proves (8.2). Proof of (8.3). By definition (5.18) and relation (5.9),

$$\mathcal{D}(\rho) = \bigcup_{\mathfrak{W} \subset \mathcal{V}(n), 1 \leq n \leq d} \Xi(\mathfrak{W}; \rho).$$

Let us estimate the volume of each intersection $\Xi(\mathfrak{W}; \rho) \cap \mathcal{A}(\rho, \delta)$. Since Lemma 5.12 implies that $\Xi(\mathbb{R}^d) \cap \mathcal{A}(\rho, \delta) = \emptyset$, we assume that $n \leq d-1$.

For all $\boldsymbol{\chi} \in \mathfrak{W}$ and $\boldsymbol{\Omega} \in \mathfrak{W}^\perp$, $|\boldsymbol{\Omega}| = 1$ denote

$$S(\boldsymbol{\chi}, \boldsymbol{\Omega}; \rho) = \{t \geq 0 : \boldsymbol{\chi} + t\boldsymbol{\Omega} \in \Xi(\mathfrak{W}; \rho)\}.$$

According to Lemma 5.22, this set is either empty, or it is a half-line of the form $[t_0, \infty)$ or (t_0, ∞) with some $t_0 \geq 0$. Due to the estimate (5.19), $S(\boldsymbol{\chi}, \boldsymbol{\Omega}; \rho)$ is empty if $|\boldsymbol{\chi}| \geq 2\rho^{\alpha_{d-1}}$. From now on we assume that $S(\boldsymbol{\chi}, \boldsymbol{\Omega}; \rho) \neq \emptyset$, so that $|\boldsymbol{\chi}| < 2\rho^{\alpha_{d-1}}$. Consider the subset

$$\text{S:eq} \quad (8.7) \quad S(\boldsymbol{\chi}, \boldsymbol{\Omega}; \rho, \delta) = \{t \in S(\boldsymbol{\chi}, \boldsymbol{\Omega}; \rho) : \rho^{2m} - \delta \leq g(\boldsymbol{\chi} + t\boldsymbol{\Omega}) \leq \rho^{2m} + \delta\}.$$

In view of (7.3), $t \asymp \rho$. By (7.4), the function $\tilde{g}(t) = g(\boldsymbol{\chi} + t\boldsymbol{\Omega})$ is strictly increasing and continuous, and hence, $S(\boldsymbol{\chi}, \boldsymbol{\Omega}; \rho, \delta)$ is an interval. The bound (7.4) also guarantees the upper bound

$$|S(\boldsymbol{\chi}, \boldsymbol{\Omega}; \rho, \delta)| \ll \delta \rho^{1-2m},$$

for the length of this interval, uniformly in $\boldsymbol{\chi}$ and $\boldsymbol{\Omega}$. Now we can estimate the volume of the intersection:

$$\begin{aligned} \text{vol}(\Xi(\mathfrak{B}; \rho) \cap \mathcal{A}(\rho, \delta)) &= \int_{|\boldsymbol{\chi}| < 2\rho^{\alpha_{d-1}}} \int_{\mathbb{S}^{d-n-1}} \int_{S(\boldsymbol{\chi}, \boldsymbol{\Omega}; \rho, \delta)} t^{d-n-1} dt d\boldsymbol{\Omega} d\boldsymbol{\chi} \\ &\ll \rho^{d-n-1} \int_{|\boldsymbol{\chi}| < 2\rho^{\alpha_{d-1}}} \int_{\mathbb{S}^{d-n-1}} |S(\boldsymbol{\chi}, \boldsymbol{\Omega}; \rho, \delta)| d\boldsymbol{\Omega} d\boldsymbol{\chi} \\ &\ll \delta \rho^{1-2m} \rho^{d-n-1} (\rho^{\alpha_{d-1}})^n \ll \delta \rho^{d-1-2m+\alpha_{d-1}}. \end{aligned}$$

Recall that the number of distinct subspaces $\mathfrak{B} \subset \mathcal{W}(r)$ does not exceed Cr^{d^2} with some universal constant C , so that

$$\text{vol} \mathcal{D}(\rho, \delta) \ll \delta \rho^{d-1-2m+\alpha_{d-1}} r^{d^2} \ll \delta \rho^{d-1-2m+\alpha_d},$$

where we have used the conditions (5.12). Now (8.3) is proved. \square

The next estimate is more subtle:

Theorem 8.2. *Let $\mathcal{B}(g; \rho, \delta)$, $\delta \in (0, \rho^{2m}/4]$, be as defined in (8.1). Let $\varepsilon > 0$ be a fixed number. If $\delta \rho^{2-2m+2\varepsilon} \rightarrow 0$ as $\rho \rightarrow \infty$, then*

$$(8.8) \quad \text{vol}(\mathcal{B}(g; \rho, \delta) \cap (\mathcal{B}(g; \rho, \delta) + \mathbf{b})) \ll \delta^2 \rho^{4-4m+d+6\varepsilon} + \delta \rho^{1-2m-\varepsilon(d-1)},$$

uniformly in \mathbf{b} , $|\mathbf{b}| \gg 1$.

This Theorem will be proved in the next section. As an immediate consequence, we can write the following estimate:

$$(8.9) \quad \text{vol} \bigcup_{\mathbf{n} \in \Gamma^+ \setminus \{0\}} (\mathcal{B}(g; \rho, \delta) \cap (\mathcal{B}(g; \rho, \delta) + \mathbf{n})) \ll \delta^2 \rho^{4-4m+2d+6\varepsilon} + \delta \rho^{1-2m+d-\varepsilon(d-1)},$$

valid under the condition $\delta \rho^{2-2m+2\varepsilon} \rightarrow 0$, $\rho \rightarrow \infty$. Indeed, to get (8.9) from (8.8) one notices that the union in the above estimate does not extend to the lattice points \mathbf{n} such that $|\mathbf{n}| \geq 3\rho$.

Another important ingredient is the following estimate on the volumes:

Lemma 8.3. *Let $\mathcal{B}(g; \rho, \delta)$, $\mathcal{D}(g; \rho, \delta)$, $\delta \in (0, \rho^{2m}/4]$, be as defined in [\(8.1\)](#). Let $\varepsilon > 0$ be some number. If $\delta \rho^{2-2m+2\varepsilon} \rightarrow 0$ as $\rho \rightarrow \infty$, then*

$$\text{vol} \bigcup_{\mathbf{n} \in \Gamma^\dagger \setminus \{0\}} \left(\mathcal{B}(g; \rho, \delta) \cap (\mathcal{D}(g; \rho, \delta) + \mathbf{n}) \right) \ll \delta^2 \rho^{4-4m+2d+6\varepsilon} + \delta \rho^{1-2m+d-\varepsilon(d-1)} + \delta \rho^{d-1-2m+\alpha_d}.$$

BD:eq (8.10)

Proof. Let us split $\mathcal{D}(\rho, \delta)$ in three disjoint sets:

$$\mathcal{D}_0(\rho, \delta) = \{\boldsymbol{\xi} \in \mathcal{D}(\rho, \delta) : \boldsymbol{\xi} + \mathbf{n} \notin \mathcal{B}(\rho, \delta), \text{ for all } \mathbf{n} \in \Gamma^\dagger \setminus \{0\}\},$$

$$\mathcal{D}_1(\rho, \delta) = \{\boldsymbol{\xi} \in \mathcal{D}(\rho, \delta) :$$

there exists a unique $\mathbf{n} = \mathbf{n}(\boldsymbol{\xi}) \in \Gamma^\dagger \setminus \{0\}$ such that $\boldsymbol{\xi} + \mathbf{n} \in \mathcal{B}(\rho, \delta)\},$

$$\mathcal{D}_2(\rho, \delta) = \mathcal{D}(\rho, \delta) \setminus \left(\mathcal{D}_0(\rho, \delta) \cup \mathcal{D}_1(\rho, \delta) \right).$$

The definition of $\mathcal{D}_0(\rho, \delta)$ immediately implies that

$$\mathcal{B}(\rho, \delta) \cap \left(\bigcup_{\mathbf{n} \in \Gamma^\dagger \setminus \{0\}} (\mathcal{D}_0(\rho, \delta) + \mathbf{n}) \right) = \emptyset.$$

CD0:eq (8.11)

For the set $\mathcal{D}_2(\rho, \delta)$ we have the inclusion

$$\bigcup_{\mathbf{n} \in \Gamma^\dagger \setminus \{0\}} (\mathcal{D}_2(\rho, \delta) + \mathbf{n}) \subset \bigcup_{\mathbf{n} \in \Gamma^\dagger \setminus \{0\}} (\mathcal{B}(\rho, \delta) + \mathbf{n}).$$

CD2:eq (8.12)

Indeed, for each $\boldsymbol{\xi} \in \mathcal{D}_2(\rho, \delta)$ there are at least two distinct lattice vectors $\mathbf{n}_1, \mathbf{n}_2 \neq 0$ such that $\boldsymbol{\xi} + \mathbf{n}_1 \in \mathcal{B}(\rho, \delta)$ and $\boldsymbol{\xi} + \mathbf{n}_2 \in \mathcal{B}(\rho, \delta)$, so that any lattice vector $\mathbf{m} \neq 0$ is distinct either from \mathbf{n}_1 or from \mathbf{n}_2 . Thus, assuming for definiteness that $\mathbf{m} \neq \mathbf{n}_1$, we get

$$\boldsymbol{\xi} + \mathbf{m} = \boldsymbol{\xi} + \mathbf{n}_1 + (\mathbf{m} - \mathbf{n}_1) \in (\mathcal{B}(\rho, \delta) + \mathbf{m} - \mathbf{n}_1) \subset \bigcup_{\mathbf{n} \in \Gamma^\dagger \setminus \{0\}} (\mathcal{B}(\rho, \delta) + \mathbf{n}).$$

This proves [\(8.12\)](#).

Now observe that by definition of $\mathcal{D}_1(\rho, \delta)$ the sets $\mathcal{D}_1(\rho, \delta) \cap (\mathcal{B}(\rho, \delta) + \mathbf{n})$ are disjoint for different $\mathbf{n} \in \Gamma^\dagger \setminus \{0\}$. Therefore

$$\begin{aligned} \text{vol} \bigcup_{\mathbf{n} \in \Gamma^\dagger \setminus \{0\}} \left((\mathcal{D}_1(\rho, \delta) + \mathbf{n}) \cap \mathcal{B}(\rho, \delta) \right) &= \sum_{\mathbf{n} \in \Gamma^\dagger \setminus \{0\}} \text{vol} \left(\mathcal{D}_1(\rho, \delta) \cap (\mathcal{B}(\rho, \delta) + \mathbf{n}) \right) \\ &\leq \text{vol} \mathcal{D}_1(\rho, \delta) \leq \text{vol} \mathcal{D}(\rho, \delta). \end{aligned}$$

Together with [\(8.11\)](#) and [\(8.12\)](#) this produces the bound

$$\text{vol} \bigcup_{\mathbf{n} \in \Gamma^\dagger \setminus \{0\}} \left(\mathcal{B}(\rho, \delta) \cap (\mathcal{D}(\rho, \delta) + \mathbf{n}) \right) \leq \text{vol} \mathcal{D}(\rho, \delta) + \text{vol} \bigcup_{\mathbf{n} \in \Gamma^\dagger \setminus \{0\}} \left(\mathcal{B}(\rho, \delta) \cap (\mathcal{B}(\rho, \delta) + \mathbf{n}) \right).$$

The estimate [\(8.10\)](#) follows from [\(8.9\)](#) and [\(8.3\)](#). \square

The next section is devoted to the proof of Theorem [8.2](#) ^{volumeCB2a}.

