Bethe - Sommerfeld conjecture for pseudo-differential perturbation

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April 22, 2008

Abstract

We consider a periodic pseudodifferential operator $H = (-\Delta)^l + A$ ($l > 0$) in $\mathbf{R}^d$ which satisfies the following conditions: (i) the symbol of H is smooth in x, and (ii) the perturbation A has order smaller than $2l - 1$. Under these assumptions, we prove that the spectrum of H contains a half-line.

AMS Subject Classification: 35P20 (58J40, 58J50, 35J10)
Keywords: Bethe-Sommerfeld conjecture, periodic problems, pseudo-differential operators, spectral gaps

1 Introduction

Let H be a periodic pseudodifferential elliptic operator in $\mathbf{R}^d$. Then its spectrum consists of spectral bands converging to $+\infty$. These bands are separated by spectral gaps, and one of the most important questions of the spectral theory of periodic operators is whether the number of these gaps is finite. There is a wide belief that for $d \geq 2$, under fairly general conditions on H the number of finite gaps is finite (often this statement is called the 'Bethe-Sommerfeld conjecture'). This statement (in the setting of periodic operators) is equivalent to stating that the whole interval $[\lambda, +\infty)$ is covered by the spectrum of H, provided λ is big enough. If we discard certain very special cases of operators H (like Schrödinger operator with potential which allows the separation of variables), then, until recently, the conjecture was known to hold only under serious restrictions on the dimension d of the Euclidean space and the order of our operator H, see [PoSk], [S1]–[S3], [Kar], [HM], [PS1], [PS2], [V1], [M]. Another type of sufficient conditions is assuming that the lattice of periods of H is rational, see [S2], [SS1], [SS2]. In the paper [P], the conjecture was proved for Schrödinger operators with smooth periodic potentials, without any assumptions on dimension $d \geq 2$, or on the lattice of periods (see also [V2] for an alternative approach to this problem). In our paper, we prove that this conjecture holds for a wide class of pseudo-differential operators.

Let us describe the results of our paper in detail. Let $h = h(x, \xi)$ be the symbol of H and let $2l$ be the order of H ($l > 0$). We assume a decomposition $h = h_p + a$, where $h_p \asymp |\xi|^{2l}$ (as $|\xi| \to \infty$) is the principal symbol of H, and $a = O(|\xi|^\alpha)$ (as $|\xi| \to \infty$, where $\alpha < 2l$) is the remainder. The symbol $h(x, \xi)$ is assumed to be periodic in x with periodicity lattice Γ. We denote by $\Gamma^\dagger$ the dual lattice, and by $\mathcal{O}$ and $\mathcal{O}^\dagger$ the respective fundamental domains. We also set $d(\Gamma) = \mathrm{vol}(\mathcal{O})$ and $d(\Gamma^\dagger) = \mathrm{vol}(\mathcal{O}^\dagger)$.

We make the following additional assumptions:

(a) $h_p(\xi) = |\xi|^{2l}$. This assumption is made mainly for simplicity of exposition; our results are likely to hold if we replace h_p by a more general principal symbol that is homogeneous in ξ. However it is essential that $h_p(\xi)$ does not depend on x (in other words, the principal part of H has constant coefficients). This latter assumption, to the best of our knowledge, is present in all approaches to proving the Bethe-Sommerfeld conjecture.

(b) We assume that the symbol $a(x, \xi)$ is smooth in x. This requirement is a major disadvantage of our method when compared, e.g., with the approach of Karpeshina (see [Kar] and references therein).

(c) $a = O(|\xi|^\alpha)$ and $\nabla_\xi a = O(|\xi|^{\alpha-1})$ uniformly in x. We emphasize that we do not assume the existence of higher (than first) derivatives of a with respect to ξ.

(d) Finally, we assume that $\alpha < 2l - 1$. This assumption is the most restrictive one. In particular, it means that the results of our paper are not applicable to the Schrödinger operator with a periodic magnetic potential. Indeed, in a forthcoming paper [PS3] it will be shown that under conditions (a)-(c) alone, the conjecture does not hold if we only assume $\alpha < 2l$.

The method of proof in this paper follows very closely that of [P]. However, there are numerous amendments which, while being not too difficult, are not straightforward either. We have written in detail most of the proofs, but occasionally we will refer the reader to [P] if the proof of some statement in our paper is (almost) identical to the corresponding proof in [P] (otherwise the size of our paper would become almost intolerable). Here is the list of all the major changes we had to make to [P] to cater for a bigger class of operators: a generalization of the main approximation lemma has been given to be able to deal with an unbounded perturbation; the definition of the resonance sets Ξ_j has been changed; the proof of the asymptotic formula for eigenvalues in the non-resonance region has been changed (we use the implicit function theorem, which is slightly easier than the method used in [P]); a complication arising from the fact that the mappings F_{ξ_1,ξ_2} in Section 5 do not any longer provide unitary equivalence, has been addressed; finally, Lemma 6.2 now includes two cases (a lying inside and outside a spherical layer of radius 2ρ; the latter case allows a much better estimate) and the rest of Section 6 incorporates this point. Some further changes have been indicated in the text.

We have also made certain changes in order to make exposition simpler. The major such change is as follows: for simplicity, we will always assume that the symbol of the perturbation $a(x, \xi)$ is a trigonometric potential in x, i.e. that

$$a(x, \xi) = \sum_{\theta \in \Gamma^\dagger,\ |\theta| < R} \hat{a}(\theta, \xi) e_\theta(x), \tag{1.1}$$

where

$$e_\xi(x) = d(\Gamma)^{-1/2} e^{ix\cdot\xi} \tag{1.2}$$

and

$$\hat{a}(\theta, \xi) = \int_{\mathcal{O}} a(x, \xi) e_{-\theta}(x) dx. \tag{1.3}$$

Here, R is a fixed number. The general case of $a(x,\xi)$ being merely smooth in x can be treated in the same way as in [P]: for each large ρ we choose $R=\rho^\tau$ with sufficiently small (but fixed) τ and consider the truncated symbol

$$a'(x,\xi)=\sum\sum_{\theta\in\Gamma^\dagger,\ |\theta|<R}\hat{a}(\theta,\xi)e_\theta(x). \tag{1.4}$$

Then the difference between eigenvalues of the original operator $H=H_0+A$ and the truncated operator $H'=H_0+A'$ (A' is the operator with symbol a') is an arbitrarily large negative power of ρ (strictly speaking, we need to replace A with A' after our first cut-off introduced in Corollary 2.6). Then the results would follow if we carefully keep tracing how all the estimates for H' depend on R. This has been done in detail in [P], so in order not to overburden our paper with extra notation, we will assume that $H=H'$, i.e. that the symbol of the perturbation has the form (1.1) for a fixed R.

Setting and notation.

We fix a lattice $\Gamma\subset\mathbf{R}^d$ and denote by $\Gamma^\dagger$ the dual lattice. We denote by $\mathcal{O}$ and $\mathcal{O}^\dagger$ the respective fundamental domains and set $d(\Gamma)=\mathrm{vol}(\mathcal{O})$, $d(\Gamma^\dagger)=\mathrm{vol}(\mathcal{O}^\dagger)$. We denote by $\mathcal{F}u(\xi)$, $\xi\in\mathbf{R}^d$, the Fourier transform of a function $u(x)$ and by $\hat{a}(\theta)$, $\theta\in\Gamma^\dagger$, the Fourier coefficients of a function $a(x)$ which is periodic with respect to the lattice Γ; that is

$$(\mathcal{F}u)(x)=(2\pi)^{-d/2}\int_{\mathbf{R}^d}u(x)e^{-ix\cdot\xi}dx\ ,\qquad \hat{a}(\theta)=d(\Gamma)^{-1/2}\int_{\mathcal{O}}a(x)e^{-ix\cdot\theta}dx.$$

By $\{\xi\}$ we denote the fractional part of a point $\xi\in\mathbf{R}^d$ with respect to the lattice $\Gamma^\dagger$, that is a unique point such that $\{\xi\}\in\mathcal{O}^\dagger$, $\xi-\{\xi\}\in\Gamma^\dagger$. By $f\ll g$ we shall mean that there exists $0<c<\infty$ such that $f\le cg$.

Let $r>0$. A linear subspace $V\subset\mathbf{R}^d$ is called a lattice subspace of dimension n, $1\le n\le d$, if it is spanned by linearly independent vectors $\theta_1,\dots,\theta_n\in\Gamma^\dagger$ each of which has length smaller than r. We denote by $\mathcal{V}(r,n)$ the set of all lattice subspaces of dimension n. We will usually take $r=6MR$ for some fixed and large M and R, so we set for simplicity $\mathcal{V}(n)=\mathcal{V}(6MR,n)$. Given a subspace V we denote by ξ_V and $\xi_V^\perp$ the orthogonal projections of $\xi\in\mathbf{R}^d$ on V and $V^\perp$ respectively. We also define

$$\Theta_j=B(jR)\cap\Gamma^\dagger\ ,\qquad \Theta_j'=\Theta_j\setminus\{0\}.$$

Given $k\in\mathcal{O}^\dagger$ and a set $U\subset\mathbf{R}^d$ we denote by $\mathcal{P}^{(k)}(U)$ the orthogonal projection in $L^2(\mathcal{O})$ onto the subspace spanned by the set $\{e_\xi\ :\ \{\xi\}=k\ ,\ \xi\in U\}$. Given a bounded below self-adjoint operator T with discrete spectrum, we denote by $\{\mu_j(T)\}$ its eigenvalues, written in increasing order and repeated according to multiplicity. By c or C we denote a generic constant whose value may change from one line to another. However the constants c_1,c_2, etc. are fixed throughout. All constants may depend not only on the parameters of the problem, i.e. the order $2l$, the lattice Γ and the symbol $a(x,\xi)$, but also on the number M.

We consider the self-adjoint operator

$$H=(-\Delta)^l+A=:H_0+A \tag{1.5}$$

on $L^2(\mathbf{R}^d)$, where $l > 0$ (not necessarily an integer) and A is a periodic PDO of order $\alpha < 2l - 1$ and periodicity lattice Γ. What we mean by this is that A has the form

$$Au(x) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} a(x,\xi) e^{ix\cdot\xi} (\mathcal{F}u)(\xi) d\xi \,,$$

where the symbol $a(x,\xi)$ is assumed to have the following properties: as a function of x it is C^∞ and periodic with periodicity lattice Γ; moreover, there exists $c > 0$ such that

$$|a(x,\xi)| \le c\langle\xi\rangle^\alpha \;, \quad |\nabla_\xi a(x,\xi)| \le c\langle\xi\rangle^{\alpha-1},$$

for all $x, \xi \in \mathbf{R}^d$; here $\langle\xi\rangle = 1 + |\xi|$. It is standard [RS] that under the above conditions the operator H admits the Bloch-Floquet decomposition: it is unitary equivalent to a direct integral,

$$H \simeq \int_\oplus H(k) dk \;; \tag{1.6}$$

the direct integral is taken over $\mathcal{O}^\dagger$, and for each $k \in \mathcal{O}^\dagger$ the operator $H(k)$ acts on $L^2(\mathcal{O})$ as follows: It has the same symbol as H and it satisfies quasi-periodic boundary conditions depending on k: its domain is given by

$$\mathrm{Dom}(H(k)) = \{u|_\mathcal{O} \;:\; u \in H^{2l}_{loc}(\mathbf{R}^d) \;\;,\;\; u(x+y) = e^{ik\cdot y} u(x) \text{ all } x \in \mathbf{R}^d,\, y \in \Gamma\}.$$

When working with the operator $H(k)$ it is convenient to use the basis $\{e_\xi\}_{\{\xi\}=k} \subset \mathrm{Dom}(H(k))$. In this respect we note that

$$H(k)e_\xi =: H_0(k)e_\xi + A(k)e_\xi = |\xi|^{2l} e_\xi + d(\Gamma)^{-1/2} \sum_{\{\eta\}=k} \hat{a}(\eta - \xi, \xi) e_\eta \,, \tag{1.7}$$

for any ξ with $\{\xi\} = k$. The domain of $H(k)$ is given, equivalently, by

$$\mathrm{Dom}(H(k)) = \{u = \sum_{\{\xi\}=k} u_\xi e_\xi \;:\; \sum_{\{\xi\}=k} |\xi|^{4l} |u_\xi|^2 < \infty\}.$$

It follows from (1.6) that $\sigma(H) = \overline{\cup_k \sigma(H(k))}$. Each $H(k)$ has discrete spectrum, $\sigma(H(k)) = \{\lambda_j(k)\}_{j=1}^\infty$, where the eigenvalues are written in increasing order and repeated according to multiplicity. By standard perturbation theory [K], each $\lambda_j(k)$ is continuous in $k \in \mathcal{O}^\dagger$ and therefore for each j the union $\cup_k \{\lambda_j(k)\}$ is a closed interval $[a_j, b_j]$, known as a spectral band. It follows that $\sigma(H) = \cup_{j=1}^\infty [a_j, b_j]$. In our main theorem we prove that there is only a finite number of spectral gaps. More precisely for $d \ge 3$, we have:

Theorem 1.1 *Suppose that ρ is large enough. Then $\lambda = \rho^{2l}$ belongs to $\sigma(H)$. Moreover, there exists $Z > 0$ such that the interval $[\rho^{2l} - Z\rho^{2l-d-1}, \rho^{2l} + Z\rho^{2l-d-1}]$ lies inside a single spectral band.*

A similar statement is valid for $d = 2$.

2 Preliminary results

2.1 Abstract results

In this subsection we present some abstract theorems about self-adjoint operators with discrete spectra. The following two lemmas have been proved in [P]. Roughly speaking, they state that that the eigenvalues of a perturbed operator $H = H_0 + A$ that lie in a specific interval J are very close to those of $\sum_P PHP$, where $\{P\}$ is a carefully chosen family of eigenprojections of H_0.

Lemma 2.1 *Let H_0, A and B be self-adjoint operators such that H_0 is bounded below and has compact resolvent, and A and B are bounded. Put $H = H_0 + A$ and $\hat{H} = H_0 + A + B$ and denote by $\mu_l = \mu_l(H)$ and $\hat{\mu}_l = \mu_l(\hat{H})$ the sets of eigenvalues of these operators. Let $\{P_j\}$ $(j = 0, \dots, n)$ be a collection of orthogonal projections commuting with H_0 such that $\sum P_j = I$, $P_j A P_k = 0$ for $|j - k| > 1$, and $B = P_n B$. Let l be a fixed number. Denote by a_j the distance from μ_l to the spectrum of $P_j H_0 P_j$. Assume that for $j \geq 1$ we have $a_j > 4a$, where $a := \|A\| + \|B\|$. Then $|\hat{\mu}_l - \mu_l| \leq 2^{2n} a^{2n+1} \prod_{j=1}^{n} (a_j - 2a)^{-2}$.*

Proof. This is Lemma 3.1 of [P].

Lemma 2.2 *Let H_0 and A be self-adjoint operators such that H_0 is bounded below and has compact resolvent and A is bounded. Let $\{P^m\}$ $(m = 0, \dots, n)$ be a collection of orthogonal projections commuting with H_0 such that if $m \neq n$ then $P^m P^n = P^m A P^n = 0$. Denote $Q := I - \sum P^m$. Suppose that each P^m is a further sum of orthogonal projections commuting with H_0: $P^m = \sum_{j=0}^{j_m} P_j^m$ such that $P_j^m A P_l^m = 0$ for $|j - l| > 1$ and $P_j^m A Q = 0$ if $j < j_m$. Let $b := \|A\|$ and let us fix an interval $J = [\lambda_1, \lambda_2]$ on the spectral axis which satisfies the following properties: the spectra of the operators QH_0Q and $P_j^k H_0 P_j^k$, $j \geq 1$ lie outside J; moreover, the distance from the spectrum of QH_0Q to J is greater than $6b$ and the distance from the spectrum of $P_j^k H_0 P_j^k$ $(j \geq 1)$ to J, which we denote by a_j^k, is greater than $16b$. Denote by $\mu_p \leq \dots \leq \mu_q$ all eigenvalues of $H := H_0 + A$ which are inside J. Then the corresponding eigenvalues $\tilde{\mu}_p, \dots, \tilde{\mu}_q$ of the operator*

$$\tilde{H} := \sum_m P^m H P^m + QH_0Q$$

are eigenvalues of $\sum_m P^m H P^m$, and they satisfy

$$|\tilde{\mu}_r - \mu_r| \leq \max_m \left[(6b)^{2j_m+1} \prod_{j=1}^{j_m} (a_j^m - 6b)^{-2} \right];$$

all other eigenvalues of $\tilde{H}$ are outside the interval $[\lambda_1 + 2b, \lambda_2 - 2b]$.

