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Lower Bound on the Density of States for Periodic Schrödinger Operators

Sergey Morozov, Leonid Parnovski, and Irina Pchelintseva

Abstract. We consider Schrödinger operators $-\Delta + V$ in $\mathbb{R}^d$ ($d \geqslant 2$) with smooth periodic potentials V, and obtain a uniform lower bound on the (non-integrated) density of states for large values of the spectral parameter. It turns out that as the spectral parameter increases, the density of states approaches the one for $-\Delta$.

1. Introduction

Let $H = -\Delta + V$ be a Schrödinger operator in $L_2(\mathbb{R}^d)$ with a smooth periodic potential V. We will assume throughout that $d \geqslant 2$. The *integrated density of states (IDS)* for H is defined as

$$N(\lambda) := \lim_{L\to\infty} L^{-d} N(\lambda; H_D^{(L)}), \quad \lambda \in \mathbb{R}. \tag{1.1}$$

Here $H_D^{(L)}$ is the restriction of H to the cube $[0, L]^d$ with the Dirichlet boundary conditions, and $N(\lambda; \cdot)$ is the counting function of the discrete spectrum below λ. The existence of the limit in (1.1) is well known, see e.g. [**RS78, Shu79**]. For $H_0 := -\Delta$ the IDS can be easily computed explicitly (e.g. using the representation (2.6) below):

$$N_0(\lambda) = \begin{cases} (2\pi)^{-d} d^{-1} \omega_d \lambda^{d/2}, & \lambda > 0; \\ 0, & \lambda \leqslant 0. \end{cases} \tag{1.2}$$

Here $\omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere S^{d-1} in $\mathbb{R}^d$.

The asymptotic behaviour of the function (1.1) for large values of the spectral parameter was recently studied in a number of publications, see [**Skr87**], [**Kar97**], [**HM98**], [**Kar00**], [**Sob05**], [**PS09**], and references therein.

Our article concerns the high–energy behaviour of the Radon–Nikodym derivative of IDS

$$g := \mathrm{d}N/\,\mathrm{d}\lambda,$$

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which is called the *density of states (DOS)* (see [**RS78**]). Our main result is that for big values of λ

$$g(\lambda) \geqslant g_0(\lambda)\big(1 - o(1)\big), \tag{1.3}$$

where

$$g_0(\lambda) = \mathrm{d}N_0(\lambda)/\,\mathrm{d}\lambda = (2\pi)^{-d}\omega_d\lambda^{(d-2)/2}/2.$$

We remark that (1.3) should be understood in the sense of measures; in particular, we do not claim that $g(\lambda)$ is everywhere differentiable.

It has been proved in [**Par08**] that the spectrum of H contains a semi-axis $[\lambda_0, +\infty)$; this statement is known as the Bethe-Sommerfeld conjecture (see the references in [**Par08**] for the history of this problem). This result has an obvious reformulation in terms of IDS: each point $\lambda \geqslant \lambda_0$ is a point of growth of N. It was also proved in [**Par08**] that for each $n \in \mathbb{N}$ and $\varepsilon = \lambda^{-n}$ we have

$$N(\lambda + \varepsilon) - N(\lambda) \ll \varepsilon\lambda^{(d-2)/2}. \tag{1.4}$$

Later, when the second author discussed the results and methods of [**Par08**] with Yu. Karpeshina, she suggested that using the technique from that paper, one should be able to prove the opposite bound

$$N(\lambda + \varepsilon) - N(\lambda) \gg \varepsilon\lambda^{(d-2)/2} \tag{1.5}$$

when λ is sufficiently large, not just with $\varepsilon = \lambda^{-n}$ (when the proof is relatively straightforward given [**Par08**]), but also uniformly over all $\varepsilon \in (0, 1]$. In our paper we prove that for big λ

$$N(\lambda + \varepsilon) - N(\lambda) \geqslant \frac{\omega_d}{2(2\pi)^d}\varepsilon\lambda^{(d-2)/2}\big(1 - o(1)\big). \tag{1.6}$$

Note that (1.6) implies the claimed bound (1.3).

The main result of the paper is

Theorem 1.1. *For sufficiently big λ and any $\varepsilon > 0$ the integrated density of states of H satisfies* (1.6).

The proof of Theorem 1.1 is heavily based on the technique of [**Par08**] and uses various statements proved therein. In order to minimise the size of the paper, we will try to quote as many results as we can from [**Par08**], possibly with some minor modifications where necessary.

The article is organized as follows. In Section 2 we introduce the necessary notation and quote the results of [**Par08**] which we need for the proof of Theorem 1.1. Sections 3 and 4 contain some auxiliary results, and the proof is finished in Section 5.

2. Preliminaries

We study the Schrödinger operator

$$H = -\Delta + V(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d \tag{2.1}$$

with the potential V being infinitely smooth and periodic with the lattice of periods Λ. We denote the lattice dual to Λ by $\Lambda^\dagger$, fundamental cells of these lattices are denoted by Ω and $\Omega^\dagger$, respectively. We introduce

$$Q := \sup\big\{|\boldsymbol{\xi}|\big|\boldsymbol{\xi} \in \Omega^\dagger\big\}. \tag{2.2}$$

Let

$$\mathbf{D} := -i\nabla, \quad \mathbf{D}(\mathbf{k}) := \mathbf{D} + \mathbf{k}. \tag{2.3}$$

The Floquet-Bloch decomposition allows to represent our operator (2.1) as a direct integral (see e.g. [**RS78**]):

$$H = \int_{\Omega^\dagger} \oplus H(\mathbf{k})\, \mathrm{d}\mathbf{k}, \tag{2.4}$$

where

$$H(\mathbf{k}) = \mathbf{D}(\mathbf{k})^2 + V(\mathbf{x}) \tag{2.5}$$

is the family of 'fibre' operators acting in $L_2(\Omega)$. The domain $\mathfrak{D}$ of each $H(\mathbf{k})$ is the set of periodic functions from $H^2(\Omega)$. The spectrum of H is the union over $\mathbf{k} \in \Omega^\dagger$ of the spectra of the operators (2.5). Let $\{\lambda_j(\mathbf{k})\}$, $j \in \mathbb{N}$ be the set of eigenvalues of $H(\mathbf{k})$ (counting with multiplicities). Then the integrated density of states (1.1) admits the following representation:

$$N(\lambda) := (2\pi)^{-d} \int_{\Omega^\dagger} \#\big\{j :\, \lambda_j(\mathbf{k}) < \lambda\big\}\, \mathrm{d}\mathbf{k}, \tag{2.6}$$

see e.g. [**RS78**].