9. ESTIMATES OF VOLUMES: PART TWO

9.1. Results and preliminary estimates for the intersection volume. Consider two continuous functions $g_j : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $g_j(\boldsymbol{\xi}) \rightarrow \infty$, $|\boldsymbol{\xi}| \rightarrow \infty$, $j = 1, 2$. Our objective is to establish upper bounds for the measure of the set

$$\mathcal{X}(g_1, g_2; \rho; \delta; \mathbf{b}_1, \mathbf{b}_2) := (\mathcal{A}(g_1; \rho; \delta) + \mathbf{b}_1) \cap (\mathcal{A}(g_2; \rho; \delta) + \mathbf{b}_2),$$

for arbitrary vectors $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^d$ such that $|\mathbf{b}_1 - \mathbf{b}_2| \gg 1$, and $\delta \in (0, \rho^{2m}/4]$. Clearly,

$$\text{vol}(\mathcal{X}(g_1, g_2; \rho; \delta; \mathbf{b}_1, \mathbf{b}_2)) = \text{vol}(\mathcal{X}(g_1, g_2; \rho; \delta; \mathbf{0}, \mathbf{b}_2 - \mathbf{b}_1)),$$

so that it suffices to study the set

$$\text{bigx:eq} \quad (9.1) \quad \mathcal{X} := \mathcal{X}(g_1, g_2; \rho; \delta; \mathbf{0}, \mathbf{b})$$

with some $\mathbf{b} \in \mathbb{R}^d$, $1 \ll |\mathbf{b}| \ll \rho$. Note that the condition $|\mathbf{b}| \ll \rho$ does not restrict generality, since for $|\mathbf{b}| \geq 3\rho$ the set \mathcal{X} is empty.

Let us make more precise assumptions about the functions g_1, g_2 . Suppose that

$$\text{gsmall:eq} \quad (9.2) \quad g_j(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{2m} + G_j(\boldsymbol{\xi}), \quad G_j \in \mathcal{C}^2(\mathbb{R}^d), \quad j = 1, 2.$$

Further conditions are imposed for the following range of values of $\boldsymbol{\xi}$:

$$\text{range:eq} \quad (9.3) \quad |\boldsymbol{\xi}| \asymp \rho, \quad |\boldsymbol{\xi} - \mathbf{b}| \asymp \rho.$$

The functions G_j 's are assumed to satisfy the following conditions:

$$\text{cond1} \quad (9.4) \quad |G_j(\boldsymbol{\xi})| \ll \rho^\gamma,$$

$$\text{cond2} \quad (9.5) \quad |\nabla G_j(\boldsymbol{\xi})| \ll \rho^\sigma,$$

$$\text{cond3} \quad (9.6) \quad |\nabla^2 G_j(\boldsymbol{\xi})| \ll \rho^\omega,$$

$$\text{cond5} \quad (9.7) \quad |G_1(\boldsymbol{\xi}) - G_2(\boldsymbol{\xi} - \mathbf{b})| \ll |\mathbf{b}|\rho^\sigma,$$

and

$$\text{cond4} \quad (9.8) \quad |\nabla G_1(\boldsymbol{\xi}) - \nabla G_2(\boldsymbol{\xi} - \mathbf{b})| \ll |\mathbf{b}|\rho^\omega,$$

for all \mathbf{b} , $1 \ll |\mathbf{b}| \ll \rho$, with some $\gamma < 2m$, $\sigma < 2m - 1$, $\omega < 2m - 2$, for all $\boldsymbol{\xi}$ satisfying [\(9.3\)](#) ^{range:eq}. The constants in these estimates are allowed to depend on the constant C in [\(9.3\)](#) ^{range:eq}.

Theorem 9.1. ^{gsmall:eq} *Let two functions g_1, g_2 be as in [\(9.2\)](#), and suppose that the conditions [\(9.4\)](#), [\(9.5\)](#), [\(9.6\)](#), [\(9.7\)](#) and [\(9.8\)](#) are satisfied. Then for any $\varepsilon > 0$, if $\delta \rho^{2-2m+2\varepsilon} \rightarrow 0$, $\rho \rightarrow \infty$, then*

$$\text{volumeCB2} \quad \text{nokiss1:eq} \quad (9.9) \quad \text{vol} \mathcal{X}(g_1, g_2; \rho; \delta; \mathbf{0}, \mathbf{b}) \ll \delta^2 \rho^{4-4m+d+6\varepsilon} + \delta \rho^{1-2m-\varepsilon(d-1)},$$

uniformly in \mathbf{b} , $1 \ll |\mathbf{b}| \ll \rho$.

Let us show how to derive Theorem [8.2](#) ^{volumeCB2a} from Theorem [9.1](#) ^{volumeCB2}:

Proof of Theorem 8.2. Extend the function g from the non-resonant set \mathcal{B} to the entire space \mathbb{R}^d by the formula (7.1), and denote the new function by g_1 . By (7.1) and (6.3), the functions g_1 and $g_2 = g_1$ satisfy the conditions (9.4), (9.5), (9.6), (9.7), (9.8) with $\gamma = \alpha\beta < 2m$, $\sigma = (\alpha - 1)\beta < 2m - 1$, $\omega = (\alpha - 2)\beta < 2m - 2$ for $\boldsymbol{\xi}$ in the range (9.3).

Then, clearly,

$$\left(\mathcal{B}(g; \rho; \delta) \cap (\mathcal{B}(g; \rho; \delta) + \mathbf{b})\right) \subset \left(\mathcal{A}(g_1; \rho; \delta) \cap (\mathcal{A}(g_1; \rho; \delta) + \mathbf{b})\right).$$

It remains to use Theorem 9.1 with $g_1 = g_2$. \square

Let us concentrate on proving Theorem 9.1. Our first observation is that it suffices to do it for $m = 1$. Indeed, introducing functions

$$\check{g}_j(\boldsymbol{\xi}) = (g_j(\boldsymbol{\xi}))^{\frac{1}{m}}, \check{G}_j(\boldsymbol{\xi}) = \check{g}_j(\boldsymbol{\xi}) - |\boldsymbol{\xi}|^2,$$

we note that under the condition (9.3) the functions \check{G}_j satisfy the bounds (9.4)-(9.8) with the parameters

$$\begin{aligned} \check{\gamma} &= \gamma + 2 - 2m < 2, \quad \check{\sigma} = \max(\sigma + 2 - 2m, \gamma + 1 - 2m) < 1, \\ \check{\omega} &= \max(\omega + 2 - 2m, \sigma + 1 - 2m, \gamma - 2m) < 0. \end{aligned}$$

One checks directly that

$$\mathcal{X}(g_1, g_2; \rho, \delta; \mathbf{0}, \mathbf{b}) \subset \mathcal{X}(\check{g}_1, \check{g}_2; \rho, \check{\delta}; \mathbf{0}, \mathbf{b}), \quad \check{\delta} = 2m^{-1}\delta\rho^{2-2m},$$

for sufficiently large ρ . Moreover, the condition $\delta\rho^{2-2m+2\epsilon} \rightarrow 0$ becomes $\check{\delta}\rho^{2\epsilon} \rightarrow 0$.

Thus, from now on until the end of this section we assume that $m = 1$.

Due to the condition $\delta\rho^{2\epsilon} \rightarrow 0$, we may assume that $\delta \in (0, 1]$. Now, in view of (9.4) we have

$$(9.10) \quad c_1\rho \leq |\boldsymbol{\xi}| \leq C_1\rho, \quad c_1\rho \leq |\boldsymbol{\xi} - \mathbf{b}| \leq C_1\rho.$$

Here, the constants $c_1, C_1, c_1 < 1 < C_1$ can be chosen arbitrarily close to 1, assuming that ρ is sufficiently large.

Below, we denote by $\phi(\boldsymbol{\xi}, \boldsymbol{\eta}) \in [0, \pi]$ the angle between arbitrary non-zero vectors $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d$. A central role in the study of the set \mathcal{X} (see (9.1)) is played by the angle $\phi(\boldsymbol{\mu}, \boldsymbol{\mu} - \mathbf{b})$ for the points $\boldsymbol{\mu} \in \mathcal{X}$. Let us establish some general facts about this angle. It is convenient to introduce new orthogonal coordinates in \mathbb{R}^d in the following way: $\boldsymbol{\xi} = (\xi_1, \hat{\boldsymbol{\xi}})$ with $\xi_1 = \boldsymbol{\xi} \cdot \mathbf{e}(\mathbf{b})$ and $\hat{\boldsymbol{\xi}} = \boldsymbol{\xi}_{\mathbf{b}^\perp}$, so that $\boldsymbol{\xi} = \xi_1\mathbf{e}(\mathbf{b}) + \hat{\boldsymbol{\xi}}$.

Lemma 9.2. *Let $m = 1$, and suppose that the functions $G_j, j = 1, 2$ satisfy conditions (9.4), (9.7), and that $\delta \in (0, 1]$, $1 \ll |\mathbf{b}| \ll \rho$. Then $\phi(\boldsymbol{\xi}, \boldsymbol{\xi} - \mathbf{b}) \gg |\mathbf{b}|\rho^{-1}$ uniformly in $\boldsymbol{\xi} \in \mathcal{X}$.*

Proof. Denote for brevity $\phi = \phi(\boldsymbol{\xi}, \boldsymbol{\xi} - \mathbf{b})$ and $\mathcal{X} = \mathcal{X}(\rho; \delta, g_1, g_2; \mathbf{0}, \mathbf{b})$. For our purposes we assume that the constants c_1, C_1 in (9.10) satisfy the bound

$$(9.11) \quad 5c_1^2/4 \geq C_1^2.$$

We consider separately two cases: $|\xi_1| < (\sqrt{3}/2)c_1\rho$ and $|\xi_1| \geq (\sqrt{3}/2)c_1\rho$.

Case 1: $|\xi_1| < \sqrt{3}/2c_1\rho$. Denote $\tilde{\phi} = \phi(\mathbf{b}, \boldsymbol{\xi})$. Since $|\boldsymbol{\xi}| \geq c_1\rho$ and $|\xi_1| < \sqrt{3}/2c_1\rho$, we have $\tilde{\phi} \in (\pi/6, 5\pi/6)$, and hence $\sin \tilde{\phi} > 1/2$. By the sine rule

$$\frac{\sin \phi}{|\mathbf{b}|} = \frac{\sin \tilde{\phi}}{|\boldsymbol{\xi} - \mathbf{b}|},$$

which implies that

$$\sin \phi = \frac{|\mathbf{b}|}{|\boldsymbol{\xi} - \mathbf{b}|} \sin \tilde{\phi} \geq C_1^{-1} \frac{|\mathbf{b}|}{2\rho}.$$

Thus $\phi \gg |\mathbf{b}|\rho^{-1}$, as claimed.