Moreover, there exists an injection G defined on the set of eigenvalues of the operator $\sum_m P^m H P^m$ (all eigenvalues are counted according to their multiplicities) and mapping them to the set of eigenvalues of H (again considered counting multiplicities) such that:

(i) *all eigenvalues of H inside $[\lambda_1 + 2b, \lambda_2 - 2b]$ have a pre-image,*

(ii) *if $\mu_j \in [\lambda_1 + 2b, \lambda_2 - 2b]$ is an eigenvalue of $\sum_m P^m H P^m$, then*

$$|G(\mu_j) - \mu_j| \leq \max_m \left[(6b)^{2j_m+1} \prod_{j=1}^{j_m} (a_j^m - 6b)^{-2} \right],$$

and
(iii) *we have* $G(\mu_j(\sum_m P^m H P^m)) = \mu_{j+l}(H)$, *where* l *is the number of eigenvalues of* QH_0Q *which are smaller than* λ_1.

Proof. This is Lemma 3.2 and Corollary 3.3 of [P].

The last two lemmas involve bounded perturbations. The next proposition is a generalization of Lemma 2.2 to the case where the perturbation is unbounded.

Proposition 2.3 *Let* H_0 *and* A *be self-adjoint operators such that* H_0 *and* $H = H_0 + A$ *are bounded below and have compact resolvents. Assume that* A *is bounded relative to* H_0 *and that there exists* $\epsilon \in (0,1)$ *and* $k_\epsilon > 0$ *such that*

$$|\langle Au, u\rangle| \leq \epsilon\langle H_0 u, u\rangle + k_\epsilon \|u\|^2, \quad u \in \mathrm{Dom}(H_0). \tag{2.8}$$

Let $P_0, \ldots, P_N$ *be orthogonal projections commuting with* H_0 *such that* $P_i A P_j = 0$ *if* $|i-j| > 1$. *Let* $P = \sum_{i=0}^N P_i$, $Q = I - P$, *and assume that* $P_i A Q = 0$ *if* $i < N$. *Assume that* AP *is bounded and let* $b = \|AP\|$. *Let* $J = [\lambda_1, \lambda_2]$ *be an interval on the spectral axis such that*

$$\mathrm{dist}(\sigma(QH_0Q), J) \geq D_1, \quad \mathrm{dist}(\sigma(P_j H_0 P_j), J) \geq D_2\,, \quad j \geq 1, \tag{2.9}$$

where $D_1 := (4b + 2(\epsilon\lambda_2 + k_\epsilon))/(1-\epsilon)$ *and* $D_2 := 20b$. *Let* $\tilde{H} = PHP + QHQ$. *Then the following holds true: given an eigenvalue* $\mu_l(H)$ *of* H *inside* J, *the corresponding eigenvalue* $\mu_l(\tilde{H})$ *of* $\tilde{H}$ *is an eigenvalue of* PHP. *Moreover,* $|\mu_l(H) - \mu_l(\tilde{H})| < 3b/4^N$.

Proof. It follows from our assumptions that $H = \tilde{H} + P_N AQ + QAP_N$. Therefore,

$$\tilde{H} - 2b(P_N + Q) \leq H \leq \tilde{H} + 2b(P_N + Q), \tag{2.10}$$

and, in particular,

$$\mu_l(\tilde{H} - 2b(P_N + Q)) \leq \mu_l(H) \leq \mu_l(\tilde{H} + 2b(P_N + Q)). \tag{2.11}$$

Notice that the operator $\tilde{H} \pm 2b(P_N + Q)$ can be decomposed as $(PHP \pm 2bP_N) \oplus (QHQ \pm 2bQ)$.

Claim. $\mu_l(\tilde{H} + 2b(P_N + Q))$ is an eigenvalue of $PHP + 2bP_N$. Indeed, suppose it is an eigenvalue of $QHQ + 2bQ$, say $\mu_l(\tilde{H} + 2b(P_N + Q)) = \mu_i(QHQ) + 2b$. Then the fact that $\mu_l(H) \in J$ implies that

$$\lambda_1 - 2b \leq \mu_i(QHQ) \leq \lambda_2 + 2b\,. \tag{2.12}$$

Moreover, from (2.8) and min-max we have

$$(1-\epsilon)\mu_i(QH_0Q) - k_\epsilon \leq \mu_i(QHQ) \leq (1+\epsilon)\mu_i(QH_0Q) + k_\epsilon. \tag{2.13}$$

Now, we have either $\mu_i(QH_0Q) \leq \lambda_1 - D_1$ or $\mu_i(QH_0Q) \geq \lambda_2 + D_1$. In the first case (2.12) and (2.13) give $\lambda_1 - 2b \leq (1+\epsilon)(\lambda_1 - D_1) + k_\epsilon$, which contradicts the definition of D_1. In the second case we obtain $(1-\epsilon)(\lambda_2 + D_1) - k_\epsilon \leq \lambda_2 + 2b$, which is also a contradiction. Hence the claim has been proved.

So $\mu_l(\tilde{H} + 2b(P_N + Q))$ is an eigenvalue of $PHP + 2bP_N$, $\mu_l(\tilde{H} + 2b(P_N + Q)) = \mu_i(PHP + 2bP_N)$, say. Let

$$a_j = \mathrm{dist}(\mu_i(PHP), \sigma(P_j H_0 P_j)) \,, \quad 1 \le j \le N.$$

From (2.10) we have $\mu_i(PHP) \in [\lambda_1 - 2b, \lambda_2 + 2b]$. Hence for $j \ge 1$, $a_j \ge D_2 - 2b = 18b$. We can now apply Lemma 2.1 to the unperturbed operator $PHP = PH_0P + PAP$ with the perturbation $B = 2bP_N$. We conclude that

$$|\mu_i(PHP) - \mu_i(PHP + 2bP_N)| \le 2^{2N}(3b)^{2N+1} \prod_{j=1}^{N} (a_j - 6b)^{-2} \le \frac{3b}{4^N},$$

completing the proof of the proposition. □

2.2 Perturbation cut-off

We now return to the operator H introduced in (1.5). We fix $\rho > 0$; our aim is to prove that if ρ is large enough then $\rho^{2l} \in \sigma(H)$. Many of the statements that follow are valid provided ρ is sufficiently large; usually this will not be mentioned explicitly.

Lemma 2.4 *Let $k \in \mathcal{O}^\dagger$ and $U, V \subset \mathbf{R}^d$ be such that* $\mathrm{dist}(U, V) > R$. *Then*

$$\mathcal{P}^{(k)}(V) A(k) \mathcal{P}^{(k)}(U) = 0 \tag{2.14}$$

Proof. We simply note that, because of (1.7), if $\xi \in U$, $\{\xi\} = k$, then $A(k)e_\xi$ is a linear combination of e_η, $\{\eta\} = k$, $\eta \in \xi + B(R)$. □

Let $\chi = \chi_\rho$ be the characteristic function of the set $\{\xi \in \mathbf{R}^d \ : \ ||\xi| - \rho| < \rho/2\}$. Define the projection $\bar{P} = \mathcal{F}^{-1}\chi\mathcal{F}$. We have

Lemma 2.5 *There exists $L > 0$ such that $\|A\bar{P}\| \le L\rho^\alpha$.*

Proof. Let $a_{\bar{P}}(x, \xi) := a(x, \xi)\chi(\xi)$ be the symbol of $A\bar{P}$. Then $A\bar{P} = \sum_{\theta \in \Gamma^\dagger} A_\theta$, where

$$A_\theta u(x) = (2\pi)^{-d/2} d(\Gamma)^{-1/2} \int_{\mathbf{R}^d} \hat{a}_{\bar{P}}(\theta, \xi) e^{i(\theta + \xi)\cdot x} (\mathcal{F}u)(\xi) d\xi \,.$$

The smoothness of $a(x, \xi)$ with respect to x implies that $|\hat{a}(\theta, \xi)| \ll \langle\theta\rangle^{-N}\langle\xi\rangle^\alpha$ for any $N \in \mathbf{N}$; hence

$$|\hat{a}_{\bar{P}}(\theta, \xi)| \ll \langle\theta\rangle^{-N}\rho^\alpha \,, \quad N \in \mathbf{N} \,.$$

It follows that for any $u \in L^2(\mathbf{R}^d)$,

$$\begin{aligned} \|A_\theta u\|^2 &= (2\pi)^{-d} d(\Gamma)^{-1} \int_{\mathbf{R}^d} \left| \int_{\mathbf{R}^d} \hat{a}_{\bar{P}}(\theta, \xi) e^{i\theta\cdot x} e^{i\xi\cdot x} (\mathcal{F}u)(\xi) d\xi \right|^2 dx \\ &= d(\Gamma)^{-1} \int_{\mathbf{R}^d} |\hat{a}_{\bar{P}}(\theta, \xi)(\mathcal{F}u)(\xi)|^2 d\xi \\ &\le c\langle\theta\rangle^{-2N}\rho^{2\alpha}\|u\|^2 \,. \end{aligned}$$

Taking $N > d$, we conclude that

$$\|A\bar{P}\| \leq \sum_{\theta\in\Gamma^\dagger} \|A_\theta\| \ll \rho^\alpha \sum_\theta \langle\theta\rangle^{-N} \ll \rho^\alpha,$$

as required. □

Let $\bar{P}(k)$ be the orthogonal projection on the linear span of the set $\{e_\xi \ :\ \{\xi\} = k\ ,\ ||\xi| - \rho| < \rho/2\}$. It is easily verified that the Floquet decomposition of $A\bar{P}$ is $\int_\oplus A(k)\bar{P}(k)dk$, that is $(A\bar{P})(k) = A(k)\bar{P}(k)$. This implies in particular that

$$\|A(k)\bar{P}(k)\| \leq \|A\bar{P}\| \leq L\rho^\alpha. \tag{2.15}$$

Let $L > 0$ be defined in Lemma 2.5 and put $J = [\rho^{2l} - 100L\rho^\alpha, \rho^{2l} + 100L\rho^\alpha]$.

Corollary 2.6 *Let $k \in \mathcal{O}^\dagger$ be fixed. Let $\bar{P} = \bar{P}(k)$ be the orthogonal projection on the linear span of $\{e_\xi \ :\ \{\xi\} = k\ ,\ ||\xi| - \rho| < \rho/2\}$ and let $Q = I - \bar{P}$. Define $\tilde{H}(k) := \bar{P}H(k)\bar{P}$. Then the following holds true for ρ large enough: given an eigenvalue $\mu_l(H(k))$ of $H(k)$ inside J, the corresponding eigenvalue $\mu_l(\bar{P}H(k)\bar{P} + QH(k)Q)$ is an eigenvalue of $\tilde{H}(k)$; moreover, there exists $c > 0$, independent of $k \in \mathcal{O}^\dagger$, such that $|\mu_l(H(k)) - \mu_l(\bar{P}H(k)\bar{P} + QH(k)Q)| < \exp(-c\rho)$.*

Proof. We shall apply Proposition 2.3 to the operator $H(k) = H_0(k) + A(k)$. We fix a natural number N (to be determined later) and we write $\bar{P} = \oplus_{j=0}^N P_j$, where P_0 is the orthogonal projection on the liner span of

$$\{e_\xi \ :\ \{\xi\} = k\ ,\ ||\xi| - \rho| < \rho/4\},$$

and P_j, $j \geq 1$, is similarly defined for

$$\{e_\xi \ :\ \{\xi\} = k\ ,\ \frac{\rho}{4} + \frac{(j-1)\rho}{4N} \leq ||\xi| - \rho| < \frac{\rho}{4} + \frac{j\rho}{4N}\},$$

It follows from Lemma 2.4 that $P_iA(k)P_j = 0$ if $|i - j| > 1$ and, similarly, $P_jA(k)Q = 0$ for $j < N$. We also note that

$$b := \|A(k)\bar{P}\| \leq L\rho^\alpha \tag{2.16}$$

by (2.15).

The inequality $s^\alpha < s^{2l} + 1$, after substituting $s = |\xi|\epsilon^{\frac{1}{2l-\alpha}}$, implies that $|\xi|^\alpha \leq \epsilon|\xi|^{2l} + \epsilon^{-\alpha/(2l-\alpha)}$ for any $\epsilon > 0$. Thus, for any ξ with $\{\xi\} = k$,

$$\Big|\sum_{\{\eta\}=k} \hat{a}(\xi - \eta, \eta)\Big| \leq \epsilon|\xi|^{2l} + c\epsilon^{-\frac{\alpha}{2l-\alpha}}.$$

This implies that (2.8) is valid with $k_\epsilon = c\epsilon^{-\frac{\alpha}{2l-\alpha}}$. Hence $D_1 \ll \rho^\alpha$. Since $\mathrm{dist}(\sigma(QH_0Q), J) \geq c\rho^{2l}$, the first relation in (2.9) is satisfied; so is the second by a similar argument.

So all assumptions of Proposition 2.3 are fulfilled. We conclude that $|\mu_l(H(k)) - \mu_l(\tilde{H}(k))| < 3b/4^N$. Taking $N = [\rho] - 1$ concludes the proof. □

This corollary shows that we can study the spectrum of $\tilde{H}(k)$ instead of $H(k)$.

3 Reduction to invariant subspaces

The Floquet decomposition and Corollary 2.6 has led us to the study of the eigenvalues of $\tilde{H}(k)$, $k \in \mathcal{O}^\dagger$, that are close to $\lambda = \rho^{2l}$. The operator $\tilde{H}(k)$ which was defined in Corollary 2.6 is a bounded perturbation of $\bar{P}H_0(k)\bar{P}$ (for a fixed ρ), so we can apply Lemma 2.2 to it. This will require a specific choice of the projections $\{P_j^k\}$. Because they have to be invariant for $H_0(k)$, they will be of the form $\mathcal{P}^{(k)}(U)$ for some carefully defined sets $U \subset \mathbf{R}^d$, localized near $|\xi| = \rho$.

We define the spherical layer

$$\mathcal{A} = \{\xi \in \mathbf{R}^d \;:\; ||\xi|^{2l} - \rho^{2l}| < 100L\rho^\alpha\}.$$

It has width of order $\rho^{\alpha-2l+1}$. Note that for all $\xi \in \mathcal{A}$ we have

$$||\xi|^2 - \rho^2| \ll \rho^{\alpha-2l+2}. \tag{3.17}$$

We fix numbers $q_0, q_1, \ldots, q_{d-1}$ and γ such that

$$0 < q_0 < q_1 < \ldots < q_{d-1} < 1 \qquad , \qquad \alpha < \gamma < 2l - 2 + q_0 \;, \tag{3.18}$$

and also

$$q_1 \geq \frac{3 + \alpha - 2l}{2}. \tag{3.19}$$

The existence of such numbers follows from our assumption $\alpha < 2l - 1$. We also define

$$\epsilon_0 := \frac{1}{100}\min\{2l - 2 + q_0 - \alpha, \gamma - \alpha, 1 - q_{d-1}, \min_s(q_s - q_{s-1})\}$$

Given a lattice subspace $V \in \mathcal{V}(n)$, we define the sets

$$\begin{aligned}
&\Xi_0(V) = \{\xi \in \mathcal{A} \;:\; |\xi_V| < \rho^{q_n}\};\\
&\Xi_1(V) = (\Xi_0(V) + V) \cap \mathcal{A}\;;\\
&\Xi_2(V) = \Xi_1(V) \setminus \Big(\bigcup_{m=n+1}^{d-1} \bigcup_{W \in \mathcal{V}(m), W \supset V} \Xi_1(W)\Big)\;;\\
&\Xi_3(V) = \{\xi = \eta + \theta \;:\; \eta \in \Xi_2(V)\;,\; \theta \in V \cap \Gamma^\dagger\;,\; ||\xi|^{2l} - \rho^{2l}| < \rho^\gamma\};\\
&\Xi(V) = \Xi_3(V) + \Theta_M\;.
\end{aligned}$$

We also define

$$\mathcal{D} = \bigcup_{m=1}^{d-1} \bigcup_{W \in \mathcal{V}(m)} \Xi_2(W), \qquad \mathcal{B} = \mathcal{A} \setminus \mathcal{D}.$$

Note that $\Xi_2(V) \subset \Xi_3(V)$ by (3.18) and $\Xi_3(\{0\}) = \mathcal{B}$. We also have

$$||\xi|^2 - \rho^2| < c\rho^{\gamma-2l+2} \tag{3.20}$$

for all $\xi \in \Xi_3(V)$.