We denote by $|\cdot|_\circ$ the surface area Lebesgue measure on the unit sphere S^{d-1} in $\mathbb{R}^d$ and put $\omega_d := |S^{d-1}|_\circ = 2\pi^{d/2}/\Gamma(d/2)$. By $\mathrm{vol}(\cdot)$ we denote the Lebesgue measure in $\mathbb{R}^d$. We write $B(R)$ for the ball of radius R centered at the origin. The identity matrix is denoted by $\mathbf{I}$. By $\lambda = \rho^2$ we denote a point on the spectral axis. We also denote by v the L_∞–norm of the potential V, and put $J := [\lambda - 20v, \lambda + 20v]$.

Any vector $\boldsymbol{\xi} \in \mathbb{R}^d$ can be uniquely decomposed as $\boldsymbol{\xi} = \mathbf{n} + \mathbf{k}$ with $\mathbf{n} \in \Lambda^\dagger$ and $\mathbf{k} \in \Omega^\dagger$. We call $\mathbf{n} =: [\boldsymbol{\xi}]$ the 'integer part' of $\boldsymbol{\xi}$ and $\mathbf{k} =: \{\boldsymbol{\xi}\}$ the 'fractional part' of $\boldsymbol{\xi}$. For $\boldsymbol{\xi} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ we define $r = r(\boldsymbol{\xi}) := |\boldsymbol{\xi}|$ and $\boldsymbol{\xi}' := \boldsymbol{\xi}/|\boldsymbol{\xi}|$. For any $h \in L_2(\Omega)$ we introduce its Fourier coefficients

$$h_{\mathbf{n}} := (\mathrm{vol}\,\Omega)^{-1/2} \int_\Omega h(\mathbf{x}) \exp\big(-i\langle \mathbf{n}, \mathbf{x}\rangle\big)\, \mathrm{d}\mathbf{x}, \quad \mathbf{n} \in \Lambda^\dagger. \tag{2.7}$$

Given two positive functions f and g, we say that $f \gg g$, or $g \ll f$, or $g = O(f)$ if the ratio g/f is bounded. We say $f \asymp g$ if $f \gg g$ and $f \ll g$. Whenever we use O, o, $\gg$, $\ll$, or $\asymp$ notation, the constants involved can depend on d and norms of the potential in various Sobolev spaces H^s; the same is also the case when we use the expression 'sufficiently large'.

Let

$$\mathcal{A} := \Big\{\boldsymbol{\xi} \in \mathbb{R}^d,\ \big||\boldsymbol{\xi}|^2 - \lambda\big| \leqslant 40v\Big\}. \tag{2.8}$$

Notice that the definition of $\mathcal{A}$ obviously implies that if $\boldsymbol{\xi} \in \mathcal{A}$, then $\big||\boldsymbol{\xi}| - \rho\big| \ll \rho^{-1}$. We put

$$R = R(\rho) := \rho^{1/(36d^2(d+2))} \tag{2.9}$$

(so that the condition stated after equation (5.15) in [**Par08**] is satisfied). For $j \in \mathbb{N}$ let

$$\Theta'_j := \Lambda^\dagger \cap B(jR) \setminus \{\mathbf{0}\}.$$

Let $M := 5d^2 + 7d$. We introduce the set

$$\mathcal{B} := \Big\{\boldsymbol{\xi} \in \mathcal{A} \Big| \big|\langle \boldsymbol{\xi}, \boldsymbol{\eta}'\rangle\big| > \rho^{1/2},\ \text{for all } \boldsymbol{\eta} \in \Theta'_{6M}\Big\}. \tag{2.10}$$

In other words, $\mathcal{B}$ consists of all points $\boldsymbol{\xi} \in \mathcal{A}$ the projections of which to the directions of all vectors $\boldsymbol{\eta} \in \Theta'_{6M}$ have lengths larger than $\rho^{1/2}$. We also denote $\mathcal{D} := \mathcal{A} \setminus \mathcal{B}$.

In the rest of the section we quote some results from [**Par08**] which will be used in this paper. Our approach is slightly different from that of [**Par08**]. In particular, we consider arbitrary lattice of periods Λ, not equal to $(2\pi\mathbb{Z})^d$. We also use a different form of the Floquet-Bloch decomposition (so that the operators on fibers (2.5) are defined on the same domain). This leads to several straightforward changes in the formulation of the results from [**Par08**]. These changes are:

(1) The lattices $(2\pi\mathbb{Z})^d$ and $\mathbb{Z}^d$ are replaced by Λ and $\Lambda^\dagger$, respectively. The 'integer' and 'fractional' parts are now defined with respect to $\Lambda^\dagger$ (see above);
(2) The matrices $\mathbf{F}$ and $\mathbf{G}$ are replaced by the unit matrix $\mathbf{I}$ throughout;
(3) The Fourier transform is now defined by (2.7), and the exponentials $e_{\mathbf{m}}$ introduced at the beginning of Section 5 in [**Par08**] are redefined as

$$e_{\mathbf{m}}(\mathbf{x}) := (\operatorname{vol}\Omega)^{-1/2}\, \mathrm{e}^{i\langle \mathbf{m}, \mathbf{x}\rangle}, \quad \mathbf{m} \in \Lambda^\dagger;$$

(4) The operators $H(\mathbf{k})$ are now given by (2.5) on the common domain $\mathfrak{D}$.