Case 2: $|\xi_1| \geq (\sqrt{3}/2)c_1\rho$. Let us show first that

$$\boxed{\text{new*}} \quad (9.12) \quad \xi_1 > 0$$

and

$$\boxed{\text{new**}} \quad (9.13) \quad \xi_1 - |\mathbf{b}| < 0.$$

Assume, on the contrary, that $\xi_1 \leq 0$, so that

$$|\boldsymbol{\xi} - \mathbf{b}|^2 - |\boldsymbol{\xi}|^2 = (\xi_1 - |\mathbf{b}|)^2 - \xi_1^2 \geq -2\xi_1|\mathbf{b}|.$$

Then

$$g_2(\boldsymbol{\xi} - \mathbf{b}) - g_1(\boldsymbol{\xi}) \geq -2\xi_1|\mathbf{b}| - |G_1(\boldsymbol{\xi}) - G_2(\boldsymbol{\xi} - \mathbf{b})|$$

By condition $\frac{\text{cond5}}{(9.7)}$,

$$|G_1(\boldsymbol{\xi}) - G_2(\boldsymbol{\xi} - \mathbf{b})| \ll |\mathbf{b}|\rho^\sigma.$$

Together with the assumption $|\xi_1| \geq (\sqrt{3}/2)c_1\rho$ this implies that

$$g_2(\boldsymbol{\xi} - \mathbf{b}) - g_1(\boldsymbol{\xi}) \gg \rho|\mathbf{b}|.$$

This contradicts the condition $g_2(\boldsymbol{\xi} - \mathbf{b}) - g_1(\boldsymbol{\xi}) < 2\delta$, and hence $\frac{\text{new*}}{(9.12)}$ is satisfied.

Assume now that $\xi_1 - |\mathbf{b}| \geq 0$. Then, similarly to the above argument,

$$|\boldsymbol{\xi}|^2 - |\boldsymbol{\xi} - \mathbf{b}|^2 = \xi_1^2 - (\xi_1 - |\mathbf{b}|)^2 = 2\xi_1|\mathbf{b}| - |\mathbf{b}|^2 \geq \xi_1|\mathbf{b}|.$$

Thus, as above,

$$g_1(\boldsymbol{\xi}) - g_2(\boldsymbol{\xi} - \mathbf{b}) \geq \xi_1|\mathbf{b}| - |G_1(\boldsymbol{\xi}) - G_2(\boldsymbol{\xi} - \mathbf{b})|$$

By condition $\frac{\text{cond5}}{(9.7)}$,

$$|G_1(\boldsymbol{\xi}) - G_2(\boldsymbol{\xi} - \mathbf{b})| \ll |\mathbf{b}|\rho^\sigma.$$

Together with the assumption $|\xi_1| \geq (\sqrt{3}/2)c_1\rho$ this implies that

$$g_1(\boldsymbol{\xi}) - g_2(\boldsymbol{\xi} - \mathbf{b}) \gg \rho|\mathbf{b}|.$$

This contradicts the condition $g_1(\boldsymbol{\xi}) - g_2(\boldsymbol{\xi} - \mathbf{b}) < 2\delta$, and hence $\frac{\text{new**}}{(9.13)}$ is satisfied.

The next step is to show that $|\mathbf{b}| \gg \rho$. Indeed, it follows from $\frac{\text{inter:eq}}{(9.10)}$ that

$$|\hat{\boldsymbol{\xi}}|^2 = |\boldsymbol{\xi}|^2 - \xi_1^2 \leq \left(C_1^2 - \frac{3}{4}c_1^2\right)\rho^2,$$

which implies that

$$\begin{aligned} |\mathbf{b}|^2 &= |\boldsymbol{\xi}|^2 + |\boldsymbol{\xi} - \mathbf{b}|^2 - 2\boldsymbol{\xi} \cdot (\boldsymbol{\xi} - \mathbf{b}) \geq |\boldsymbol{\xi}|^2 + |\boldsymbol{\xi} - \mathbf{b}|^2 - 2|\hat{\boldsymbol{\xi}}|^2 \\ &\geq 2\left(c_1^2 - C_1^2 + \frac{3}{4}c_1^2\right)\rho^2 = 2\left(\frac{7}{4}c_1^2 - C_1^2\right)\rho^2 \geq c_1^2\rho^2. \end{aligned}$$

Here we have used [\(9.11\)](#) as well as [\(9.12\)](#) and [\(9.13\)](#). On the other hand,

$$\begin{aligned} |\mathbf{b}|^2 &= |\boldsymbol{\xi}|^2 + |\boldsymbol{\xi} - \mathbf{b}|^2 - 2\cos\phi|\boldsymbol{\xi}||\boldsymbol{\xi} - \mathbf{b}| \\ &= (|\boldsymbol{\xi}| - |\boldsymbol{\xi} - \mathbf{b}|)^2 + 2(1 - \cos\phi)|\boldsymbol{\xi}||\boldsymbol{\xi} - \mathbf{b}| \\ &\leq (C_1 - c_1)^2\rho^2 + 4C_1^2\sin^2\left(\frac{\phi}{2}\right)\rho^2. \end{aligned}$$

Using this, together with the lower bound $|\mathbf{b}| \geq c_1\rho$, we arrive at

$$4C_1^2\sin^2\left(\frac{\phi}{2}\right) \geq c_1^2 - (C_1 - c_1)^2 = C_1(2c_1 - C_1) \geq \frac{1}{2}c_1C_1.$$

This means that $\sin(\phi/2) \gg 1 \gg |\mathbf{b}|\rho^{-1}$, which means that $\phi \gg |\mathbf{b}|\rho^{-1}$, as claimed.

The proof of the Lemma is complete. \square

The next result is proved for those $\boldsymbol{\mu} \in \mathcal{X}$ which satisfy the relations

$$\nabla G_1(\boldsymbol{\mu}) = \nabla G_2(\boldsymbol{\mu} - \mathbf{b}) = 0. \tag{9.14}$$

Lemma 9.3. *Let $m = 1$, and let the functions $G_1, G_2 \in \mathcal{C}^2(\mathbb{R}^d)$ satisfy the conditions [\(9.4\)](#), [\(9.6\)](#), and let $\delta \in (0, 1]$, $1 \ll |\mathbf{b}| \ll \rho$. Suppose that there exists a point $\boldsymbol{\mu} \in \mathcal{X}$ such that $\pi - \phi(\boldsymbol{\mu}, \boldsymbol{\mu} - \mathbf{b}) \leq l\rho^{-1}$ with $0 < l \leq 1$, and [\(9.14\)](#) is satisfied. Then under the condition $\delta l^{-2} \rightarrow 0$, $\rho \rightarrow \infty$, we have*

1. $\mathcal{X} \subset \{\boldsymbol{\xi} \in \mathbb{R}^d : |\hat{\boldsymbol{\xi}}| < 4C_1^2c_1^{-1}l\}$, and
2. $\text{vol}\mathcal{X} \ll \delta l^{(d-1)}\rho^{-1}$.

Proof. First note some useful inequalities for $\boldsymbol{\mu}, \mathbf{b}$. Denote $\phi_0 := \phi(\boldsymbol{\mu}, \boldsymbol{\mu} - \mathbf{b})$, $\phi_1 := \phi(\boldsymbol{\mu}, \mathbf{b})$. Since $\pi - \phi_0 \leq l\rho^{-1}$, we have $\phi_0 \geq \pi/2$, so that $\cos\phi_0 < 0$ and $\cos\phi_1 > 0$. Recalling [\(9.10\)](#), we conclude that

$$|\mathbf{b}|^2 = |\boldsymbol{\mu}|^2 + |\boldsymbol{\mu} - \mathbf{b}|^2 - 2\cos\phi_0|\boldsymbol{\mu}||\boldsymbol{\mu} - \mathbf{b}| \geq 2c_1^2\rho^2.$$

This also gives $|\boldsymbol{\mu}| < |\mathbf{b}|$ and $|\boldsymbol{\mu} - \mathbf{b}| < |\mathbf{b}|$. Furthermore, it follows from the sine rule that

$$\sin\phi_1 = \sin\phi_0 \frac{|\boldsymbol{\mu} - \mathbf{b}|}{|\mathbf{b}|} \leq \frac{lC_1\rho}{\rho c_1\rho} = \frac{C_1 l}{c_1 \rho}.$$

This leads to the bounds

$$|\hat{\boldsymbol{\mu}}| = |\boldsymbol{\mu}|\sin\phi_1 \leq c_2l, \quad c_2 = C_1^2c_1^{-1}, \quad \mu_1 = |\boldsymbol{\mu}|\cos\phi_1 > 0.$$

Similarly, by considering $\phi_2 := \phi(\boldsymbol{\mu} - \mathbf{b}, \mathbf{b})$ we can prove that $|\mathbf{b}| - \mu_1 > 0$.

Now let $\boldsymbol{\xi}$ be an arbitrary element of \mathcal{X} and let us prove that $|\hat{\boldsymbol{\xi}}| < 4c_2l$. Suppose that, on the contrary, $|\hat{\boldsymbol{\xi}}| \geq 4c_2l$, and let $\boldsymbol{\eta} = \boldsymbol{\xi} - \boldsymbol{\mu}$. Clearly, $|\hat{\boldsymbol{\eta}}| \geq 3c_2l \geq 3|\hat{\boldsymbol{\mu}}|$, and hence

$$\boxed{\text{newtilde}} \quad (9.15) \quad |\hat{\boldsymbol{\xi}}|^2 - |\hat{\boldsymbol{\mu}}|^2 = |\hat{\boldsymbol{\eta}}|^2 + 2\hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{\eta}} \geq |\hat{\boldsymbol{\eta}}|(|\hat{\boldsymbol{\eta}}| - 2|\hat{\boldsymbol{\mu}}|) \geq \frac{|\hat{\boldsymbol{\eta}}|^2}{3}.$$

Let us now assume that $\eta_1 \geq 0$. Then, combining $\boxed{\text{newtilde}}$ (9.15) with the identity

$$\boxed{+:\text{eq}} \quad (9.16) \quad \xi_1^2 - \mu_1^2 = \eta_1^2 + 2\mu_1\eta_1,$$

we obtain

$$|\boldsymbol{\xi}|^2 - |\boldsymbol{\mu}|^2 \geq \frac{1}{3}|\boldsymbol{\eta}|^2.$$

At the same time, due to $\boxed{\text{nulli:eq}}$ (9.14) and $\boxed{\text{cond3}}$ (9.6),

$$|G_1(\boldsymbol{\xi}) - G_1(\boldsymbol{\mu})| \ll |\boldsymbol{\eta}|^2 \rho^\omega,$$

so that

$$g_1(\boldsymbol{\xi}) - g_1(\boldsymbol{\mu}) \geq |\boldsymbol{\xi}|^2 - |\boldsymbol{\mu}|^2 - C|\boldsymbol{\eta}|^2 \rho^\omega \geq \left(\frac{1}{3} - C\rho^\omega\right)|\boldsymbol{\eta}|^2 \gg l^2,$$

and hence $g_1(\boldsymbol{\xi}) \geq g_1(\boldsymbol{\mu}) + Cl^2 \geq \rho^2 - \delta + Cl^2$. Since $\delta l^{-2} \rightarrow 0$ as $\rho \rightarrow \infty$, it follows that $g_1(\boldsymbol{\xi}) > \rho^2 + \delta$ for large ρ . This means that $\boldsymbol{\xi} \notin \mathcal{A}(g_1; \rho; \delta)$, so that, by contradiction, $|\hat{\boldsymbol{\xi}}| < 4c_2l$.

