SPECTRAL ASYMPTOTICS OF LAPLACE OPERATORS ON SURFACES WITH CUSPS

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1 Introduction

Let M be a connected surface with cusps (i.e., M is a compact perturbation of a surface with constant negative curvature and finite volume). Let Δ be the selfadjoint extention of the positive Laplace operator on M. Then (see [8] and references there) the spectrum of Δ consists of:

(i) the finite number of eigenvalues $0 = \lambda_0 < \lambda_1 \leq ... \leq \lambda_l < 1/4$,

(ii) the absolutely continuous spectrum $[1/4, +\infty)$

(iii) the eventual eigenvalues $1/4 \leq \lambda_{l+1} \leq ...$ (in finite number or not) which are embedded in the continuous spectrum.

Let $N_d(T)$ and $N_c(T)$ be the counting functions of the discrete and continuous spectra correspondingly (see Section 2 for the precise definition). Because of the complicated structure of spectrum (existence of embedded eigenvalues), it is hard to compute the asymptotics of $N_d(T)$ or $N_c(T)$ separately. However, it is possible to study the asymptotics of the sum

$$N_d(T) + N_c(T). \tag{1.1}$$

In the case of a surface with constant negative curvature three terms of the asymptotics of (1.1) can be obtained using methods of the theory of automorphic forms [6]:

$$N_d(T) + N_c(T) = \frac{|M|}{4\pi}T^2 - \frac{k}{\pi}T\ln T + \frac{k}{\pi}(1 - \ln 2)T + o(T) \tag{1.2}$$

($|M|$ is the area of M and k is the number of cusps). In the general case only the existence of the leading term in the asymptotics (1.2) was known [8]. In the first part of our paper we study the asymptotic behaviour of (1.1) for arbitrary surfaces with cusps and prove the following

Theorem 1.1

Let M be a surface with cusps. Then the following asymptotic formula holds:

$$N_d(T) + N_c(T) = \frac{|M|}{4\pi}T^2 - \frac{k}{\pi}T\ln T + O(T). \tag{1.3}$$

Suppose in addition that the Liouville measure of the periodic trajectories of the geodesic flow on M equals zero. Then formula (1.2) *is valid.*

Remark 1

The nonperiodicity condition in this theorem is very essential; if it is not satisfied, then formula (1.2) is not valid even for closed surfaces ($k = 0$).

Remark 2

Since for surfaces with constant negative curvature of finite volume the nonperiodicity condition is satisfied, we thus give another proof of (1.2) for such surfaces, without using the Selberg trace formula.

There exist two natural approaches to this problem. The first one is to use the cut-off Laplacian Δ_a [3]. If $N_{\Delta_a}(T)$ denotes the number of eigenvalues of Δ_a less than T^2, one can prove in the same way as in [5] that

$$N_d(T) + N_c(T) - N_{\Delta_a}(T) = O(T).$$

In order to compute $N_{\Delta_a}(T)$, one can use, for example, Dirichlet-Neumann bracketing (in the spirit of [2]). This approach has the advantage of being direct and simple, it is easy to check that the contribution from the cusp to $N_{\Delta_a}(\lambda)$, computed via bracketing, equals exactly (4.12). After proceeding further like we do in this paper, one arrives at the two-term asymptotics, which is quite sufficient to obtain formulae (1.8) and (1.9) for resonances. The disadvantage of this method is the bad estimate of the remainder term. Using bracketing it is never possible to obtain something better than $O(T)$ in the remainder, and so it seems difficult to obtain (1.2) using the cut-off Laplacian. In order to obtain three terms of $N_d(T) + N_c(T)$, we will use the approach of Vassiliev [12, 13]. But since his method was created for compact manifolds (and pure discrete spectrum), we should change it a little.

Suppose first that M is compact and denote $T^2 = \lambda$. Then Vassiliev's method computes $N_d(T) = \mathrm{Tr}E_\lambda = \int_M e(\lambda, x, x)dx$, where E_λ is a spectral projector, i.e., the orthogonal projector onto the subspace of $L^2(M)$, generated by the eigenfunctions of Δ with eigenvalues less than λ, and $e(\lambda, x, y)$ is the integral kernel of E_λ. In case M contains cusps, the trace of the spectral projector is infinite. But suppose that M_Y is big enough compact part of M (more precisely M_Y is defined via (4.1)); then the cut-off trace $\int\limits_{M_Y} e(\lambda, x, x)dx$ is finite and may be computed in the same way as in [9]. This results in obtaining (1.2) for general surfaces with cusps and non-periodic geodesic flow and (1.3) in the general case.

Another question we would like to discuss in this paper is the asymptotic behaviour of the resonance set of Δ. Let $\varphi(s)$ be the determinant of the scattering matrix and R be the set of all poles of $\varphi(s)$ (all but a finite number of elements of R lie in the left half-plane $\mathrm{Re}\, s < 1/2$). Let $N_p(T)$ be the number of poles ρ of $\varphi(s)$, counted with the order, such that $|\rho - 1/2| < T$. The resonance set $\sigma(M)$ of M is the union of three sets:

a) R,

b) the set of points $s_j \in \mathbb{C}$ such that $s_j(1 - s_j)$ is an eigenvalue of Δ,

c) $1/2$.

Each point $\nu \in \sigma(M)$ occurs with certain multiplicity $m(\nu)$ (see [8] for the precise definition). The counting function of the resonance set

$$N_\rho(T) = \sum_{\nu \in \sigma(M),\, |\nu - 1/2| < T} m(\nu) \tag{1.4}$$

is, up to small terms, equal to $2(N_d(T) + \frac{1}{2}N_p(T))$. We want to obtain the asymptotics of (1.4) by means of (1.3). In order to do this we have to estimate the difference

$$R(T) := N_c(T) - \frac{1}{2}N_p(T).$$

Thus, the second half of our paper is devoted to the estimations of $R(T)$.

In [8] it was shown that

$$R(T) = o(T^2), \tag{1.5}$$

but later a gap in the proof was found. In case when M has constant curvature it is known [11] that all the poles $\rho \in R$ of $\varphi(\lambda)$ lie in the strip $|\mathrm{Re}\, \rho| < C$ and

$$R(T) = O(T),$$

which results in obtaining two terms of the asymptotics of $N_\rho(T)$.

