

## Spectral Theory

### Homework 7

1. Let  $g \in C([0, 1])$  be a given function. Consider the operator  $A \in \mathcal{B}(L_2([0, 1]))$  defined by the formula

$$(Au)(s) = g(s)u(s), \quad s \in [0, 1].$$

Find the operator  $A^*$ . Under what condition on  $g$  is the operator  $A$  self-adjoint?

2. Let  $k \in C([0, 1] \times [0, 1])$  be a given function. Consider the operator  $B \in \mathcal{B}(L_2([0, 1]))$  defined by the formula

$$(Bu)(s) = \int_0^1 k(s, t)u(t)dt, \quad s \in [0, 1].$$

Find the operator  $B^*$ . Under what condition on  $k$  is the operator  $B$  self-adjoint?

3. Let  $B$  be defined by

$$(Bf)(t) = tf(1 - t^3), \quad \forall f \in L_2([0, 1]), \quad \forall t \in [0, 1].$$

Prove that  $B \in \mathcal{B}(L_2([0, 1]))$  and find  $B^*$ ,  $BB^*$  and  $B^*B$ .

4. Find the numerical range of the operator  $R : l^2 \rightarrow l^2$  defined by

$$Rx = (0, x_1, x_2, \dots), \quad x = (x_1, x_2, \dots) \in l^2.$$

5. Let  $P$  be a non-trivial orthogonal projection ( $P \neq 0, I$ ). Find its numerical range.