Consider now the case $\eta_1 < 0$. Then instead of $\boxed{+:\text{eq}}$ (9.16) we use

$$|\xi_1 - |\mathbf{b}||^2 - |\mu_1 - |\mathbf{b}||^2 = |\eta_1|^2 + 2(\mu_1 - |\mathbf{b}|)\eta_1.$$

Since $\mu_1 - |\mathbf{b}| < 0$, combining this with $\boxed{\text{newtilde}}$ (9.15), we obtain

$$|\boldsymbol{\xi} - \mathbf{b}|^2 - |\boldsymbol{\mu} - \mathbf{b}|^2 \geq \frac{1}{3}|\boldsymbol{\eta}|^2.$$

At the same time, due to $\boxed{\text{nulli:eq}}$ (9.14) and $\boxed{\text{cond3}}$ (9.6),

$$|G_2(\boldsymbol{\xi} - \mathbf{b}) - G_2(\boldsymbol{\mu} - \mathbf{b})| \ll |\boldsymbol{\eta}|^2 \rho^\omega,$$

so that

$$g_2(\boldsymbol{\xi} - \mathbf{b}) - g_2(\boldsymbol{\mu} - \mathbf{b}) \geq |\boldsymbol{\xi} - \mathbf{b}|^2 - |\boldsymbol{\mu} - \mathbf{b}|^2 - C|\boldsymbol{\eta}|^2 \rho^\omega \gg l^2,$$

and hence $g_2(\boldsymbol{\xi} - \mathbf{b}) \geq g_2(\boldsymbol{\mu} - \mathbf{b}) + Cl^2 \geq \rho^2 - \delta + Cl^2$. Since $\delta l^{-2} \rightarrow 0$ as $\rho \rightarrow \infty$, it follows that $g_2(\boldsymbol{\xi} - \mathbf{b}) > \rho^2 + \delta$ for large ρ . This means that $\boldsymbol{\xi} \notin (\mathcal{A}(g_2; \rho; \delta) + \mathbf{b})$, so that, by contradiction, $|\hat{\boldsymbol{\xi}}| < 4c_2l$. This completes the proof of Part 1.

Let us fix $\hat{\boldsymbol{\xi}} : |\hat{\boldsymbol{\xi}}| < c_1\rho/2$. For all $\boldsymbol{\xi} = (\xi_1, \hat{\boldsymbol{\xi}})$, $|\boldsymbol{\xi}| > c_1\rho$ we have

$$\xi_1^2 = |\boldsymbol{\xi}|^2 - |\hat{\boldsymbol{\xi}}|^2 \geq c_1^2\rho^2 - \frac{1}{4}c_1^2\rho^2 = \frac{3}{4}c_1^2\rho^2.$$

By conditions $\boxed{\text{cond3}}$ (9.6) and $\boxed{\text{nulli:eq}}$ (9.14),

$$|\nabla G_1(\boldsymbol{\xi})| = |\nabla G_1(\boldsymbol{\xi}) - \nabla G_1(\boldsymbol{\mu})| \ll |\boldsymbol{\eta}| \rho^\omega \ll \rho^{1+\omega},$$

so that

$$|\partial_{\xi_1} g_1(\xi_1, \hat{\boldsymbol{\xi}})| = |2\xi_1 + \partial_{\xi_1} G_1(\xi_1, \hat{\boldsymbol{\xi}})| \gg \rho.$$

In particular, the function $g(\cdot, \hat{\boldsymbol{\xi}})$ is strictly monotone, and the set

$$I_{\hat{\boldsymbol{\xi}}} = \{\xi_1 : |g_1(\xi_1, \hat{\boldsymbol{\xi}}) - \rho^2| \leq \delta\}$$

is a closed interval of length $|I_{\hat{\boldsymbol{\xi}}}| \ll \delta \rho^{-1}$. By Part 1,

$$\text{vol}(\mathcal{X}(g_1, g_2; \rho, \delta, \mathbf{b})) \leq \int_{|\hat{\boldsymbol{\xi}}| < 4c_2 l} |I_{\hat{\boldsymbol{\xi}}}| d\hat{\boldsymbol{\xi}} \ll \delta l^{d-1} \rho^{-1},$$

as claimed. \square

Let us now consider the case when $\phi_0 = \phi(\boldsymbol{\mu}, \boldsymbol{\mu} - \mathbf{b})$ is separated away from π . The following elementary observation will be useful:

lem:angle1

Lemma 9.4. *Let $\mathbf{n}_1, \mathbf{n}_2 \in \mathbb{R}^2$ be two unit vectors. Then for any other unit vector $\mathbf{n} \in \mathbb{R}^2$ one has*

$$|\mathbf{n} \cdot \mathbf{n}_1|^2 + |\mathbf{n} \cdot \mathbf{n}_2|^2 \geq 1 - |\mathbf{n}_1 \cdot \mathbf{n}_2|.$$

Proof. The result follows from the following elementary trigonometric calculation for arbitrary $\psi, \phi \in \mathbb{R}$:

$$\begin{aligned} \cos^2 \psi + \cos^2(\psi - \phi) &= 1 + \frac{1}{2} \left(\cos(2\psi) + \cos(2(\psi - \phi)) \right) \\ &= 1 + \cos(2\psi - \phi) \cos \phi \geq 1 - |\cos \phi|. \end{aligned}$$

\square

volumeCB2.2

Lemma 9.5. *Let $m = 1$ and $\delta \in (0, 1]$. $\frac{1}{l} \ll \frac{|\mathbf{b}|}{\rho} \ll \frac{\rho}{l}$. Assume that two functions $G_1, G_2 \in \mathcal{C}^2(\mathbb{R}^d)$ satisfy the conditions $\frac{\text{cond1}}{(9.4)}$, $\frac{\text{cond3}}{(9.6)}$, $\frac{\text{cond5}}{(9.7)}$. Suppose that there exists a point $\boldsymbol{\mu} \in \mathcal{X}$ such that $\frac{\text{multi:eg}}{(9.14)}$ holds and $\pi - \phi_0 \geq l\rho^{-1}$, $\phi_0 := \phi(\boldsymbol{\mu}, \boldsymbol{\mu} - \mathbf{b})$, with some $0 < l \leq 1$. Then for any $\varepsilon_2 > 0$*

area1

$$(9.17) \quad \text{vol}(\mathcal{X} \cap B(\boldsymbol{\mu}, l\rho^{-\varepsilon_2})) \ll l^{d-2} \rho^{-2-\varepsilon_2(d-2)} \frac{\delta^2}{\sin^2 \phi_0}.$$

Proof. First of all, notice that assumptions of this Lemma together with Lemma $\frac{\text{polozh:lem}}{9.2}$ imply

newhat

$$(9.18) \quad \sin \phi_0 \gg l\rho^{-1}.$$

Let $\boldsymbol{\xi} \in B(\boldsymbol{\mu}, l\rho^{-\varepsilon_2})$, so that $|\boldsymbol{\xi}| \gg \rho$. Due to $\frac{\text{multi:eg}}{(9.14)}$ and $\frac{\text{cond3}}{(9.6)}$ we have

$$\begin{aligned} |\nabla G_1(\boldsymbol{\xi})| &= |\nabla G_1(\boldsymbol{\xi}) - \nabla G_1(\boldsymbol{\mu})| \ll l\rho^{\omega-\varepsilon_2}, \\ |\nabla G_2(\boldsymbol{\xi} - \mathbf{b})| &= |\nabla G_2(\boldsymbol{\xi} - \mathbf{b}) - \nabla G_2(\boldsymbol{\mu} - \mathbf{b})| \ll l\rho^{\omega-\varepsilon_2}, \end{aligned}$$

for all $\boldsymbol{\xi} \in B(\boldsymbol{\mu}, l\rho^{-\varepsilon_2})$, and hence, by elementary trigonometric argument, we have the following upper bounds:

$$(9.19) \quad \begin{cases} \phi(\boldsymbol{\xi}, \boldsymbol{\mu}) = O(l\rho^{-1-\varepsilon_2}), \\ \phi(\boldsymbol{\xi} - \mathbf{b}, \boldsymbol{\mu} - \mathbf{b}) = O(l\rho^{-1-\varepsilon_2}), \\ \phi(\boldsymbol{\xi}, \nabla g_1(\boldsymbol{\xi})) = \phi(\boldsymbol{\xi}, 2\boldsymbol{\xi} + \nabla G_1(\boldsymbol{\xi})) = O(l\rho^{-1+\omega-\varepsilon_2}), \\ \phi(\boldsymbol{\xi} - \mathbf{b}, \nabla g_2(\boldsymbol{\xi} - \mathbf{b})) = \phi(\boldsymbol{\xi} - \mathbf{b}, 2(\boldsymbol{\xi} - \mathbf{b}) + \nabla G_2(\boldsymbol{\xi} - \mathbf{b})) = O(l\rho^{-1+\omega-\varepsilon_2}). \end{cases}$$

Since $\nabla g_1(\boldsymbol{\mu}) = 2\boldsymbol{\mu}$, $\nabla g_2(\boldsymbol{\mu} - \mathbf{b}) = 2(\boldsymbol{\mu} - \mathbf{b})$ and $\omega < 0$, it follows that

$$\phi(\nabla g_1(\boldsymbol{\mu}), \nabla g_1(\boldsymbol{\xi})) = O(l\rho^{-1-\varepsilon_2}), \quad \phi(\nabla g_2(\boldsymbol{\mu} - \mathbf{b}), \nabla g_2(\boldsymbol{\xi} - \mathbf{b})) = O(l\rho^{-1-\varepsilon_2}).$$

The above bounds imply that

$$|\phi(\boldsymbol{\xi}, \boldsymbol{\xi} - \mathbf{b}) - \phi_0| \ll l\rho^{-1-\varepsilon_2}, \quad \forall \boldsymbol{\xi} \in B(\boldsymbol{\mu}, l\rho^{-\varepsilon_2}).$$

Since $l \leq 1$, together with [\(9.18\)](#) ^{newhat} this means that

$$\sin \phi(\boldsymbol{\xi}, \boldsymbol{\xi} - \mathbf{b}) \gg \sin \phi_0 \gg l\rho^{-1}.$$

Thus, the vectors $\boldsymbol{\xi}$, \mathbf{b} span a two-dimensional space. From now on we represent every vector $\boldsymbol{\xi} \in B(\boldsymbol{\mu}, l\rho^{-\varepsilon_2})$ as follows: $\boldsymbol{\xi} = (z, \Theta_1, \hat{\boldsymbol{\Theta}})$, where $z = |\boldsymbol{\xi}|$, $\Theta_1 \in [0, \pi]$ is the angle between $\boldsymbol{\xi}$ and \mathbf{b} , and $\hat{\boldsymbol{\Theta}} = \hat{\boldsymbol{\xi}}|\hat{\boldsymbol{\xi}}|^{-1} \in \mathbb{S}^{d-2}$. We denote the plane spanned by $\boldsymbol{\xi}$ and \mathbf{b} by $\mathfrak{V}_{\hat{\boldsymbol{\Theta}}}$.