Although we are not able to obtain the second term in the general case, we can prove (1.5) and even improve it a little. Namely, we prove the following

Theorem 1.2

a) For arbitrary $\varepsilon > 0$ the following estimate holds:

$$R(T) = o(T^{3/2+\varepsilon}). \tag{1.6}$$

b) Suppose that all poles $\rho \in R$ of $\varphi(\lambda)$ lie in the strip $|\mathrm{Re}\, \rho| < C$. Then for arbitrary $\varepsilon > 0$

$$R(T) = o(T^{1+\varepsilon}). \tag{1.7}$$

Corollary 1.1

a) For arbitrary $\varepsilon > 0$ the following estimate holds:

$$N_d(T) + \frac{1}{2}N_p(T) = \frac{|M|}{4\pi}T^2 + o(T^{3/2+\varepsilon}). \tag{1.8}$$

b) Suppose that all the poles $\rho \in R$ of $\varphi(\lambda)$ lie in the strip $|\mathrm{Re}\, \rho| < C$. Then for arbitrary $\varepsilon > 0$

$$N_d(T) + \frac{1}{2}N_p(T) = \frac{|M|}{4\pi}T^2 + o(T^{1+\varepsilon}). \tag{1.9}$$

The proof of these two statements is based on the pigeonhole principle; the monotonicity of $N_d(T) + N_p(T)$ is very essential.

After our paper was written two new results concerning the above mentioned problems appeared. Namely, in [4] the example of manifold with cusps is constructed, for which the condition of part b) of theorem 1.2 is not satisfied. And in [14] the estimate

$$R(T) = O(T^{5/3})$$

were obtained, using methods, quite different from that of us.

The rest of the paper is organized in the following way: in Section 2 we introduce some preliminary notions and results; in Section 3 we give a brief description of Vassiliev's method for compact manifolds; in Section 4 we prove theorem 1.1 for general surfaces with cusps, and in Section 5 we prove theorem 1.2.

2 Preliminaries

For more details we refer the reader to [3, 7, 8].

Let M be the surface with cusps, i.e., M is a complete orientable two-dimentional manifold with metric g, and M admits a decomposition

$$M = M_0 \cup Z_1 \cup ... \cup Z_k.$$

Here M_0 is compact,

$$Z_i \simeq S^1 \times [a_i, \infty), \quad i = 1, ..., k, \tag{2.1}$$

$a_i > 0$, and the metric on Z_i in coordinates $z := (x, y) \in S^1 \times [a_i, \infty)$ is

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

If the metric g on M has constant negative curvature -1, then M is of the form $\Gamma\backslash H^2$, where H^2 is hyperbolic plane and Γ is a torsion-free discrete subgroup of $PSL(2, \mathbb{R})$.

Next, we consider the positive Laplace operator $\Delta : C_0^\infty(M) \to C^\infty(M)$, generated by the metric g. This operator, regarded as an operator in $L^2(M)$, has a unique self-adjoint extension, for which we save the notation Δ.

The spectrum of Δ consists of:

(i) the finite number of eigenvalues $0 = \lambda_0 < \lambda_1 \le ... \le \lambda_l < 1/4$,

(ii) the absolutely continuous spectrum $[1/4, +\infty)$ of multiplicity k (k is the number of cusps of our surface),

(iii) the eventual eigenvalues $1/4 \le \lambda_{l+1} \le ...$ (in finite number or not) which are embedded in the continuous spectrum (as it is shown in [3], for generic metric on M the set (iii) is empty).

The generalized eigenfunctions, which belong to the continuous spectrum, are Eisenstein functions $E_j(z, s)$ ($j = 1, ..., k$). Each $E_j(z, s)$ is a meromorphic function of $s \in \mathbb{C}$ with poles contained in the union of the half-plane $\operatorname{Re} s < 1/2$ and the interval $(1/2, 1]$; $E_j(z, s)$ is a smooth function of $z \in M$ and

$$\Delta E_j(z, s) = s(1 - s)E_j(z, s). \tag{2.2}$$

The zeroth Fourier coefficient of $E_j(z, s)$ on the cusp Z_i equals

$$\delta_{i,j} y_i^s + \phi_{i,j}(s) y_i^{1-s}. \tag{2.3}$$

Let

$$\phi(s) = \big(\phi_{i,j}(s)\big), \ \varphi(s) = \det \phi(s).$$

Then $\phi(s)$ is called the scattering matrix and satisfies

$$\phi(s)\phi(1 - s) = \mathrm{Id}, \quad \overline{\phi(s)} = \phi(\bar{s}), \quad \phi(s)^* = \phi(\bar{s}).$$

The poles of $\phi(s)$ (and of $\varphi(s)$) are contained in the union of the half-plane $\operatorname{Re} s < 1/2$ and the interval $(1/2, 1]$.

Let

$$N_d(T) = \sum_{\lambda_j < T^2} 1$$

and

$$N_c(T) := -\frac{1}{4\pi}\int\limits_{-T}^{T}\frac{\varphi'}{\varphi}(1/2+ir)dr$$

be the counting functions of the discrete and continuous spectra correspondingly.

Let u_j be an eigenfunction of Δ, corresponding to λ_j. We may assume that $\{u_j\}$ form an orthonormal basis in the subspace of $L^2(M)$, generated by the eigenfunctions of Δ. The Fourier expansion of the function $f \in C_0^\infty(M)$ has the form

$$f(z) = \sum_j (f,u_j)u_j(z) + \frac{1}{4\pi}\sum_{j=1}^{k}\int\limits_{-\infty}^{\infty} E_j(1/2+ir,\, z)\int\limits_M f(w)E_j(1/2-ir,\, w)d\mu(w)dr, \tag{2.4}$$

where $d\mu(w)$ is the volume form, generated by g.

3 Vassiliev's approach

Here, for the convenience of the reader, we give a brief description of the method used by Vassiliev in [12, 13] to compute the spectral asymptotics of elliptic operators on compact connected manifolds with boundaries (for the sake of simplicity we consider Laplace operators only).