We now proceed to establish further properties of these sets. Let us stress again that in what follows we shall often implicitly assume that ρ is sufficiently large.

Lemma 3.1 *Let $V \in \mathcal{V}(n)$, $0 \leq n \leq d - 1$. Then for $\xi \in \Xi_1(V)$ we have $|\xi_V| < 2\rho^{q_n}$.*

Proof. Let $\xi' \in \Xi_0(V)$ be such that $\xi - \xi' \in V$. Then

$$|\xi_V|^2 \leq ||\xi_V|^2 - |\xi'_V|^2| + |\xi'_V|^2 = ||\xi|^2 - |\xi'|^2| + |\xi'_V|^2 < c\rho^{\alpha-2l+2} + \rho^{2q_n} ,$$

from which the result follows. □

Lemma 3.2 *Let $V \in \mathcal{V}(n)$, $0 \leq n \leq d-1$. Then for $\xi \in \Xi_3(V)$ we have $|\xi_V| < 2\rho^{q_n}$.*

Proof. Let us write $\xi = \eta + \theta$ with $\eta \in \Xi_2(V)$, $\theta \in V \cap \Gamma^\dagger$. Then, by (3.17),

$$|\xi_V^\perp|^2 = |\eta|^2 - |\eta_V|^2 \geq \rho^2 - c\rho^{\alpha-2l+2} - \rho^{2q_n},$$

and therefore, by (3.20),

$$|\xi_V|^2 = |\xi|^2 - |\xi_V^\perp|^2 \leq (\rho^2 + c\rho^{\gamma-2l+2}) - (\rho^2 - c\rho^{\alpha-2l+2} - \rho^{2q_n}) \leq 2\rho^{2q_n},$$

and the result follows. □

Corollary 3.3 *If $\xi \in \Xi(V)$, then $|\xi_V| \ll \rho^{q_n}$.*

Lemma 3.4 *Let $V \in \mathcal{V}(n)$, $0 \leq n \leq d-1$. Let $\xi \in \Xi_3(V)$, $\xi = \eta + \theta$ with $\eta \in \Xi_2(V)$ and $\theta \in V \cap \Gamma^\dagger$. Then $|\theta| \ll \rho^{q_n}$.*

Proof. We have $|\theta| = |\theta_V| \leq |\xi_V| + |\eta_V| \leq c\rho^{q_n}$. □

Lemma 3.5 *If $\xi \in \Xi_3(V)$ then*

$$\text{(i)} \ \ ||\xi| - \rho| < c\rho^{\gamma-2l+1} \quad , \quad \text{(ii)} \ \ ||\xi_V^\perp| - \rho| < c\rho^{2q_n-1} . \tag{3.21}$$

Proof. Part (i) follows directly from the definition of $\Xi_3(V)$. Part (ii) follows from (i) and Lemma 3.2. □

The following geometric lemma will be used repeatedly in what follows.

Lemma 3.6 *We have*

$$|\xi_{V_1+V_2}| \ll |\xi_{V_1}| + |\xi_{V_2}|, \tag{3.22}$$

for any two subspaces V_1 and V_2 generated by vectors in $\Gamma^\dagger \cap B(R)$.

Proof. Suppose that V_1 and V_2 are two lattice subspaces such that $W \neq V_1$ and $W \neq V_2$, where $W := V_1 + V_2$ (otherwise the estimate is trivial). Let ϕ be the angle between V_1 and V_2. This means that ϕ is a minimum of angles between ξ_1 and ξ_2 where $\xi_j \in U_j := W \ominus V_j$ (the orthogonal complement). Then a simple geometry shows that

$$|\xi_{V_1+V_2}| \leq \frac{|\xi_{V_1}| + |\xi_{V_2}|}{\sin(\phi/2)}. \tag{3.23}$$

Since the number of pairs (V_1, V_2) is finite, this proves the statement. □

Remark 3.7 *The key part of extending the proof to the case when the symbol a is just smooth in x (and is no longer a trigonometric polynomial in x) is checking how the constant in (3.22) depends on R. This has been done in detail in section 4 of [P].*

Lemma 3.8 *Let $V_i \in \mathcal{V}(n_i)$, $i = 1, 2$, be two lattice subspaces such that neither of them is contained in the other and assume that $n_2 \geq n_1$. Then for any $\xi_i \in \Xi_2(V_i)$, $i = 1, 2$, we have: $|\xi_1 - \xi_2| > \rho^{q_{n_2}+\epsilon_0}$.*

Proof. Suppose to the contrary that $|\xi_1 - \xi_2| \leq \rho^{q_{n_2}+\epsilon_0}$. From Lemma 3.1 we have $|(\xi_i)_{V_i}| \leq 2\rho^{q_{n_i}}$ and therefore

$$|(\xi_1)_{V_2}| \leq |\xi_1 - \xi_2| + |(\xi_2)_{V_2}| \leq \rho^{q_{n_2}+\epsilon_0} + 2\rho^{q_{n_2}} \leq 2\rho^{q_{n_2}+\epsilon_0} \ .$$

Letting $W = V_1 + V_2$ and $m = \dim(W)$ we have $m > n_2$ and hence, by (3.22),

$$|(\xi_1)_W| < c(|(\xi_1)_{V_1}| + |(\xi_1)_{V_2}|) < c(\rho^{q_{n_1}} + 2\rho^{q_{n_2}+\epsilon_0}) < \rho^{q_m}.$$

Hence $\xi_1 \in \Xi_1(W)$, which is a contradiction. □

Lemma 3.9 *Let $V_i \in \mathcal{V}(n_i)$, $i = 1, 2$, be two lattice subspaces such that neither of them is contained in the other. Then for any $\xi_i \in \Xi_3(V_i)$, $i = 1, 2$, there holds $|\xi_1 - \xi_2| > \max\{\rho^{q_{n_1}}, \rho^{q_{n_2}}\}$.*

Proof. Assume that $n_2 \geq n_1$. Writing $\xi_i = \eta_i + \theta_i$ with $\eta_i \in \Xi_2(V_i)$ and $\theta_i \in V_i \cap \Gamma^\dagger$, we have from Lemmas 3.4 and 3.8,

$$|\xi_1 - \xi_2| \geq |\eta_1 - \eta_2| - |\theta_1| - |\theta_2| \geq \rho^{q_{n_2}+\epsilon_0} - c\rho^{q_{n_2}} \geq \rho^{q_{n_2}} \ .$$

Proposition 3.10 *If $V_i \in \mathcal{V}(n_i)$, $i = 1, 2$, are two different lattice subspaces, then $(\Xi(V_1) + \Theta_M) \cap (\Xi(V_2) + \Theta_M) = \emptyset$.*

Proof. If neither of V_1, V_2 is contained in the other, then the result follows from Lemma 3.9, so we assume that $V_1 \subset V_2$. We consider $\xi_i \in \Xi_3(V_i)$ and shall prove that the difference $\theta = \xi_1 - \xi_2$ cannot belong to Θ_{4M}. Let $\eta_i \in \Xi_2(V_i)$ and $\theta_i \in V_i \cap \Gamma^\dagger$ be such that $\xi_i = \eta_i + \theta_i$. We distinguish two cases:
(i) $\theta \in V_2$. In this case we write $\eta_2 = \tilde{\eta} + a$ with $\tilde{\eta} \in \Xi_0(V_2)$ and $a \in V_2$. We then obtain $\eta_1 = \tilde{\eta} + (a + \theta_2 + \theta - \theta_1) \in \Xi_1(V_2)$, which contradicts the fact that $\eta_1 \in \Xi_2(V_1)$.
(ii) $\theta \notin V_2$. We argue again by contradiction, assuming that $\theta \in \Theta_{4M}$. Then, in particular, $|\theta| \gg 1$. We claim that $|\eta_2 \cdot \theta| > \rho^{q_{n_2}+\epsilon_0}|\theta|$; indeed, if this were not the case then we would have with U being a linear span of V_2 and θ (and thus a lattice subspace):

$$\begin{aligned} |(\eta_2)_U| &\leq c(|(\eta_2)_{V_2}| + |\eta_2 \cdot \theta|) \\ &\leq c(\rho^{q_{n_2}} + \rho^{q_{n_2}+\epsilon_0}) \\ &\leq \rho^{q_{n_2+1}}. \end{aligned}$$

Therefore, $\eta_2 \in \Xi_1(U)$, which is a contradiction. Hence

$$|\xi_2 \cdot \theta| \geq |\eta_2 \cdot \theta| - |\theta_2 \cdot \theta| \geq \rho^{q_{n_2}+\epsilon_0} - 12R\rho^{q_{n_2}} \geq \rho^{q_{n_2}+\epsilon_0}/2 \ ,$$

and therefore

$$\begin{aligned} ||\xi_1|^{2l} - \rho^{2l}| &\geq ||\xi_2 + \theta|^{2l} - |\xi_2|^{2l}| - ||\xi_2|^{2l} - \rho^{2l}| \\ &\geq c\rho^{2l-2+q_{n_2}+\epsilon_0} - \rho^\gamma \\ &\geq \rho^\gamma \ , \end{aligned}$$

which is a contradiction. □

Proposition 3.10 is one of the main results of this section. We now state some additional lemmas which will also be useful in what follows.

Lemma 3.11 *Let $\xi \in \Xi_3(V)$, $V \in \mathcal{V}(n)$, and $\theta \in \Theta_{2M}$, $\theta \notin V$. Then $||\xi+\theta|^{2l} - \rho^{2l}| > \rho^{2l-2+q_n}$.*

Proof. Let $\xi = \eta + \theta'$, $\eta \in \Xi_2(V)$, $\theta' \in V \cap \Gamma^\dagger$. Then $|\xi \cdot \theta| > \rho^{q_n+\epsilon_0}$ since otherwise we would have

$$|\eta \cdot \theta| \leq |\xi \cdot \theta| + |\theta' \cdot \theta| \leq \rho^{q_{n_2}+\epsilon_0} + c\rho^{q_n} \leq 2\rho^{q_n+\epsilon_0}|\theta|$$

and therefore $\eta \in \Xi_1(V + \{t\theta : t \in \mathbf{R}\})$ by (3.22). Hence

$$\begin{aligned} ||\xi+\theta|^{2l} - \rho^{2l}| &\geq ||\xi+\theta|^{2l} - |\xi|^{2l}| - ||\xi|^{2l} - \rho^{2l}| \\ &\geq c\rho^{2l-2}\rho^{q_n+\epsilon_0} - \rho^{\gamma} \\ &> \rho^{2l-2+q_n}. \end{aligned}$$

□

Lemma 3.12 *Let $V \in \mathcal{V}(n)$ and let $\xi \in \Xi_3(V)$ and $\theta \in V \cap \Gamma^\dagger$. If $\xi + \theta \notin \Xi_3(V)$ then $||\xi+\theta|^{2l} - \rho^{2l}| \geq \rho^{\gamma}$.*

Proof. Let $\xi = \eta' + \theta'$, $\eta' \in \Xi_2(V)$, $\theta' \in V \cap \Gamma^\dagger$. Then $\xi + \theta = \eta' + (\theta + \theta') \in \Xi_2(V) + (V \cap \Gamma^\dagger)$, and the result follows from the definition of $\Xi_3(V)$. □

Corollary 3.13 *Let $\xi \in \Xi_3(V)$ and $\theta \in \Theta_{2M}$. If $\xi + \theta \notin \Xi_3(V)$ then $||\xi+\theta|^{2l} - \rho^{2l}| > \rho^{\gamma}$.*

Proof. The result follows from Lemma 3.11 if $\theta \in V$ and from Lemma 3.12 if $\theta \notin V$. □

We can now use the results obtained so far and apply Lemma 2.2 in our context. Let $k \in \mathcal{O}^\dagger$ be fixed and let the projection $\bar{P} = \bar{P}(k)$ be as in Corollary 2.6. Given a lattice subspace $V \in \mathcal{V}(n)$, $0 \leq n \leq d-1$, we set $P(V) = \mathcal{P}^{(k)}(\Xi(V))$. The above statements imply that $P(V)\bar{P} = P(V)$. The next proposition provides information about the proximity of the parts of $\sigma(H(k))$ (or $\sigma(\bar{P}H(k)\bar{P})$) and $\sigma(\sum_V P(V)H(k)P(V))$ that lie near ρ. The sum is taken over all lattice subspaces V.

Proposition 3.14 *There exists a map G from the set of all eigenvalues of the operator $\bar{H}(k) := \sum_V P(V)H(k)P(V)$ that lie in J into the set of all eigenvalues of $\bar{P}H(k)\bar{P}$ such that whenever $\mu_l(\sum_V PH(k)P) \in J$, we have:*

$$|\mu_l(\sum_V P(V)H(k)P(V)) - G(\mu_l(\sum_V P(V)H(k)P(V)))| \leq c\rho^{-2M(\gamma-\alpha)+\alpha}.$$

This mapping is an injection and all eigenvalues of $\bar{P}H(k)\bar{P}$ inside $J_1 := [\lambda - 90L, \lambda + 90L]$ have a pre-image under G.

Proof. We apply Lemma 2.2 to the operator $\bar{P}H(k)\bar{P} = \bar{P}H_0(k)\bar{P} + \bar{P}A(k)\bar{P}$. We use the projections $\{P(V)\}$, where V ranges over all possible lattice subspaces of dimension smaller than d; these are orthogonal by Proposition 3.10. Each $P(V)$ is further writen as a sum of orthogonal and invariant (for $H_0(k)$) projections, $P(V) = \sum_{j=0}^{M} P_j(V)$, where

$$\begin{aligned} &P_0(V) = \mathcal{P}^{(k)}(\Xi_3(V)), \\ &P_j(V) = \mathcal{P}^{(k)}((\Xi_3(V) + \Theta_j) \setminus (\Xi_3(V) + \Theta_{j-1})), \quad j = 1, \ldots, M. \end{aligned}$$

It follows from Lemma 2.4 that

$$P(V_1)A(k)P(V_2) = 0 \text{ if } V_1 \neq V_2 \,,$$
$$P_j(V)A(k)P_l(V) = 0 \text{ if } |j-l| > 1 \,,$$
$$P_j(V)A(k)(\bar{P} - \sum_V P(V)) = 0 \text{ if } j < M \,.$$

Let us also check that the remaining two conditions of Lemma 2.2 are satisfied. We have

$$\begin{aligned} & \sigma((\bar{P} - \sum_V P(V))H_0(k)(\bar{P} - \sum_V P(V))) \\ = \;& \{|\xi|^{2l} \;:\; \{\xi\} = k \,,\; ||\xi|^{2l} - \rho^{2l}| < \rho^{2l}/2 \,,\; \xi \notin \cup_V \Xi(V)\} \\ \subset \;& \{|\xi|^{2l} \;:\; \xi \notin \mathcal{A}\}, \end{aligned}$$

and therefore

$$\mathrm{dist}(\sigma((\bar{P} - \sum_V P(V))H_0(k)(\bar{P} - \sum_V P(V))), J) \geq 6\|\bar{P}A(k)\bar{P}\|$$

We also note that for $1 \leq j \leq M$,

$$\begin{aligned} \sigma(P_j(V)H_0(k)P_j(V)) &= \{|\xi|^{2l} \;:\; \{\xi\} = k \,,\; \xi \in (\Xi_3(V) + \Theta_j) \setminus (\Xi_3(V) + \Theta_{j-1})\} \\ &\subset \{|\xi|^{2l} \;:\; \xi \in (\Xi_3(V) + \Theta_M) \setminus \Xi_3(V)\} \,. \end{aligned}$$

Corollary 3.13 together with the fact that $\alpha < \gamma$ imply that $a_j(V) \geq c\rho^{\gamma}$ and, in particular,

$$a_j(V) := \mathrm{dist}(\sigma(P_j(V)H_0(k)P_j(V)), J) \geq 16\|\bar{P}A(k)\bar{P}\| \,,\;\; j \geq 1 \,.$$

Hence Lemma 2.2 can be applied. We conclude that there exists an injection G from the set of all eigenvalues of $\sum_V P(V)H(k)P(V)$ that lie in J into the set of all eigenvalues of $\bar{P}H(k)\bar{P}$ such that for any $\mu_l(\sum_V P(V)H(k)P(V)) \in J$ there holds

$$\begin{aligned} & |\mu_l(\sum_V P(V)H(k)P(V)) - G(\mu_l(\sum_V P(V)H(k)P(V)))| \\ \leq \;& \max_V \left[(6\|\bar{P}A(k)\bar{P}\|)^{2M+1} \prod_{j=1}^{M} (a_j(V) - 6\|\bar{P}A(k)\bar{P}\|)^{-2} \right] \\ \leq \;& c\rho^{-2M(\gamma-\alpha)+\alpha}; \end{aligned}$$

this completes the proof of the proposition. □

Let us briefly describe the aim of the next two sections. Proposition 3.14 has led to the study of the operators $P^{(k)}(\Xi(V))H(k)P^{(k)}(\Xi(V))$, where V is a lattice subspace and $k \in \mathcal{O}^{\dagger}$. It will be proved that for fixed k and V we have a direct sum decomposition

$$P^{(k)}(\Xi(V))H(k)P^{(k)}(\Xi(V)) = \bigoplus_{\xi} H(\xi)$$

where the sum is taken over certain $\xi \in \Xi_2(V)$ with $\{\xi\} = k$ and $H(\xi)$ are certain operators. Rather than keeping k fixed, we intend to study the spectrum of $H(\xi)$ as ξ varies continuously. For each ξ we shall choose a specific eigenvalue $\tilde{g}(\xi)$; the choice is such that $\tilde{g}(\xi) = |\xi|^{2l}$ in the unperturbed case. We then study how $\tilde{g}(\xi)$ varies as ξ varies. This turns out to depend on the location of ξ. If $\xi \in \mathcal{B}$ (resonance region), then $\tilde{g}(\xi)$ varies smoothly with ξ. If however $\xi \in \mathcal{D}$ (non-resonance region), then we do not have this good dependence anymore. In this case a new function $g(\xi)$ is introduced, which is very close to $\tilde{g}(\xi)$ and is smooth along one direction only.