Let $\boldsymbol{\xi} \in \mathfrak{V}_{\hat{\boldsymbol{\Theta}}}$ with some $\hat{\boldsymbol{\Theta}} \in \mathbb{S}^{d-2}$. By Lemma [9.4](#) ^{lem:angle1}, for any unit vector $\mathbf{e} \in \mathfrak{V}_{\hat{\boldsymbol{\Theta}}}$ we have

$$|\mathbf{e}(\boldsymbol{\xi}) \cdot \mathbf{e}|^2 + |\mathbf{e}(\boldsymbol{\xi} - \mathbf{b}) \cdot \mathbf{e}|^2 \geq 1 - |\cos \phi(\boldsymbol{\xi}, \boldsymbol{\xi} - \mathbf{b})| \geq \frac{1}{2} \sin^2 \phi(\boldsymbol{\xi}, \boldsymbol{\xi} - \mathbf{b}) \gg \sin^2 \phi_0,$$

which implies that at least one of the following estimates hold:

$$(i) \quad |\mathbf{e} \cdot \mathbf{e}(\boldsymbol{\xi})| \gg \sin \phi_0,$$

or

$$(ii) \quad |\mathbf{e} \cdot \mathbf{e}(\boldsymbol{\xi} - \mathbf{b})| \gg \sin \phi_0.$$

Since $\sin \phi_0 \gg l\rho^{-1}$, in view of [\(9.19\)](#) ^{eq:angle2}, we also have that at least one of the following estimates hold for all $\boldsymbol{\xi} \in B(\boldsymbol{\mu}, l\rho^{-\varepsilon_2}) \cap \mathfrak{V}_{\hat{\boldsymbol{\Theta}}}$:

$$\begin{cases} |\mathbf{e} \cdot \mathbf{e}(\nabla g_1(\boldsymbol{\xi}))| \gg \sin \phi_0, & \text{for case (i);} \\ |\mathbf{e} \cdot \mathbf{e}(\nabla g_2(\boldsymbol{\xi} - \mathbf{b}))| \gg \sin \phi_0, & \text{for case (ii).} \end{cases}$$

Let us fix another vector $\boldsymbol{\eta} \in B(\boldsymbol{\mu}, l\rho^{-\varepsilon_2}) \cap \mathfrak{V}_{\hat{\boldsymbol{\Theta}}}$ and use (i) or (ii) for

$$\mathbf{e} = \mathbf{e}(\boldsymbol{\eta} - \boldsymbol{\xi}) = \mathbf{e}((\boldsymbol{\eta} - \mathbf{b}) - (\boldsymbol{\xi} - \mathbf{b})).$$

If the condition (i) holds, then

$$(9.20) \quad |\mathbf{e}(\boldsymbol{\eta}) - \mathbf{e}(\boldsymbol{\xi})| \gg |\boldsymbol{\eta} - \boldsymbol{\xi}| \inf_{\boldsymbol{\chi}} |\mathbf{e} \cdot \nabla g_1(\boldsymbol{\chi})| \gg |\boldsymbol{\eta} - \boldsymbol{\xi}| \rho \sin \phi_0,$$

where the infimum is taken over $\boldsymbol{\chi} \in B(\boldsymbol{\mu}, l\rho^{-\varepsilon_2}) \cap \mathfrak{V}_{\hat{\Theta}}$. Here we have used that $|\nabla g_1(\boldsymbol{\chi})| \gg \rho$. Analogously, if the condition (ii) holds, then

$$\boxed{\text{cond(ii)}} \quad (9.21) \quad |g_2(\boldsymbol{\eta} - \mathbf{b}) - g_2(\boldsymbol{\xi} - \mathbf{b})| \gg |\boldsymbol{\eta} - \boldsymbol{\xi}| \rho \sin \phi_0.$$

Suppose in addition that $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathcal{X} \cap B(\boldsymbol{\mu}, l\rho^{-\varepsilon_2})$. Then, if condition (i) holds, by definition of \mathcal{X} we get from (9.20):

$$|\boldsymbol{\eta} - \boldsymbol{\xi}| \ll |g_1(\boldsymbol{\eta}) - g_1(\boldsymbol{\xi})| \frac{1}{\rho \sin \phi_0} \ll \delta \frac{1}{\rho \sin \phi_0}.$$

Similarly, if condition (ii) holds, then (9.21) implies

$$|\boldsymbol{\eta} - \boldsymbol{\xi}| \ll |g_2(\boldsymbol{\eta} - \mathbf{b}) - g_2(\boldsymbol{\xi} - \mathbf{b})| \frac{1}{\rho \sin \phi_0} \ll \delta \frac{1}{\rho \sin \phi_0}.$$

Therefore, for any $\boldsymbol{\xi} \in \mathcal{X} \cap B(\boldsymbol{\mu}, l\rho^{-\varepsilon_2}) \cap \mathfrak{V}_{\hat{\Theta}}$ we have $\mathcal{X} \cap B(\boldsymbol{\mu}, l\rho^{-\varepsilon_2}) \cap \mathfrak{V}_{\hat{\Theta}} \subset B(\boldsymbol{\xi}, C\delta(\rho \sin \phi_0)^{-1})$, and hence

$$\boxed{2\text{dimarea:eq}} \quad (9.22) \quad \text{vol}_2(\mathcal{X} \cap B(\boldsymbol{\mu}, l\rho^{-\varepsilon_2}) \cap \mathfrak{V}_{\hat{\Theta}}) \ll \delta^2 \frac{1}{\rho^2 \sin^2 \phi_0},$$

where vol_2 denotes the area on the plane $\mathfrak{V}_{\hat{\Theta}}$.

Integrating in the coordinates $(z, \Theta_1, \hat{\Theta})$, introduced previously, we can estimate:

$$\text{vol}(\mathcal{X} \cap B(\boldsymbol{\mu}, l\rho^{-\varepsilon_2})) \ll l^{d-2} \rho^{-\varepsilon_2(d-2)} \sup_{\hat{\Theta}} \text{vol}_2(\mathcal{X} \cap B(\boldsymbol{\mu}, l\rho^{-\varepsilon_2}) \cap \mathfrak{V}_{\hat{\Theta}}),$$

which completes the proof upon using (9.22). \square

9.2. Proof of Theorem 9.1 for $m = 1$. In the previous Lemmas the volume of \mathcal{X} was estimated under the assumptions that (9.14) holds. Although this condition cannot be expected to hold for arbitrary functions g_1, g_2 , one can always satisfy (9.14) locally, by "adjusting" the functions g_1, g_2 appropriately. Afterwards, one can use the above Lemmas. This strategy is implemented in the proof of Theorem 9.1.

Proof of Theorem 9.1 for $m = 1$. Pick a vector $\boldsymbol{\mu} \in \mathcal{X}$, and denote $\mathbf{v}_1 := \frac{1}{2}(\nabla G_1(\boldsymbol{\mu}))$ and $\mathbf{v}_2 := \frac{1}{2}(\nabla G_2(\boldsymbol{\mu} - \mathbf{b}))$. In view of (9.8),

$$\boxed{\text{v1v2:eq}} \quad (9.23) \quad |\mathbf{v}_1 - \mathbf{v}_2| \ll |\mathbf{b}| \rho^\omega.$$

Define $\tilde{g}_j(\boldsymbol{\xi}) := g_j(\boldsymbol{\xi} - \mathbf{v}_j)$ and $\tilde{G}_j(\boldsymbol{\xi}) := \tilde{g}_j(\boldsymbol{\xi}) - |\boldsymbol{\xi}|^2$, so that

$$\begin{aligned} \tilde{G}_j(\boldsymbol{\xi}) &= |\boldsymbol{\xi} - \mathbf{v}_j|^2 - |\boldsymbol{\xi}|^2 + G_j(\boldsymbol{\xi} - \mathbf{v}_j) = -2\boldsymbol{\xi} \cdot \mathbf{v}_j + |\mathbf{v}_j|^2 + G_j(\boldsymbol{\xi} - \mathbf{v}_j), \\ \nabla \tilde{G}_j(\boldsymbol{\xi}) &= -2\mathbf{v}_j + \nabla G_j(\boldsymbol{\xi} - \mathbf{v}_j). \end{aligned}$$

It is easily checked that the functions \tilde{G}_j satisfy the conditions (9.4), (9.5) and (9.6) with new parameters $\tilde{\gamma} = \max(\gamma, 1 + \sigma)$, $\tilde{\sigma} = \max(1 + \omega, \sigma)$, $\tilde{\omega} = \omega$. We also introduce

$$\mathbf{a} := \mathbf{b} + \mathbf{v}_1 - \mathbf{v}_2$$

and notice that in view of (9.23) , we have $|\mathbf{a}| \asymp |\mathbf{b}|$. Now, writing

$$\tilde{G}_1(\boldsymbol{\xi}) - \tilde{G}_2(\boldsymbol{\xi} - \mathbf{a}) = -2\boldsymbol{\xi} \cdot (\mathbf{v}_1 - \mathbf{v}_2) - 2\mathbf{a} \cdot \mathbf{v}_2 + |\mathbf{v}_1|^2 - |\mathbf{v}_2|^2 + G_1(\boldsymbol{\xi} - \mathbf{v}_1) - G_2(\boldsymbol{\xi} - \mathbf{v}_1 - \mathbf{b}),$$

and using (9.7) and (9.8) , (9.23) , we make the following estimate:

$$|\tilde{G}_1(\boldsymbol{\xi}) - \tilde{G}_2(\boldsymbol{\xi} - \mathbf{a})| \ll |\mathbf{b}|\rho^{1+\omega} + |\mathbf{a}|\rho^\sigma + |\mathbf{b}|\rho^{\sigma+\omega} + |\mathbf{b}|\rho^\sigma \ll |\mathbf{a}|\rho^{\tilde{\sigma}}.$$

Thus, the condition (9.7) is also satisfied with \mathbf{b} replaced by \mathbf{a} . Moreover, by definition of \tilde{G}_j ,

$$\nabla \tilde{G}_1(\boldsymbol{\nu}) = 0, \quad \nabla \tilde{G}_2(\boldsymbol{\nu} - \mathbf{a}) = 0, \quad \boldsymbol{\nu} = \boldsymbol{\mu} + \mathbf{v}_1,$$

so that (9.14) is fulfilled. By definition of \tilde{g}_1, \tilde{g}_2 ,

$$\mathcal{X}(\rho; \delta; g_1, g_2; \mathbf{0}, \mathbf{b}) = \mathcal{X}(\rho; \delta; \tilde{g}_1, \tilde{g}_2; -\mathbf{v}_1, -\mathbf{v}_2 + \mathbf{b}),$$

and consequently,

$$\text{vol}(\mathcal{X}(\rho; \delta; g_1, g_2; \mathbf{0}, \mathbf{b})) = \text{vol}(\mathcal{X}(\rho; \delta; \tilde{g}_1, \tilde{g}_2; \mathbf{0}, \mathbf{a})).$$

Denote $\tilde{\mathcal{X}} = \mathcal{X}(\rho; \delta, \tilde{g}_1, \tilde{g}_2; \mathbf{0}, \mathbf{a})$.

Now, depending on the value of $\phi(\boldsymbol{\nu}, \boldsymbol{\nu} - \mathbf{a})$ we use Lemma 9.3 or Lemma 9.5 with $l = \rho^{-\varepsilon}$, $\varepsilon > 0$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon$. Note that Lemma 9.3 can be used since $\delta l^{-2} = \delta \rho^{2\varepsilon} \rightarrow 0$ as $\rho \rightarrow \infty$.