Let M be a compact connected n-dimensional manifold with smooth boundary Γ. Suppose, $0 < \lambda_1 < \lambda_2 \leq ...$ are eigenvalues of Δ (with Dirichlet boundary conditions) and $u_i(z)$ are corresponding eigenvectors. Let $m \in \mathbb{N}$ be any number greater than $n/2$. We are going to prove the asymptotic formula

$$N_{\Delta^m}(T) := \mathrm{card}\{\lambda_j < T^{2/m}\} = C_n|M|T^{n/m} - \tilde{C}_n|\Gamma|T^{(n-1)/m} + o(T^{(n-1)/m}) \tag{3.1}$$

in case, when Liouville measure of the union of periodic trajectories of the geodesic flow equals zero, and

$$N_{\Delta^m}(T) = C_n|M|T^{n/m} + O(T^{(n-1)/m}) \tag{3.2}$$

in the general case ($|\Gamma|$ is the $(n-1)$-dimensional volume of Γ, $C_n = (2\pi)^{-n}\omega_n$ and $\hat{C}_n = (2\pi)^{1-n}\omega_{n-1}/4$, ω_n being the volume of a unit ball in R^n). Obviously, from (3.1) and (3.2) the asymptotics of $N_\Delta(\lambda)$ follows at once.

First of all we introduce the coordinates $z := (r, r_\Gamma)$ on M such that r measures the distance from z to Γ and $r_\Gamma = (r_1, ..., r_{n-1})$ are coordinates along the boundary. Next we denote $\lambda = T^2$ and introduce the partition of M into three regions:

(i) inner region $r > \delta$ (δ is some small positive number),

(ii) intermediate region $\lambda^{(\varepsilon-1)/2m} < r < \delta \qquad (0 < \varepsilon < 1/4)$,

(iii) boundary region $r < \lambda^{(\varepsilon-1)/2m}$.

Let $\chi_+(r) \in C^\infty(R^+)$ be a real function such that $\chi_+(r) = 0$ whenever $r < 1/2$, $d\chi_+(r)/dr \geq 0$, $\chi_+(1-r) = 1-\chi_+(1+r)$; we also denote $\chi_0(r) := \chi_+(r) - \chi_+(r/2)$, $\chi_-(r) := 1-\chi_+(r/2)$ and $|M|_\epsilon = \int\limits_M \chi_\epsilon$. ($\epsilon = +,-,0$).

Let p be the smallest positive number such that $\delta/2^{p+1} < \lambda^{(\varepsilon-1)/2m}$. We set $\delta_l = \delta/2^l$, $l = 1, ..., p+1$. One may easily see that

$$N_{\Delta^m}(T) = \mathrm{Tr}E_\lambda = \mathrm{Tr}(\chi_+(r/\delta)E_\lambda) + \sum_{l=1}^{p}\mathrm{Tr}(\chi_0(r/\delta_l)E_\lambda) + \mathrm{Tr}\big(\chi_-(r/\delta_{p+1})E_\lambda\big), \tag{3.3}$$

where E_λ is the spectral projector of Δ^m, i.e.,

$$E_\lambda(v) = \sum_{\lambda_k^m < \lambda}(v, u_k)u_k.$$

Now using microlocal analysis as well as the wave equation method it is possible to prove the following asymptotics in the inner and intermediate regions:

Proposition 3.1

As $\lambda \to \infty$,

$$\mathrm{Tr}\big(\chi_+(r/\delta)E_\lambda\big) = C_n|M|_+\lambda^{n/2m} + O(\lambda^{(n-1)/2m}). \tag{3.4}$$

Suppose that the Liouville measure of the union of periodic trajectories of the geodesic flow on M *equals zero. Then, as* $\lambda \to \infty$,

$$\mathrm{Tr}\big(\chi_+(r/\delta)E_\lambda\big) = C_n|M|_+\lambda^{n/2m} + o(\lambda^{(n-1)/2m}). \tag{3.5}$$

Proposition 3.2

As $\lambda \to \infty$, *the following asymptotic formula holds uniformly for* $\lambda^{(\varepsilon-1)/2m} < \delta \le d_0$ $(d_0 > 0)$:

$$\mathrm{Tr}\big(\chi_0(r/\delta)E_\lambda\big) = C_n|M|_0\lambda^{n/2m} + O(\delta^{\varepsilon/32}\lambda^{(n-1)/2m}). \tag{3.6}$$

Since the boundary region is very small, it is sufficient to compute the contribution from there very roughly, namely, we can apply the usual method of freezing coefficients. Let us fix arbitrary point $r_\Gamma \in \Gamma$ and consider the new problem:

$$\Delta_\Gamma^m u(r) = \lambda u(r), \tag{3.7}$$

$$\frac{d^{2j}}{dr^{2j}}u(r)\Big|_{r=0} = 0 \quad (j = 0, ..., m-1),$$

where Δ_Γ^m is obtained from Δ^m via the following procedure (which we call freezing of the coefficients):

a) in the expression for Δ^m in coordinates (r, r_Γ) we save only terms with $2m$ derivatives;

b) all derivatives along the boundary $\partial/\partial r_k$ $(k = 1, ..., n-1)$ we change into $i\xi_k$;

c) we put $z = (0, r_\Gamma)$ into coefficients.

Let $R(\mu)$ be the resolvent of the initial Dirichlet problem, and $R_\Gamma(\mu, \xi_\Gamma)$ $(\xi_\Gamma = (\xi_1, ..., \xi_{n-1}))$ be the resolvent of the problem (3.7). Then the following formula holds:

$$Tr\Big(\chi_-(r/\delta)\big(R(\mu) - (2\pi)^{1-n}\int\limits_{\mathbb{R}^{n-1}} R_\Gamma(\mu, \xi_\Gamma)d\xi_\Gamma\big)\Big) = o(\lambda^{\varepsilon(n-1)/2m-1}), \tag{3.8}$$

whenever $|\arg\mu| \ge c\lambda^{-\varepsilon/2m}$ ($c > 0$ being arbitrary). Let λ_c be a complex number such that $\arg\lambda_c = c(\lambda^{1/2m}\delta)^{-1}$, $|\lambda_c| = |\lambda|$, and $L(\lambda_c)$ be a circle arc which leads from λ_c to $\bar{\lambda}_c$ and does not intersect the real positive λ-axis. In order to compute the contribution from the boundary region we use the Pleijel formula [1]:

$$\Big|\ \mathrm{Tr}\big(\chi_-(r/\delta)E_\lambda\big) - (-2\pi i)^{-1}\int\limits_{L(\lambda_c)} \mathrm{Tr}\big(\chi_-(r/\delta)\mathbf{R}(\mu)\big)d\mu\ \Big|$$
$$\le (1+\pi^{-2})^{1/2}(\mathrm{Im}\lambda_c)\ \Big|\ \mathrm{Tr}\big(\chi_-(r/\delta)\mathbf{R}(\lambda_c)\big)\ \Big|\ . \tag{3.9}$$

