Spectral Theory

Homework 7

1. Let $g \in C([0,1])$ be a given function. Consider the operator $A \in \mathcal{B}(L^2([0,1]))$ defined by the formula

$$(Au)(s) = g(s)u(s), \quad s \in [0,1].$$

Find the operator $A^*$. Under what condition on $g$ is the operator $A$ self-adjoint?

2. Let $k \in C([0,1] \times [0,1])$ be a given function. Consider the operator $B \in \mathcal{B}(L^2([0,1]))$ defined by the formula

$$(Bu)(s) = \int_0^1 k(s,t)u(t)dt, \quad s \in [0,1].$$

Find the operator $B^*$. Under what condition on $k$ is the operator $B$ self-adjoint?

3. Let $B$ be defined by

$$(Bf)(t) = tf(1-t^3), \quad \forall f \in L^2([0,1]), \quad \forall t \in [0,1].$$

Prove that $B \in \mathcal{B}(L^2([0,1]))$ and find $B^*$, $BB^*$ and $B^*B$.

4. Find the numerical range of the operator $R : l^2 \to l^2$ defined by

$$Rx = (0, x_1, x_2, \ldots), \quad x = (x_1, x_2, \ldots) \in l^2.$$