If $\pi - \phi(\boldsymbol{\nu}, \boldsymbol{\nu} - \mathbf{a}) \leq \rho^{-\varepsilon-1}$, then by Lemma 9.3

$$(9.24) \quad \text{vol } \tilde{\mathcal{X}} \ll \delta \rho^{-1-\varepsilon(d-1)}.$$

Assume now that for all points $\boldsymbol{\mu} \in \mathcal{X}$ we have the bound $\pi - \phi(\boldsymbol{\nu}, \boldsymbol{\nu} - \mathbf{a}) \geq \rho^{-\varepsilon-1}$. It follows again from definition of \tilde{g}_j that

$$\text{vol}(\mathcal{X}(\rho; \delta; g_1, g_2; \mathbf{0}, \mathbf{b}) \cap B(\boldsymbol{\mu}, \rho^{-2\varepsilon})) = \text{vol}(\tilde{\mathcal{X}}(\rho; \delta; \tilde{g}_1, \tilde{g}_2; \mathbf{0}, \mathbf{a}) \cap B(\boldsymbol{\nu}, \rho^{-2\varepsilon})).$$

Since $\pi - \phi(\boldsymbol{\nu}, \boldsymbol{\nu} - \mathbf{a}) \geq \rho^{-\varepsilon-1}$, according to Lemma 9.5,

$$\text{vol}(\mathcal{X} \cap B(\boldsymbol{\mu}, \rho^{-2\varepsilon})) = \text{vol}(\tilde{\mathcal{X}} \cap B(\boldsymbol{\nu}, \rho^{-2\varepsilon})) \ll \delta^2 \rho^{-2\varepsilon(d-3)}.$$

As $\mathcal{X} \subset B(0, 3\rho)$, one needs $\ll \rho^{d(1+2\varepsilon)}$ balls of radius $\rho^{-2\varepsilon}$ to cover \mathcal{X} . Thus,

$$\text{vol } \mathcal{X} \ll \delta^2 \rho^{-2\varepsilon(d-3)} \rho^{d(1+2\varepsilon)} = \delta^2 \rho^{d+6\varepsilon}.$$

Adding this bound with (9.24) produces (9.9) . □

As explained earlier, Theorem 9.1 for $m = 1$ implies itself for all $m > 0$.

10. PROOF OF THE BETHE-SOMMERFELD CONJECTURE

In this section, we prove the Main Theorem 2.1. We do it in a few steps. First we prove it for the model operator A defined (6.1) with conditions (6.3) satisfied. After that we invoke Theorem 4.3, which states that the original operator H can be reduced to the model operator up to controllable error terms. At the second step we show that these errors do not destroy the spectral band overlap, obtained for the model operator.

10.1. **Theorem 2.1 for the model operator** ^{main:thm} (6.1). Our proof of the spectral band overlap for the operator A relies on the following elementary Intermediate Value Theorem type result for the function $g(\boldsymbol{\xi})$ defined in Section ^{global:sect} 7. As before we assume that $\lambda = \rho^{2m}$.

Lemma 10.1. *Let $\boldsymbol{\xi} = \boldsymbol{\xi}(t) \subset \mathcal{B}, t \in [t_1, t_2], t_1 < t_2$, be a continuous path. Suppose that $g(\boldsymbol{\xi}(t_1)) \leq \lambda - \delta, g(\boldsymbol{\xi}(t_2)) \geq \lambda + \delta$ with some $\delta \in (0, \lambda/4)$, and for each $t \in [t_1, t_2]$ the number $g(\boldsymbol{\xi}(t))$ is a simple eigenvalue of $A(\mathbf{k}), \mathbf{k} = \{\boldsymbol{\xi}(t)\}$. Then there exists a $t_0 \in (t_1, t_2)$ such that $\lambda = g(\boldsymbol{\xi}(t_0))$, so that $\lambda \in \sigma(A)$. Moreover, $\zeta(\lambda; A) \geq \delta$.*

Proof. Recall that on the set \mathcal{B} the function $g(\boldsymbol{\xi})$ is defined by ^{ginb:eq} (7.1), so that $g(\boldsymbol{\xi}(t)) = |\boldsymbol{\xi}(t)|^{2m} + b^o(\boldsymbol{\xi}(t))$. Since $g(\boldsymbol{\xi}(t))$ is a simple eigenvalue of $A(\mathbf{k}), \{\boldsymbol{\xi}(t)\} = \mathbf{k}$ for each $t \in [t_1, t_2]$, we have $g(\boldsymbol{\xi}(t)) = \lambda_j(A(\mathbf{k}))$ with j independent of the choice of t . Since g is continuous on \mathcal{B} , the function $g(\boldsymbol{\xi}(t))$ is a continuous function of $t \in [t_1, t_2]$, and hence the intermediate value theorem implies that there is a $t_0 \in (t_1, t_2)$ such that $\lambda_j(A(\{\boldsymbol{\xi}(t_0)\})) = \lambda$. The bound $\zeta(\lambda; A) \geq \delta$ follows from the definition ^{zeta:eq} (2.13) of $\zeta(\lambda; A)$. \square

Our next step is to prove that there is a path with the properties required in Lemma 10.1. In fact we shall prove that the required properties will hold for an interval $I(\boldsymbol{\Omega}; \rho, \delta) \subset (0, \infty)$ (see ^{iomd:eq} (8.4)) with some $\boldsymbol{\Omega} \in T(\rho)$.

Lemma 10.2. *There exists a constant $Z \geq 1$ with the following property. Suppose that for some $\boldsymbol{\Omega} \in T(\rho)$ and some $t \in I(\boldsymbol{\Omega}; \rho, \delta), \delta \in (0, \rho^{2m}/4]$, the number $g(\boldsymbol{\eta}), \boldsymbol{\eta} = t\boldsymbol{\Omega}$ is a multiple eigenvalue of $A(\mathbf{k}), \mathbf{k} = \{\boldsymbol{\eta}\}$. Then for any $\tau \in I(\boldsymbol{\Omega}; \rho, \delta)$ there exists a vector $\mathbf{n} \in \Gamma^\dagger \setminus \{\mathbf{0}\}$ such that $\tau\boldsymbol{\Omega} + \mathbf{n} \in \mathcal{A}(\rho, Z\delta)$.*

Proof. Since the number $g(\boldsymbol{\eta}), \boldsymbol{\eta} = t\boldsymbol{\Omega}$, is a multiple eigenvalue, by definition of the function $g(\cdot)$, there is a vector $\mathbf{p} \in \Gamma^\dagger \setminus \{\mathbf{0}\}$ such that $g(\boldsymbol{\eta}) = g(\boldsymbol{\eta} + \mathbf{p})$. In view of ^{twosided:eq} (7.3), $|\boldsymbol{\eta} + \mathbf{p}| \asymp \rho$. Thus by Lemma ^{partialofs2} 7.3, for any $\tau \in I(\boldsymbol{\Omega}; \rho, \delta)$ there exist two vectors $\mathbf{m}_1, \mathbf{m}_2 \in \Gamma^\dagger, \mathbf{m}_1 \neq \mathbf{m}_2$ such that, with $\boldsymbol{\xi} = \tau\boldsymbol{\Omega}$,

$$(10.1) \quad \begin{cases} |g(\boldsymbol{\eta}) - g(\boldsymbol{\xi} + \mathbf{m}_1)| \ll \rho^{2m-1} |\boldsymbol{\eta} - \boldsymbol{\xi}| \ll \delta, \\ |g(\boldsymbol{\eta} + \mathbf{p}) - g(\boldsymbol{\xi} + \mathbf{m}_2)| \ll \rho^{2m-1} |\boldsymbol{\eta} - \boldsymbol{\xi}| \ll \delta. \end{cases}$$

Here we have used the bound $|t - \tau| \ll \delta \rho^{1-2m}$, which follows from ^{iomd1:eq} (8.5). As $\mathbf{m}_1 \neq \mathbf{m}_2$, one of these vectors is not zero. Denote this vector by \mathbf{n} . Since $g(\boldsymbol{\eta}) = g(\boldsymbol{\eta} + \mathbf{p})$, it follows from ^{inthezone:eq} (10.1) that

$$|g(\boldsymbol{\xi} + \mathbf{n}) - g(\boldsymbol{\eta})| \ll \delta,$$

so that $\boldsymbol{\xi} + \mathbf{n} \in \mathcal{A}(\rho, Z\delta)$ with some constant Z independent of $\boldsymbol{\xi}$ and ρ , as required. \square

The next Lemma is the cornerstone of our argument: it shows that at least for one $\boldsymbol{\Omega} \in T(\rho)$ the interval $I(\boldsymbol{\Omega}; \rho, \delta)$ consists entirely of the points t such that $g(t\boldsymbol{\Omega})$ is a simple eigenvalue.

Lemma 10.3. *There exists a vector $\boldsymbol{\Omega} \in T(\rho)$ and a number $c_3 > 0$ such that for $\delta = c_3 \rho^{2m-4-d-12(d-1)^{-1}}$ and each $t \in I(\boldsymbol{\Omega}; \rho, \delta)$ the number $g(\boldsymbol{\xi}), \boldsymbol{\xi} = t\boldsymbol{\Omega}$ is a simple eigenvalue of $A(\{\boldsymbol{\xi}\})$. Moreover, $\zeta(\rho^{2m}; A) \geq \delta$.*

Proof. Suppose the contrary, i.e. if ρ is sufficiently large, then for any $\Omega \in T(\rho)$ there is a $t \in I(\Omega; \rho, \delta)$ such that $g(t\Omega)$ is a multiple eigenvalue of $A(t\Omega)$. Then due to formula (8.6), Lemma 10.2 implies that

$$\boxed{\text{cover0}} \quad (10.2) \quad \tilde{\mathcal{B}}(\rho, \delta) \subset \bigcup_{\mathbf{n} \in \Gamma^+ \setminus \{0\}} (\mathcal{A}(\rho, \delta_1) + \mathbf{n})$$

with $\delta_1 := Z\delta$. Since $\tilde{\mathcal{B}}(\rho, \delta) \subset \mathcal{B}(\rho, \delta_1)$, we can re-write (10.2) as

$$\boxed{\text{cover1}} \quad (10.3) \quad \begin{aligned} \tilde{\mathcal{B}}(\rho, \delta) &\subset \bigcup_{\mathbf{n} \in \Gamma^+ \setminus \{0\}} \left((\mathcal{A}(\rho, \delta_1) + \mathbf{n}) \cap \mathcal{B}(\rho, \delta_1) \right) \\ &= \bigcup_{\mathbf{n} \in \Gamma^+ \setminus \{0\}} \left((\mathcal{B}(\rho, \delta_1) + \mathbf{n}) \cap \mathcal{B}(\rho, \delta_1) \right) \cup \bigcup_{\mathbf{n} \in \Gamma^+ \setminus \{0\}} \left((\mathcal{D}(\rho, \delta_1) + \mathbf{n}) \cap \mathcal{B}(\rho, \delta_1) \right). \end{aligned}$$

Let us estimate the volumes of sets on both sides of this inclusion. For a fixed $\varepsilon > 0$, whose value is chosen a few lines down, assume that $\delta\rho^{2-2m+2\varepsilon} \rightarrow 0$, $\rho \rightarrow \infty$, we can use (8.9) and (8.10) for the volume of the right hand side. For the left hand side we use (8.2), so that (10.3) results in the estimate

$$\delta\rho^{d-2m} \ll \delta^2\rho^{4-4m+2d+6\varepsilon} + \delta\rho^{1-2m+d-\varepsilon(d-1)} + \delta\rho^{d-1-2m+\alpha_d},$$

which simplifies to

$$1 \ll \delta\rho^{4-2m+d+6\varepsilon} + \rho^{1-\varepsilon(d-1)} + \rho^{-1+\alpha_d}.$$

Choose $\varepsilon = 2(d-1)^{-1}$ and $\delta = c_3\rho^{2m-4-d-6\varepsilon}$ with a suitably small c_3 . Then for large ρ the right hand side is less than the left hand side, which produces a contradiction, thus proving the Lemma. \square

10.2. Proof of the Main Theorem. We assume that the conditions of Theorem 2.1 are satisfied. The proof uses the reduction of the operator H to A_1 , established in Theorem 4.3. The first step is to show that the spectrum of A_1 is well approximated by that of the model operator (6.1) with B replaced with X , i.e.

$$A = H_0 + X^o + X^{\mathcal{R}}.$$

Let numbers $\alpha_j < 1, j = 1, 2, \dots, d$ be as defined in Subsection 5.3.