Formulae (3.8) and (3.9) imply that in order to obtain the contribution to the asymptotics from the boundary region we should investigate the one-dimension problem (3.7). After doing this we arrive at

Proposition 3.3

As $\lambda \to \infty$,

$$\mathrm{Tr}\big(\chi_-(r/\delta)E_\lambda\big) = C_n|M|_-\lambda^{n/2m} + \tilde{C}_n|\Gamma|\lambda^{(n-1)/2m} + o(\lambda^{(n-1)/2m}). \tag{3.10}$$

Formulae (3.1) and (3.2) follow directly from (3.4)–(3.6) and (3.10).

Before proceeding further we make some remarks. As it turns out, the condition $2m > n$ is purely technical and was introduced only in order to make all the integrals absolutely convergent; however, if not to care much about the reasoning of passing to the limit $c \to 0$ in (4.11), one may put $m = 1$, which leads to the same answer but reduces computations reasonably. Since our computations are complicated enough even without additional fractional powers of λ, in what follows we put $m = 1$ and do not pay much attention to the fact that some integrals fail to converge absolutely; we also refer the reader to [9], where the complete computations are made in a slightly different setting.

4 Improved Asymptotics of Counting Function

Here we are going to give a proof of theorem 1.1.

Let $\lambda := T^2 + 1/4$. Since $|\sqrt{\lambda} - T| = O(T^{-1})$, it is sufficient to prove the asymptotic formulas for $N_d(\sqrt{1/4 + T^2}) + N_c(T)$, equvivalent to (1.2) and (1.3).

Suppose for simplicity that M contains only one cusp. Then, if we denote by E_λ the spectral projector of Δ, from (2.4) it follows that

$$E_\lambda v = \sum_{\lambda_j < \lambda} (v, u_j) u_j + \frac{1}{4\pi} \int_{-T}^{T} \big(v, E(1/2 + ir, \cdot)\big) E(1/2 + ir, \cdot) dr,$$

where $(\cdot,\cdot)$ stands for the scalar product in $L^2(M)$. The spectral function (i.e., the integral kernel of E_λ) is then equal to

$$e(\lambda, z, w) = \sum_{\lambda_j < \lambda} \overline{u_j(z)} u_j(w) + \frac{1}{4\pi} \int_{-T}^{T} \overline{E(1/2 + ir, z)} E(1/2 + ir, w) dr.$$

Unfortunately,

$$\int_M e(\lambda, z, z) d\mu(z)$$

is infinite (if $\lambda > 1/4$), and so E_λ is not of trace class. Thus, we should introduce the cut-off trace which is finite. Suppose that $Y > a_1$ and let

$$M_Y = M_0 + Z_{1,Y}, \tag{4.1}$$

where under the isometry (2.1) $Z_{1,Y}$ maps onto $S^1 \times [a_1, Y]$.

We define the cut-off trace of E_λ to be

$$\mathrm{Tr}_Y E_\lambda := \int_{M_Y} e(\lambda, z, z) d\mu(z) = \int_{M_Y} \big[\sum_{\lambda_j < \lambda} |u_j(z)|^2 + \frac{1}{4\pi} \int_{-T}^{T} |E(1/2 + ir, z)|^2 dr\big] d\mu(z). \tag{4.2}$$

Employing the same method as in [5], one can show that

$$\int_{M_Y} |E(1/2 + ir, z)|^2 d\mu(z) = 2 \ln Y - \frac{\varphi'}{\varphi}(1/2 + ir) + b_1(r, Y) - b_2(r, Y), \tag{4.3}$$

where

$$b_1(r, Y) = O(1/r)$$

uniformly on Y and

$$b_2(r, Y) = \int_{M \setminus M_Y} \left| \, E(1/2 + ir, z) - y^{1/2 + ir} - \varphi(1/2 + ir) y^{1/2 - ir} \, \right|^2 d\mu(z).$$

It is obvious that

$$\int_{M_Y} \big[\sum_{\lambda_j < \lambda} |u_j(z)|^2\big] d\mu(z) = N_d(\sqrt{\lambda}) - b_3(\lambda, Y),$$

where

$$b_3(\lambda, Y) = \int_{M \setminus M_Y} \sum_{\lambda_j < \lambda} |u_j(z)|^2 d\mu(z).$$

Integrating (4.3), we obtain:

$$\int_{-T}^{T} dr \int_{M_Y} |E(1/2+ir,z)|^2 d\mu(z) = 4T\ln Y - \int_{-T}^{T} \frac{\varphi'}{\varphi}(1/2+ir)dr + O(\ln T) - \int_{-T}^{T} b_2(r,Y)dr. \tag{4.4}$$

This formula implies

$$0 < \int_{-T}^{T} b_2(r,a_1)dr + b_3(\lambda,a_1) < -\int_{-T}^{T} \frac{\varphi'}{\varphi}(1/2+ir)dr + N_d(\lambda) + O(T) = O(T^2)$$

(the last estimate follows from [8]). Now, using the fact that Fourier expansions of $u_j(z)$ and $\big(E(1/2+ir,z) - y^{1/2+ir} - \varphi(1/2+ir)y^{1/2-ir}\big)$ contain no constant term, one can show that if

$$Y \sim T^3, \tag{4.5}$$

then

$$\int_{-T}^{T} b_2(r,Y)dr + b_3(\lambda,Y) = o(1), \quad T\to\infty. \tag{4.6}$$

So, now we should compute the left side of (4.2) when $Y \sim \lambda^{3/2}$, and the time has come to employ Vassiliev's method. Suppose, $a=\lambda^{2/5}$. Then, if we chose the inner region to be M_0 and the intermediate one $S^1\times[a_1,a]$, propositions 3.1 and 3.2 are valid without any changes also in our case (one may simply repeat Vassiliev's proof of proposition 3.1; in order to prove proposition 3.2 one can use the same approach as in [10] together with the explicit form of the fundamental solution of the wave equation at the cusp). However, proposition 3.3 fails to be true. Thus what we have to do is to compute the contribution from the "boundary" region, i.e.