4 Non-resonance region

Lemma 4.1 *If $\xi \in \mathcal{B}$ and $\theta \in \Theta'_M$ then $||\xi + \theta|^{2l} - \rho^{2l}| \gg \rho^{2l-2+q_1}$.*

Proof. We have $|\xi \cdot \theta| \geq \rho^{q_1}$, since otherwise $\xi \in \Xi_1(\{t\theta : t \in \mathbf{R}\})$. Hence, since $\alpha < 2l - 2 + q_1$,

$$\begin{aligned} ||\xi + \theta|^{2l} - \rho^{2l}| &\geq ||\xi + \theta|^{2l} - |\xi|^{2l}| - ||\xi|^{2l} - \rho^{2l}| \\ &\geq c\rho^{2l-2+q_1} - 100L\rho^{\alpha} \\ &\geq c\rho^{2l-2+q_1}, \end{aligned}$$

as required. □

Let us fix a $\xi \in \mathcal{B}$ with $\{\xi\} =: k$. From (2.15) we have

$$\|\mathcal{P}^{(k)}(\xi + \Theta_M)A(k)\mathcal{P}^{(k)}(\xi + \Theta_M)\| \leq L\rho^{\alpha}. \tag{4.24}$$

Lemma 4.2 *There exists a unique eigenvalue $\tilde{g}(\xi)$ of $\mathcal{P}^{(k)}(\xi + \Theta_M)H(k)\mathcal{P}^{(k)}(\xi + \Theta_M)$ which lies within distance $L\rho^{\alpha}$ of $|\xi|^{2l}$.*

Proof. The existence of such an eigenvalue follows from (4.24) and the min-max principle. To prove its uniqueness we argue by contradiction: let us assume that there exist two such eigenvalues. Then there exists an eigenvalue of $\mathcal{P}^{(k)}(\xi + \Theta_M)H_0(k)\mathcal{P}^{(k)}(\xi + \Theta_M)$ which is different from $|\xi|^{2l}$ and which is within distance $2L\rho^{\alpha}$ of $|\xi|^{2l}$. Hence $||\xi + \theta|^{2l} - |\xi|^{2l}| < 2L\rho^{\alpha}$ for some $\theta \in \Theta'_M$. This implies that

$$||\xi + \theta|^{2l} - \rho^{2l}| \leq (102L)\rho^{\alpha} ,$$

contradicting Lemma 4.1. □

We shall obtain some more information on the eigenvalues $\tilde{g}(\xi)$, $\xi \in \mathcal{B}$. First, we observe that the matrix of $\mathcal{P}^{(k)}(\xi + \Theta_M)H(k)\mathcal{P}^{(k)}(\xi + \Theta_M)$ (with respect to the basis $\{e_{\xi+\theta}\}_{\theta \in \Theta_M}$ of $\mathrm{Ran}\mathcal{P}^{(k)}(\xi + \Theta_M)$ and for some ordering $0 = \theta_0, \theta_1, \theta_2, \dots$ of Θ_M) has the form

$$\begin{pmatrix} a_{00}(\xi) & a_{0\theta_1}(\xi) & a_{0\theta_2}(\xi) & \dots \\ a_{\theta_1 0}(\xi) & a_{\theta_1\theta_1}(\xi) & a_{\theta_1\theta_2}(\xi) & \dots \\ a_{\theta_2 0}(\xi) & a_{\theta_2\theta_1}(\xi) & a_{\theta_2\theta_2}(\xi) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4.25}$$

where (cf. (1.7))

$$a_{\theta\theta'}(\xi) = \begin{cases} |\xi+\theta|^{2l} + (2\pi)^{-d/2}\hat{a}(0,\xi+\theta), & \theta = \theta', \\ (2\pi)^{-d/2}\hat{a}(\theta-\theta',\xi+\theta'), & \theta \neq \theta'. \end{cases}$$

The size of this matrix is fixed and does not depend on ρ. Expanding the determinant we find that the characteristic polynomial $p(\mu)$ can be written as

$$p(\mu) = \Big(\prod_{\theta\in\Theta'_M} (a_{\theta\theta}(\xi)-\mu) \Big) \Big(a_{00}(\xi) - \mu + I(\xi,\mu) \Big). \tag{4.26}$$

The function $I(\xi,\mu)$ is a (finite) sum $I(\xi,\mu) = I_1 + I_2 + \tilde{I}_2 + \dots$ where each I_n is a linear combination of terms of the form

$$T_n(\xi,\mu) = \frac{P_{n+1}(\xi;\theta_1,\theta_2,\dots)}{(a_{\theta_{i_1}\theta_{i_1}}(\xi)-\mu)(a_{\theta_{i_2}\theta_{i_2}}(\xi)-\mu)\dots(a_{\theta_{i_n}\theta_{i_n}}(\xi)-\mu)} \tag{4.27}$$

and $\tilde{I}_n$ is a linear combination of terms of the form

$$\tilde{T}_n(\xi,\mu) = \frac{(a_{00}(\xi)-\mu)P_n(\xi;\theta_1,\theta_2,\dots)}{(a_{\theta_{i_1}\theta_{i_1}}(\xi)-\mu)(a_{\theta_{i_2}\theta_{i_2}}(\xi)-\mu)\dots(a_{\theta_{i_n}\theta_{i_n}}(\xi)-\mu)}\ ; \tag{4.28}$$

here P_k stands for a polynomial of degree k in the off-diagonal terms $a_{\theta\theta'}(\xi)$, $\theta \neq \theta'$, of the above matrix. We restrict our attention to μ inside the interval $J_\xi := [|\xi|^{2l} - L\rho^\alpha, |\xi|^{2l} + L\rho^\alpha]$ where we already know that the equation $p(\mu) = 0$ has $\tilde{g}(\xi)$ as its unique solution.

Lemma 4.3 *For $\theta \in \Theta'_M$ and $\mu \in J_\xi$ we have $|a_{\theta\theta}(\xi) - \mu| \gg \rho^{2l-2+q_1}$.*

Proof. From Lemma 4.1 we have

$$\begin{aligned} |a_{\theta\theta}(\xi)-\mu| &\geq ||\xi+\theta|^{2l} - \rho^{2l}| - |\rho^{2l} - |\xi|^{2l}| - ||\xi|^{2l} - \mu| - (2\pi)^{-d/2}|\hat{a}(0,\xi+\theta)| \\ &\geq c\rho^{2l-2+q_1} - 100L\rho^\alpha - L\rho^\alpha - c\rho^\alpha \\ &\geq c\rho^{2l-2+q_1}\ . \end{aligned}$$

□

It follows from (4.26) and Lemma 4.3 that $\tilde{g}(\xi)$ is the (unique in J_ξ) solution of

$$a_{00}(\xi) - \mu + I(\xi,\mu) = 0\ . \tag{4.29}$$

Lemma 4.4 *We have $|\partial I/\partial\mu| \ll \rho^{-(2l-2-\alpha+q_1)}$ uniformly over all $\mu \in J_\xi$.*

Proof. Let T_n and $\tilde{T}_n$ be as in (4.27) and (4.28) respectively. Using Lemma 4.3 we obtain by a direct computation that

$$\Big|\frac{\partial T_n}{\partial\mu}\Big| \leq c\rho^{-(n+1)(2l-2-\alpha+q_1)} \quad , \quad \Big|\frac{\partial \tilde{T}_n}{\partial\mu}\Big| \leq c\rho^{-(n+1)(2l-2-\alpha+q_1)}\ ;$$

the result follows. □

Proposition 4.5 *We have*

$$\tilde{g}(\xi) = |\xi|^{2l} + G(\xi) \ , \ \ \xi \in \mathcal{B} \, , \tag{4.30}$$

where G is a differentiable function satisfying

$$\begin{array}{ll} \text{(i)} & |G(\xi)| \ll \rho^{\alpha} \, ; \\ \text{(ii)} & |\nabla G(\xi)| \ll \rho^{\alpha-1} . \end{array}$$

Proof. We shall only prove (ii), the proof of part (i) being similar and simpler. Let us define G by (4.30). From (4.29) we have

$$I(\xi, |\xi|^{2l} + G(\xi)) - G(\xi) + (2\pi)^{-d/2} \hat{a}(0, \xi) = 0 \, . \tag{4.31}$$

Defining

$$F(\xi, t) = I(\xi, |\xi|^{2l} + t) - t + (2\pi)^{-d/2} \hat{a}(0, \xi) \ , \qquad \xi \in \mathcal{B} \ , \ |t| < L\rho^{\alpha} \ ,$$

we thus obtain $F(\xi, G(\xi)) = 0$ on $\mathcal{B}$. From Lemma 4.4 we have $|\partial F/\partial t| \geq 1/2$, so an application of the implicit function theorem yields that G is differentiable and

$$|\nabla G| \leq 2 \, |\nabla_{\xi} F| \ .$$

Hence it remains to estimate the partial derivatives $\partial F/\partial \xi_i$. Note that $I_n(\xi, |\xi|^2 + t)$ is a linear combination of terms of the form

$$T_n(\xi, |\xi|^{2l} + t) = \frac{P_{n+1}(\xi; \theta_1, \theta_2, \ldots)}{\prod_{j=1}^{n} \Big(|\xi + \theta_{i_j}|^{2l} - |\xi|^{2l} + (2\pi)^{-d/2} \hat{a}(0, \xi + \theta_{i_j}) - t \Big)} . \tag{4.32}$$

By Lemma 4.3 each factor in the denominator is larger in absolute value than $c\rho^{2l-2+q_1}$. Also, the derivative of each such factor with respect to ξ_i does not exceed $c\rho^{2l-2}$. Similarly we have $|P_{n+1}| \leq c\rho^{(n+1)\alpha}$ and $|\partial P_{n+1}/\partial \xi_i| \leq c\rho^{(n+1)\alpha-1}$. These facts imply that the partial derivatives with respect to ξ_i of the RHS of (4.32) are estimated by $c\rho^{-(2l-2+q_1-\alpha)n+\alpha-q_1}$. The argument is similar for $\tilde{I}_n(\xi, |\xi|^2 + t)$ which is a linear combination of terms

$$\tilde{T}_n(\xi, |\xi|^{2l} + t) = \frac{((2\pi)^{-d/2} \hat{a}(0, \xi) - t) P_n(\xi; \theta_1, \theta_2, \ldots)}{\prod_{j=1}^{n} \Big(|\xi + \theta_{i_j}|^{2l} - |\xi|^{2l} + \hat{a}(0, \xi + \theta_{i_j}) - t \Big)} \ .$$

Similar calculations show that the partial derivatives with respact to ξ of this expression are also smaller than $c\rho^{-(2l-2+q_1-\alpha)n+\alpha-q_1}$. The worst estimate corresponds to $n = 1$; recalling (3.19) completes the proof of (ii). □

5 Resonance region

We shall now study the eigenvalues of $P(V)H(k)P(V)$, where $V \in \mathcal{V}(n)$, $1 \leq n \leq d-1$, is fixed. Let $\xi \in \Xi_2(V)$ be given and let $k = \{\xi\}$. We define

$$\begin{aligned} \tilde{Y}(\xi) &= (\xi + (V \cap \Gamma^{\dagger})) \cap \Xi_3(V) \ , & Y(\xi) &= \tilde{Y}(\xi) + \Theta_M \ , \\ P(\xi) &= \mathcal{P}^{(k)}(Y(\xi)) \ , & H(\xi) &= P(\xi)H(k)P(\xi) \ , \\ H_0(\xi) &= P(\xi)H_0(k)P(\xi) \ , & A(\xi) &= P(\xi)A(k)P(\xi) \ . \end{aligned}$$

Recalling the decomposition $\xi = \xi_V + \xi_V^\perp$, $\xi_V \in V$, $\xi_V^\perp \in V^\perp$, we also define

$$r(\xi) = |\xi_V^\perp| \quad , \quad \xi_V' = \xi_V^\perp / r(\xi) \; .$$

We note that $r(\xi) \asymp \rho$ by (3.21). The triple $(r(\xi), \xi_V', \xi_V)$ can be thought of as cylindrical coordinates of the point $\xi \in \Xi_2(V)$.

Lemma 5.1 (i) *The sets $Y(\xi)$, $\xi \in \Xi_2(V)$, either coincide or are disjoint.*
(ii) *If $Y(\xi_1) = Y(\xi_2)$ then $\xi_1 - \xi_2 \in V$ and, in particular, $r(\xi_1) = r(\xi_2)$.*

Proof. Assume that $Y(\xi_1) \cap Y(\xi_2) \neq \emptyset$. Then there exist $\xi_j \in \Xi_3(V)$ and $\theta_j \in \Theta_M$, $j = 1, 2$, such that $\xi_1 + \theta_1 = \xi_2 + \theta_2$. We claim that the difference $\theta_1 - \theta_2$ lies in V. Indeed, suppose it does not. Then the relation $\xi_2 = \xi_1 + (\theta_1 - \theta_2)$ together with Lemma 3.11 yields $||\xi_2|^{2l} - \rho^{2l}| > \rho^{2l-2+q_n} \geq \rho^\gamma$, contradicting the fact that $\xi_2 \in \Xi_3(V)$. Hence $\xi_2 - \xi_1 \in V$, and both (i) and (ii) follow. □

Part (i) of Lemma 5.1 points to an equivalence relation defined on $\Xi_2(V)$, whereby $\xi_1 \sim \xi_2$ if and only if $Y(\xi_1) = Y(\xi_2)$. We thus have for each $k \in \mathcal{O}^\dagger$ the direct sum decomposition

$$P(V)H(k)P(V) = \bigoplus H(\xi) \, , \tag{5.33}$$

where the sum is taken over all equivalence classes of this relation with $\{\xi\} = k$.