Lemma 10.4. *Suppose that the conditions of Theorem 2.1 are satisfied. Let A_1 be the operator (4.27), and let $r = \rho^\varkappa$ with a number $\varkappa > 0$, satisfying (5.12) and the inequality*

$$\boxed{\text{vark:eq}} \quad (10.4) \quad d^2\varkappa < (2m - \alpha\beta)\alpha_d.$$

Then for any $L > 0$ there exists an M (i.e. the number of steps in Theorem 4.3) such that

$$\boxed{\text{loca:eq}} \quad (10.5) \quad N(\mu - \rho^{-L}, A(\mathbf{k})) \leq N(\mu, A_1(\mathbf{k})) \leq N(\mu + \rho^{-L}, A(\mathbf{k}))$$

for all $\mu \in ((1 - c_4)^{2m}\rho^{2m}, (1 + c_4)^{2m}\rho^{2m})$ with any $c_4 < 1/32$.

Proof. By Theorem [4.3](#), $\|R_{M+1}\| \ll \rho^{\beta\epsilon_{M+1}}$ uniformly in $b : |b|^{(\alpha)} \ll 1$ (see [\(4.16\)](#) for definition of ϵ_{M+1}). The condition [\(2.15\)](#) is equivalent to $\sigma < 1$, so that $\epsilon_j \rightarrow -\infty$ as $j \rightarrow \infty$. Thus for sufficiently large $M = M(L)$ we have $\|R_{M+1}\| \ll \rho^{-L}/2$. As a consequence,

$$N(\mu - \rho^{-L}/2, \tilde{A}_1(\mathbf{k})) \leq N(\mu, A_1(\mathbf{k})) \leq N(\mu + \rho^{-L}/2, \tilde{A}_1(\mathbf{k})),$$

$$\tilde{A}_1 = A + X^{\mathcal{SE}, \mathcal{LE}, \mathcal{LF}},$$

loca2:eq

for all $\mu \in \mathbb{R}$. Due to [\(6.9\)](#), the operator \tilde{A}_1 can be represented in the block-matrix form:

$$\tilde{A}_1 = \bigoplus_{\mathfrak{W} \in \mathcal{W}(r)} \mathcal{P}(\Xi(\mathfrak{W})) A_{\mathfrak{W}} \mathcal{P}(\Xi(\mathfrak{W})) + \bigoplus_{\mathfrak{W}, \mathfrak{W}' \in \mathcal{W}(r)} \mathcal{P}(\Xi(\mathfrak{W})) X^{\mathcal{SE}, \mathcal{LE}, \mathcal{LF}} \mathcal{P}(\Xi(\mathfrak{W}')).$$

Since the number of distinct subspaces $\mathfrak{W} \in \mathcal{W}(r)$ is bounded above by Cr^{d^2} with some universal constant $C > 0$, the second term satisfies the two-sided estimate

$$-Cr^{d^2} \bigoplus_{\mathfrak{W} \in \mathcal{W}(r)} \mathcal{P}(\Xi(\mathfrak{W})) |X|^{\mathcal{SE}, \mathcal{LE}, \mathcal{LF}} \mathcal{P}(\Xi(\mathfrak{W})) \leq \bigoplus_{\mathfrak{W}, \mathfrak{W}' \in \mathcal{W}(r)} \mathcal{P}(\Xi(\mathfrak{W})) X^{\mathcal{SE}, \mathcal{LE}, \mathcal{LF}} \mathcal{P}(\Xi(\mathfrak{W}'))$$

$$\leq Cr^{d^2} \bigoplus_{\mathfrak{W} \in \mathcal{W}(r)} \mathcal{P}(\Xi(\mathfrak{W})) |X|^{\mathcal{SE}, \mathcal{LE}, \mathcal{LF}} \mathcal{P}(\Xi(\mathfrak{W})).$$

Here we have denoted $|X|^{\mathcal{SE}, \mathcal{LE}, \mathcal{LF}} = |X^{\mathcal{SE}}| + |X^{\mathcal{LE}}| + |X^{\mathcal{LF}}|$, see [\[3\]](#), Chapter 8, p. 183 for definition of the modulus and polar decomposition of a closed operator. Note that in the above estimate the left- and rightmost terms contain only *single* orthogonal sums in contrast to the *double* sum in the middle term. This bound follows from the following inequality for the quadratic form of an arbitrary self-adjoint bounded operator K and any two orthogonal projections P_1, P_2 :

$$2|(KP_1u, P_2u)| \leq (P_1|K|P_1u, u) + (P_2|K|P_2u, u).$$

This follows from the polar representation of K upon applying the elementary inequality $2ab \leq a^2 + b^2$ for $a, b \geq 0$.

Now it follows that

$$\tilde{A}_- \leq \tilde{A}_1 \leq \tilde{A}_+$$

with

$$\tilde{A}_{\pm} = \bigoplus_{\mathfrak{W} \in \mathcal{W}(r)} \mathcal{P}(\Xi(\mathfrak{W})) (A_{\mathfrak{W}} \pm Cr^{d^2} |X|^{\mathcal{SE}, \mathcal{LE}, \mathcal{LF}}) \mathcal{P}(\Xi(\mathfrak{W})).$$

Since \tilde{A}_{\pm} are orthogonal sums, the problem is reduced to estimating the counting functions of $\tilde{A}_{\pm}(\mathbf{k})$ on each invariant subspace $\mathfrak{H}(\mathbf{k}; \Xi(\mathfrak{W}))$. From now on we assume that \mathfrak{W} is fixed and omit it from the notation.

If $\mathfrak{W} \in \mathcal{V}(r, d)$, i.e. $\mathfrak{W} = \mathbb{R}^d$, then $\Xi = \Xi(\mathfrak{W}) \subset B(0, 2\rho^{\alpha d})$, see Lemma [5.12](#). Clearly, $\|H_0\mathcal{P}(\Xi)\| \leq \rho^{2m\alpha d}$. Also, by [\(3.22\)](#),

$$|x^o|^{(\alpha)} + |x^{\mathcal{SE}}|^{(\alpha)} + |x^{\mathcal{LE}}|^{(\alpha)} + |x^{\mathcal{LF}}|^{(\alpha)} \ll |b|^{(\alpha)},$$

and hence, by Lemma [formbound:lem](#) [3.2](#),

$$\|X^o\mathcal{P}(\Xi)\| + r^{d^2}\|\mathcal{P}(\Xi)|X|^{\mathcal{SE},\mathcal{LE},\mathcal{LF}}\mathcal{P}(\Xi)\| \ll r^{d^2}\rho^{\alpha\beta\alpha_d}.$$

In view of [\(10.4\)](#), [\(vark:eq\)](#) the right hand side does not exceed $\rho^{2m\alpha_d}$. Consequently, $\|\tilde{A}_\pm\mathcal{P}(\Xi)\| \ll \rho^{2m\alpha_d}$, which implies that $N(\mu, \tilde{A}_\pm(\mathbf{k}); \Xi)$ and $N(\mu, A_{\mathfrak{V}}(\mathbf{k}); \Xi)$ are independent of μ, \mathbf{k} for all $\mu \geq (\rho/2)^{2m}$, and $N(\mu, \tilde{A}_\pm(\mathbf{k}); \Xi) = N(\mu, A_{\mathfrak{V}}(\mathbf{k}); \Xi)$ if $\mu \geq (\rho/2)^{2m}$.

Now, let us fix $\mathfrak{V} \in \mathcal{V}(r, n), n \leq d-1$, and prove the bounds

$$\boxed{\text{loca1:eq}} \quad (10.7) \quad N(\mu - \rho^{-L}/2, A_{\mathfrak{V}}(\mathbf{k}); \Xi) \leq N(\mu, \tilde{A}_\pm(\mathbf{k}); \Xi) \leq N(\mu + \rho^{-L}/2, A_{\mathfrak{V}}(\mathbf{k}); \Xi),$$

for sufficiently large ρ . Split Ξ into three disjoint sets:

$$\Xi = \mathcal{C}_< \cup \mathcal{C}_0 \cup \mathcal{C}_>, \quad \mathcal{C}_0 = \{\xi \in \Xi : 7\rho/8 \leq |\xi_{\mathfrak{V}^\perp}| \leq 17\rho/16\},$$

$$\mathcal{C}_< = \{\xi \in \Xi : |\xi_{\mathfrak{V}^\perp}| < 7\rho/8\}, \quad \mathcal{C}_> = \{\xi \in \Xi : 17\rho/16 < |\xi_{\mathfrak{V}^\perp}|\}.$$

Note that by definition of the operator $A_{\mathfrak{V}}$ (see [\(6.4\)](#)) all three subspaces $\mathcal{H}(\mathcal{C}_0), \mathcal{H}(\mathcal{C}_<), \mathcal{H}(\mathcal{C}_>)$ (see Subsection [2.3](#)) are invariant for $A_{\mathfrak{V}}$. Since $|\xi_{\mathfrak{V}^\perp}| < 2\rho^{\alpha_d-1}$ (see Lemma [5.12](#)), we have

$$\Xi \cap B(0, 7\rho/8) \subset \mathcal{C}_< \subset B(0, 29\rho/32),$$

$$\Xi \cap B(17\rho/16) \subset (\mathcal{C}_< \cup \mathcal{C}_0) \subset B(0, 9\rho/8).$$

Therefore, by Lemma [smallorthog:lem](#) [3.5](#),

$$\mathcal{P}(\Xi)|X^{\mathcal{SE}}|\mathcal{P}(\Xi) = \mathcal{P}(\mathcal{C}_<)|X^{\mathcal{SE}}|\mathcal{P}(\mathcal{C}_<), \quad \mathcal{P}(\Xi)|X^{\mathcal{LE}}|\mathcal{P}(\Xi) = \mathcal{P}(\mathcal{C}_>)|X^{\mathcal{LE}}|\mathcal{P}(\mathcal{C}_>).$$

Thus $\tilde{A}_\pm\mathcal{P}(\Xi)$ can be rewritten as

$$\tilde{A}_\pm\mathcal{P}(\Xi) = F_\pm \pm Cr^{d^2}(\mathcal{P}(\Xi)|X|^{\mathcal{LF}}\mathcal{P}(\Xi) - \mathcal{P}(\mathcal{C}_>)|X|^{\mathcal{LF}}\mathcal{P}(\mathcal{C}_>)),$$

with

$$F_\pm = \mathcal{P}(\mathcal{C}_<)(A_{\mathfrak{V}} \pm Cr^{d^2}|X^{\mathcal{SE}}|)\mathcal{P}(\mathcal{C}_<) \oplus \mathcal{P}(\mathcal{C}_0)A_{\mathfrak{V}}\mathcal{P}(\mathcal{C}_0) \\ \oplus \mathcal{P}(\mathcal{C}_>)(A_{\mathfrak{V}} \pm Cr^{d^2}|X|^{\mathcal{LF},\mathcal{LE}})\mathcal{P}(\mathcal{C}_>).$$