The main result we will need follows from Corollary 7.15 of [**Par08**]:

Proposition 2.1. *There exist mappings $f, g : \mathcal{A} \to \mathbb{R}$ which satisfy the following properties:*

(i) $f(\boldsymbol{\xi})$ is an eigenvalue of $H(\mathbf{k})$ with $\{\boldsymbol{\xi}\} = \mathbf{k}$; $\big|f(\boldsymbol{\xi}) - |\boldsymbol{\xi}|^2\big| \leqslant 2v$. f is an injection (if we count all eigenvalues with multiplicities) and all eigenvalues of $H(\mathbf{k})$ inside J are in the image of f.

(ii) If $\boldsymbol{\xi} \in \mathcal{A}$, then $\big|f(\boldsymbol{\xi}) - g(\boldsymbol{\xi})\big| \leqslant \rho^{-d-3}$.

(iii) For any $\boldsymbol{\xi} \in \mathcal{B}$

$$\begin{aligned} g(\boldsymbol{\xi}) &= |\boldsymbol{\xi}|^2 \\ &+ \sum_{j=1}^{2M} \sum_{\boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_j \in \Theta'_M} \sum_{2 \leqslant n_1 + \dots + n_j \leqslant 2M} C_{n_1,\dots,n_j} \langle \boldsymbol{\xi}, \boldsymbol{\eta}_1\rangle^{-n_1} \dots \langle \boldsymbol{\xi}, \boldsymbol{\eta}_j\rangle^{-n_j}. \end{aligned} \tag{2.11}$$

Remark 2.2. Formula (2.11) implies that

$$\partial g / \partial r(\boldsymbol{\xi}) \asymp \rho, \quad \text{for any} \quad \boldsymbol{\xi} \in \mathcal{B}. \tag{2.12}$$

For each positive $\delta \leqslant v$ we denote by $\mathcal{A}(\delta)$, $\mathcal{B}(\delta)$, and $\mathcal{D}(\delta)$ the intersections of $g^{-1}\big([\rho^2 - \delta, \rho^2 + \delta]\big)$ with $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$, respectively.

It is proved in Lemma 8.1 of [**Par08**] that

$$\operatorname{vol}\big(\mathcal{D}(\delta)\big) \ll \rho^{d-7/3}\delta. \tag{2.13}$$

The following statement (Corollary 8.5 of [**Par08**]) gives a sufficient condition for the continuity of f:

Lemma 2.3. *There exists a constant C_1 with the following properties. Let*

$$I := \big\{\boldsymbol{\xi}(t) : t \in [t_{min}, t_{max}]\big\} \subset \mathcal{B}(v).$$

be a straight interval of length $L < \rho^{-1}\delta$. Suppose that there is a point $t_0 \in [t_{min}, t_{max}]$ with the property that for each non-zero $\mathbf{n} \in \Lambda^\dagger$ $g\big(\boldsymbol{\xi}(t_0) + \mathbf{n}\big)$ is either outside the interval

$$\Big[g\big(\boldsymbol{\xi}(t_0)\big) - C_1\rho^{-d-3} - C_1\rho L,\ g\big(\boldsymbol{\xi}(t_0)\big) + C_1\rho^{-d-3} + C_1\rho L\Big]$$

or not defined. Then $f\big(\boldsymbol{\xi}(t)\big)$ *is a continuous function of* t.

By inspection of the proof of Lemma 8.3 of [**Par08**] we obtain

Lemma 2.4. *For large enough* ρ *and* $\delta < \rho^{-1}$ *the following estimates hold uniformly over* $\mathbf{a} \in \Lambda^\dagger \setminus \{\mathbf{0}\}$*: if* $d \geqslant 3$*,*

$$\operatorname{vol}\Big(\mathcal{B}(\delta) \cap \big(\mathcal{B}(\delta) + \mathbf{a}\big)\Big) \ll (\delta^2\rho^{d-3} + \delta\rho^{-d}); \tag{2.14}$$

if $d = 2$*,*

$$\operatorname{vol}\Big(\mathcal{B}(\delta) \cap \big(\mathcal{B}(\delta) + \mathbf{a}\big)\Big) \begin{cases} \ll \delta^{3/2}, & |\mathbf{a}| \leqslant 2\rho - 1, \\ \ll \delta^{3/2} + \delta\rho^{-2}, & \big||\mathbf{a}| - 2\rho\big| < 1, \\ = 0, & |\mathbf{a}| \geqslant 2\rho + 1. \end{cases} \tag{2.15}$$

3. Prevalence of regular directions

In this section we prove that for most directions $\boldsymbol{\xi}'$ the image of the function f of Proposition 2.1 is an isolated eigenvalue of $H\big(\{\boldsymbol{\xi}\}\big)$ continuously depending on $|\xi|$ if it belongs to a neighborhood of ρ^2.

Lemma 3.1. *For* ρ *big enough and*

$$0 < \delta \leqslant \rho^{-d-3}$$

there exists a set $\mathcal{F} = \mathcal{F}(\rho)$ *on the unit sphere* S^{d-1} *in* $\mathbb{R}^d$ *with*

$$|\mathcal{F}|_\circ \geqslant \omega_d\big(1 - o(1)\big) \tag{3.1}$$

such that $f(\boldsymbol{\xi})$ *is a simple eigenvalue of* $H\big(\{\boldsymbol{\xi}\}\big)$ *continuously depending on* $r := |\boldsymbol{\xi}|$ *for every* $\boldsymbol{\xi} = (r, \boldsymbol{\xi}') \in f^{-1}\big([\rho^2 - \delta, \rho^2 + \delta)\big)$ *with* $\boldsymbol{\xi}' := \boldsymbol{\xi}/|\boldsymbol{\xi}| \in \mathcal{F}$.