$$\mathrm{Tr}[\chi_{a,Y}E_\lambda] = \int_M \chi_{a,Y}(z)\Big[\sum_{\lambda_j<\lambda}|u_j(z)|^2 + \frac{1}{4\pi}\int_{-T}^{T}|E(1/2+ir,z)|^2dr\Big]d\mu(z), \tag{4.7}$$

where

$$\chi_{a,Y}(z) = \begin{cases} 1, & \text{if } a<y(z)<Y \\ 0, & \text{otherwise}\end{cases}$$

(the fact that $\chi_-(z)=\chi_{a,Y}(z)$ is no longer smooth does not play any role; see [9] for more details). Just as before, in order to compute (4.7) we use the Pleijel formula

$$\begin{aligned} &|\mathrm{Tr}(\chi_{a,Y}E_\lambda) - (-2\pi i)^{-1}\int_{L(\lambda_c)} d\mu\,\mathrm{Tr}\big(\chi_{a,Y}\mathbf{R}(\mu)\big)| \\ &\le (1+\pi^{-2})^{1/2}(\mathrm{Im}\lambda_c)\,\big|\,\mathrm{Tr}\big(\chi_{a,Y}\mathbf{R}(\lambda_c)\big)\,\big|, \end{aligned} \tag{4.8}$$

where

$$\arg\lambda_c = c(\lambda^{1/2m}\delta)^{-1}, \quad |\lambda_c| = |\lambda|, \tag{4.9}$$

and $L(\lambda_c)$ is a circle arc which leads from λ_c to $\bar\lambda_c$ and does not intersect the real positive λ-axis.

Now we again employ the method of freezing coefficients. Let $\mathbf{R}_{\mathbf{fr}}(\mu)$ be the resolvent of the Laplacian with frozen coefficients on the circle S^1 (later we compute $\mathbf{R}_{\mathbf{fr}}(\mu)$ precisely). Changing slightly considerations of [12], it is easy to prove that

$$\mathrm{Tr}\Big[\chi_{a,Y}\big(\mathbf{R}(\mu) - (2\pi)^{-1}\int_{\mathbb{R}}\mathbf{R}_{\mathbf{fr}}(\mu)d\xi\big)\Big] = O(\lambda^{-4/5}).$$

So, it is sufficient to compute

$$\mathbf{I} := (-2\pi i)^{-1} \int\limits_{L(\lambda_c)} \int\limits_{\mathbb{R}} \mathrm{Tr}\big(\chi_{a,Y}\mathbf{R}_{fr}(\mu)\big) d\xi d\mu \tag{4.10}$$

and use (4.8) in order to obtain the "boundary" contribution (4.7) (the estimation of the right side of (4.8) may be handled in the same way and will be skipped).

Let us proceed with direct computations. Suppose, we fixed a point $y \in [a, Y]$. Then the operator with frozen coefficients on $S^1 = \mathbb{R}/\mathbb{Z}$ is equal ([12] or [9]) to

$$\Delta_{fr} f(x) := -y^2 \frac{d^2}{dx^2} f(x) + y^2\xi^2 f(x)$$

with periodical boundary conditions on the interval $x \in [0,1]$. Using the Fourier expansion of a periodic function, we can find the kernel of the resolvent of Δ_{fr} to be equal to

$$R_{fr}(y, \xi; \lambda; x_1, x_2) = \sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n(x_1 - x_2)}}{y^2\xi^2 + 4\pi^2 y^2 n^2 - \lambda}.$$

Thus, the contribution (4.10) from the boundary region equals

$$\begin{aligned}
\mathbf{I} &= (-2\pi i)^{-1}(2\pi)^{-1} \int\limits_{-\infty}^{\infty} d\xi \int\limits_{a}^{Y} dy \int\limits_{0}^{1} dx \int\limits_{L(\lambda_c)} d\mu R_{fr}(y, \xi; \mu; x, x) \\
&= (-2\pi i)^{-1}(2\pi)^{-1} \sum_n \int\limits_{-\infty}^{\infty} d\xi \int\limits_{a}^{Y} dy \int\limits_{L(\lambda_c)} d\mu (y^2\xi^2 + 4\pi^2 y^2 n^2 - \mu)^{-1}.
\end{aligned} \tag{4.11}$$

Now we may pass to the limit $c \to 0$ in (4.11), which results in $\lambda_c \to \lambda \in \mathbb{R}$. Since

$$\lim_{\lambda_c \to \lambda} \int\limits_{L(\lambda_c)} \frac{d\mu}{A - \mu} = \begin{cases} 0, & A > \lambda \\ -2\pi i, & A < \lambda \end{cases},$$

we see that

$$\begin{aligned}
\mathbf{I} &= (2\pi)^{-1} \sum_n \int\limits_{a}^{Y} dy \int\limits_{y^2\xi^2 + 4\pi^2 y^2 n^2 < \lambda} d\xi = 2(2\pi)^{-1} \sum_{n=-[\sqrt{\lambda}/(2\pi a)]}^{[\sqrt{\lambda}/(2\pi a)]} \int\limits_{a}^{Y} \frac{\sqrt{(\lambda - 4\pi^2 y^2 n^2)^+}}{y} dy \\
&= \pi^{-1}\sqrt{\lambda}(\ln Y - \ln a) + 2\pi^{-1} \sum_{n=1}^{[\sqrt{\lambda}/(2\pi a)]} \int\limits_{a}^{\sqrt{\lambda}/(2\pi n)} \frac{\sqrt{\lambda - 4\pi^2 y^2 n^2}}{y} dy,
\end{aligned} \tag{4.12}$$

where $(\cdot)^+$ stands for the positive part and $[\cdot]$ for the greatest integer part ((4.5) implies $Y > \sqrt{\lambda}/(2\pi)$).