By [\(3.25\)](#), [\(shar:eq\)](#)

$$r^{d^2}\|\mathcal{P}(\mathcal{C}_< \cup \mathcal{C}_0)|X^{\mathcal{LF}}|\| + r^{d^2}\|\mathcal{P}(\mathcal{C}_>)|X^{\mathcal{LF}}|\| \ll r^{d^2+p-l}\rho^{\beta\max(\alpha,0)},$$

for any $p > d$ and $l \geq p$ uniformly in b satisfying $|b|^{(\alpha)} \ll 1$. As $r = \rho^\varkappa, \varkappa > 0$, by choosing a sufficiently large l , we can guarantee that the right hand side is bounded by $\rho^{-L}/2$. This leads to the bounds

$$\boxed{\text{loca4:eq}} \quad (10.8) \quad N(\mu - \rho^{-L}/2, F_\pm(\mathbf{k}); \Xi) \leq N(\mu, \tilde{A}_\pm(\mathbf{k}); \Xi) \leq N(\mu + \rho^{-L}/2, F_\pm(\mathbf{k}); \Xi),$$

for all $\mu \in \mathbb{R}$. Consequently, [\(10.7\)](#) will be proved if we show that

$$\boxed{\text{loca3:eq}} \quad (10.9) \quad N(\mu, F_\pm; \Xi) = N(\mu, A_{\mathfrak{V}}; \Xi), \quad \left(\frac{15\rho}{16}\right)^{2m} \leq \mu \leq \left(\frac{33\rho}{32}\right)^{2m}.$$

To this end note first that the definition of $\mathcal{C}_<$ and $\mathcal{C}_>$ implies

$$(10.10) \quad H_0\mathcal{P}(\mathcal{C}_<) \leq (29\rho/32)^{2m}\mathcal{P}(\mathcal{C}_<), \quad H_0\mathcal{P}(\mathcal{C}_>) \geq (17\rho/16)^{2m}\mathcal{P}(\mathcal{C}_>).$$

Also, by Lemma [3.5](#), [smallorthog:lem](#)

$$\|X^o\| + \|X_{\mathfrak{Y}}^{\mathfrak{R}}\| + Cr^{d^2}\|X^{\mathfrak{SE}}\| \ll r^{d^2}\rho^{\beta\max(\alpha,0)}.$$

Under the condition [\(10.4\)](#) [vark:eq](#) the right hand side of this estimate is bounded by $o(\rho^{2m})$, $\rho \rightarrow \infty$ uniformly in b . Together with [\(10.10\)](#) [partial:eq](#), this entails that

$$(10.11) \quad N(\mu, A_{\mathfrak{Y}} \pm Cr^{d^2}\mathcal{P}(\mathcal{C}_<)|X^{\mathfrak{SE}}|\mathcal{P}(\mathcal{C}_<); \mathcal{C}_<) = N(\mu, A_{\mathfrak{Y}}; \mathcal{C}_<), \quad \mu \geq \left(\frac{15\rho}{16}\right)^{2m}.$$

Furthermore, in view of [\(3.22\)](#) [subord:eq](#) and [\(3.2\)](#) [formbound1:eq](#),

$$\mathcal{P}(\mathcal{C}_>)(|X^o| + |X_{\mathfrak{Y}}^{\mathfrak{R}}| + Cr^{d^2}|X|^{\mathcal{L}\mathcal{F},\mathcal{L}\mathcal{E}})\mathcal{P}(\mathcal{C}_>) \ll r^{d^2}(H_0 + I)^\gamma\mathcal{P}(\mathcal{C}_>), \quad \gamma = \frac{\alpha\beta}{2m}.$$

Using again [\(10.4\)](#) [vark:eq](#) and remembering [\(10.10\)](#) [partial:eq](#), we conclude that the right hand side is estimated above by $o(1)H_0\mathcal{P}(\mathcal{C}_>)$, $\rho \rightarrow \infty$, uniformly in b . Together with [\(10.10\)](#) [partial:eq](#) this implies that

$$(10.12) \quad N(\mu, A_{\mathfrak{Y}} \pm Cr^{d^2}\mathcal{P}(\mathcal{C}_>)|X|^{\mathcal{L}\mathcal{F},\mathcal{L}\mathcal{E}}\mathcal{P}(\mathcal{C}_>); \mathcal{C}_>) = 0, \quad \mu \leq \left(\frac{33\rho}{32}\right)^{2m}.$$

Putting together [\(10.11\)](#) [c<:eq](#) and [\(10.12\)](#) [c>:eq](#), we arrive at [\(10.9\)](#) [loca3:eq](#). In combination with [\(10.8\)](#) [loca4:eq](#) this leads to [\(10.7\)](#) [loca1:eq](#). Together with [\(10.6\)](#) [loca2:eq](#) they yield [\(10.5\)](#) [loca:eq](#). \square

Proof of the Main Theorem. By Theorem [4.3](#) [reduction:thm](#), it suffices to prove that $\zeta(\rho^{2m}, A_1) > c\rho^S$ with some S for sufficiently large ρ . It follows from Lemma [10.3](#) [lem:simple4](#) that $\zeta(\rho^{2m}; A) \geq c\rho^S$ with $S = 2m - 4 - d - 12(d - 1)^{-1}$. Using the bounds [\(10.5\)](#) [loca:eq](#) with $L > -S$, we get the required estimate $\zeta(\rho^{2m}, A_1) \gg \rho^S$ from the definition [\(2.14\)](#) [zetal:eq](#). This completes the proof of Theorem [2.1](#) [main:thm](#). \square

REFERENCES

- [BarPar](#) 1. G. Barbatis and L. Parnovski, *Bethe - Sommerfeld conjecture for pseudo-differential perturbation*, Comm.P.D.E. **34**, Issue 4(2009), 383 - 418.
- [BS](#) 2. A. Sommerfeld and H. Bethe, *Elektronentheorie der Metalle*, in H. Geiger and K. Scheel, eds., Handbuch der Physik, Volume **24**, Part 2, 333-622 (Springer, 1933). Later edition: *Elektronentheorie der Metalle*, Springer, 1967.
- [BirSol](#) 3. M.Ş. Birman and M. Z. Solomyak, *Spectral theory of self-adjoint operators in Hilbert space*, Reidel, 1987.
- [Cas](#) 4. J.W.S.Cassels, *An introduction to the geometry of numbers*, Springer-Verlag, Berlin, 1959.
- [DahlTru](#) 5. B.E.J. Dahlberg, E. Trubowitz, *A remark on two dimensional periodic potentials*, Comment. Math. Helvetici **57** (1982), 130-134.
- [E](#) 6. M.S.P. Eastham, *The spectral theory of periodic differential equations*, Scottish Academic Press, 1973.
- [FKTO](#) 7. J. Feldman, H. Knörrer, E. Trubowitz, *The perturbatively stable spectrum of a periodic Schrödinger operator*, Invent. Math. **100** (1990), 259-300.

- FKT 8. J. Feldman, H. Knörrer, E. Trubowitz, *Perturbatively unstable eigenvalues of a periodic Schrödinger operator*, Comment. Math. Helvetici **66** (1991), 557-579.
- HelMoh 9. B. Helffer and A. Mohamed, *Asymptotics of the density of states for the Schrödinger operator with periodic electric potential*, Duke Math. J. **92** (1998), 1–60.
- Karnew 10. Y. E. Karpeshina, *Perturbation series for the Schrödinger operator with a periodic potential near planes of diffraction*, Comm. Anal. Geom. **4** (1996), no. 3, 339–413.
- Kar 11. Y. E. Karpeshina, *Perturbation theory for the Schrödinger operator with a periodic potential*, Lecture Notes in Math. vol 1663, Springer Berlin 1997.
- Karp2 12. Y. E. Karpeshina, *Spectral properties of periodic magnetic Schrödinger operator in the high-energy region. Two-dimensional case*, Comm. Math. Phys. **251**(3) (2004), 473 – 514.
- K 13. T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1980.
- Kuch 14. P. Kuchment, *Floquet theory for partial differential equations*, Birkhäuser, Basel, 1993.
- Moh 15. A. Mohamed, *Asymptotic of the density of states for the Schrödinger operator with periodic electromagnetic potential*, J. Math. Phys. **38** no. 8 (1997), 4023-4051.
- P 16. L. Parnowski, *Bethe-Sommerfeld conjecture*, Annales H. Poincaré, **9**(3)(2008), 457-508.
- PS1 17. L. Parnowski and A. V. Sobolev, *Bethe-Sommerfeld conjecture for polyharmonic operators*, Duke Math. J. **107** no.2 (2001), 209–238.
- PS2 18. L. Parnowski and A. V. Sobolev, *Perturbation theory and the Bethe-Sommerfeld conjecture*, Annales H. Poincaré **2**(2001), 573–581.
- PopSkr 19. V.N. Popov and M. Skriganov, *A remark on the spectral structure of the two dimensional Schrödinger operator with a periodic potential*, Zap. Nauchn. Sem. LOMI AN SSSR **109** (1981), 131–133 (Russian).
- RS 20. M. Reed M. and B. Simon, *Methods of Modern Mathematical Physics, IV*, Academic Press, New York, 1975.
- R 21. G. V. Rozenbljum, *Near-similarity of operators and the spectral asymptotic behavior of pseudodifferential operators on the circle*, Trudy Moskov. Mat. Obsch. **36** (1978), 59–84 (Russian).
- Skr0 22. M. Skriganov, *Proof of the Bethe-Sommerfeld conjecture in dimension two*, Soviet Math. Dokl. **20** no. 1 (1979), 89–90.
- Skr1 23. M. Skriganov, *Geometrical and arithmetical methods in the spectral theory of the multi-dimensional periodic operators*, Proc. Steklov Math. Inst. Vol. 171, 1984.
- Skr2 24. M. Skriganov, *The spectrum band structure of the three-dimensional Schrödinger operator with periodic potential*, Inv. Math. **80** (1985), 107–121.
- SkrSob1 25. M. Skriganov and A. Sobolev, *Asymptotic estimates for spectral bands of periodic Schrödinger operators*, St Petersburg Math. J. **17** no. 1 (2006), 207–216.
- SkrSob2 26. M. Skriganov and A. Sobolev, *Variation of the number of lattice points in large balls*, Acta Arith. **120**(3)(2005), 245–267.
- Sob0 27. A.V. Sobolev, *Asymptotics of the integrated density of states for periodic elliptic pseudo-differential operators in dimension one*, Rev. Mat. Iberoamericana **22** (2006), no.1, 55-92.
- Sob 28. A.V.Sobolev, *Integrated Density of States for the Periodic Schrödinger Operator in Dimension Two*, Ann. Henri Poincaré **6**(2005), 31–84.
- Sob1 29. A.V.Sobolev, *Recent results on the Bethe-Sommerfeld conjecture*, Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday, ix–xii, Proc. Sympos. Pure Math., **76**, Part 1, Amer. Math. Soc., Providence, RI, 2007.
- Vel0 30. O.A. Veliev, *Asymptotic formulas for the eigenvalues of the periodic Schrödinger operator and the Bethe-Sommerfeld conjecture*, Functional Anal. Appl. **21** no. 2 (1987), 87–100.
- Vel2 31. O.A.Veliev, *On the spectrum of multidimensional periodic operators*, Theory of Functions, functional analysis and their applications, Kharkov University, **49**, (1988), 17–34 (in Russian).

- Vel 32. O.A. Veliev, *Perturbation theory for the periodic multidimensional Schrödinger operator and the Bethe-Sommerfeld Conjecture*, Int. J. Contemp. Math. Sci., **2** no. 2 (2007), 19–87.

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