Proof. It is enough to consider $\delta := \rho^{-d-3}$. For each $\boldsymbol{\xi}' \in S^{d-1}$ let

$$I_{\boldsymbol{\xi}'}(\delta) := \{r\boldsymbol{\xi}', r > 0\} \cap \mathcal{B}(\delta). \tag{3.2}$$

Let $\mathcal{F}_1 := \big\{\boldsymbol{\xi}' \in S^{d-1} \big| I_{\boldsymbol{\xi}'}(\delta) \neq \emptyset,\ \overline{I_{\boldsymbol{\xi}'}(\delta)} \cap \mathcal{D}(\delta) = \emptyset\big\}$.

For any $\boldsymbol{\eta} \in \Theta'_{6M}$ the area of the set of points $\boldsymbol{\xi}' \in S^{d-1}$ satisfying

$$\big|\langle r\boldsymbol{\xi}', \boldsymbol{\eta}'\rangle\big| \leqslant \rho^{1/2}$$

is evidently $O(\rho^{-1/2})$ if $r \geqslant \rho/2$ (the latter is true for all $r\boldsymbol{\xi}' \in \mathcal{A}$). Since the number of elements in Θ'_{6M} is $O(R^d)$, by (2.9) and (2.10) we have

$$|S^{d-1} \setminus \mathcal{F}_1|_\circ = o(1). \tag{3.3}$$

By definition $\mathcal{B}(\delta) = \mathcal{B} \cap g^{-1}\big([\rho^2 - \delta, \rho^2 + \delta]\big)$, hence (2.12) implies that for big ρ the length $l_{\boldsymbol{\xi}'}(\delta)$ of $I_{\boldsymbol{\xi}'}(\delta)$ satisfies

$$l_{\boldsymbol{\xi}'}(\delta) \asymp \delta\rho^{-1}, \quad \boldsymbol{\xi}' \in \mathcal{F}_1. \tag{3.4}$$

Let

$$\mathcal{F} := \big\{\boldsymbol{\xi}' \in \mathcal{F}_1 \,\big|\, f \text{ is continuous on } I_{\boldsymbol{\xi}'}(\delta)\big\},$$

and

$$\mathcal{E}(\delta) := \big\{\boldsymbol{\xi} \in \mathcal{B}(\delta) \big| \boldsymbol{\xi}' \in \mathcal{F}_1 \setminus \mathcal{F}\big\}.$$

Lemma 2.3 tells us that for each point $\boldsymbol{\xi} \in \mathcal{E}(\delta)$ there is a non-zero vector $\mathbf{n} \in \Lambda^\dagger$ such that

$$\big|g(\boldsymbol{\xi} + \mathbf{n}) - g(\boldsymbol{\xi})\big| \leqslant C_1\big(\rho^{-d-3} + \rho l_{\boldsymbol{\xi}'}(\delta)\big) \ll (\rho^{-d-3} + \delta). \tag{3.5}$$

Since $\big|g(\boldsymbol{\xi})-\rho^2\big|\leqslant\delta$, this implies

$$\big|g(\boldsymbol{\xi}+\mathbf{n})-\rho^2\big|\leqslant C_2(\rho^{-d-3}+\delta)=:\delta_1\ll\rho^{-d-3}=\delta,$$

and thus $\boldsymbol{\xi}+\mathbf{n}\in\mathcal{A}(\delta_1)$; notice that $C_2>1$ and so $\delta_1>\delta$. Therefore, each point $\boldsymbol{\xi}\in\mathcal{E}(\delta)$ also belongs to the set $\big(\mathcal{A}(\delta_1)-\mathbf{n}\big)$ for a non-zero $\mathbf{n}\in\Lambda^\dagger$; obviously, $|\mathbf{n}|\ll\rho$. In other words,

$$\mathcal{E}(\delta)\subset\bigcup_{\mathbf{n}\in\Lambda^\dagger\cap B(C\rho),\,\mathbf{n}\neq 0}\big(\mathcal{A}(\delta_1)-\mathbf{n}\big)=\bigcup_{\mathbf{n}\neq 0}\big(\mathcal{B}(\delta_1)-\mathbf{n}\big)\cup\bigcup_{\mathbf{n}\neq 0}\big(\mathcal{D}(\delta_1)-\mathbf{n}\big). \tag{3.6}$$

To proceed further, we need more notation. Denote $\mathcal{D}_0(\delta_1)$ to be the set of all points $\boldsymbol{\nu}$ from $\mathcal{D}(\delta_1)$ for which there is no non-zero $\mathbf{n}\in\Lambda^\dagger$ satisfying $\boldsymbol{\nu}-\mathbf{n}\in\mathcal{B}(\delta)$; $\mathcal{D}_1(\delta_1)$ to be the set of all points $\boldsymbol{\nu}$ from $\mathcal{D}(\delta_1)$ for which there is a unique non-zero $\mathbf{n}\in\Lambda^\dagger$ satisfying $\boldsymbol{\nu}-\mathbf{n}\in\mathcal{B}(\delta)$; and $\mathcal{D}_2(\delta_1)$ to be the rest of the points from $\mathcal{D}(\delta_1)$ (i.e. $\mathcal{D}_2(\delta_1)$ consists of all points $\boldsymbol{\nu}$ from $\mathcal{D}(\delta_1)$ for which there exist at least two different non-zero vectors $\mathbf{n}_1,\mathbf{n}_2\in\Lambda^\dagger$ satisfying $\boldsymbol{\nu}-\mathbf{n}_j\in\mathcal{B}(\delta)$). Then Lemma 8.7 of [**Par08**] implies that we can rewrite (3.6) as

$$\mathcal{E}(\delta)\subset\bigcup_{\mathbf{n}\neq\mathbf{0}}\big(\mathcal{B}(\delta_1)-\mathbf{n}\big)\cup\bigcup_{\mathbf{n}\neq\mathbf{0}}\big(\mathcal{D}_1(\delta_1)-\mathbf{n}\big). \tag{3.7}$$

From this we conclude that

$$\mathcal{E}(\delta)\subset\bigcup_{\mathbf{n}\neq\mathbf{0}}\Big(\big(\mathcal{B}(\delta_1)-\mathbf{n}\big)\cap\mathcal{B}(\delta)\Big)\cup\bigcup_{\mathbf{n}\neq\mathbf{0}}\Big(\big(\mathcal{D}_1(\delta_1)-\mathbf{n}\big)\cap\mathcal{B}(\delta)\Big), \tag{3.8}$$

since $\mathcal{E}(\delta)\subset\mathcal{B}(\delta)$.