Hence we intend to study the operators $H(\xi)$, $\xi \in \Xi_2(V)$. In fact, we shall compare the eigenvalues of two such operators $H(\xi_1)$ and $H(\xi_2)$; this will be carried out using auxiliary operators denoted by $H(\xi, U)$, where $\xi \in \Xi_2(V)$ and U is a subset of $\Xi_2(V)$ containing ξ. We therefore introduce some additional definitions: given $\xi_1, \xi_2 \in \Xi_2(V)$ and letting $k_1 = \{\xi_1\}$ we set

$$\begin{aligned} &Y(\xi_1, \xi_2) = Y(\xi_1) \cup (Y(\xi_2) - \xi_2 + \xi_1) \, , \quad && P(\xi_1, \xi_2) = \mathcal{P}^{(k_1)}(Y(\xi_1, \xi_2)) \, , \\ &H(\xi_1, \xi_2) = P(\xi_1, \xi_2) H(k_1) P(\xi_1, \xi_2) \, , && H_0(\xi_1, \xi_2) = P(\xi_1, \xi_2) H_0(k_1) P(\xi_1, \xi_2) \, , \\ &A(\xi_1, \xi_2) = P(\xi_1, \xi_2) A(k_1) P(\xi_1, \xi_2) \, . \end{aligned}$$

Finally, given a set $U \subset \Xi_2(V)$ containing ξ we define (with $k := \{\xi\}$)

$$\begin{aligned} &Y(\xi, U) = \cup_{\xi_1 \in U} Y(\xi, \xi_1) \, , && P(\xi, U) = \mathcal{P}^{(k)}(Y(\xi, U)) \, , \\ &H(\xi, U) = P(\xi, U) H(k) P(\xi, U) \, , \quad && H_0(\xi, U) = P(\xi, U) H_0(k) P(\xi, U) \, , \\ &A(\xi, U) = P(\xi, U) A(k) P(\xi, U) \, . \end{aligned}$$

We also define the isometry $F_{\xi_1, \xi_2} : \mathrm{Ran} P(\xi_1, U) \to \mathrm{Ran} P(\xi_2, U)$ by

$$F_{\xi_1, \xi_2} e_\eta = e_{\eta + \xi_2 - \xi_1} \, , \;\; \eta \in Y(\xi_1, U).$$

Lemma 5.2 *Let $\xi_1, \xi_2 \in \Xi_2(V)$ satisfy $|\xi_1 - \xi_2| \leq c\rho^{\alpha - 2l + 1}$. Then for any $\xi \in Y(\xi_1, \xi_2) \setminus Y(\xi_1)$ we have $||\xi|^{2l} - \rho^{2l}| > \rho^\gamma$.*

Proof. Let $\xi' = \xi + \xi_2 - \xi_1$. Then $\xi' \in Y(\xi_2)$ and therefore $\xi' = \eta + \theta$ where $\eta \in \Xi_3(V)$, $\theta \in \Theta_M$ and $\eta - \xi_2 \in V \cap \Gamma^\dagger$. We distinguish two cases.
(i) $\theta \notin V$. In this case Lemma 3.11 gives $||\xi'|^{2l} - \rho^{2l}| > \rho^{2l-2+q_n}$. Therefore

$$||\xi|^{2l} - \rho^{2l}| \geq ||\xi'|^{2l} - \rho^{2l}| - ||\xi'|^{2l} - |\xi|^{2l}| > \rho^{2l-2+q_n} - c\rho^{2l-1}|\xi_2 - \xi_1| > \rho^\gamma.$$

(ii) $\theta \in V$. Then $\xi - \xi_1 = (\eta - \xi_2) + \theta \in V \cap \Gamma^\dagger$. Therefore, since $\xi_1 \in \Xi_2(V)$ and $\xi \notin Y(\xi_1) \supset \Xi_3(V)$, it follows that $||\xi|^{2l} - \rho^{2l}| > \rho^\gamma$, which completes the proof in this case. □

Lemma 5.3 *Let $\xi \in \Xi_2(V)$ and let $U \subset \Xi_2(V)$ be a set of diameter smaller than $c\rho^{\alpha-2l+1}$ containing ξ. Then there exists an injection G from the set of all eigenvalues of $H(\xi)$ into the set of all eigenvalues of $H(\xi, U)$, such that each eigenvalue of $H(\xi, U)$ in $J_1 := [\rho^{2l} - 98L, \rho^{2l} + 98L]$ is in the range of G. Moreover for any $\mu_i(H(\xi)) \in J_1$ we have*

$$|\mu_i(H(\xi)) - G(\mu_i(H(\xi)))| < c\rho^{-2M(\gamma-\alpha)+\alpha},$$

and

$$G(\mu_i(H(\xi))) = \mu_{i+l}(H(\xi, U)),$$

where $l =: l(\xi, U)$ is the number of points $\eta \in Y(\xi, U) \setminus Y(\xi)$ such that $|\eta| < \rho$.

Proof. The proof is an application of Lemma 2.2 so let us verify that all its conditions are satisfied. The lemma is applied to the operator $H(\xi, U)$ which is the sum of $H_0(\xi, U)$ and the perturbation $A(\xi, U)$; we note that $\|A(\xi, U)\| \le L\rho^\alpha$ by Lemma 2.5. We apply the lemma with $n = 0$. The projection P^0 is $P^0 = P(\xi)$ and is decomposed as a sum of orthogonal and invariant projections, $P^0 = \sum_{j=0}^M P_j^0$, where $P_0^0 = \mathcal{P}^{(k)}(\tilde{Y}(\xi))$ and

$$P_j^0 = P(\xi, U)\mathcal{P}^{(k)}((\tilde{Y}(\xi) + \Theta_j) \setminus (\tilde{Y}(\xi) + \Theta_{j-1}))P(\xi, U), \quad 1 \le j \le M.$$

The fact that

$$\sigma((P(\xi, U) - P(\xi))H_0(k)(P(\xi, U) - P(\xi))) = \\ = \{|\eta|^{2l} \ : \ \{\eta\} = k \ , \ \eta \in Y(\xi, U) \setminus Y(\xi)\}$$

together with Lemma 5.2 yields

$$\mathrm{dist}(\sigma((P(\xi, U) - P(\xi))H_0(k)(P(\xi, U) - P(\xi))), J) \ge \rho^\gamma.$$

Similarly, Corollary 3.13 yields

$$a_j := \mathrm{dist}(\sigma(P_j^0 H_0(k_1) P_j^0), J) \ge \rho^\gamma, \ \ j \ge 1.$$

The above imply that Lemma 2.2 can be applied. We conclude that there exists a map G from the set of all eigenvalues of $H(\xi)$ into the set of all eigenvalues of $H(\xi, U)$, such that each eigenvalue of $H(\xi, U)$ in J_1 is in the range of G and for any $\mu_i(H(\xi)) \in J_1$

$$\begin{aligned} & |\mu_i(H(\xi)) - G(\mu_i(H(\xi)))| \\ \le \ & (6\|A(\xi, U)\|)^{2M+1} \prod_{j=1}^M (a_j - 6\|A(\xi, U)\|)^{-2} \\ \le \ & c\rho^{-2M(\gamma-\alpha)+\alpha}, \end{aligned}$$

as required. □

Remark. In order to apply Lemma 2.2 we were forced to consider the smaller interval $J_1 \subset J$. There will be more occasions where our spectral interval shall need to be reduced. Strictly speaking, this will require the introduction of several intervals

$J_1 \supset J_2 \supset \ldots$. In order not to overburden our notation, we shall not make this explicit from now on and we shall always use the symbol J for the (possibly slightly reduced) spectral integral in hand.

Let $\{\eta_1, \ldots, \eta_p\} \subset \Theta_M$ be a complete set of representatives of Θ_M modulo V, that is for each $\theta \in \Theta_M$ there exist unique $\eta_j \in \{\eta_1, \ldots, \eta_p\}$ and $a \in V$ such that $\theta = \eta_j + a$. Letting $V_j = \eta_j + V$ and

$$\Psi_j(\xi) = (\xi + (V_j \cap \Gamma^\dagger)) \cap Y(\xi),$$

it follows that for each $\xi \in \Xi_2(V)$ the sets $\Psi_j(\xi)$, $j = 1, \ldots, p$, are pairwise disjoint and

$$Y(\xi) = \bigcup_{j=1}^{p} \Psi_j(\xi) \, .$$

Let $U \subset \Xi_2(V)$ be a set of diameter smaller than $c\rho^{\alpha - 2l + 1}$ containing ξ. We shall consider the matrix elements of $H(\xi, U)$ with respect to the basis $\{e_\eta \, : \, \eta \in Y(\xi, U)\}$ of $\operatorname{Ran} P(\xi, U)$. So let $\eta \in Y(\xi, U)$. Then there exist a unique η_k and a unique $\mu \in V \cap \Gamma^\dagger$ such that

$$\eta = \xi + \mu + \eta_k = r(\xi)\xi'_V + \xi_V + \mu + \eta_k \, . \tag{5.34}$$

Using Taylor's expansion we then have

$$\begin{aligned} |\eta|^{2l} &= |r(\xi)\xi'_V + \xi_V + \mu + \eta_k|^{2l} \\ &= r(\xi)^{2l} + \sum_{s=1}^{\infty} r(\xi)^{2l-s} b_s(\xi, \eta), \end{aligned}$$

where the function $b_s(\xi, \eta)$ has the form (using standard multi-index notation) $b_s(\xi, \eta) = \sum_{|\alpha|=s} c_\alpha P_\alpha(\xi'_V)(\xi_V + \mu + \eta_k)^\alpha$; here P_α is a polynomial of degree $|\alpha|$. Hence

$$H(\xi, U) = r(\xi)^{2l} I + \sum_{s=1}^{\infty} r(\xi)^{2l-s} B_s(\xi, U) + A(\xi, U) \, , \tag{5.35}$$

where the operator $B_s(\xi, U)$ is given by $B_s(\xi, U)e_\eta = b_s(\xi, \eta)e_\eta$, $\eta \in Y(\xi, U)$. We note that Corollary 3.3 together with the fact that $\operatorname{diam}(U) \ll \rho^{\alpha - 2l + 1}$ give

$$|\xi_V + \mu + \eta_k| = |\eta_V + (\eta_k)_V^\perp| \ll \rho^{q_n}.$$

Therefore

$$\|B_s(\xi, U)\| = \max_\eta |b_s(\xi, \eta)| \ll \rho^{q_n s} \, . \tag{5.36}$$

We also note that for $s = 1$ we have

$$b_1(\xi, \eta) = 2l\xi'_V \cdot (\xi_V + \mu + \eta_k), \tag{5.37}$$

so that $\|B_1(\xi, U)\| \ll 1$. Concerning $A(\xi, U)$, we note that for $\xi, a \in U$ and $\eta \in Y(\xi, U)$,

$$A(\xi, U)e_\eta - F_{a,\xi} A(a, U) F_{\xi,a} e_\eta = (2\pi)^{-d/2} \sum_{\eta' \in Y(\xi, U)} [\hat{a}(\eta' - \eta, \eta) - \hat{a}(\eta' - \eta, \eta + a - \xi)] e'_\eta \, .$$

Since $|\hat{a}(\eta' - \eta, \eta) - \hat{a}(\eta' - \eta, \eta + a - \xi)| \leq c\rho^{\alpha-1}|a - \xi|$, we can use the argument in the proof of Lemma 2.5 to obtain

$$\|A(\xi, U) - F_{a,\xi} A(a, U) F_{\xi,a}\| \leq c\rho^{\alpha-1}|a - \xi| \,. \tag{5.38}$$

We shall now use the above considerations to study how the eigenvalues of $H(\xi, U)$ change as ξ varies. Because the operators $H(\xi, U)$ act on different spaces, we shall use the unitary operators $F_{\xi,a}$ to move between $\mathrm{Ran}P(\xi, U)$ and $\mathrm{Ran}P(a, U)$. Let us denote by $\{\lambda_j(\xi, U)\}$ the eigenvalues of $H(\xi, U)$ in increasing order and repeated according to multiplicity. By (5.35)

$$\lambda_j(\xi, U) = r(\xi)^{2l} + \nu_j(\xi, U),$$

where $\{\nu_j(\xi, U)\}$ are the eigenvalues of the operator

$$D(\xi, U) := \sum_{s=1}^{\infty} r(\xi)^{2l-s} B_s(\xi, U) + A(\xi, U).$$

We first consider how $\lambda_j(\xi, U)$ varies as $r(\xi)$ varies.

Lemma 5.4 *Let $\xi \in U \subset \Xi_2(V)$ where $\mathrm{diam}(U) < \rho^{\alpha-2l+1}$. Assume that for t close enough to $r(\xi)$ the point $a(t) := t\xi'_V + \xi_V$ belongs in U. Let $\{\lambda_j(t)\}$ be the eigenvalues of $H(a(t), U)$. Then for t such that $a(t) \in U$,*

$$\lambda_j(t) = t^{2l} + \nu_j(t)$$

where $\nu_j(t)$ is a function satisfying

$$\frac{d\nu_j(t)}{dt} = O(\rho^{2l-2+q_n}) \,. \tag{5.39}$$

Proof. Replacing ξ by $a(t)$ in (5.35) yields

$$H(a(t), U) = t^{2l} I + \sum_{s=1}^{\infty} t^{2l-s} B_s(a(t), U) + A(a(t), U) \,. \tag{5.40}$$

Hence $\lambda_j(t) = t^{2l} + \nu_j(t)$, where $\{\nu_j(t)\}$ are the eigenvalues of the operator $\sum t^{2l-s} B_s(a(t), U) + A(a(t), U)$. Now a simple computation shows that

$$F_{a(t),\xi} B_s(a(t), U) F_{\xi,a(t)} = B_s(\xi, U),$$

hence

$$F_{a(t),\xi} \Big(\sum_{s=1}^{\infty} t^{2l-s} B_s(a(t), U) \Big) F_{\xi,a(t)} = \sum_{s=1}^{\infty} t^{2l-s} B_s(\xi, U) =: B(t).$$

We also have for $\eta \in Y(\xi, U)$,

$$F_{a(t),\xi} A(a(t), U) F_{\xi,a(t)} e_\eta = (2\pi)^{-d/2} \sum_{\eta' \in Y(\xi,U)} \hat{a}(\eta' - \eta, \eta + a(t) - \xi) e_{\eta'} =: A(t) e_\eta \,.$$

We note that both $B(t)$ and $A(t)$ act on the same space which is t-independent – namely $\mathrm{Ran}\, P(\xi, U)$. Therefore $\nu_j(t) = \mu_j(B(t) + A(t))$. Letting $\{\phi_j(t)\}$ be an orthonormal set of eigenfunctions of $B(t)+A(t)$ and recalling (5.36) we use standard perturbation theory to obtain

$$\begin{aligned}\Big|\frac{d\nu_j(t)}{dt}\Big| &= \Big|\langle\frac{dB(t)}{dt}\phi_j(t),\phi_j(t)\rangle+\langle\frac{dA(t)}{dt}\phi_j(t),\phi_j(t)\rangle\Big| \\ &\le \|\frac{dB(t)}{dt}\|+\|\frac{dA(t)}{dt}\| \\ &\ll \sum_{s=1}^{\infty}|2l-s|t^{2l-s-1}\|B_s(\xi,U)\|+\rho^{\alpha-1} \\ &\ll \rho^{2l-2+q_n}\ ;\end{aligned}$$

here we also used (5.38) to estimate $\|dA(t)/dt\|$. This completes the proof. □

Remark. We can slightly improve estimate (5.39) if we use (5.37); this however would not be of any use in what follows. Notice that this lemma is yet another place in our paper where the proof is more complicated than in [P]. Indeed, in [P] the mappings $F_{a(t_1),a(t_2)}$ provide unitary equivalence between $D(a(t_1),U)$ and $D(a(t_2),U)$, whereas in our paper this is no longer the case.

We next examine the case where $r(\xi)$ is fixed.