After computing the integral

$$\int u^{-1}\sqrt{1 - u^2} du = \sqrt{1 - u^2} + \frac{1}{2} \ln \frac{1 - \sqrt{1 - u^2}}{1 + \sqrt{1 - u^2}} =: F(u) \tag{4.13}$$

we may transform $\mathbf{I}$ as

$$\mathbf{I} = \pi^{-1}\sqrt{\lambda}(\ln Y - \ln a) + 2\pi^{-1}\mathbf{J}, \tag{4.14}$$

where

$$\mathbf{J} = \sum_{n=1}^{[\alpha]} \int\limits_{a}^{\sqrt{\lambda}/(2\pi n)} \frac{\sqrt{\lambda - 4\pi^2 y^2 n^2}}{y} dy = 2\pi a \sum_{n=1}^{[\alpha]} \int\limits_{n}^{\alpha} \frac{\sqrt{\alpha^2 - w^2}}{w} dw$$

$$= 2\pi a \alpha \sum_{n=1}^{[\alpha]} \big[F(1) - F(n/\alpha)\big] = -\sqrt{\lambda} \sum_{n=1}^{[\alpha]} F(n/\alpha) \tag{4.15}$$

(we have put $\alpha := \sqrt{\lambda}/(2\pi a)$ and made the substitution $w = ny/a$).

So, the problem is reduced to the computing of the asymptotic behaviour of $\mathbf{J}$ as $\alpha \to \infty$. Since the expression in the right side of (4.15) is just α times integral sum for $F(u)$ on [0,1], we obviously have:

$$\mathbf{J} \sim -\sqrt{\lambda}\alpha \int\limits_0^1 F(u) du =: C_1 \sqrt{\lambda}\alpha, \tag{4.16}$$

which gives us the leading term of the asymptotics. Now let us compute the following terms. First, we note that

$$F(u) = \ln u + G(u), \tag{4.17}$$

where

$$G(1) = 0, \quad G(0) = 1 - \ln 2. \tag{4.18}$$

Then we have:

$$\mathbf{J} - C_1\sqrt{\lambda}\alpha = -\sqrt{\lambda}\Big[\sum_{n=1}^{[\alpha]}\big(F(n/\alpha) - \alpha \int\limits_{(n-1)/\alpha}^{n/\alpha} F(u)du\big) - \alpha \int\limits_{[\alpha]/\alpha}^{1} F(u)du\Big]$$

$$= \sqrt{\lambda}\alpha\Big[\sum_{n=1}^{[\alpha]} \int\limits_{(n-1)/\alpha}^{n/\alpha} \big(\ln u - \ln(n/\alpha)\big)du\Big] + \sqrt{\lambda}\alpha\Big[\sum_{n=1}^{[\alpha]} \int\limits_{(n-1)/\alpha}^{n/\alpha} \big(G(u) - G(n/\alpha)\big)du\Big] + O(1)$$

$$=: \mathbf{J_1} + \mathbf{J_2} + O(1). \tag{4.19}$$

It is not difficult to compute $\mathbf{J_2}$, we just have to notice that

$$\int\limits_{(n-1)/\alpha}^{n/\alpha} \big(G(u) - G(n/\alpha)\big)du = \frac{G\big((n-1)/\alpha\big) - G(n/\alpha)}{2\alpha} + O(\alpha^{-3}), \tag{4.20}$$

and thus

$$\mathbf{J_2} = \sqrt{\lambda}\alpha \sum_{n=1}^{[\alpha]} \frac{G\big((n-1)/\alpha\big) - G(n/\alpha)}{2\alpha} + O(1) = \sqrt{\lambda}G(0)/2 + O(1) = \frac{1 - \ln 2}{2}\sqrt{\lambda} + O(1). \tag{4.21}$$

In order to compute $\mathbf{J_1}$, we use Stirling's formula:

$$\mathbf{J_1} = \sqrt{\lambda}\big([\alpha]\ln([\alpha]/\alpha) - [\alpha]\big) - \sqrt{\lambda}\big(\sum_{n=1}^{[\alpha]} \ln(n/\alpha)\big) = \sqrt{\lambda}\big([\alpha]\ln([\alpha]/\alpha) - [\alpha]\big) - \sqrt{\lambda}\big(\ln \frac{[\alpha]!}{\alpha^{[\alpha]}}\big)$$

$$= \sqrt{\lambda}\big([\alpha]\ln([\alpha]/\alpha) - [\alpha]\big) - \sqrt{\lambda}\Big([\alpha]\ln([\alpha]/\alpha) - [\alpha] + 1/2\ln(2\pi[\alpha]) + \ln\big(1 + o(1)\big)\Big)$$

$$= -\frac{\sqrt{\lambda}}{4}\ln\lambda + \frac{\sqrt{\lambda}}{2}\ln a + o(\lambda). \tag{4.22}$$

Comparing formulas (4.14)-(4.22), we have:

$$\mathbf{I} = \pi^{-1}\sqrt{\lambda}\ln Y + 4C_1\lambda/a - \frac{1}{2\pi}\sqrt{\lambda}\ln\lambda + \frac{1-\ln 2}{\pi}\sqrt{\lambda} + o(\lambda). \tag{4.23}$$

Now from (4.2)–(4.6) and (4.23) as well as (3.4)-(3.6) the desired asymptotic formula follows immediately, since

$$\ln Y(\sqrt{\lambda} - \sqrt{\lambda - 1/4}) \sim \frac{\ln\lambda}{\sqrt{\lambda}} = o(1)$$

(we do not need to compute C_1, since the coefficient at the first term is already known [8]).

5 Resonance set

Here we give a proof of theorem 1.2. The main tool for our investigations will be the formula

$$R(T) = \frac{1}{2\pi} \sum_{\substack{\rho\in R \\ \mathrm{Re}_{\rho<1/2}}} \mathrm{arctg}\Big(\frac{(1-2\mathrm{Re}\,\rho)T}{|\rho-1/2|^2 - T^2}\Big) + O(T), \tag{5.1}$$

due to W.Müller [8].