The definition of the set $\mathcal{D}_1(\delta_1)$ and (2.13) imply that

$$\begin{aligned}&\operatorname{vol}\left(\bigcup_{\mathbf{n}\neq\mathbf{0}}\Big(\big(\mathcal{D}_1(\delta_1)-\mathbf{n}\big)\cap\mathcal{B}(\delta)\Big)\right)\leqslant\operatorname{vol}\big(\mathcal{D}_1(\delta_1)\big)\\&\leqslant\operatorname{vol}\big(\mathcal{D}(\delta_1)\big)\ll\delta_1\rho^{d-7/3}\ll\delta\rho^{d-7/3}.\end{aligned} \tag{3.9}$$

For $d\geqslant 3$ Lemma 2.4, inequality $\delta<\delta_1$, and the fact that the union in (3.8) consists of no more than $C\rho^d$ terms imply

$$\operatorname{vol}\left(\bigcup_{\mathbf{n}\neq\mathbf{0}}\Big(\big(\mathcal{B}(\delta_1)-\mathbf{n}\big)\cap\mathcal{B}(\delta)\Big)\right)\ll\rho^d(\delta_1^2\rho^{d-3}+\delta_1\rho^{-d})\ll\delta(\rho^{d-6}+1). \tag{3.10}$$

For $d=2$ we obtain by Lemma 2.4

$$\begin{aligned}&\operatorname{vol}\left(\bigcup_{\mathbf{n}\in\Lambda^\dagger\setminus\{\mathbf{0}\}}\Big(\mathcal{B}(\delta)\cap\big(\mathcal{B}(\delta_1)+\mathbf{n}\big)\Big)\right)\\&\leqslant\sum_{\substack{\mathbf{n}\in\Lambda^\dagger\setminus\{\mathbf{0}\}\\|\mathbf{n}|\leqslant 2\rho-1}}\operatorname{vol}\Big(\mathcal{B}(\delta)\cap\big(\mathcal{B}(\delta_1)+\mathbf{n}\big)\Big)\\&+\sum_{\substack{\mathbf{n}\in\Lambda^\dagger\setminus\{\mathbf{0}\}\\||\mathbf{n}|-2\rho|<1}}\operatorname{vol}\Big(\mathcal{B}(\delta)\cap\big(\mathcal{B}(\delta_1)+\mathbf{n}\big)\Big)\\&\ll\delta_1^{3/2}\rho^2+\rho(\delta_1^{3/2}+\delta_1\rho^{-2})\ll\delta\rho^{-1/2},\end{aligned} \tag{3.11}$$

where we have used that

$$\#\Big\{\mathbf{n}\in\Lambda^\dagger\Big|\big||\mathbf{n}|-2\rho\big|<1\Big\}\ll\rho.$$

Applying (3.9), (3.10), and (3.11) to (3.8) we obtain for all $d \geqslant 2$

$$\operatorname{vol}\mathcal{E}(\delta) \ll \delta\rho^{d-7/3}. \tag{3.12}$$

By definition,

$$\mathcal{E}(\delta) = \bigcup_{\boldsymbol{\xi}' \in \mathcal{F}_1 \setminus \mathcal{F}} I_{\boldsymbol{\xi}'}(\delta).$$

Hence by (3.4)

$$|\mathcal{F}_1 \setminus \mathcal{F}|_\circ \ll \delta^{-1}\rho^{2-d} \operatorname{vol}\mathcal{E}(\delta). \tag{3.13}$$

Combining (3.12) and (3.13) we conclude that for big ρ

$$|\mathcal{F}_1 \setminus \mathcal{F}|_\circ = o(1). \tag{3.14}$$

We have

$$|S^{d-1} \setminus \mathcal{F}|_\circ = |S^{d-1} \setminus \mathcal{F}_1|_\circ + |\mathcal{F}_1 \setminus \mathcal{F}|_\circ. \tag{3.15}$$

Substituting (3.3) and (3.14) into (3.15) we obtain (3.1).

Now we notice that for every $\boldsymbol{\xi}' \in \mathcal{F}$ the interval $I_{\boldsymbol{\xi}'}(\delta)$ has the following property: for each point $\boldsymbol{\xi} \in I_{\boldsymbol{\xi}'}(\delta)$ and each non-zero vector $\mathbf{n} \in \Lambda^\dagger$ such that $\boldsymbol{\xi} + \mathbf{n} \in \mathcal{A}$ we have $\big|g(\boldsymbol{\xi}+\mathbf{n}) - g(\boldsymbol{\xi})\big| > 2\rho^{-d-3}$. This implies $f(\boldsymbol{\xi}+\mathbf{n}) - f(\boldsymbol{\xi}) \neq 0$. Therefore, $f(\boldsymbol{\xi})$ is a simple eigenvalue of $H\big(\{\boldsymbol{\xi}\}\big)$ for each $\boldsymbol{\xi} \in I_{\boldsymbol{\xi}'}(\delta)$. The lemma is proved. □

4. Some properties of operators on the fibers

In this section we discuss some properties of operators $H(\mathbf{k})$, $\mathbf{k} \in \Omega^\dagger$. In Lemma 4.1 we prove that the Fourier coefficients of the eigenfunctions of these operators satisfy certain decay estimates if the corresponding eigenvalues are big enough. Using this, we obtain an estimate on the rate of change of such eigenvalues with $\mathbf{k}$ in Lemma 4.2.