Lemma 5.5 *Let $\xi,a\in U\subset\Xi_2(V)$ be such that $r(\xi)=r(a)=:r$ and $|\xi-a|<c\rho^{\alpha-2l+1}$. Then*

$$|\lambda_j(\xi,U)-\lambda_j(a,U)|\ll\rho^{2l-2+q_n}|\xi-a|\,. \tag{5.41}$$

Proof. Let $\eta\in Y(\xi,U)$. Using the same notation as in (5.34) we have

$$\begin{aligned}&|B_s(\xi,U)e_\eta-F_{a,\xi}B_s(a,U)F_{\xi,a}e_\eta| \\ &\quad= |b_s(\xi,\eta)-b_s(a,\eta+a-\xi)| \\ &\quad= \Big|\sum_{|\alpha|=s}c_\alpha\Big[P_\alpha(\xi_V')(\xi_V+\mu+\eta_k)^\alpha-P_\alpha(a_V')(a_V+\mu+\eta_k)^\alpha\Big]\Big| \\ &\quad\ll \rho^{q_n(s-1)}|\xi-a|\,.\end{aligned}$$

For $s=1$ we can do better because of (5.37): we have

$$|b_1(\xi,\eta)-b_1(a,\eta+a-\xi)|=2l|(\xi_V'-a_V')\cdot\eta_k|=\frac{2l}{r}|(\xi_V^\perp-a_V^\perp)\cdot\eta_k|\ll\frac{|\xi-a|}{\rho}\,.$$

Therefore, using also (5.38), we obtain:

$$\begin{aligned}&\|F_{\xi,a}D(\xi,U)F_{a,\xi}-D(a,U)\| \\ &\quad\le \sum_{s=1}^{\infty}r^{2l-s}\|F_{\xi,a}B_s(\xi,U)F_{a,\xi}-B_s(a,U)\|+\|F_{\xi,a}A(\xi,U)F_{a,\xi}-A(a,U)\| \\ &\quad\ll \Big(\rho^{2l-2}+\sum_{s=2}^{\infty}[r^{2l-s}\rho^{q_n(s-1)}]+\rho^{\alpha-1}\Big)|\xi-a| \\ &\quad\ll \rho^{2l-2+q_n}|\xi-a|\,.\end{aligned}$$

The result follows. □

Combining the last two lemmas we have

Lemma 5.6 *Let $U \subset \Xi_2(V)$ be a set with $\mathrm{diam}(U) \ll \rho^{\alpha-2l+1}$. Assume that U contains a piecewise C^1 curve joining ξ_1, ξ_2, of length smaller than $c|\xi_1 - \xi_2|$. Suppose that $\mu_i(H(\xi_1, U)) \in J$. Then*

$$|\mu_i(H(\xi_1, U)) - \mu_i(H(\xi_2, U))| \ll \rho^{2l-1}|\xi_1 - \xi_2|. \tag{5.42}$$

Suppose now in addition that $(\xi_1)_V = (\xi_2)_V$ and $(\xi_1)'_V = (\xi_2)'_V$. Then

$$\mu_i(H(\xi_1, U)) - \mu_i(H(\xi_2, U)) = \{2l\rho^{2l-1} + O(\rho^{2l-2+q_n})\}(r(\xi_1) - r(\xi_2)). \tag{5.43}$$

Proof. Suppose first that $(\xi_1)_V = (\xi_2)_V$ and $(\xi_1)'_V = (\xi_2)'_V$. Then by Lemma 5.4 there exists t between $r(\xi_1)$ and $r(\xi_2)$ such that

$$\mu_i(H(\xi_1, U)) - \mu_i(H(\xi_2, U)) = [2lt^{2l-1} + O(t^{2l-2+q_n})](r(\xi_1) - r(\xi_2)).$$

Since $t = \rho + O(\rho^{2q_n-1})$ (cf. (3.21)) estimate (5.43) follows. Suppose next that $r(\xi_1) = r(\xi_2)$. From Lemma 5.5 we obtain

$$|\mu_i(H(\xi_1, U)) - \mu_i(H(\xi_2, U))| \le c\rho^{2l-2+q_n}|\xi_1 - \xi_2|.$$

Combining these two cases we obtain (5.42). □

We now proceed with some more definitions. Let $\xi \in \Xi_2(V)$ be given and $k := \{\xi\}$. We label the elements of $\sigma(H_0(\xi)) = \{|\eta|^{2l} \;:\; \eta \in Y(\xi)\}$ in increasing order; if there are two different points $\eta_1, \eta_2 \in Y(\xi)$ with $|\eta_1| = |\eta_2|$, then we order them in the lexicographic order of their coordinates. Hence to each $\eta \in Y(\xi)$ we have associated a natural number $j(\eta)$ such that

$$|\eta|^{2l} = \mu_{j(\eta)}(H_0(\xi)) \;, \quad \eta \in Y(\xi).$$

We then define $\tilde{g}(\xi) = \mu_{j(\xi)}(H(\xi))$. It follows from Lemma 2.5 that

$$|\tilde{g}(\xi) - |\xi|^{2l}| \le L\rho^{\alpha}. \tag{5.44}$$

Let us next define for $\xi \in \Xi_2(V)$,

$$X(\xi) = \{\eta \in \Xi_2(V) \;:\; \eta_V = \xi_V \;,\; \eta'_V = \xi'_V\}.$$

Clearly $X(\xi)$ is a union of at most finitely many intervals; without any loss of generality we assume that $X(\xi)$ itself is an interval. If $\eta_1, \eta_2 \in X(\xi)$, then (cf. (3.17))

$$|\eta_1 - \eta_2| = |r(\eta_1) - r(\eta_2)| = \frac{||\eta_1|^2 - |\eta_2|^2|}{r(\eta_1) + r(\eta_2)} \le c\rho^{\alpha-2l+1},$$

so $X(\xi)$ has length smaller than $c\rho^{\alpha-2l+1}$.

We label the elements of $\sigma(H_0(\xi, X(\xi))) = \{|\eta|^{2l} \;:\; \eta \in Y(\xi, X(\xi))\}$ in the same way as above. Hence to each $\eta \in Y(\xi, X(\xi))$ is associated an integer $i(\eta)$ such that

$$|\eta^{2l}| = \mu_{i(\eta)}(H_0(\xi, X(\xi))) \;, \quad \eta \in Y(\xi, X(\xi)).$$

We then define $g(\xi) = \mu_{i(\xi)}(H(\xi, X(\xi)))$. Clearly $|g(\xi) - |\xi|^{2l}| \le L\rho^{\alpha}$ for all $\xi \in \Xi_2(V)$.

Lemma 5.7 *For each $\xi \in \Xi_2(V)$ the function $i(\cdot)$ is constant on $X(\xi)$.*

Proof. Let $\xi_1 \in X(\xi)$. We must show that the number of points of the set $\{\eta \in Y(\xi, X(\xi)) : |\eta| < |\xi|\}$ coincides with the number of points of the set $\{\eta_1 \in Y(\xi_1, X(\xi)) : |\eta_1| < |\xi_1|\}$. Let $\eta \in Y(\xi, X(\xi))$ be given and define $\eta_1 = \eta + \xi_1 - \xi$; then $\eta_1 \in Y(\xi_1, X(\xi))$. We claim that $|\eta| < |\xi|$ if and only if $|\eta_1| < |\xi_1|$. To prove this we distinguish two cases:
(i) $\xi - \eta \in V$. In this case $\xi_1 - \eta_1 \in V$, therefore

$$|\xi_1|^2 - |\eta_1|^2 = |(\xi_1)_V|^2 - |(\eta_1)_V|^2 = |\xi_V|^2 - |\eta_V|^2 = |\xi|^2 - |\eta|^2,$$

and the claim follows.
(ii) $\xi - \eta \notin V$. We shall prove that in this case

$$||\eta|^{2l} - \rho^{2l}| > \rho^\gamma. \tag{5.45}$$

If $\eta \notin Y(\xi)$ then (5.45) follows from Lemma 5.2, so let us assume that $\eta \in Y(\xi)$. We then have $\eta = \bar{\eta} + \theta$ for some $\bar{\eta} \in \Xi_3(V)$ with $\bar{\eta} - \xi \in V \cap \Gamma^\dagger$ and some $\theta \in \Theta_M$. Then $\theta \notin V$ and (5.45) follows from Lemma 3.11. Similarly we have $||\eta_1|^{2l} - \rho^{2l}| > \rho^\gamma$. Suppose now that $|\xi| < |\eta|$. Then $|\eta|^{2l} > \rho^{2l} + \rho^\gamma$. Hence we have

$$|\eta_1|^{2l} \geq |\eta|^{2l} - c\rho^{2l-1}|\eta - \eta_1| \geq |\eta|^{2l} > \rho^{2l} + \rho^\gamma - c\rho^\alpha$$

and therefore, since $\xi_1 \in \mathcal{A}$, we conclude that $|\eta_1| > |\xi_1|$. This completes the proof. □

Lemma 5.8 *Let $\xi \in \Xi_2(V) \cap \mathcal{A}$. Then:*

(i) $|g(\xi) - \tilde{g}(\xi)| \ll \rho^{-2M(\gamma-\alpha)+\alpha}$;

(ii) $g(\xi) = r(\xi)^{2l} + s(\xi)$ *where $s(\xi)$ is differentiable with respect to $r = r(\xi)$ and* $\dfrac{\partial s}{\partial r} = O(\rho^{2l-2+q_n})$.

Proof. (i) We apply Lemma 5.3 with $U = X(\xi)$. We conclude that that there exists an injection G from the set of all eigenvalues of $H(\xi)$ into the set of eigenvalues of $H(\xi, X(\xi))$ such that each eigenvalue of $H(\xi, X(\xi))$ inside J belongs in the range of G and for each $\mu_i(H(\xi)) \in J$ we have $|G(\mu_i(H(\xi))) - \mu_i(H(\xi))| < c\rho^{-2M(\gamma-\alpha)+\alpha}$ and moreover

$$G(\mu_i(H(\xi))) = \mu_{i+m}(H(\xi, X(\xi))) \tag{5.46}$$

where m is the number of eigenvalues of $[P(\xi, X(\xi)) - P(\xi)]H_0(k)[P(\xi, X(\xi)) - P(\xi)]$ that are smaller than ρ^{2l}. Now, it follows from the above definitions that the difference $i(\xi) - j(\xi)$ is equal to the number of points $\eta \in Y(\xi, X(\xi)) \setminus Y(\xi)$ such that $|\eta| \leq |\xi|$. Because of Lemma 5.2, this can be rephrased as

$$i(\xi) - j(\xi) = \#\{\eta \in Y(\xi, X(\xi)) \setminus Y(\xi) \ : \ |\eta| \leq \rho\} = m\,. \tag{5.47}$$

Choosing $i = j(\xi)$ in (5.46) proves (i).
(ii) This is a direct consequence of Lemma 5.4 (applied for $U = X(\xi)$) and Lemma 5.7.
□

The fact that g does not exhibit good behaviour in $\mathcal{D}$ except in (locally) one direction, prevents us from estimating $|g(b) - g(a)|$ in terms of $|b - a|$. The next lemma compensates for this; it establishes the existence of a conjugate point $b + n$, $n \in \Gamma^\dagger$, which can be used in the place of b.

Lemma 5.9 *Let $[a,b] \subset \Xi_2(V) \cap \mathcal{A}$ be a segment of length $|b-a| < c\rho^{\alpha-2l+1}$. Then there exists $n \in \Gamma^\dagger$ such that $|g(b+n)-g(a)| \le c\rho^{2l-1}|b-a| + O(\rho^{-2M(\gamma-\alpha)+\alpha})$. Suppose now in addition that there exists $m \in \Gamma^\dagger \setminus \{0\}$ such that $[a+m, b+m] \subset \Xi_2(V) \cap \mathcal{A}$. Then there exists $n_1 \in \Gamma^\dagger$, $n_1 \ne n$, such that $|g(b+n_1)-g(a+m)| \le c\rho^{2l-1}|b-a| + O(\rho^{-2M(\gamma-\alpha)+\alpha})$.*

Proof. The proof is almost identical to the proof of lemma 7.11 from [P], so we will skip it. □

We can now state the following lemma, which collects together the previous results.

Lemma 5.10 *Let $V \in \mathcal{V}(n)$, $1 \le n \le d-1$, and $M > 0$ be given. There exist mappings $g, \tilde{g} : \Xi_2(V) \to \mathbf{R}$ with the following properties:*

(i) *$\tilde{g}(\xi)$ is an eigenvalue of $P(V)H(k)P(V)$, where $k := \{\xi\}$. Moreover, for each k, all eigenvalues of $P(V)H(k)P(V)$ inside J are in the image of $\tilde{g}$;*

(ii) *If $\xi \in \mathcal{A}$, then*

 (a) $|\tilde{g}(\xi) - g(\xi)| \ll \rho^{-2M(\gamma-\alpha)+\alpha}$;

 (b) $|g(\xi) - |\xi|^{2l}| \le 2L\rho^\alpha$;

(iii) *$g(\xi) = r(\xi)^{2l} + s(\xi)$ where $s(\xi)$ is differentiable with respect to $r = r(\xi)$ and $\dfrac{\partial s}{\partial r} = O(\rho^{2l-2+q_n})$.*

Proof. The first statement of (i) follows immediately from the definition of $\tilde{g}$ and (5.33). Parts (ii)(a) and (iii) are contained in Lemma 5.8. Finally (ii)(b) follows from (5.44) and (ii)(a). □

We now proceed to combine the results obtained so far in this section with those of Section 4. For this we shall need to extend the definition of $g(\xi)$ for $\xi \in \Xi_2(\{0\}) = \mathcal{B}$. We recall the $\tilde{g}(\xi)$ has already been defined for such ξ (cf. Lemma 4.2). We extend g in $\mathcal{B}$ defining

$$g(\xi) = \tilde{g}(\xi) \quad , \quad \xi \in \mathcal{B}\,. \tag{5.48}$$

Hence g is now a function defined on the whole of the spherical layer $\mathcal{A}$.

We shall define one more function f on $\mathcal{A}$; this will have values in $\sigma(H)$. Let $\xi \in \mathcal{A}$ be given and $\{\xi\} =: k$. Then there exists a unique lattice subspace V containing ξ (so $V \in \mathcal{V}(n)$ for some $n \in \{0, 1, \dots, d-1\}$; if $\xi \in \mathcal{B}$ then $n = 0$, while if $\xi \in \mathcal{D}$ then $n \ge 1$). As we have seen $\tilde{g}(\xi)$ is an eigenvalue of $H(\xi)$; hence (cf. (5.33)) it is an eigenvalue of $\sum_V P(V)H(k)P(V) + QH(k)Q$. Ordering the eigenvalues of $\sum_V P(V)H(k)P(V) + QH(k)Q$ in the usual way determines a number $\tau(\xi) \in \mathbf{N}$ such that $\tilde{g}(\xi) = \mu_{\tau(\xi)}(\sum_V P(V)H(k)P(V) + QH(k)Q)$. We then define

$$f(\xi) = \mu_{\tau(\xi)}(H(k))\,.$$

We have the following

Proposition 5.11 *Let $N > 0$ be given. There exist two mappings $f, g : \mathcal{A} \to \mathbf{R}$ with* *the following properties:*

(i) $f(\xi)$ *is an eigenvalue of* $H(k)$, *where* $k := \{\xi\}$;

(ii) *For any* k, *all eigenvalues of* $H(k)$ *inside* J *are in the range of* f;

(iii) *If* $\xi \in \mathcal{A}$ *then* $|f(\xi) - g(\xi)| \leq \rho^{-N}$;

(iv) $|f(\xi) - |\xi|^{2l}| \leq c\rho^{\alpha}$;

(v) *Considering the disjoint union* $\mathcal{A} = \mathcal{B} \cup \bigcup_{n=1}^{d-1} \bigcup_{V \in \mathcal{V}(n)} \Xi_2(V)$ *we have:*

(a) If $\xi \in \mathcal{B}$ then $g(\xi) = |\xi|^{2l} + G(\xi)$, where $|\nabla G(\xi)| \leq c\rho^{\alpha-1}$;

(b) If $\xi \in \Xi_2(V)$ then $g(\xi) = r(\xi)^{2l} + s(\xi)$, where $\left|\frac{\partial s}{\partial r}\right| \leq c\rho^{2l-2+q_n}$.

Note. We do not claim – and indeed it is not the case in general – that either f or g is continuous in $\mathcal{A}$.

Proof. Part (i) is trivial. Part (ii) follows from Lemma 2.2. The same lemma together with Corollary 2.6 implies that

$$|f(\xi) - \tilde{g}(\xi)| \ll \rho^{-2M(\gamma-\alpha)+\alpha}. \tag{5.49}$$

This, together with Lemma 5.10 (ii)(a) (if $\xi \in \mathcal{D}$) or (5.48) (if $\xi \in \mathcal{B}$), implies (iii) if we choose M sufficiently large so that $-2M(\gamma - \alpha) + \alpha < -N$. Part (iv) follows from (iii) and Lemma 5.10 (ii). Finally parts (v) (a) and (b) follow from Proposition 4.5 and Lemma 5.8 (ii) respectively. □

The next lemma is a global version of Lemma 5.9; once again, the proof is almost identical to the proof of Lemma 7.14 from [P], so we will skip it.

Lemma 5.12 *Let* $[a, b] \subset \mathcal{A}$ *be a segment of length* $|b - a| < c\rho^{\alpha-2l+1}$. *Then there exists* $n \in \Gamma^{\dagger}$ *such that* $|g(b+n) - g(a)| \leq c\rho^{2l-1}|b-a| + O(\rho^{-2M(\gamma-\alpha)+\alpha})$. *Suppose now in addition that there exists* $m \in \Gamma^{\dagger} \setminus \{0\}$ *such that* $[a+m, b+m] \subset \mathcal{A}$. *Then there exists* $n_1 \in \Gamma^{\dagger}$, $n_1 \neq n$, *such that* $|g(b+n_1) - g(a+m)| \leq c\rho^{2l-1}|b-a| + O(\rho^{-2M(\gamma-\alpha)+\alpha+d})$.

6 Proof of the main theorem

Let δ, $0 < \delta \leq \rho^{2l-3}$, be a parameter, the precise value of which will be determined later on. We denote by $\mathcal{A}(\delta)$, $\mathcal{B}(\delta)$ and $\mathcal{D}(\delta)$ the intersections of $g^{-1}([\rho^{2l} - \delta, \rho^{2l} + \delta])$ with $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ respectively.