Lemma 5.1

Suppose that for some $\delta > 0$

$$R(T) = O(T^{3/2+\delta}).$$

Then there exists a constant C_1 such that for all sufficiently big T there exists $T_0 \in [T, T + T^{1/2+\delta/3}]$ such that

$$|R(T_0)| < C_1 T_0^{3/2+\delta/3}.$$

Lemma 5.2

Suppose that all the poles $\rho \in R$ of $\varphi(s)$ lie in the strip $|\mathrm{Re}\ \rho| < C$. Suppose also that for some $\delta > 0$

$$R(T) = O(T^{1+\delta}).$$

Then there exists C_2 such that for all sufficiently big T there exists $T_0 \in [T, T + T^{2\delta/3}]$ such that

$$|R(T_0)| < C_2 T_0^{1+2\delta/3}.$$

Lemma 5.3

Suppose that there exist $\delta_1, \delta_2 > 0$ and C_3 such that for all sufficiently big T there exists $T_0 \in [T, T + T^{\delta_1}]$ such that

$$|R(T_0)| < C_3 T_0^{1+\delta_2}.$$

Then

$$R(T) = O(T^{1+\max(\delta_1,\delta_2)}).$$

First suppose we have proved these lemmas. Since we know already that

$$R(T) = O(T^2),$$

we can apply lemma 5.1 and show that for all sufficiently big T there exists $T_0 \in [T, T^{2/3}]$ such that $R(T_0) < C_1 T_0^{5/3}$. After applying lemma 5.3 we see that $R(T) = O(T^{5/3})$. Now we can apply lemma 5.1 again. After sufficiently many steps we arrive at (1.6). Formula (1.7) follows in the same way from lemmas 5.2-5.3. So, it remains only to prove lemmas 5.1-5.3.

Proof of lemma 5.1

Let T be so big that the number of poles $\rho \in R$ such that $|\rho - 1/2| \in [T, T + T^{1/2+\delta}]$ is less than $C_4 T^{3/2+\delta}$. Now we cut $[T, T + T^{1/2+\delta/3}]$ into $[T^{2\delta/3}]$ smaller intervals, each of length $\sim T^{1/2-\delta/3}$. By the pigeonhole principle one of these smaller intervals (say, $[T_1, \tilde{T}_1]$, where $\tilde{T}_1 - T_1 \sim T^{1/2-\delta/3}$) contains modules of not more than $C_5 T^{3/2+\delta/3}$ poles $\rho \in R$. We chose T_0 to be the middle point of $[T_1, \tilde{T}_1]$. Then we have:

$$\begin{aligned} R(T_0) &= \frac{1}{2\pi}\Big(\sum_{|\rho-1/2|\in[T_1,\tilde{T}_1]} + \sum_{|\rho-1/2|\notin[T_1,\tilde{T}_1]} \Big) \mathrm{arctg}\Big(\frac{(1-2\mathrm{Re}\,\rho)T_0}{|\rho-1/2|^2 - T_0^2} \Big) + O(T) \\ &=: I_1 + I_2 + O(T). \end{aligned}$$

Since the number of terms in I_1 is smaller than $C_5 T^{3/2+\delta/3}$ and $\mathrm{arctg}\, x < \pi/2$, obviously

$$I_1 = O(T^{3/2+\delta/3}).$$

The estimations of I_2 may be carried in the following way:

$$|I_2| \le C_6 \sum_{|\rho-1/2|\notin[T_1,\tilde{T}_1]} \Big(\frac{(1-2\mathrm{Re}\,\rho)}{|\rho-1/2|^2} T_0 \mid 1 - \frac{T_0^2}{|\rho-1/2|^2} \mid^{-1} \Big) \le C_7 T^{3/2+\delta/3},$$

since [8]

$$\sum_{\rho\in R} \Big(\frac{(1-2\mathrm{Re}\,\rho)}{|\rho-1/2|^2} \Big) < \infty.$$

This completes the proof of lemma 4.1.

Proof of lemma 5.2

The proof is based on the same ideas as the previous one, but is more complicated.

First, we suppose that T is so big that the number of poles $\rho \in R$ such that $|\rho - 1/2| \in [T - T^\delta, T + T^\delta]$ is less than $C_7 T^{1+\delta}$. Next, we cut $[T, T + T^{2\delta/3}]$ onto $[T^{\delta/3}]$ smaller intervals each of length $\sim T^{\delta/3}$. By the pigeonhole principle there exists one of these smaller intervals $[T_1, \tilde{T}_1]$ $(\tilde{T}_1 - T_1 \sim T^{\delta/3})$ such that $T_1 \ne T$ (i.e. $T_1 - T \ge T^{\delta/3}$), $\tilde{T}_1 \ne T + T^{2\delta/3}$ and the number of points $\rho \in R$ such that $|\rho - 1/2| \in [T_1, \tilde{T}_1]$ is less than $C_7 T^{1+\delta}/(T^{\delta/3} - 2) < 2C_7 T^{1+2\delta/3}$. Continuing this process we obtain the sequence of intervals $[T_i, \tilde{T}_i]$ $(i = 1, ..., n = [3\delta^{-1}] + 4)$ such that $\tilde{T}_i - T_i \sim T^{\delta-(i+1)\delta/3}$, $T_i - T_{i-1} \ge T^{\delta-(i+1)\delta/3}$, $\tilde{T}_{i-1} - \tilde{T}_i \ge T^{\delta-(i+1)\delta/3}$ and the number of points $\rho \in R$ such that $|\rho - 1/2| \in [T_i, \tilde{T}_i]$ is less than $2^i C_7 T^{1+\delta-i\delta/3}$. Let now $T_0 = (T_n + \tilde{T}_n)/2$. Then we have:

$$\begin{aligned} R(T_0) &= (2\pi)^{-1} \sum_{\rho\in R} \mathrm{arctg}\Big(\frac{(1-2\mathrm{Re}\,\rho)T_0}{|\rho-1/2|^2 - T_0^2} \Big) + O(T_0) \\ &\le C_8 T_0 \sum_{|\rho-1/2|>2T_0} \frac{1-2\mathrm{Re}\,\rho}{|\rho|^2} + C_9 \Big(\sum_{|\rho-1/2|\in[0,2T_0]\setminus[T_0-T_0^\delta, T_0+T_0^\delta]} \\ &+ \sum_{|\rho-1/2|\in[T_0-T_0^\delta, T_0+T_0^\delta]\setminus[T_1,\tilde{T}_1]} + \sum_{i=1}^{n-1} \sum_{|\rho-1/2|\in[T_i,\tilde{T}_i]\setminus[T_{i+1},\tilde{T}_{i+1}]} \Big) \frac{1}{|\rho-1/2| - T_0} \\ &+ C_{10} \mathrm{card}\{\rho \in R \mid |\rho - 1/2| \in [T_n, \tilde{T}_n]\} + O(T) \end{aligned} \tag{5.2}$$