For $m \in \mathbb{R}$ let

$$V^{(m)} := \Big(\sum_{\mathbf{n} \in \Lambda^\dagger} |\mathbf{n}|^{2m} |V_{\mathbf{n}}|^2 \Big)^{1/2}.$$

Since V is smooth, $V^{(m)}$ is finite for any $m \geqslant 0$. Recall that Q is defined by (2.2).

Lemma 4.1. *Fix $m \in \mathbb{N}$ and $\varkappa \in (0,1)$. For $\mathbf{k} \in \Omega^\dagger$ let ψ be a normalized eigenfunction of $H(\mathbf{k})$:*

$$H(\mathbf{k})\psi = \zeta\psi \tag{4.1}$$

with the eigenvalue

$$\zeta \geqslant \max\big\{36Q^2\varkappa^{-2}, (1+m\varkappa)^{2/(d-1)}\varkappa^{-2d/(d-1)}\big\}. \tag{4.2}$$

Then there exists $M_m = M_m(d,\Lambda,V) \in \mathbb{R}_+$ such that for all $\mathbf{n} \in \Lambda^\dagger$ with

$$|\mathbf{n}| \geqslant (1+m\varkappa)\sqrt{\zeta} \tag{4.3}$$

the Fourier coefficients of ψ satisfy

$$|\psi_{\mathbf{n}}| < M_m \varkappa^{-m} |\mathbf{n}|^{-(3m+1)/2}. \tag{4.4}$$

Proof. We proceed by induction. Suppose that either $m = 1$, or $m > 1$ and the statement is proved for $m - 1$. Substituting the Fourier series

$$\psi(\mathbf{x}) = (\operatorname{vol}\Omega)^{-1/2} \sum_{\mathbf{n}\in\Lambda^\dagger} \psi_{\mathbf{n}} \exp\bigl(i\langle \mathbf{n}, \mathbf{x}\rangle\bigr), \quad \mathbf{x} \in \Omega$$

into (4.1) and equating the coefficients at $\exp\bigl(i\langle \mathbf{n}, \mathbf{x}\rangle\bigr)$ on both sides, we obtain by (2.5):

$$|\mathbf{n} + \mathbf{k}|^2 \psi_{\mathbf{n}} + \sum_{\mathbf{l}\in\Lambda^\dagger} V_{\mathbf{n}-\mathbf{l}} \psi_{\mathbf{l}} = \zeta\psi_{\mathbf{n}}. \tag{4.5}$$

Since $|\mathbf{k}| \leqslant Q$, by (4.2) and (4.3) we have

$$2|\mathbf{n}||\mathbf{k}| \leqslant \varkappa|\mathbf{n}|^2/6 + 6\varkappa^{-1}Q^2 \leqslant \varkappa|\mathbf{n}|^2/3. \tag{4.6}$$

For $\varkappa \in (0, 1)$, it follows from (4.3) that

$$|\mathbf{n}|^2 - \zeta \geqslant \bigl(1 - (1 + \varkappa)^{-2}\bigr)|\mathbf{n}|^2 = \varkappa(2 + \varkappa)(1 + \varkappa)^{-2}|\mathbf{n}|^2 \geqslant \varkappa|\mathbf{n}|^2/2. \tag{4.7}$$

Combining (4.6) and (4.7) we obtain

$$|\mathbf{n} + \mathbf{k}|^2 - \zeta \geqslant |\mathbf{n}|^2 - 2|\mathbf{n}||\mathbf{k}| - \zeta \geqslant \varkappa|\mathbf{n}|^2/6,$$

and thus by (4.5)

$$|\psi_{\mathbf{n}}| < 6\varkappa^{-1}|\mathbf{n}|^{-2} \sum_{\mathbf{l}\in\Lambda^\dagger} |V_{\mathbf{n}-\mathbf{l}}\psi_{\mathbf{l}}|. \tag{4.8}$$

If $m = 1$ we estimate the sum on the r. h. s. by $V^{(0)}$ using Cauchy–Schwarz inequality (since ψ is normalized) and obtain (4.4) with $M_1 := 6V^{(0)}$.

If $m > 1$, we estimate

$$\sum_{\mathbf{l}\in\Lambda^\dagger\colon |\mathbf{l}-\mathbf{n}|\leqslant|\mathbf{n}|^{1/d}} |V_{\mathbf{n}-\mathbf{l}}\psi_{\mathbf{l}}| \leqslant \sup_{\mathbf{m}\colon |\mathbf{m}|\geqslant|\mathbf{n}|-|\mathbf{n}|^{1/d}} |\psi_{\mathbf{m}}| \sum_{\mathbf{l}\in\Lambda^\dagger\colon |\mathbf{l}|\leqslant|\mathbf{n}|^{1/d}} |V_{\mathbf{l}}|. \tag{4.9}$$

By (4.3), (4.2), and monotonicity of the function $q(t) = t - t^{1/d}$ for $t > 1$ we have

$$|\mathbf{n}| - |\mathbf{n}|^{1/d} \geqslant (1 + m\varkappa)\sqrt{\zeta} - \bigl((1 + m\varkappa)\sqrt{\zeta}\bigr)^{1/d} \geqslant \bigl(1 + (m - 1)\varkappa\bigr)\sqrt{\zeta}.$$

According to the induction hypothesis

$$\sup_{\mathbf{m}\colon |\mathbf{m}|\geqslant|\mathbf{n}|-|\mathbf{n}|^{1/d}} |\psi_{\mathbf{m}}| \leqslant \varkappa^{1-m} M_{m-1}\bigl(1 - |\mathbf{n}|^{(1-d)/d}\bigr)^{1-3m/2}|\mathbf{n}|^{1-3m/2}. \tag{4.10}$$