Lemma 6.1 *There holds*

$$\begin{aligned} &\text{(i)} \quad \mathrm{vol}(\mathcal{A}(\delta)) \asymp \delta\rho^{d-2l}\ , \\ &\text{(ii)} \quad \mathrm{vol}(\mathcal{B}(\delta)) \asymp \delta\rho^{d-2l}\ , \\ &\text{(iii)} \quad \mathrm{vol}(\mathcal{D}(\delta)) \leq \delta\rho^{d-2l-\epsilon_0}\ , \end{aligned}$$

provided ρ *is large enough.*

Proof. It is enough to prove (ii) and (iii). Let us consider a point $\xi \in \mathcal{B}$. We write $\xi = r\xi'$ where $r > 0$ and $|\xi'| = 1$. Definition (5.48) together with Proposition 4.5 implies that

$$\frac{\partial g}{\partial r} \asymp \rho^{2l-1},$$

uniformly over $\xi \in \mathcal{B}$. Hence for each ξ' the segment

$$\{\xi = r\xi' \in \mathcal{B} \ : \ r > 0\ ,\ g(\xi) \in [\rho^{2l} - \delta, \rho^{2l} + \delta]\} \tag{6.50}$$

is an interval of length $\asymp \delta\rho^{-2l+1}$. Integration over all $\xi' \in S^{d-1}$ yields (ii). To prove (iii), let us consider a point $\xi \in \Xi_2(V)$ and let (r, ξ'_V, ξ_V) be the corresponding cylindrical coordinates. For $\theta \in \Theta'_{6M}$ let

$$\mathcal{D}_\theta(\delta) = \{\xi \in \mathcal{A}(\delta) \ : \ |\xi \cdot \theta| \le \rho^{1-\epsilon_0}|\theta|\}.$$

It follows from (v) of Proposition 5.11 that

$$\frac{\partial g}{\partial r} \asymp \rho^{2l-1}.$$

Thus, the intersection of $\mathcal{D}_\theta(\delta)$ with the semi-infinite interval $\{\xi = (r, \xi'_V, \xi_V),\ r > 0\}$, with (ξ'_V, ξ_V) being fixed, is an interval of length smaller than $c\delta\rho^{-2l+1}$. Therefore, $\mathrm{vol}(\mathcal{D}_\theta(\delta)) \ll (\delta\rho^{-2l+1})\rho^{1-\epsilon_0}\rho^{d-2}$. The number of points $\theta \in \Theta'_{6M}$ is fixed. Hence, since $\mathcal{D}(\delta) \subset \cup_{\theta\in\Theta'_{6M}}\mathcal{D}_\theta(\delta)$, we conclude that

$$\mathrm{vol}(\mathcal{D}(\delta)) \le \sum_{\theta\in\Theta'_{6M}} \mathrm{vol}(\mathcal{D}_\theta(\delta)) \ll \delta\rho^{d-2l-\epsilon_0}\,,$$

which implies (iii). □

The next lemma is crucial for the proof of the main theorem. It gives an upper estimate on the volume of intersections of translates of $\mathcal{B}(\delta)$. Recall that when $\xi \in \mathcal{A}$, we have $g(\xi) = |\xi|^{2l} + G(\xi)$, where $|G(\xi)| = O(\rho^\alpha)$ and $|\nabla G(\xi)| = O(\rho^{\alpha-1})$.

Lemma 6.2 *(i) Let $d \ge 3$. Then*

$$\mathrm{vol}\Big(\mathcal{B}(\delta) \cap (\mathcal{B}(\delta) + a)\Big) \ll (\delta^2\rho^{-4l+d+1} + \delta\rho^{(\alpha-2l+1)d-\alpha})\,, \tag{6.51}$$

uniformly over all $a \in \mathbf{R}^d$ with $|a| \ge C$ for any positive constant C. In addition, there exists $c_4 > 0$ such that if a satisfies $||a| - 2\rho| \ge c_4$, then

$$\mathrm{vol}\Big(\mathcal{B}(\delta) \cap (\mathcal{B}(\delta) + a)\Big) \le c\delta^2\rho^{-4l+d+1}. \tag{6.52}$$

(ii) If $d = 2$, then

$$\mathrm{vol}\Big(\mathcal{B}(\delta) \cap (\mathcal{B}(\delta) + a)\Big) \ll \delta^{3/2}\rho^{3-3l} + \delta\rho^{\alpha-4l+2}. \tag{6.53}$$

In addition, there exists $c_4 > 0$ such that if a satisfies $||a| - 2\rho| \ge c_4$, then

$$\mathrm{vol}\Big(\mathcal{B}(\delta) \cap (\mathcal{B}(\delta) + a)\Big) \ll \delta^{3/2}\rho^{3-3l}. \tag{6.54}$$

Proof. First of all, we notice that it is enough to prove this lemma assuming $l = 1$. Indeed, suppose we have established Lemma 6.2 for $l = 1$. In the general case, we introduce a new function $\tilde{g}(\xi) := (g(\xi))^{1/l}$. Then a simple calculation shows that $\tilde{g}(\xi) =$

$|\xi|^2 + \tilde{G}(\xi)$, where $|\tilde{G}(\xi)| = O(\rho^{\tilde{\alpha}})$ and $|\nabla\tilde{G}(\xi)| = O(\rho^{\tilde{\alpha}-1})$, with $\tilde{\alpha} = \alpha + 2 - 2l < 1$. Moreover,

$$\mathcal{B}(\delta) \subset \tilde{\mathcal{B}}(\tilde{\delta}) := \{\xi \in \mathcal{B},\, \tilde{g}(\xi) \in [\rho^2 - \tilde{\delta}, \rho^2 + \tilde{\delta}]\}, \tag{6.55}$$

with $\tilde{\delta} = \delta\rho^{2-2l}$. Thus, applying (6.51) for $\tilde{g}$ (with $l = 1$), we obtain:

$$\mathrm{vol}\Big(\mathcal{B}(\delta) \cap (\mathcal{B}(\delta) + a)\Big) \ll (\tilde{\delta}^2\rho^{d-3} + \tilde{\delta}\rho^{(\tilde{\alpha}-1)d-\tilde{\alpha}}). \tag{6.56}$$

After inserting expressions defining $\tilde{\delta}$ and $\tilde{\alpha}$, we obtain (6.51). The rest of estimates is similar.

Thus, from now on we assume without loss of generality that $l = 1$. In this case we need to prove the following estimates:

$$\mathrm{vol}\Big(\mathcal{B}(\delta) \cap (\mathcal{B}(\delta) + a)\Big) \ll (\delta^2\rho^{d-3} + \delta\rho^{(\alpha-1)d-\alpha}),\ \ d \geq 3; \tag{6.57}$$

$$\mathrm{vol}\Big(\mathcal{B}(\delta) \cap (\mathcal{B}(\delta) + a)\Big) \ll \delta^{3/2} + \delta\rho^{\alpha-2},\ \ d = 2, \tag{6.58}$$

and if $||a| - 2\rho| \geq c_4$, then

$$\mathrm{vol}\Big(\mathcal{B}(\delta) \cap (\mathcal{B}(\delta) + a)\Big) \ll \delta^2\rho^{d-3},\ \ d \geq 3, \tag{6.59}$$

$$\mathrm{vol}\Big(\mathcal{B}(\delta) \cap (\mathcal{B}(\delta) + a)\Big) \ll \delta^{3/2},\ \ d = 2. \tag{6.60}$$

Denote by C_2 a constant such that

$$|G(\xi)| \leq C_2\rho^{\alpha},\ \ \ |\nabla G(\xi)| \leq C_2\rho^{\beta}, \tag{6.61}$$

where we have denoted

$$\beta := \alpha - 1 < 0. \tag{6.62}$$

We need to estimate the volume of the set

$$\mathcal{X} := \{\xi \in \big(\mathcal{B} \cap ((\mathcal{B}) + a)\big),\qquad g(\xi) \in [\rho^2 - \delta, \rho^2 + \delta], \tag{6.63}$$
$$g(\xi - a) \in [\rho^2 - \delta, \rho^2 + \delta]\}. \tag{6.64}$$

First, we will estimate the 2-dimensional area of the intersection of $\mathcal{X}$ with arbitrary 2-dimensional plane containing the origin and vector a; the volume of $\mathcal{X}$ then can be obtained using the integration in cylindrical coordinates. So, let V be any 2-dimensional plane containing the origin and a, and let us estimate the area of $\mathcal{X}_V := V \cap \mathcal{X}$. Let us introduce cartesian coordinates in V so that $\xi \in V$ has coordinates (ν_1, ν_2) with ν_1 going along a, and ν_2 being orthogonal to a. For any $\xi \in \mathcal{X}_V$ we have

$$g(\xi) = \nu_1^2 + \nu_2^2 + G(\xi), \tag{6.65}$$

and so

$$2\delta \geq |g(\xi) - g(\xi - a)| = |\nu_1^2 - (\nu_1 - |a|)^2 + G(\xi) - G(\xi - a)| = |a|\ |(|a| - 2\nu_1)| + |a|O(\rho^{\beta}). \tag{6.66}$$

This implies that $|\frac{|a|}{2} - \nu_1| = O(\rho^{\beta}) + O(\rho^{-1})$ and, therefore,

$$\frac{|a|}{3} < \nu_1 < \frac{2|a|}{3}, \tag{6.67}$$

whenever $\xi \in \mathcal{X}_V$.

$$\frac{\partial g(\xi)}{\partial \nu_1} \gg |a| \,. \tag{6.68}$$

Thus, for any fixed $t \in \mathbf{R}$, the intersection of the line $\nu_2 = t$ with $\mathcal{X}_V$ is an interval of length $\ll \rho^\beta$ (we can assume without loss of generality that $\beta > -1/2$).

Let us cut $\mathcal{X}_V$ into two parts: $\mathcal{X}_V = \mathcal{X}_V^1 \cup \mathcal{X}_V^2$ with $\mathcal{X}_V^1 := \{\xi \in \mathcal{X}_V,\ |\nu_2| \le 2C_2\rho^\beta\}$, $\mathcal{X}_V^2 = \mathcal{X}_V \setminus \mathcal{X}_V^1$, and estimate the volumes of these sets (C_2 is the constant from (6.61)). We start from $\mathcal{X}_V^1$. Suppose that $\mathcal{X}_V^1$ is non-empty, say $\xi = (\nu_1, \nu_2) \in \mathcal{X}_V^1$ (note that $|\nu_2| \ll \rho^\beta$). Denote $\eta := (\nu_1, 0)$. Then

$$g(\eta) = g(\xi) + O(\rho^{2\beta}) = \rho^2 + O(\rho^{2\beta}). \tag{6.69}$$

Similarly,

$$g(\eta - a) = g(\xi - a) + O(\rho^{2\beta}) = \rho^2 + O(\rho^{2\beta}). \tag{6.70}$$

Thus, if $\mathcal{X}_V^1$ is non-empty, then $|a| \sim \rho$ and the first coordinate of any point $\xi \in \mathcal{X}_V^1$ satisfies $\nu_1 \sim \rho$. Therefore, we have $\frac{\partial g}{\partial \nu_1}(\xi) \gg \rho$. Let s_1 denote the unique positive solution of the equation $g(s_1, 0) = \rho^2$; similarly, let s_2 be the unique positive solution of the equation $g(-s_2, 0) = \rho^2$. Our conditions on g imply that $s_j = \rho + O(1)$. Estimate (6.69) implies $\nu_1 = s_1 + O(\rho^{2\beta-1})$; similarly, estimate (6.70) implies $\nu_1 - |a| = -s_2 + O(\rho^{2\beta-1})$. Thus, if $\mathcal{X}_V^1$ is non-empty, then we have

$$|a| = s_1 + s_2 + O(\rho^{2\beta-1}) = 2\rho + O(1). \tag{6.71}$$

Let us now fix t, $0 \le t \le 2C_2\rho^\beta$. Since $\frac{\partial g}{\partial \nu_1}(\xi) \gg \rho$, the length of the intersection of $\mathcal{X}_V^1$ with the line $\nu_2 = t$ is $\ll \delta\rho^{-1}$. This implies that the area of $\mathcal{X}_V^1$ is $\ll \rho^{\beta-1}\delta$. Now we define the 'rotated' set $\mathcal{X}^1$ which consists of the points from $\mathcal{X}$ which belong to $\mathcal{X}_V^1$ for some V. Computing the volume of this set using integration in the cylindrical coordinates, we obtain

$$\mathrm{vol}(\mathcal{X}^1) \ll \rho^{(d-1)\beta-1}\delta. \tag{6.72}$$

We now consider $\mathcal{X}_V^2$. Let us decompose $\mathcal{X}_V^2 = \overline{\mathcal{X}_V^2} \cup \underline{\mathcal{X}_V^2}$, where

$$\overline{\mathcal{X}_V^2} = \{\xi \in \mathcal{X}_V^2 : \nu_2 > 0\} \tag{6.73}$$

and

$$\underline{\mathcal{X}_V^2} = \{\xi \in \mathcal{X}_V^2 : \nu_2 < 0\}. \tag{6.74}$$

Notice that for any $\xi \in \overline{\mathcal{X}}_V^2$, formula ((6.61)) implies

$$\frac{\partial g(\xi)}{\partial \nu_2} \gg \nu_2. \tag{6.75}$$

Let $\xi^l = (\nu_1^l, \nu_2^l)$ be the point in the closure of $\overline{\mathcal{X}_V^2}$ with the smallest value of the first coordinate: $\nu_1^l \le \nu_1$ for any $\xi = (\nu_1, \nu_2) \in \overline{\mathcal{X}_V^2}$. Analogously, we define ξ^r to be the point in the closure of $\overline{\mathcal{X}_V^2}$ with the largest first coordinate, ξ^t the point with the largest second coordinate, and ξ^b the point with the smallest second coordinate. Note that $\nu_2^t \ll \rho$.

Let us prove that

$$\nu_1^r - \nu_1^l \ll \delta. \tag{6.76}$$

Indeed, suppose first that $\nu_2^r \geq \nu_2^l$. Let $\xi^{rl} := (\nu_1^r, \nu_2^l)$. Then, since g is an increasing function of ν_2 when $\nu_2 > 2C_2\rho^\beta$, we have $g(\xi^{rl}) \leq g(\xi^r) \leq \rho^2 + \delta$. Therefore, $g(\xi^{rl}) - g(\xi^l) \leq 2\delta$. Since g is increasing with respect to ν_1, estimate (6.68) implies (6.76).

Suppose now that $\nu_2^r \leq \nu_2^l$. Let $\xi^{lr} := (\nu_1^l, \nu_2^r)$. Then $g(\xi^{lr} - a) \leq g(\xi^l - a) \leq \rho^2 + \delta$. Therefore, $g(\xi^{lr} - a) - g(\xi^r - a) \leq 2\delta$. Since $g(\cdot - a)$ is decreasing with respect to ν_1, (6.67) and (6.68) imply (6.76).

Thus, we have estimated the width of $\overline{\mathcal{X}_V^2}$. Let us estimate its height (i.e. $\nu_2^t - \nu_2^b$). Let us assume that $\nu_1^t \geq \nu_1^b$; otherwise, we use the same trick as in the previous paragraph and consider $g(\cdot - a)$ instead of g. Let $\xi^{bt} := (\nu_1^b, \nu_2^t)$. Then $g(\xi^{bt}) \leq g(\xi^t) \leq \rho^2 + \delta$. Therefore, $g(\xi^{bt}) - g(\xi^b) \leq 2\delta$. Now, (6.75) implies

$$(\nu_2^t)^2 - (\nu_2^b)^2 = 2\int_{\nu_2^b}^{\nu_2^t} \nu_2 d\nu_2 \ll \int_{\nu_2^b}^{\nu_2^t} \frac{\partial g}{\partial \nu_2}(\nu_1^b, \nu_2) d\nu_2 \leq 2\delta. \tag{6.77}$$

Therefore, we have the following estimate for the height of $\overline{\mathcal{X}_V^2}$:

$$\nu_2^t - \nu_2^b \ll \frac{\delta}{\nu_2^t + \nu_2^b}. \tag{6.78}$$

Now, we can estimate the volume of $\mathcal{X}^2 := \mathcal{X} \setminus \mathcal{X}^1$ using estimates (6.76) and (6.78). Cylindrical integration produces the following:

$$\mathrm{vol}(\mathcal{X}^2) \ll \frac{\delta^2}{\nu_2^t + \nu_2^b}(\nu_2^t)^{d-2} \leq \delta^2(\nu_2^t)^{d-3} \leq \delta^2\rho^{d-3}. \tag{6.79}$$

Equations (6.72) and (6.79) imply (6.57). If $d = 2$, we have to notice that (6.77) implies $\nu_2^t - \nu_2^b \ll \delta^{1/2}$ and then use (6.72) and (6.76). Finally, the remark before (6.71) implies (6.58) and (6.60).