The estimate of every term of the right side in (5.2) exept the sum over

$$|\rho - 1/2| \in [0, 2T_0] \setminus [T_0 - T_0^\delta, T_0 + T_0^\delta] \tag{5.3}$$

follows immediately from the construction of T_0. In order to estimate sum over (5.3), we notice that whenever $1 > \delta_1 > \delta$, the number of poles $\rho \in R$ in the interval $[T - T^{\delta_1}, T + T^{\delta_1}]$ is $O(T^{1+\delta_1})$. So, writing our sum as

$$\sum_{j=1}^{[3\delta^{-1}]-3} \sum_{T_0^{\delta+(j-1)\delta/3} < |\rho - 1/2 - T_0| \le T_0^{\delta+j\delta/3}}$$

we obtain the desired estimate and finish the proof of lemma 4.2.

Proof of lemma 5.3

We may suppose that $\max(\delta_1, \delta_2) < 1$. Obviously,

$$N_d(T) + \frac{1}{2}N_p(T) + R(T) = N_d(T) + N_c(T) = \frac{|M|}{4\pi}T^2 + O(T\ln T).$$

Suppose that T is big enough. Then there exist $T_0 \in [T, T + T^{\delta_1}]$ and $T_1 \in [T - 2T^{\delta_1}, T]$ such that $|R(T_0)| < C_3 T_0^{1+\delta_2}$, $|R(T_1)| < C_3 T_1^{1+\delta_2}$. Since $N_d(T) + N_p(T)$ is monotonious, we have:

$$\begin{aligned} R(T) &= \frac{|M|}{4\pi}T^2 - N_d(T) - \frac{1}{2}N_p(T) + O(T\ln T) \\ &\geq \frac{|M|}{4\pi}T^2 - \frac{|M|}{4\pi}T_0^2 + \Big(\frac{|M|}{4\pi}T_0^2 - N_d(T_0) - \frac{1}{2}N_p(T_0)\Big) + O(T\ln T) \\ &\geq \frac{|M|}{4\pi}T^2 - \frac{|M|}{4\pi}(T + T^{\delta_1})^2 + R(T_0) + O(T\ln T) \\ &\geq -2\frac{|M|}{4\pi}T^{1+\delta_1} - 4C_3T^{1+\delta_2} + O(T\ln T). \end{aligned}$$

The opposite estimate may be proved in the same way. This finishes the proof of lemma 5.3 and of theorems 1.2.

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References

[1] Agmon, S.: Asymptotic formulas with remainder estimates for eigenvalues of elliptic operators. Arch. Ration. Mech. and Analysis **28**, 165-183 (1968)

[2] van den Berg, M.: Dirichlet-Neumann bracketing for horn-shaped regions. J. Funct. Anal. **104**, 110-120 (1992)

[3] Colin de Verdière, Y.: Pseudo-Laplacians II. Ann. Inst. Fourier **33**, 87-113 (1983)

[4] Froese, R. and Zworski, M.: Finite volume surfaces with resonances far from the unitary axis. Int. Math. Res. Notes **10**, 275-277 (1993)

[5] Lax, P., Phillips, R.: Scattering theory for automorphic functions. Princeton Univ. Press, 1976.

[6] Levitan, B.M.: On the asymptotic behaviour of the spectral function and discrete spectrum of the Laplace operator on a fundamental domain of Lobachevsky's plane. Rendiconti del Seminario Matematico e Fisiko di Milano **52**, 195-219 (1982)

[7] Müller, W.: Spectral theory for Riemannian manifolds with cusps and a related trace formula. Math. Nachr. **. 111**, 197-288 (1983)

[8] Müller, W.: Spectral geometry and scattering theory for certain complete surfaces of finite volume. Invent. Math. **109** , 265-305 (1992)

[9] Parnovski, L.B.: Asymptotics of Dirichlet spectrum on some class of noncompact domains. Preprint.

[10] Seeley, R.: An estimate near the boundary for the spectral function of the Laplace operator. Amer. J. of Math. **102**, 869-902 (1980)

[11] Selberg, A.: Remarks on the distribution of poles of Eisenstein series. In: Gelbart S., Howe R., Sarnak P. (eds) Israel Math. Conf. Proc., vol. 3, part II. Festschrift in honor of I.I. Piatetski-Shapiro pp. 251-278. Jerusalem: Weizman 1990.

[12] Vassiliev, D. G.: Asymptotics of the spectrum of a boundary value problem. Transactions of the Moscow Mathematical Society, **49**, 173-245 (1986)

[13] Vassiliev, D. G.: Binomial asymptotics of the spectrum of a boundary-value problem. Funct. Anal. Appl. **17** (4), 309-311 (1983)

[14] Vodev, G.: Asymptotics on the number of scattering poles for degenerate perturbations of the Laplacian. Preprint.

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Summary

Let M be a surface with cusps of finite volume. We consider the sum of the counting functions of the discrete and continuous spectrum of M. Using the method of Vasssiliev we obtain three terms of asymptotics of this sum:

$$N_d(T) - \frac{1}{4\pi}\int_{-T}^{T} \frac{\varphi'}{\varphi}(1/2 + ir)dr = \frac{|M|}{4\pi}T^2 - \frac{k}{\pi}T\ln T + \frac{k}{\pi}(1 - \ln 2)T + o(T)$$

($N_d(T)$ being the counting function of the discrete spectrum, φ the determinant of the scattering matrix, and k the number of cusps), if the Liouville measure of the periodic trajectories of the geodesic flow on M equals zero; in the general case $o(T)$ must be substituted by $O(T)$.

Using this formula we improve the remainder term in the asymptotics of the counting function of the resonance set $N_\rho(\lambda)$. We prove that

$$N_\rho(T) = \frac{|M|}{4\pi}T^2 + R(T),$$

where

$$R(T) = O(T^{3/2+\varepsilon})$$

and, if the real parts of all resonances are bounded, then

$$R(T) = O(T^{1+\varepsilon}),$$

$\varepsilon > 0$ being arbitrary.