Since $\varkappa \in (0, 1)$, from (4.3) and (4.2) we conclude

$$|\mathbf{n}| \geqslant (1 + m\varkappa)\sqrt{\zeta} \geqslant (\varkappa^{-1} + m)^{d/(d-1)} > 2^{d/(d-1)},$$

hence

$$\bigl(1 - |\mathbf{n}|^{(1-d)/d}\bigr)^{1-3m/2} < 2^{3m/2-1}. \tag{4.11}$$

Let

$$W := \sup_{r>1} r^{-d}\#\bigl\{\mathbf{l} \in \Lambda^\dagger \bigm| |\mathbf{l}| \leqslant r\bigr\}.$$

Clearly, $W < \infty$. By Cauchy–Schwarz inequality

$$\sum_{\mathbf{l}\in\Lambda^\dagger\colon |\mathbf{l}|\leqslant|\mathbf{n}|^{1/d}} |V_{\mathbf{l}}| \leqslant W^{1/2}V^{(0)}|\mathbf{n}|^{1/2}. \tag{4.12}$$

Substituting (4.10), (4.11), and (4.12) into (4.9) we get

$$\sum_{\mathbf{l}\in\Lambda^\dagger:\,|\mathbf{l}-\mathbf{n}|\leqslant|\mathbf{n}|^{1/d}} |V_{\mathbf{n}-\mathbf{l}}\psi_{\mathbf{l}}| < 2^{3m/2-1}\varkappa^{1-m}W^{1/2}V^{(0)}M_{m-1}|\mathbf{n}|^{3(1-m)/2}. \tag{4.13}$$

On the other hand, since $\|\psi\|=1$, applying Cauchy–Schwarz inequality we obtain

$$\begin{aligned}\sum_{\mathbf{l}\in\Lambda^\dagger:\,|\mathbf{l}-\mathbf{n}|>|\mathbf{n}|^{1/d}} |V_{\mathbf{n}-\mathbf{l}}\psi_{\mathbf{l}}| &< |\mathbf{n}|^{3(1-m)/2}\sum_{\mathbf{l}\in\Lambda^\dagger:\,|\mathbf{l}|>|\mathbf{n}|^{1/d}} |\mathbf{l}|^{3(m-1)d/2}|V_{\mathbf{l}}||\psi_{\mathbf{n}-\mathbf{l}}| \\ &\leqslant |\mathbf{n}|^{3(1-m)/2}\Big(\sum_{\mathbf{l}\in\Lambda^\dagger:\,|\mathbf{l}|>|\mathbf{n}|^{1/d}} |\mathbf{l}|^{3(m-1)d}|V_{\mathbf{l}}|^2\Big)^{1/2} \leqslant V^{(3(m-1)d/2)}|\mathbf{n}|^{3(1-m)/2}.\end{aligned} \tag{4.14}$$

Inserting (4.13) and (4.14) into (4.8) we arrive at (4.4) with

$$M_m := 6(2^{3m/2-1}W^{1/2}V^{(0)}M_{m-1} + V^{(3(m-1)d/2)}).$$

□

Lemma 4.2. *For any $\eta\in(0,1)$ there exists $\zeta_0>0$ such that if $\zeta(\mathbf{k})\geqslant\zeta_0$ is a simple eigenvalue of $H(\mathbf{k})$ for some $\mathbf{k}\in\Omega^\dagger$ then*

$$|\nabla_{\mathbf{k}}\zeta| \leqslant 2(1+\eta)\sqrt{\zeta}. \tag{4.15}$$

Proof. Let $\psi(\mathbf{k})$ be the eigenfunction corresponding to $\zeta(\mathbf{k})$ with

$$\big\|\psi(\mathbf{k})\big\| = 1. \tag{4.16}$$

Then

$$\nabla_{\mathbf{k}}\zeta(\mathbf{k}) = \nabla_{\mathbf{k}}\big(\psi(\mathbf{k}), H(\mathbf{k})\psi(\mathbf{k})\big) = \Big(\psi(\mathbf{k}), \big(\nabla_{\mathbf{k}}H(\mathbf{k})\big)\psi(\mathbf{k})\Big). \tag{4.17}$$

By (2.5) and (2.3),

$$\nabla_{\mathbf{k}}H(\mathbf{k}) = 2\mathbf{D}(\mathbf{k}).$$

Substituting this into (4.17) we obtain:

$$\big|\nabla_{\mathbf{k}}\zeta(\mathbf{k})\big| \leqslant 2\big\|\mathbf{D}(\mathbf{k})\psi(\mathbf{k})\big\| = 2\Big(\sum_{\mathbf{n}\in\Lambda^\dagger}|\mathbf{n}+\mathbf{k}|^2\big|\psi_{\mathbf{n}}(\mathbf{k})\big|^2\Big)^{1/2}. \tag{4.18}$$

Let

$$m := \big[(d+1)/3\big]+1 \tag{4.19}$$

and

$$\varkappa := \eta/(2m+1). \tag{4.20}$$

We assume that

$$\zeta := \zeta(\mathbf{k}) \geqslant \max\big\{36Q^2\varkappa^{-2}, (1+m\varkappa)^{2/(d-1)}\varkappa^{-2d/(d-1)}\big\}. \tag{4.21}$$

Since by (2.2) $|\mathbf{k}|\leqslant Q$, by (4.16), (4.21), and (4.20) we have

$$\sum_{|\mathbf{n}|<(1+m\varkappa)\sqrt{\zeta}} |\mathbf{n}+\mathbf{k}|^2\big|\psi_{\mathbf{n}}(\mathbf{k})\big|^2 < \big(1+(m+1/6)\varkappa\big)^2\zeta < (1+\eta/2)^2\zeta. \tag{4.22}$$