□

Let N be an arbitrary natural number, the precise value of which will be established later (as well as the precise value of δ). We construct the mappings f, g according to Proposition 5.11.

Lemma 6.3 *Let $\xi \in \mathcal{B}(\delta)$ be a point of discontinuity of f. Then there exists $n \in \Gamma^\dagger\backslash\{0\}$ such that $\xi + n \in \mathcal{A}$ and*

$$|g(\xi + n) - g(\xi)| \leq 2\rho^{-N}. \tag{6.80}$$

Proof. Let $\xi \in \mathcal{B}(\delta)$ be a point of discontinuity of f. Since f is bounded, there exist two sequences (ξ_j) and $(\tilde{\xi}_j)$ in $\mathcal{B}(\delta)$ both converging to ξ and such that the limits $\lambda := \lim f(\xi_j)$ and $\tilde{\lambda} := \lim f(\tilde{\xi}_j)$ both exist in $\mathbf{R}$ and $\lambda \neq \tilde{\lambda}$. Let $k := \{\xi\}$, $k_j := \{\xi_j\}$. Since $f(\xi_j)$ are eigenvalues of $H(k_j)$, the limit λ is an eigenvalue of $H(k)$ [K]; similarly $\tilde{\lambda}$ is an eigenvalue of $H(k)$. Since $\lambda \neq \tilde{\lambda}$ at least one of $\lambda, \tilde{\lambda}$ is different from $f(\xi)$, say $\tilde{\lambda} \neq f(\xi)$. The fact that $\tilde{\lambda}$ is an eigenvalue of $H(k)$ inside J implies, by (ii) of Proposition 5.11, that $\tilde{\lambda} = f(\tilde{\xi})$ for some $\tilde{\xi} \in \mathcal{A}$ with $\{\tilde{\xi}\} = k$. Using the continuity of g in $\mathcal{B}$ and (iii) of Proposition 5.11, we conclude that

$$|g(\tilde{\xi}) - g(\xi)| \leq |g(\tilde{\xi}) - f(\tilde{\xi})| + \lim |f(\tilde{\xi}_j) - g(\tilde{\xi}_j)| \leq 2\rho^{-N},$$

which is (6.80). We have $\tilde{\xi} \neq \xi$ since otherwise we would obtain $f(\xi) = f(\tilde{\xi}) = \tilde{\lambda}$. □

Lemma 6.4 *There exists a constant* $c_2 > 0$ *with the following property: suppose that* $I \subset \mathcal{B}(\delta)$ *is a straight segment of length* $T < \delta\rho^{-2l+1}$. *Suppose also that there exists a point* $\xi_0 \in I$ *such that for each* $n \in \Gamma^\dagger \setminus \{0\}$ *we have*

$$\begin{array}{rll} \textit{either} & \text{(i)} & \xi_0 + n \notin \mathcal{A} \\ \textit{or} & \text{(ii)} & |g(\xi_0 + n) - g(\xi_0)| \geq c_2(T\rho^{2l-1} + \rho^{-N}). \end{array}$$

Then the restriction $f|_I$ *is continuous.*

Proof. We argue by contradiction. So let us assume the contrary: for any $c_2 > 0$ there exists a segment $I \subset \mathcal{B}(\delta)$ of length $T < \delta\rho^{-2l+1}$ and a point $\xi_0 \in I$ such that for any $n \in \Gamma^\dagger \setminus \{0\}$ either (i) of (ii) is true but the restriction $f|_I$ is discontinuous at some $\xi_1 \in I$. By Lemma 6.3 there exists $n_0 \in \Gamma^\dagger \setminus \{0\}$, such that $\xi_1 + n_0 \in \mathcal{A}$ and $|g(\xi_1 + n_0) - g(\xi_1)| < 2\rho^{-N}$. It follows in particular that $I + n_0 \subset \mathcal{A}$ hence (ii) above is true by our assumptions.

We now apply Lemma 5.12 with $a = \xi_1$ and $b = \xi_0$ (we may assume that $\alpha > 2l-3$). We conclude that there exist $m_1, m_2 \in \Gamma^\dagger$, $m_1 \neq m_2$, such that

$$|g(\xi_0 + m_1) - g(\xi_1)| \leq c(\rho^{2l-1}T + \rho^{-N}) \ \ , \ \ |g(\xi_0 + m_2) - g(\xi_1 + n_0)| \leq c(\rho^{2l-1}T + \rho^{-N}).$$

At least one of the m_1, m_2 is non-zero. If $m_1 \neq 0$ then, using also estimate (ii) of Proposition 4.5, we have

$$\begin{aligned} |g(\xi_0 + m_1) - g(\xi_0)| &\leq |g(\xi_0 + m_1) - g(\xi_1)| + |g(\xi_1) - g(\xi_0)| \\ &\leq c(\rho^{2l-1}T + \rho^{-N}) + c\rho^{2l-1}T \\ &\leq c(\rho^{2l-1}T + \rho^{-N}). \end{aligned}$$

This contradicts (ii) (provided c_2 has been chosen to be large enough). Suppose now that $m_2 \neq 0$. Then

$$\begin{aligned} |g(\xi_0 + m_2) - g(\xi_0)| &\leq |g(\xi_0 + m_2) - g(\xi_1 + n_0)| + |g(\xi_1 + n_0) - g(\xi_1)| + \\ &\qquad + |g(\xi_1) - g(\xi_0)| \\ &\leq c(\rho^{2l-1}T + \rho^{-N}) + c\rho^{-N} + c\rho^{2l-1}T \\ &\leq c(\rho^{2l-1}T + \rho^{-N}), \end{aligned}$$

which again contradicts (ii). This completes the proof of the lemma. □

Let us now write $\mathcal{D}(\delta)$ as a disjoint union, $\mathcal{D}(\delta) = \mathcal{D}_0(\delta) \cup \mathcal{D}_1(\delta) \cup \mathcal{D}_2(\delta)$, where: $\mathcal{D}_0(\delta)$ contains all $\xi \in \mathcal{D}(\delta)$ for which there does not exist any $n \in \Gamma^\dagger \setminus \{0\}$ such that $\xi - n \in \mathcal{B}(\delta)$; $\mathcal{D}_1(\delta)$ contains those $\xi \in \mathcal{D}(\delta)$ for which there exists exactly one such n; and $\mathcal{D}_2(\delta)$ contains those $\xi \in \mathcal{D}(\delta)$ for which there exist at least two (different) such points n.

Lemma 6.5 *Suppose $d \geq 3$. Then there holds*

(i) $$\mathcal{B}(\delta) \bigcap \Big(\bigcup_{n \in \Gamma^\dagger \setminus \{0\}} (\mathcal{D}_0(\delta) - n) \Big) = \emptyset \ ;$$

(ii) $$\bigcup_{n \in \Gamma^\dagger \setminus \{0\}} (\mathcal{D}_2(\delta) - n) \subset \bigcup_{n \in \Gamma^\dagger \setminus \{0\}} (\mathcal{B}(\delta) - n) \ ;$$

(iii) $$\mathrm{vol}\Big(\bigcup_{n \in \Gamma^\dagger \setminus \{0\}} \Big((\mathcal{B}(\delta) - n) \cap \mathcal{B}(\delta) \Big) \Big) \leq c(\delta^2 \rho^{-4l+2d+1} + \delta \rho^{(\alpha-2l+1)d-\alpha+d-1}) \ ;$$

(iv) $$\mathrm{vol}\Big(\bigcup_{n \in \Gamma^\dagger \setminus \{0\}} \Big((\mathcal{D}_1(\delta) - n) \cap \mathcal{B}(\delta) \Big) \Big) \leq \delta \rho^{d-2l-\epsilon_0} \ .$$

If $d = 2$ then the same estimates are valid provided (iii) is replaced by

(iii′) $$\mathrm{vol}\Big(\bigcup_{n \in \Gamma^\dagger \setminus \{0\}} \Big((\mathcal{B}(\delta) - n) \cap \mathcal{B}(\delta) \Big) \Big) \leq c(\delta^{3/2} \rho^{5-3l} + \delta \rho^{\alpha-4l+3}) \ .$$

Proof. Parts (i) and (ii) follow easily from the definition of the sets $\mathcal{D}_0(\delta)$ and $\mathcal{D}_2(\delta)$. To prove (iii) we first note that from Lemma 6.2 we have that for any $n \in \Gamma^\dagger \setminus \{0\}$ there holds

$$\mathrm{vol}((\mathcal{B}(\delta) - n) \cap \mathcal{B}(\delta)) \leq c(\delta^2 \rho^{-4l+d+1} + \delta \rho^{(\alpha-2l+1)d-\alpha}); \tag{6.81}$$

moreover, if in addition $n \in \Gamma^\dagger \setminus \{0\}$ satisfies $||n| - 2\rho| \geq c$, then we can do better, namely

$$\mathrm{vol}((\mathcal{B}(\delta) - n) \cap \mathcal{B}(\delta)) \leq c\delta^2 \rho^{-4l+d+1}. \tag{6.82}$$

But the number of $n \in \Gamma^\dagger \setminus \{0\}$ for which $||n| - 2\rho| \leq c$ is smaller than $c\rho^{d-1}$; and the number of $n \in \Gamma^\dagger \setminus \{0\}$ for which $(\mathcal{B}(\delta) - n) \cap \mathcal{B}(\delta)$ is non-empty is smaller than $c\rho^d$. Hence (iii) follows. Finally, we have

$$\begin{aligned}
&\mathrm{vol}\Big(\bigcup_{n \in \Gamma^\dagger \setminus \{0\}} \Big((\mathcal{D}_1(\delta) - n) \cap \mathcal{B}(\delta) \Big) \Big) \\
&\quad \leq \sum_{n \in \Gamma^\dagger \setminus \{0\}} \mathrm{vol}(\{\xi \in \mathcal{B}(\delta) \ : \ \xi + n \in \mathcal{D}_1(\delta)\}) \\
&\quad = \sum_{n \in \Gamma^\dagger \setminus \{0\}} \mathrm{vol}(\{\eta \in \mathcal{D}_1(\delta) \ : \ \eta - n \in \mathcal{B}(\delta)\}) \\
&\quad = \mathrm{vol}(\{\eta \in \mathcal{D}_1(\delta_1) \ : \ \eta - n \in \mathcal{B}(\delta) \, , \text{ for some } n \in \Gamma^\dagger \setminus \{0\}\}) \\
&\quad = \mathrm{vol}(\mathcal{D}_1(\delta)) \\
&\quad \leq \rho^{d-2l-\epsilon_0} \, ,
\end{aligned}$$

which is (iv). □

It is now the time to choose precise values of N and δ. We put $\delta = c_3 \rho^{2l-d-1}$, where c_3 is a (small) constant to be determined later, and $N = d + 2$, so that for large ρ we have

$$2\rho^{-N} \leq \delta \ . \tag{6.83}$$

For any unit vector $\eta \in \mathbf{R}^d$ let us define $I_\eta = \{r\eta \ : \ r > 0 \ , \ \ r\eta \in \mathcal{A}(\delta)\}$. We then have

Lemma 6.6 *Let $\delta = c_3\rho^{2l-d-1}$ if $d \geq 3$ and $\delta = c_3\rho^{2l-6}$ if $d = 2$. If c_3 is small enough then there exists at least one $\eta \in \mathbf{R}^d$, $|\eta| = 1$, such that $I_\eta \subset \mathcal{B}$ and the restriction $f|_{I_\eta}$ is continuous.*

Proof. Let us assume the contrary. Lemma 6.4 then implies that for any such interval I_η and for any $\xi \in I_\eta$ there exists an $n \in \Gamma^\dagger \setminus \{0\}$ such that

$$\xi + n \in \mathcal{A} \quad , \quad |g(\xi + n) - g(\xi)| \leq c_2(T\rho^{2l-1} + \rho^{-N}). \tag{6.84}$$

Since all such intervals I_η cover $\mathcal{B}(\delta)$, the existence of an $n \in \Gamma^\dagger \setminus \{0\}$ satisfying (6.84) is in fact true for any $\xi \in \mathcal{B}(\delta)$. But (cf. (6.50)) the length of each such interval I_η is $\asymp \delta\rho^{-2l+1}$, hence (6.84) gives $|g(\xi + n) - g(\xi)| \leq c(\delta + \rho^{-N})$. It follows that

$$\begin{aligned} |g(\xi+n) - \rho^{2l}| &\leq |g(\xi + n) - g(\xi)| + |g(\xi) - \rho^{2l}| \\ &\leq c(\delta + \rho^{-N}) + \delta \\ &\leq (1 + c)\delta \\ &=: \delta_1 . \end{aligned}$$

Hence we have proved that for any $\xi \in \mathcal{B}(\delta)$ there exists $n \in \Gamma^\dagger \setminus \{0\}$ such that $\xi + n \in \mathcal{A}(\delta_1)$, that is $\mathcal{B}(\delta) \subset \bigcup_{n \in \Gamma^\dagger \setminus \{0\}} (\mathcal{A}(\delta_1) - n)$. Recalling that $\mathcal{A}(\delta_1) = \mathcal{B}(\delta_1) \cup \mathcal{D}(\delta_1)$ and using (i) and (ii) of Lemma 6.5 we obtain

$$\mathcal{B}(\delta) \subset \bigcup_{n \in \Gamma^\dagger \setminus \{0\}} (\mathcal{B}(\delta_1) - n) \bigcup \bigcup_{n \in \Gamma^\dagger \setminus \{0\}} (\mathcal{D}(\delta_1) - n). \tag{6.85}$$

Combining this with (i) and (ii) of Lemma 6.5 gives

$$\mathcal{B}(\delta) = \bigcup_{n \in \Gamma^\dagger \setminus \{0\}} \Big((\mathcal{B}(\delta_1) - n) \cap \mathcal{B}(\delta) \Big) \bigcup \bigcup_{n \in \Gamma^\dagger \setminus \{0\}} \Big((\mathcal{D}_1(\delta_1) - n) \cap \mathcal{B}(\delta) \Big). \tag{6.86}$$

We now consider the respective volumes in (6.86). Assume that $d \geq 3$. Using part (ii) of Lemma 6.1 and parts (iii) and (iv) of Lemma 6.5 we conclude that

$$\begin{aligned} \delta\rho^{d-2l} &\leq c(\delta_1^2\rho^{-4l+2d+1} + \delta_1\rho^{(\alpha-2l+1)d-\alpha+d-1}) + \delta_1\rho^{d-2l-\epsilon_0} \\ &\leq c(\delta^2\rho^{-4l+2d+1} + \delta\rho^{(\alpha-2l+1)d-\alpha+d-1} + \delta\rho^{d-2l-\epsilon_0}) \\ &=: I_1 + I_2 + I_3 . \end{aligned}$$

We have $I_3 = o(\delta\rho^{d-2l})$. Since $d > 1$ we also have $I_2 = o(\delta\rho^{d-2l})$. Hence we conclude that

$$\delta\rho^{d-2l} \leq c\delta^2\rho^{-4l+2d+1}.$$

Recalling that $\delta = c_3\rho^{2l-d-1}$, we reach a contradiction if c_3 is small enough. The same argument works when $d = 2$. □

Theorem 6.7 *Let $d \geq 3$. Suppose that ρ is large enough. Then $\lambda = \rho^{2l}$ belongs in $\sigma(H)$. Moreover, there exists $c_3 > 0$ such that the interval $[\rho^{2l} - c_3\rho^{2l-d-1}, \rho^{2l} + c_3\rho^{2l-d-1}]$ lies inside a single spectral band. If $d = 2$ then the same is true, but the respective interval is $[\rho^{2l} - c_3\rho^{2l-6}, \rho^{2l} + c_3\rho^{2l-6}]$.*

Proof. Let $d \geq 3$. We may assume that $\alpha \geq 2l - d - 1$. Let I_η be an interval with the properties specified in Lemma 6.6. Then the value of f at the one end of I_η is $\rho^{2l} + c_3\rho^{2l-d-1}$, and the value at the other end is $\rho^{2l} - c_3\rho^{2l-d-1}$. Since $f|_{I_\eta}$ is continuous, it takes the value ρ^{2l}. When $d = 2$ we argue similarly. □

Acknowledgment. The first author acknowledges the warm hospitality of the Department of Mathematics of University College London where part of this work was carried out. The work of the second author was partially supported by the Leverhulme trust and by the EPSRC grant EP/F029721/1.

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