For $|\mathbf{n}|\geqslant(1+m\varkappa)\sqrt{\zeta}$ we apply Lemma 4.1 obtaining

$$\sum_{|\mathbf{n}|\geqslant(1+m\varkappa)\sqrt{\zeta}} |\mathbf{n}+\mathbf{k}|^2\big|\psi_{\mathbf{n}}(\mathbf{k})\big|^2 \leqslant M_m^2\varkappa^{-2m}\sum_{|\mathbf{n}|\geqslant(1+m\varkappa)\sqrt{\zeta}} |\mathbf{n}+\mathbf{k}|^2|\mathbf{n}|^{-3m-1}. \tag{4.23}$$

By (4.19) the r. h. s. of (4.23) is finite and is $O(\zeta^{-1/2})$. Thus, choosing ζ_0 big enough, by (4.18), (4.22), and (4.23) we obtain (4.15). □

5. Proof of Theorem 1.1

We are now ready to finish the proof of the main result. It is enough to prove

Theorem 5.1. *For any* $\alpha \in (0,1)$ *there exists* $\rho_0 > 0$ *big enough such that for all* $\rho \geqslant \rho_0$

$$N(\rho^2+\delta) - N(\rho^2-\delta) \geqslant (1-\alpha)(2\pi)^{-d}\omega_d\delta\rho^{d-2} \tag{5.1}$$

for any

$$0 < \delta \leqslant \rho^{-d-3}. \tag{5.2}$$

Indeed, the original statement of Theorem 1.1 can be obtained by partitioning of the interval $[\lambda, \lambda+\varepsilon]$ into subintervals with lengths not exceeding $2\lambda^{(-d-3)/2}$ and adding up estimates (5.1) on these subintervals (with ρ^2 being respective middle points).

Proof. We first express the growth of IDS in terms of the function f of Proposition 2.1(i) using (2.6):

$$N(\rho^2+\delta) - N(\rho^2-\delta) = (2\pi)^{-d}\operatorname{vol}\big(f^{-1}[\rho^2-\delta, \rho^2+\delta)\big). \tag{5.3}$$

We can write

$$\operatorname{vol}\big(f^{-1}[\rho^2-\delta, \rho^2+\delta)\big) = \int_{S^{d-1}}\int_0^\infty \chi(r, \boldsymbol{\xi}')r^{d-1}\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\xi}', \tag{5.4}$$

where χ is the indicator function of $f^{-1}\big([\rho^2-\delta, \rho^2+\delta)\big)$. To obtain a lower bound we can restrict the integration in (5.4) to $\boldsymbol{\xi}' \in \mathcal{F}$ defined in Lemma 3.1. Then for any $\eta \in (0,1)$ there exists $\rho_0 > 0$ such that for any $\rho \geqslant \rho_0$ we have

$$|\mathcal{F}|_\circ \geqslant (1-\eta)\omega_d, \tag{5.5}$$

and for any $\boldsymbol{\xi}' \in \mathcal{F}$ the support of $\chi(\cdot, \boldsymbol{\xi}')$ contains an interval $[r_1, r_2]$ with

$$(1-\eta)\rho \leqslant r_1 < r_1 + (1-\eta)\rho^{-1}\delta \leqslant r_2. \tag{5.6}$$

Indeed, the first inequality in (5.6) follows from Proposition 2.1(ii),(iii). The last inequality in (5.6) follows from Lemmata 3.1 and 4.2.

Thus for all $\rho \geqslant \rho_0$ by (5.5) and (5.6) we obtain

$$\begin{aligned}\int_{S^{d-1}}\int_0^\infty \chi(r, \boldsymbol{\xi}')r^{d-1}\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\xi}' &\geqslant \int_{\mathcal{F}} (1-\eta)^d\rho^{d-2}\delta\,\mathrm{d}\boldsymbol{\xi}' \\ &\geqslant (1-\eta)^{d+1}\omega_d\rho^{d-2}\delta,\end{aligned} \tag{5.7}$$

Combining (5.3), (5.4), and (5.7), and choosing η small enough we arrive at (5.1). The theorem is proved. □

References

[HM98] Bernard Helffer and Abderemane Mohamed, *Asymptotic of the density of states for the Schrödinger operator with periodic electric potential*, Duke Math. J. **92** (1998), no. 1, 1–60. MR MR1609321 (99e:35166)

[Kar97] Yulia E. Karpeshina, *Perturbation theory for the Schrödinger operator with a periodic potential*, Lecture Notes in Mathematics, vol. 1663, Springer-Verlag, Berlin, 1997. MR MR1472485 (2000i:35002)

[Kar00] ———, *On the density of states for the periodic Schrödinger operator*, Ark. Mat. **38** (2000), no. 1, 111–137. MR MR1749362 (2001g:47088)

[Par08] Leonid Parnovski, *Bethe-Sommerfeld conjecture*, Ann. Henri Poincaré **9** (2008), no. 3, 457–508. MR MR2419769

[PS09] Leonid Parnovski and Roman Shterenberg, *Asymptotic expansion of the integrated density of states of a two-dimensional periodic Schrödinger operator*, Invent. Math. **176** (2009), no. 2, 275–323.

[RS78] Michael Reed and Barry Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978. MR MR0493421 (58 #12429c)

[Shu79] M. A. Shubin, *Spectral theory and the index of elliptic operators with almost-periodic coefficients*, Uspekhi Mat. Nauk **34** (1979), no. 2(206), 95–135. MR MR535710 (81f:35090)

[Skr87] M. M. Skriganov, *Geometric and arithmetic methods in the spectral theory of multidimensional periodic operators*, Proc. Steklov Inst. Math. (1987), no. 2(171), vi+121, A translation of Trudy Mat. Inst. Steklov. **1**71 (1985); MR0798454 (87h:47110). MR MR905202 (88g:47038)

[Sob05] Alexander V. Sobolev, *Integrated density of states for the periodic Schrödinger operator in dimension two*, Ann. Henri Poincaré **6** (2005), no. 1, 31–84. MR MR2119355 (2006b:35056)

Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, UK

E-mail address: `morozov@math.ucl.ac.uk`

E-mail address: `leonid@math.ucl.ac.uk`

E-mail address: `i.pchelintseva@ucl.ac.uk`